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ON THE PRODUCTS OF COMMUTATORS OF REAL J-SYMMETRIES[*]

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Abstract. Let $J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. A $2n$-by-$2n$ complex matrix A is said to be *symplectic* if $A^T JA = J$. If A is symplectic and $\text{rank}(A-I) = 1$, then A is called a *J-symmetry.* It is known that every $2n$-by-$2n$ complex symplectic matrix can be written as a product of $n+1$ commutators of J-symmetries. We consider the real case and study the properties of real J-symmetries and commutators of real J-symmetries. We prove that if A is a $2n$-by-$2n$ real symplectic matrix, with $\text{rank}(A-I) = m$, then A is a product of $\frac{3m}{2} - 2\lfloor \frac{m}{4} \rfloor$ commutators of real J-symmetries if $J(A-I)$ is skew-symmetric, and A is a product of $3\lceil \frac{m}{2} \rceil$ commutators of real J-symmetries if $J(A-I)$ is not skew-symmetric.

Key words. Commutators, Symplectic, J-symmetries, Householder, Matrix decompositions.

AMS subject classifications. 15A21, 15A23, 15B10.

1. Introduction.

Let $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, where I_n is the n-by-n identity matrix. An $A \in M_{2n}(\mathbb{C})$ is said to be *symplectic* if $A^T JA = J$. If A is symplectic and $\text{rank}(A-I) = 1$, then A is called a *J-symmetry*. It can be shown that a matrix A is a J-symmetry if and only if A is a *J-Householder matrix*, that is, $A = I + uu^T J$ for some nonzero vector $u \in \mathbb{F}^{2n}$. If there are J-symmetries X and Y such that $A = XYX^{-1}Y^{-1}$, then A is said to be a *commutator* of J-symmetries.

Complex J-symmetries and their commutators have been studied extensively. In [5], Dela Rosa, Merino, and Paras presented the properties of J-Householder matrices and the possible Jordan Canonical Forms of matrices that can be written as products of two J-Householder matrices. They also proved that every complex symplectic matrix can be written as a product of J-Householder matrices by showing that each symplectic operation matrix can be written as a product of three J-Householder matrices. In [4], de la Cruz, Dela Rosa, Merino, and Paras proved that a complex symplectic matrix is a J-symmetry if and only if it is a J-Householder matrix. In [3], de la Cruz and Dela Rosa presented all the possible Jordan Canonical Forms of commutators of complex J-symmetries. They also proved that every $2n$-by-$2n$ symplectic matrix can be written as a product of commutators of J-symmetries, and the number of factors needed is $n+1$.

Factorizations of symplectic matrices as products of matrices with special forms or properties have also been studied. In [2], de la Cruz proved that each complex symplectic matrix of size greater than two can be written as a product of four symplectic involutions (that is, a symplectic matrix A such that $A^2 = I$). In [1], where the real case is considered, Awa and de la Cruz proved that every 4-by-4 real symplectic matrix is a product of four real symplectic involutions; while in general, every $2n$-by-$2n$ real symplectic matrix is a product of a finite number of real symplectic involutions. In their discussion, they have also shown that

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while in the complex case, a symplectic matrix A and its inverse A^{-1} are similar if and only if they are symplectically similar, this property is no longer true in the real case. Under real symplectic similarity, a real symplectic matrix A is real symplectically similar to A^{-1} if and only if A is a product of two *real* symplectic involutions. The authors also presented and discussed some canonical forms for 2-by-2 and 4-by-4 symplectic matrices under real symplectic similarity. In the discussion of 2-by-2 real symplectic similarity, the authors also introduced the real symplectic similarity of a 2-by-2 nonscalar real symplectic matrix B to $L_{B,\mu} = \begin{bmatrix} \mathrm{tr}\ B & \mu \\ -\mu & 0 \end{bmatrix}$, where $\mu \in \{-1, 1\}$.

In [7] and [8], Hou discussed the decomposition of symplectic matrices into products of commutators of symplectic involutions. In [7], it is proved that every complex symplectic matrix of size greater than 2 can be written as a product of three commutators of symplectic involutions. An analogous result is obtained for the 4-by-4 real case in [8]. This result is then used to show that real symplectic matrices of larger sizes can be written as a product of a finite number of commutators of real symplectic involutions. In the process of proving the 4-by-4 case, Hou also improved the discussion of the $L_{B,\mu}$ mapping by giving a stronger result for the cases where $-2 \leq \mathrm{tr}\ B < 2$.

In this paper, we prove an analog of the characterization of J-symmetries of de la Cruz and Dela Rosa [3] for the real setting (see Lemma 2.8). We do this in Chapter 2 alongside proving some preliminary results. We then prove the following result, which is the main goal of this work, in Chapter 3.

THEOREM 1.1. *Let P be a $2n$-by-$2n$ real symplectic matrix, and let $m = \mathrm{rank}(P - I_{2n})$.*

(a) If $J(P-I_{2n})$ is skew-symmetric, then P is a product of $\frac{3m}{2} - 2\lfloor \frac{m}{4} \rfloor$ commutators of real J-symmetries.
(b) If $J(P-I_{2n})$ is not skew-symmetric, then P is a product of $3\lceil \frac{m}{2} \rceil$ commutators of real J-symmetries.

2. Preliminaries.

We develop some preliminary results in this chapter before we prove our main result in Chapter 3.

When working with symplectic matrices, it helps to make use of their special block structure. Observe that if we write $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, where each $A_i \in M_n(\mathbb{R})$, then A is symplectic if and only if $A_3A_4^T$, $A_1A_2^T$ are symmetric, and $A_1A_4^T - A_2A_3^T = I_n$. Thus, if $B \in M_n(\mathbb{C})$ is nonsingular, then $B \oplus B^{-T}$ is symplectic. Also, the matrix $\begin{bmatrix} I_n & X \\ 0 & I_n \end{bmatrix}$ is symplectic if and only if X is symmetric. The previous examples will be used thoroughly in the paper. Note that if A is symplectic, then so are A^T, $-A$, and A^{-1}, where $A^{-1} = \begin{bmatrix} A_4^T & -A_2^T \\ -A_3^T & A_1^T \end{bmatrix}$. If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we denote the set of $2n$-by-$2n$ symplectic matrices with entries in $\mathbb{F}$ by $\mathrm{Sp}(2n, \mathbb{F})$. It is known that $\mathrm{Sp}(2n, \mathbb{F})$ is a subgroup of the special linear group of order $2n$.

The direct sum of orthogonal matrices is orthogonal, but the same property does not hold for symplectic matrices. We define a different operation that preserves the property of being symplectic. If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, where $A_{ij} \in M_n(\mathbb{R})$ and $B_{ij} \in M_m(\mathbb{R})$ for $i, j \in \{1, 2\}$, the *expanding sum* of A and B is defined as

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$$A \boxplus B = \begin{bmatrix} A_{11} \oplus B_{11} & A_{12} \oplus B_{12} \\ A_{21} \oplus B_{21} & A_{22} \oplus B_{22} \end{bmatrix}.$$

Note that if $A, C \in M_{2m}(\mathbb{R})$ and $B, D \in M_{2n}(\mathbb{R})$, then $(A \boxplus B)(C \boxplus D) = (AC) \boxplus (BD)$. Also, $A \oplus B$ is similar to $A \boxplus B$.

Symplectic similarity also plays a central role in this work. We say that a symplectic matrix A is *(complex) symplectically similar* to B if there exists a (complex) symplectic matrix P for which $B = PAP^{-1}$. If we can take P to be a real symplectic matrix, then we say that A is *real symplectically similar* to B.

If $A \in M_{2n}(\mathbb{C})$ is a J-symmetry, then A is symplectically similar to $J_2(1) \boxplus I_{2n-2}$ (see [5]). Hence, any two complex J-symmetries are symplectically similar. In the following, we show that the real J-symmetry $J_2(1) \boxplus I_{2n-2}$ is not real symplectically similar to the real J-symmetry $J_2(1)^{-1} \boxplus I_{2n-2}$.

LEMMA 2.1. *Let n be a positive integer. Then $J_2(1) \boxplus I_{2n-2}$ is not real symplectically similar to $J_2(1)^{-1} \boxplus I_{2n-2}$.*

Proof. Suppose there exists a real symplectic matrix $P = \left[p_{(i,j)}\right] \in \mathrm{Sp}(2n, \mathbb{R})$ such that $P(J_2(1) \boxplus I_{2n-2})P^{-1} = J_2(1)^{-1} \boxplus I_{2n-2}$. If we let $E_{(i,j)}$ be the matrix whose (i,j)-entry is 1 and the rest are 0, then we have $J_2(1) \boxplus I_{2n-2} = I_{2n} + E_{(1,n+1)}$ and $J_2(1)^{-1} \boxplus I_{2n-2} = I_{2n} - E_{(1,n+1)}$. Thus,

$$\begin{aligned} P\left(J_2(1) \boxplus I_{2n-2}\right) &= \left(J_2(1)^{-1} \boxplus I_{2n-2}\right) P \\ P\left(I_{2n} + E_{(1,n+1)}\right) &= \left(I_{2n} - E_{(1,n+1)}\right) P \\ P + PE_{(1,n+1)} &= P - E_{(1,n+1)}P \\ PE_{(1,n+1)} &= -E_{(1,n+1)}P. \end{aligned}$$

Note that the only nonzero column of the matrix product on the left is its $(n+1)$-th column, which is a copy of the first column of P. On the other hand, the only nonzero row of the matrix product on the right is its first row, which is the negative of the $(n+1)$-th row of P. Hence, we must have the following: $p_{(n+1,s)} = 0$ for $s \neq (n+1)$, $p_{(t,1)} = 0$ for $t \neq 1$, and $p_{(n+1,n+1)} = -p_{(1,1)}$.

Write $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$, where $P_1, P_2, P_3, P_4 \in M_n(\mathbb{R})$. Since P is symplectic, we must have $P_1P_4^T - P_2P_3^T = I_n$. The $(1,1)$-entry of the preceding matrix equation gives us

$$\begin{aligned} 1 &= \sum_{r=1}^{n} p_{(1,r)}p_{(n+1,n+r)} - \sum_{r=1}^{n} p_{(1,n+r)}p_{(n+1,r)} \\ &= p_{(1,1)}p_{(n+1,n+1)} + \sum_{r=2}^{n} p_{(1,r)}p_{(n+1,n+r)} - \sum_{r=1}^{n} p_{(1,n+r)}p_{(n+1,r)} \\ &= p_{(1,1)}(-p_{(1,1)}) + \sum_{r=2}^{n} p_{(1,r)}(0) - \sum_{r=1}^{n} p_{(1,n+r)}(0) \\ &= -(p_{(1,1)})^2, \end{aligned}$$

which is not possible since P is a real symplectic matrix. Hence, the matrices $J_2(1) \boxplus I_{2n-2}$ and $J_2(1)^{-1} \boxplus I_{2n-2}$ are not real symplectically similar. ☐

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As a result of Lemma 2.1, there are at least two distinct classes of real J-symmetries under real symplectic similarity: those that are real symplectically similar to $\mathcal{J}_n := J_2(1) \boxplus I_{2n-2}$, and those that are real symplectically similar to $\mathcal{J}_n^{-1} := J_2(1)^{-1} \boxplus I_{2n-2}$. We will prove in Lemma 2.3 that every real J-symmetry belongs to one of these two classes. Henceforth, we let $\mathcal{S}_{\mathcal{J}_n}$ be the set of all real J-symmetries that are real symplectically similar to $\mathcal{J}_n$, and $\mathcal{S}_{\mathcal{J}_n^{-1}}$ be the set of all real J-symmetries that are real symplectically similar to $\mathcal{J}_n^{-1}$.

Let $0 \neq u \in \mathbb{C}^{2n}$ be given. The J-Householder matrix corresponding to u is $H_u := I_{2n} - uu^T J_{2n}$. By [4, Theorem 1.2], a complex symplectic matrix is a J-symmetry if and only if it is a J-Householder matrix. More properties of complex J-Householder matrices are presented in the following (see also [5, Lemma 2]):

Lemma 2.2. *Let $0 \neq u \in \mathbb{C}^{2n}$, and let H_u be the J-Householder matrix corresponding to u.*

(a) Then $H_u^{-1} = H_{iu}$.
(b) If $Q \in \mathrm{Sp}(2n, \mathbb{C})$, then $QH_uQ^{-1} = H_{Qu}$
(c) If H_v is a J-Householder matrix corresponding to some $0 \neq v \in \mathbb{C}^{2n}$, then $H_vH_uH_v^{-1} = H_{H_vu}$ is also a J-Householder matrix corresponding to $H_vu = u - (v^TJu)v$. If $v^TJu = 0$, then $H_vu = u$, and H_u and H_v commute.

Proof. One can prove $H_uH_{iu} = I$ using the fact that $u^TJu = 0$, which gives (a). Recall that if Q is symplectic, then $Q^TJQ = J$, and using this fact, one can prove (b) by direct calculation. For (c), note that for any nonzero vector $v \in \mathbb{C}^{2n}$, H_v is a symplectic matrix. Hence, a direct application of (b) gives $H_vH_uH_v^{-1} = H_{H_vu}$. Note that

$$H_vu = (I - vv^TJ)u = u - vv^TJu = u - (v^TJu)v.$$

If $v^TJu = 0$, then $H_vu = u-(0)v = u$, so $H_vH_uH_v^{-1} = H_u$. Right-multiplication by H_v gives $H_vH_u = H_uH_v$. Hence, if $v^TJu = 0$, then H_u and H_v commute. $\square$

Let $e_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T \in \mathbb{R}^{2n}$. Then $H_{e_1} = J_2(1)^{-1} \boxplus I_{2n-2}$, and $H_{ie_1} = H_{e_1}^{-1} = J_2(1) \boxplus I_{2n-2}$. Hence, H_{e_1} and H_{ie_1} are real J-symmetries. Suppose that $u \in \mathbb{R}^{2n}$ is an arbitrary nonzero vector. It turns out that H_u and H_{iu} are also real J-symmetries. Conversely, if $P \in \mathrm{Sp}(2n, \mathbb{R})$ is a real J-symmetry, we show in the following lemma that P is a J-Householder matrix that corresponds to a real vector, or a vector whose nonzero components are pure imaginary.

Lemma 2.3. *Let $P \in \mathrm{Sp}(2n, \mathbb{R})$ be a real J-symmetry.*

(a) Then $P = H_u$ for some $u \in \mathbb{R}^{2n}$ or $u \in (i\mathbb{R})^{2n}$.
(b) P is real symplectically similar to exactly one of $\mathcal{J}_n := J_2(1) \boxplus I_{2n-2}$ or $\mathcal{J}_n^{-1} := J_2(1)^{-1} \boxplus I_{2n-2}$, i.e., the set of real J-symmetries is $\mathcal{S}_{\mathcal{J}_n} \dot{\cup} \mathcal{S}_{\mathcal{J}_n^{-1}}$, where $\mathcal{S}_{\mathcal{J}_n} = \{H_{iu} | u \in \mathbb{R}^{2n}\}$ and $\mathcal{S}_{\mathcal{J}_n^{-1}} = \{H_v | v \in \mathbb{R}^{2n}\}$.
(c) In particular, $P \in \mathcal{S}_{\mathcal{J}_n}$ if and only if $P^{-1} \in \mathcal{S}_{\mathcal{J}_n^{-1}}$.

Proof. (a) Suppose $P = H_v$, where $v = \begin{bmatrix} v_1 & v_2 & \dots & v_{2n} \end{bmatrix} \in \mathbb{C}^{2n}$ is nonzero. Since $P = H_v = I_{2n} - vv^TJ$ is a real symplectic matrix, it follows that $vv^T = (I_{2n} - P)(-J)$ is a real matrix. Since the (j,k)-entry of vv^T is v_jv_k, it follows that $v_jv_k \in \mathbb{R}$ for all j, k. In particular, each $v_j^2 \in \mathbb{R}$. Since $v_j^2 \in \mathbb{R}$, we have that $v_j \in \mathbb{R} \cup i\mathbb{R}$. Take j such that $v_j \neq 0$. Then $v_jv_k \in \mathbb{R}$ implies that if $v_j \in \mathbb{R}$, $v_k \in \mathbb{R}$ for every k; similarly, if $v_j \in i\mathbb{R}$, $v_k \in i\mathbb{R}$ for every k.

(b) If $P = H_v$ is a real J-symmetry, then, from (a), $v \in \mathbb{R}^{2n}$ or $v = iu$ for some $u \in \mathbb{R}^{2n}$.

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Suppose that $v \in \mathbb{R}^{2n}$. By [1, Theorem 9], there exists a real symplectic matrix $Q \in \mathrm{Sp}(2n,\mathbb{R})$ such that $Qv = e_1$. By Lemma 2.2(b), $\mathcal{J}_n^{-1} = H_{e_1} = H_{Qv} = QH_vQ^{-1}$. Hence, if $v \in \mathbb{R}^{2n}$, then $P = H_v$ is real symplectically similar to $\mathcal{J}_n^{-1}$.

Suppose that $v = iu$, where $u \in \mathbb{R}^{2n}$. There exists a real symplectic matrix $D \in \mathrm{Sp}(2n,\mathbb{R})$ such that $Du = e_1$. Then $Dv = D(iu) = i(Du) = ie_1$. By Lemma 2.2(b), $\mathcal{J}_n = H_{ie_1} = H_{Dv} = DH_vD^{-1}$. Hence, if $v = iu$, where $u \in \mathbb{R}^{2n}$, then $P = H_v$ is real symplectically similar to $\mathcal{J}_n$.

The remaining part of the claim follows from Lemma 2.1.

(c) This follows from (a), (b), and Lemma 2.2(a). □

Suppose that $P \in \mathrm{Sp}(2n,\mathbb{R}) \setminus \{I_{2n}\}$ is a commutator of real J-symmetries, that is, $P = ABA^{-1}B^{-1}$ for some real J-symmetries A and B. Since A^{-1} is a real J-symmetry, and B is real symplectic, then $BA^{-1}B^{-1}$ is a real J-symmetry by Lemma 2.2(b). Hence, we can view a real symplectic commutator P as a product of two real J-symmetries.

We express this result as a real analog of [3, Lemma 4].

Lemma 2.4. *Let P be real symplectic and $P \neq I_{2n}$. The following are equivalent:*

(a) P is a commutator of real J-symmetries.
(b) $P = H_xH_yH_x^{-1}H_y^{-1} = H_{H_xy}H_{iy} = H_xH_{H_y(ix)}$ for some $x, y \in \mathbb{R}^{2n} \cup (i\mathbb{R})^{2n}$ with $x^TJy \neq 0$.
(c) $P = H_sH_{it}$ or $P = H_{is}H_t$ for some $s, t \in \mathbb{R}^{2n}$ with $s^TJt > 0$.

Proof. Let P be symplectic such that $P \neq I_{2n}$.

(a) ⇒ (b): Let P be a commutator of real J-symmetries. By Lemma 2.3(a), we can write $P = H_xH_yH_x^{-1}H_y^{-1} = (H_xH_yH_x^{-1})H_y^{-1}$ for some nonzero $x, y \in \mathbb{R}^{2n} \cup (i\mathbb{R})^{2n}$. By Lemma 2.2(a) and (b), $P = H_{H_xy}H_{iy}$ and $P = H_xH_{H_y(ix)}$.

From Lemma 2.2(c), if $x^TJy = 0$, then H_x and H_y commute, resulting to $P = H_xH_y(H_yH_x)^{-1} = I_{2n}$. Hence, $x^TJy \neq 0$.

Since real J-symmetries are symplectic, we have $H_{H_xy} = H_xH_yH_x^{-1}$ and $H_{H_y(ix)} = H_yH_x^{-1}H_y^{-1}$ are real J-symmetries.

(b) ⇒ (c): Suppose $P = H_{H_xy}H_{iy}$ for some $x, y \in \mathbb{R}^{2n} \cup (i\mathbb{R})^{2n}$ with $x^TJy \neq 0$. If $y \in \mathbb{R}^{2n}$ then $P = H_{H_xy}H_{iy}$ where $H_xy \in \mathbb{R}^{2n}$. If $y \in i\mathbb{R}^{2n}$ then $P = H_{iH_x(-iy)}H_{iy}$ where $H_xy \in i\mathbb{R}^{2n}$ and $iy \in \mathbb{R}^{2n}$. Finally,

$$i(H_x(-iy))^TJ(iy) = (H_xy)^TJy = y^TJ(y - xx^TJy) = -(x^TJy)y^TJx = (x^TJy)^2 > 0.$$

(c) ⇒ (a): Let $P = H_sH_{it}$ for some $s, t \in \mathbb{R}^{2n}$ with $s^TJt > 0$. We seek $x, y \in \mathbb{R}^{2n} \cup (i\mathbb{R})^{2n}$ such that $H_sH_{it} = H_{H_xy}H_{iy} = H_xH_yH_x^{-1}H_y^{-1}$ and $x^TJy \neq 0$. If $x = \frac{1}{\sqrt{t^TJs}}(t - s)$ and $y = t$, then $x \in \mathbb{R}^{2n} \cup (i\mathbb{R})^{2n}$, $y \in \mathbb{R}^{2n}$, $(x^TJy)^2 = t^TJs \neq 0$, and $H_xy = y - (x^TJy)x = t - (t - s) = s$. Hence, P is a commutator of real J-symmetries. □

In particular, Lemma 2.4 says that if $I_{2n} \neq P$ is a commutator of real J-symmetries, then we can write P as a product of two real J-symmetries which are not real symplectically similar. Conversely, if $I_{2n} \neq P = H_uH_v$ is a product of two real J-symmetries, then P is a commutator of real J-symmetries if (i) $u, iv \in \mathbb{R}^{2n}$, or (ii) $iu, v \in \mathbb{R}^{2n}$.

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Lemma 2.5. *Let $u, v \in \mathbb{R}^{2n}$ be nonzero, and suppose $a, b \in \mathbb{C}$. Then*

$$\text{tr}\,(H_{au}H_{bv}) = 2n - (ab)^2(u^TJv)^2.$$

Proof. Suppose $u, v \in \mathbb{R}^{2n}$ and are nonzero. Since for every $w \in \mathbb{R}^{2n}$, ww^tJ is nilpotent and the trace is a linear function, we have that

$$\begin{aligned}\text{tr}\,(H_{au}H_{bv}) &= \text{tr}\,(I - a^2uu^TJ - b^2vv^TJ + a^2b^2(u^TJv)uv^TJ)\\ &= 2n + (a^2b^2u^TJv)(v^TJu)\\ &= 2n - (ab)^2(u^TJv)^2.\end{aligned}$$

□

The following can easily be proven.

Lemma 2.6. *Let $P, Q \in \text{Sp}(2n, \mathbb{R})$ such that Q is real symplectically similar to P. Then*

(a) P is a commutator of real J-symmetries if and only if Q is a commutator of real J-symmetries.
(b) P is a product of $m \in \mathbb{N}$ commutators of real J-symmetries if and only if Q is a product of m commutators of real J-symmetries.
(c) P is a commutator of real J-symmetries if and only if P^{-1} is a commutator of real J-symmetries; and
(d) P is a product of m commutators of real J-symmetries if and only if P^{-1} is a product of m commutators of real J-symmetries.

Note that I_{2n} is a commutator of real J-symmetries. Hence, we also have that if P is a product of m commutators of real J-symmetries, then P is a product of $t \geq m$ commutators of real J-symmetries.

If P is a commutator of real J-symmetries, then, by Lemma 2.4, P is a product of two J-Householder matrices. It is discussed in [5, Lemma 6] that $0 \leq \text{rank}(P - I_{2n}) \leq 2$. Hence, by Lemma 2.5 and [5, Lemma 6], we can obtain a characterization of commutators of real J-symmetries. We also use the following, which is a consequence of the canonical form for real symplectic matrices under real symplectic similarity in [6, Theorem 1] (see also [9]).

Lemma 2.7. *Let $A \in \text{Sp}(2n, \mathbb{R})$.*

(a) If $\lambda \in \mathbb{R}\backslash\{-1, 0, 1\}$ is an eigenvalue of A, then so is λ^{-1}. Moreover, if $\dim(\text{Ker}(A - \lambda I_{2n})) = 1$, then A is real symplectically similar to

$$\text{diag}(\lambda, \lambda^{-1}) \boxplus B,$$

where $\lambda, \lambda^{-1} \notin \sigma(B)$.
(b) If $\lambda = \pm 1$ is an eigenvalue of A, then A is real symplectically similar to $A_1 \boxplus A_2 \boxplus \cdots \boxplus A_j \boxplus B$, where each A_j is of the form

$$\begin{bmatrix} J_{k_j}(\lambda)^{-1} & C(k_j, \alpha_j, \lambda) \\ 0 & J_{k_j}(\lambda)^T \end{bmatrix},$$

with $C(k_j, \alpha_j, \lambda) = J_{k_j}(\lambda)^{-1}\text{diag}(0, 0, ..., \alpha_j)$ and $\alpha_j \in \{-1, 0, 1\}$; and $\pm 1 \notin \sigma(B)$. The values of k_j are determined by $\text{rank}(A - \lambda I_{2n})^l$ for $l \geq 1$.

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(c) *If $e^{i\theta}$ is a nonreal eigenvalue of A for some $\theta \in (-\pi, \pi)\backslash\{0\}$, then so is $e^{-i\theta}$. Moreover, if $e^{i\theta}$ has algebraic multiplicity 1, then A is real symplectically similar to*

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \boxplus B,$$

where $e^{i\theta}, e^{-i\theta} \notin \sigma(B)$.

We now give the characterization of the commutators of real J-symmetries.

LEMMA 2.8. *Let $P \in \mathrm{Sp}(2n,\mathbb{R})$ such that $P \neq I_{2n}$. Then P is a commutator of real J-symmetries if and only if P is real symplectically similar to $\mathrm{diag}(\lambda, \lambda^{-1}) \boxplus I_{2n-2}$, where $\lambda \in \mathbb{R}^{+}\backslash\{1\}$.*

Proof. Let $P \in \mathrm{Sp}(2n,\mathbb{R})$ such that $P \neq I_{2n}$. Suppose P is a commutator of real J-symmetries. By Lemma 2.4, we can suppose that $P = H_{iu}H_v$ or $P = H_uH_{iv}$, where $u, v \in \mathbb{R}^{2n}$ and $u^TJv \neq 0$. We consider the former case, as the other case is proven similarly. By Lemma 2.5, tr $(H_{iu}H_v) = 2n + (u^TJv)^2 > 2n$, since $u^TJv \neq 0$. Since P is a product of two J-Householder matrices and $P \neq I_{2n}$, we have $0 < \mathrm{rank}(P - I_{2n}) \leq 2$.

If $\mathrm{rank}(P - I_{2n}) = 1$, then P is a real J-symmetry. By Lemma 2.3(c), $\mathrm{tr}\, P = 2n$, which is a contradiction. Thus, $\mathrm{rank}(P - I_{2n}) = 2$. By the rank-nullity theorem, $\dim(\mathrm{Ker}(P - I_{2n})) = 2n - 2$, and the algebraic multiplicity of 1 as an eigenvalue of P is at least $2n - 2$. Since $\mathrm{tr}\, P > 2n$, there exists $\lambda \in \sigma(P)$ such that $\lambda \in \mathbb{R}$ and $\lambda \neq 1$. If $\lambda = -1$, then since P is symplectic, -1 has algebraic multiplicity 2, and 1 has algebraic multiplicity $2n - 2$, which is a contradiction. Hence, $\lambda \neq \lambda^{-1}$, and 1 has algebraic multiplicity $2n - 2$. Now, $\mathrm{tr}\, P = \lambda + \lambda^{-1} + (2n - 2)$, so $\lambda + \lambda^{-1} > 2$. Hence, λ is positive. Moreover, $\dim(\mathrm{Ker}(P - \lambda I_{2n})) = 1$ and $\dim(\mathrm{Ker}(P - \lambda^{-1}I_{2n})) = 1$. By Lemma 2.7(a), P is real symplectically similar to $\mathrm{diag}(\lambda, \lambda^{-1}) \boxplus I_{2n-2}$.

Conversely, let P be real symplectically similar to $Q := \mathrm{diag}(\lambda, \lambda^{-1}) \boxplus I_{2n-2}$, where λ is a positive real number not equal to 1. We first write Q as a product of two real J-symmetries, one of which is real symplectically similar to $\mathcal{J}_n$ and the other is real symplectically similar to $\mathcal{J}_n^{-1}$. Observe that

$$Q = \left(\begin{bmatrix}2\lambda(1+\lambda)^{-1} & (1-\lambda)^2\\ -(1+\lambda)^{-2} & 2(1+\lambda)^{-1}\end{bmatrix} \boxplus I_{2n-2}\right)\left(\begin{bmatrix}2\lambda(1+\lambda)^{-1} & -\lambda^{-1}(1-\lambda)^2\\ \lambda(1+\lambda)^{-2} & 2(1+\lambda)^{-1}\end{bmatrix} \boxplus I_{2n-2}\right). \tag{2.1}$$

Now let

$$A = \begin{bmatrix}1-\lambda & -1\\ (1+\lambda)^{-1} & \lambda(1-\lambda^2)^{-1}\end{bmatrix} \boxplus I_{2n-2} \text{ and } R = \begin{bmatrix}0 & -\sqrt{(\lambda+\lambda^{-1}-2)^{-1}}\\ \sqrt{\lambda+\lambda^{-1}-2} & 1\end{bmatrix} \boxplus I_{2n-2},$$

which are real and symplectic, since λ is a positive real number not equal to 1. Then

$$A(J_2(1) \boxplus I_{2n-2})A^{-1} = \begin{bmatrix}2\lambda(1+\lambda)^{-1} & (1-\lambda)^2\\ -(1+\lambda)^{-2} & 2(1+\lambda)^{-1}\end{bmatrix} \boxplus I_{2n-2}, \tag{2.2}$$

and

$$(AR)(J_2(1)^{-1} \boxplus I_{2n-2})(AR)^{-1} = \begin{bmatrix}2\lambda(1+\lambda)^{-1} & -\lambda^{-1}(1-\lambda)^2\\ \lambda(1+\lambda)^{-2} & 2(1+\lambda)^{-1}\end{bmatrix} \boxplus I_{2n-2}. \tag{2.3}$$

Hence, Q is a product of a matrix similar to $\mathcal{J}_n$ and a matrix similar to $\mathcal{J}_n^{-1}$. By Lemma 2.3 and 2.4, Q is a commutator of real J-symmetries. Since P is real symplectically similar to Q, we have P is also a commutator of real J-symmetries by Lemma 2.6. $\square$

The following lemma follows from the fact that if A is a J-symmetry, then so is $A \boxplus I_{2t}$.

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LEMMA 2.9. *If $A \in \mathrm{Sp}(2n,\mathbb{R})$ is a product of m commutators of real J-symmetries, then $A \boxplus I_{2t}$ is a product of m commutators of real J-symmetries for each positive integer t.*

Suppose we have a real symplectic matrix $A \in \mathrm{Sp}(2n,\mathbb{R})$, which we want to write as a product of commutators of real J-symmetries. By Lemma 2.6, we may replace A with a matrix that is real symplectically similar to it. In particular, to determine such factorizations for 2-by-2 and 4-by-4 real sympletic matrices, it suffices to consider some canonical forms for these sympletic matrices under real symplectic similarity. The following is in [1, Lemma 5] and [8, Lemma 2.10, 2.11].

LEMMA 2.10. *Let $B = [b_{ij}] \in \mathrm{Sp}(2,\mathbb{R})$ be nonscalar.*

(a) If B is not a diagonal matrix, then B is real symplectically similar to $L_{B,\mu} := \begin{bmatrix} \mathrm{tr}B & \mu \\ -\mu & 0 \end{bmatrix}$ *for some $\mu \in \{-1,1\}$.*
(b) If $|\mathrm{tr}\ B| > 2$, then B is real symplectically similar to $L_{B,1}$ and $L_{B,-1}$. In particular, B is real symplectically similar to $\mathrm{diag}(\lambda,\lambda^{-1})$ for some $\lambda \in \mathbb{R}\backslash\{-1,0,1\}$.
(c) If $|\mathrm{tr}\ B| < 2$ and $b_{12} > 0$, then B is real symplectically similar to $L_{B,1}$.
If $|\mathrm{tr}\ B| < 2$ and $b_{12} < 0$, then B is real symplectically similar to $L_{B,-1}$.
(d) If $\mathrm{tr}\ B = -2$, then B is real symplectically similar to $\begin{bmatrix} -1 & \mu \\ 0 & -1 \end{bmatrix}$ *for some $\mu \in \{-1,1\}$.*

The following is a direct consequence of Lemma 2.10 and the normal forms for symplectic matrices in [6, Theorem 1]. See also [9].

LEMMA 2.11. *Let $Q \in \mathrm{Sp}(2,\mathbb{R})$. Then Q is real symplectically similar to one of the following.*

(a) $\mathcal{Q}_1 = \mathrm{diag}(\lambda,\lambda^{-1})$, *where* $\lambda \in \mathbb{R}\backslash\{-1,0,1\}$
(b) $\mathcal{Q}_2(\mu,\alpha) = \begin{bmatrix} \mu & \alpha\mu \\ 0 & \mu \end{bmatrix}$, *where* $\mu \in \{-1,1\}$, $\alpha \in \{-1,0,1\}$
(c) $R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, *where* $\theta \in (-\pi,\pi)\backslash\{0\}$

Finally, the following result is for 4-by-4 real symplectic matrices.

LEMMA 2.12. *[1, Theorem 4] Each $P \in \mathrm{Sp}(4,\mathbb{R})$ is real symplectically similar to one of the following:*

(a) $\mathcal{P}_1 = J_2(\lambda)^{-1} \oplus J_2(\lambda)^T$, *where* $\lambda \in \mathbb{R}\backslash\{0,-1,1\}$
(b) $\mathcal{P}_2(\mu,\alpha) = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & \alpha\mu & \alpha \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 1 & \mu \end{bmatrix}$, *where* $\alpha \in \{-1,0,1\}$ *and* $\mu \in \{-1,1\}$
(c) $\mathcal{P}_3 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \oplus \begin{bmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ -\frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix}$, *where* $a, b \in \mathbb{R}$ *such that* $a^2+b^2 \neq 1$
(d) $\mathcal{P}_4 = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & \alpha & -\alpha\cot\theta \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix}$, *where* $\theta \in (-\pi,\pi)\backslash\{0\}$ *and* $\alpha \in \{-1,0,1\}$
(e) $A \boxplus B$ *for some* $A, B \in \mathrm{Sp}(2,\mathbb{R})$

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3. Main results. The Cartan–Dieudonné–Scherk Theorem [4] guarantees that a symplectic matrix A is a product of J-symmetries, and this is proved for orthogonal matrices over a field $\mathbb{F}$ of characteristic not 2. In [10], Scherk showed that the number of factors needed is $\operatorname{rank}(A-I)$ if $J(A-I)$ is not skew-symmetric, and $\operatorname{rank}(A-I)+2$ otherwise.

Let A be symplectic such that $A \neq -I_{2n}$ and suppose that $A=\prod_{i=1}^{2k} A_i$, where A_i is a J-symmetry. Then $A=\prod_{j=1}^{k} B_j$, where B_j is a product of 2 J-symmetries for each j. Furthermore, by [5, Lemma 6], we have that $0 \leq \operatorname{rank}(B_j - I) \leq 2$ for each j. We use these results, along with the following, to prove Theorem 1.1.

Lemma 3.1. *Let $B \in \operatorname{Sp}(2n,\mathbb{R})$ be symplectic. If B is a product of 2 real J-symmetries, then B can only be real symplectically similar to one of the following:*

(a) $I_{2n-2} \boxplus \operatorname{diag}(\lambda,\lambda^{-1})$, *where* $\lambda \in \mathbb{R}\backslash\{0\}$
(b) $I_{2n-2} \boxplus J_2(1)^{\alpha}$, *where* $\alpha \in \{-1,1\}$
(c) $I_{2n-2} \boxplus J_2(-1)^{\alpha}$, *where* $\alpha \in \{-1,1\}$
(d) $I_{2n-2} \boxplus R(\theta)$, *where* $\theta \in (-\pi,\pi)\backslash\{0\}$
(e) $I_{2n-4} \boxplus J_2(1)^{\alpha} \boxplus J_2(1)^{\beta}$, *where* $\alpha,\beta \in \{-1,1\}$
(f) $I_{2n-4} \boxplus \mathcal{P}_2(1,0)$

Proof. Suppose that B is a product of 2 real J-symmetries. By [5, Lemma 6], $0 \leq \operatorname{rank}(B-I_{2n}) \leq 2$.

If $\operatorname{rank}(B-I_{2n})=0$, then $B=I_{2n}$.

If $\operatorname{rank}(B-I_{2n})=1$, then B is a real J-symmetry. By Lemma 2.3, B is real symplectically similar to $I_{2n-2} \boxplus J_2(1)^{\alpha}$, where $\alpha \in \{-1,1\}$.

Suppose $\operatorname{rank}(B-I_{2n})=2$. If $n=1$, then $\sigma(B)=\{\lambda,\lambda^{-1}\}$ for some $\lambda \neq 1$. Hence, B is real symplectically similar to one of the following:

- $\operatorname{diag}(\lambda,\lambda^{-1})$, where $\lambda \in \mathbb{R}\backslash\{0,1\}$
- $J_2(-1)^{\alpha}$, where $\alpha \in \{-1,1\}$
- $R(\theta)$, where $\theta \in (-\pi,\pi)\backslash\{0\}$

If $n>1$, note that B can only have at most 3 distinct eigenvalues, one of which is 1. Otherwise, if B has more than 2 distinct eigenvalues not equal to 1, then $\operatorname{rank}(B-I_{2n})>2$. Hence, we can suppose that $\sigma(B)=\{1,r_1,r_2\}$, and consider cases for r_1 and r_2. Note that since B is symplectic, we have $\det(B)=1$, so $r_1r_2=1$. However, it is possible that r_1 and r_2 are nonreal eigenvalues.

If $r_1,r_2 \neq 1$ and $r_1 \neq r_2$, then $r_2=r_1^{-1}$, where $r_1 \in \mathbb{C}\backslash\{-1,0,1\}$. If r_1 and r_1^{-1} are real eigenvalues of B, with $\dim(\operatorname{Ker}(B-r_1I_{2n}))=1$, then, by Lemma 2.7(a), B is real symplectically similar to $I_{2n-2}\boxplus\operatorname{diag}(\lambda,\lambda^{-1})$ for some $\lambda \in \mathbb{R}\backslash\{-1,0,1\}$. Otherwise, we have that B is real symplectically similar to $I_{2n-2}\boxplus R(\theta)$ for some $\theta \in (-\pi,\pi)\backslash\{0\}$ by Lemma 2.7(c).

If $r_1,r_2 \neq 1$ and $r_1=r_2$, then $r_1=r_2=-1$. If $\operatorname{rank}(B+I_{2n})=2n-1$, then B is real symplectically similar to $I_{2n-2} \boxplus J_2(-1)^{\alpha}$, where $\alpha \in \{-1,1\}$. If $\operatorname{rank}(B+I_{2n})=2n-2$, then B is real symplectically similar to $I_{2n-2} \boxplus -I_2$.

If $r_1=r_2=1$, then in view of Lemmas 2.1 and 2.7, we have that B is real symplectically similar to one of the following:

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- $I_{2n-4} \boxplus J_2(1) \boxplus J_2(1)$
- $I_{2n-4} \boxplus J_2(1) \boxplus J_2(1)^{-1}$
- $I_{2n-4} \boxplus J_2(1)^{-1} \boxplus J_2(1)^{-1}$
- $I_{2n-4} \boxplus \mathcal{P}_2(1,\alpha)$, where $\alpha \in \{-1,0,1\}$

However, note that for $\alpha = \pm 1$, $\operatorname{rank}((I_{2n-2} \boxplus \mathcal{P}_2(1,\alpha)) - I_{2n}) = 3 > 2$. Hence, $\alpha = 0$. □

The bulk of the technical work come from showing that the blocks in Lemma 3.1 are products of either 2 or 3 commutators of real J-symmetries. Observe that the matrices from Lemma 3.1 (a)-(d) are of the form $I \boxplus A$, where A is 2×2, while the remaining blocks are of the form $I \boxplus A$, where A is 4×4. We prove the technical results in the following subsections before we go back to the proof of Theorem 1.1.

3.1. The matrices from (a) to (d). The following matrix will be used quite often as a commutator factor.

LEMMA 3.2. *If $\beta \in \mathbb{R}$ such that $\beta > 2$ and $\mu \in \{-1,1\}$, then $C_{\beta,\mu} = \begin{bmatrix} \beta & \mu \\ -\mu & 0 \end{bmatrix}$ is a commutator of real J-symmetries.*

Proof. Let $k := \frac{1}{2}\left(\beta + \sqrt{\beta^2 - 4}\right)$. Observe that $k^{-1} = \frac{1}{2}\left(\beta - \sqrt{\beta^2-4}\right)$, and $k + k^{-1} = \beta$. Since $0 < \sqrt{\beta^2 - 4} < \sqrt{\beta^2} = \beta$, we have that k and k^{-1} are positive and not equal to 1. By Lemma 2.10(b), $C_{\beta,\mu} = \begin{bmatrix} \beta & \mu \\ -\mu & 0 \end{bmatrix}$ is real symplectically similar to $\operatorname{diag}(k, k^{-1})$. By Lemma 2.8, $C_{\beta,\mu}$ is a commutator of real J-symmetries. □

The following three lemmas will show that the matrices in Lemma 2.11, other than $-I_2$, are products of 2 commutators of real J-symmetries.

LEMMA 3.3. *If $\lambda \in \mathbb{R}\backslash\{-1,0,1\}$, then $\mathcal{Q}_1 := \operatorname{diag}(\lambda, \lambda^{-1})$ is a product of 2 commutators of real J-symmetries.*

Proof. If $\lambda > 0$ and not equal to 1, then $\mathcal{Q}_1$ is a commutator of real J-symmetries by Lemma 2.8. Suppose $\lambda < 0$. Observe that for $\beta_1, \beta_2 > 2$,

$$\begin{bmatrix} 0 & -\lambda \\ \lambda^{-1} & \beta_1 \end{bmatrix} \begin{bmatrix} \beta_2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ \lambda^{-1}\beta_2 - \beta_1 & \lambda^{-1} \end{bmatrix}. \tag{3.4}$$

If we let $K = \begin{bmatrix} 0 & -\sqrt{-\lambda} \\ \sqrt{-\lambda^{-1}} & 0 \end{bmatrix}$, then

$$KC_{\beta_1,1}K^{-1} = \begin{bmatrix} 0 & -\lambda \\ \lambda^{-1} & \beta_1 \end{bmatrix}. \tag{3.5}$$

It follows that the first factor on the left-hand side of (3.4) is a commutator of real J-symmetries by Lemmas 2.6 and 3.2. The second factor is of the form $C_{\beta_2,1}$ in Lemma 3.2, so it is a commutator of real J-symmetries. Finally, the matrix on the right-hand side of (3.4) is similar to $\mathcal{Q}_1$ by Lemma 2.10(b). It follows from Lemma 2.6 that $\mathcal{Q}_1$ is a product of two commutators of real J-symmetries. □

LEMMA 3.4. *If $\mu, \alpha \in \{-1,1\}$, then $\mathcal{Q}_2(\mu,\alpha) := \begin{bmatrix} \mu & \alpha\mu \\ 0 & \mu \end{bmatrix}$ is a product of 2 commutators of real J-symmetries.*

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Proof. If $\mu = 1$ and $\lambda > 1$, then $\mathcal{Q}_2(1,\alpha) = \begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}\begin{bmatrix}\lambda^{-1} & \alpha\lambda^{-1}\\ 0 & \lambda\end{bmatrix}$. By Lemma 2.10(b), each factor on the right-hand side is real symplectically similar to $\operatorname{diag}(\lambda,\lambda^{-1})$, so each one is a commutator of real J-symmetries.

Let $\mu = -1$ and $\beta_1,\beta_2 > 2$. Note that

$$\begin{bmatrix}-1 & \alpha(\beta_1+\beta_2)\\ 0 & -1\end{bmatrix} = \begin{bmatrix}\beta_1 & \alpha\\ -\alpha & 0\end{bmatrix}\begin{bmatrix}0 & \alpha\\ -\alpha & \beta_2\end{bmatrix}.$$

Since $\beta_1+\beta_2 > 0$, the matrix on the left-hand side is similar to $\mathcal{Q}_2(-1,\alpha)$. The first factor on the right-hand side is $C_{\beta_1,\alpha}$, so it is a commutator of real J-symmetries by Lemma 3.2. If we let $K = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, then

$$KC_{\beta_2,\alpha}K^{-1} = \begin{bmatrix}0 & \alpha\\ -\alpha & \beta_2\end{bmatrix}. \tag{3.6}$$

Hence, the second factor on the right-hand side is real symplectically similar to $C_{\beta_2,\alpha}$. By Lemma 3.2, it is also a commutator of real J-symmetries. Thus, if $\mu,\alpha \in \{-1,1\}$, then $\mathcal{Q}_2(\mu,\alpha)$ is a product of two commutators of real J-symmtries. □

Lemma 3.5. *If $\theta \in (-\pi,\pi)\backslash\{0\}$, then $R(\theta) := \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$ is a product of 2 commutators of real J-symmetries.*

Proof. Let $2 < \beta_1,\beta_2 \in \mathbb{R}$. Then for any $\theta \in (-\pi,\pi)\backslash\{0\}$, $(\beta_1+\beta_2)^2 - 8\cos\theta - 8 > 0$, so $d := -\frac{1}{2}(\beta_1+\beta_2) \pm \frac{1}{2}\sqrt{(\beta_1+\beta_2)^2 - 8\cos\theta - 8} \in \mathbb{R}$. Observe that $d < 0$ since $8\cos\theta + 8 > 0$.

If $\theta \in (0,\pi)$, then $R(\theta)$ is real symplectically similar to $\begin{bmatrix}2\cos\theta & 1\\ -1 & 0\end{bmatrix}$ by Lemma 2.10(c). Now observe that

$$\begin{bmatrix}-d & d^2+\beta_1 d+1\\ -1 & \beta_1+d\end{bmatrix} C_{\beta_2,1} = \begin{bmatrix}-d^2-(\beta_1+\beta_2)d-1 & -d\\ -d-(\beta_1+\beta_2) & -1\end{bmatrix}. \tag{3.7}$$

The trace of the matrix on the right-hand side is

$$\begin{aligned}-d^2-(\beta_1+\beta_2)d-2 &= \frac{1}{4}(\beta_1+\beta_2)^2 - 2 - (d+\frac{1}{2}(\beta_1+\beta_2))^2\\ &= \frac{1}{4}(8\cos\theta) = 2\cos\theta.\end{aligned}$$

Since $-d > 0$, the matrix on the right-hand side of (3.7) is real symplectically similar to $\begin{bmatrix}2\cos\theta & 1\\ -1 & 0\end{bmatrix}$ by Lemma 2.10(c). Next, if we let $K = \begin{bmatrix}d & -1\\ 1 & 0\end{bmatrix}$, then

$$KC_{\beta_1,1}K^{-1} = \begin{bmatrix}-d & d^2+\beta_1 d+1\\ -1 & \beta_1+d\end{bmatrix}, \tag{3.8}$$

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so the first factor on the left-hand side of (3.7) is a commutator of real J-symmetries. Thus, if $\theta \in (0,\pi)$, then $R(\theta)$ is a product of 2 commutators of real J-symmetries.

If $\theta \in (-\pi, 0)$, then $R(\theta)$ is real symplectically similar to $\begin{bmatrix} 2\cos\theta & -1 \\ 1 & 0 \end{bmatrix}$. By Lemma 2.10(c), $\begin{bmatrix} 2\cos\theta & -1 \\ 1 & 0 \end{bmatrix}$ is real symplectically similar to $\begin{bmatrix} 0 & -1 \\ 1 & 2\cos\theta \end{bmatrix} = \begin{bmatrix} 2\cos\theta & 1 \\ -1 & 0 \end{bmatrix}^{-1}$. Hence, by Lemma 2.6(d), $\begin{bmatrix} 2\cos\theta & -1 \\ 1 & 0 \end{bmatrix}$ is also a product of 2 commutators of real J-symmetries. ☐

Lemma 3.6. *Let $A \in \mathrm{Sp}(2,\mathbb{R})$ such that $A \neq -I_2$. If n is a positive integer, then $A \boxplus I_{2n}$ can be written as a product of 2 commutators of real J-symmetries.*

Proof. By Lemmas 2.11, 3.3, 3.4, and 3.5, we have A is a product of two commutators of real J-symmetries. By Lemma 2.9, $A \boxplus I_{2n}$ is also a product of 2 commutators of real J-symmetries. ☐

It is only natural to ask what happens when $A = -I_2$. To this end, we have the following.

Lemma 3.7. *$-I_2$ is a product of 3 commutators of real J-symmetries, and no fewer.*

Proof. Observe that if $\lambda > 1$, then $-I_2 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} -\lambda^{-1} & 0 \\ 0 & -\lambda \end{bmatrix}$. The first factor is a commutator of real J-symmetries by Lemma 2.8. The second factor is a product of two commutators of real J-symmetries by Lemma 3.3. Hence, $-I_2$ can be expressed as a product of 3 commutators of real J-symmetries.

Suppose now that $I_2 \neq A \in \mathrm{Sp}(2,\mathbb{R})$ is a commutator of real J-symmetries. Then A is nonsingular, and the unique matrix X for which $AX = -I_2$ is $X = -A^{-1}$. Since A is a commutator of real J-symmetries, then $\sigma(A) = \{\lambda, \lambda^{-1}\}$ for some $\lambda > 1$. But this means that $\sigma(X) = \sigma(-A^{-1}) = \{-\lambda, -\lambda^{-1}\}$. Since $-\lambda$ and $-\lambda^{-1}$ are negative real numbers, X cannot be a commutator of real J-symmetries by Lemma 2.8. It follows that $-I_2$ cannot be written as a product of 2 commutators of real J-symmetries. ☐

3.2. The matrices from (d) to (f). The next six lemmas will discuss factorizations for 4-by-4 real symplectic matrices.

Lemma 3.8. *Let $1 \neq \lambda$ be a positive real number, and let $a, b, c \in \mathbb{R}$. Then H_1, H_1^T, H_2, and H_2^T are commutators of real J-symmetries, where*

$$H_1 = \begin{bmatrix} \lambda & a & b & c \\ 0 & 1 & \lambda^{-1}c & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & -\lambda^{-1}a & 1 \end{bmatrix} \text{ and } H_2 = \begin{bmatrix} 1 & 0 & 0 & \lambda^{-1}b \\ a & \lambda & b & c \\ 0 & 0 & 1 & -\lambda^{-1}a \\ 0 & 0 & 0 & \lambda^{-1} \end{bmatrix}.$$

Proof. Let $P_1 = \begin{bmatrix} 1 & \frac{a}{1-\lambda} & \frac{b}{\lambda^{-1}-\lambda} & \frac{c}{1-\lambda} \\ 0 & 1 & \frac{c}{1-\lambda} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{a}{1-\lambda} & 1 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 1 & 0 & 0 & \frac{b}{1-\lambda} \\ \frac{a}{1-\lambda} & 1 & \frac{b}{1-\lambda} & \frac{c}{\lambda^{-1}-\lambda} \\ 0 & 0 & 1 & -\frac{a}{1-\lambda} \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Note that P_1 and P_2 are real symplectic matrices. Then

$$P_1 \mathrm{diag}(\lambda, 1, \lambda^{-1}, 1) P_1^{-1} = H_1, \tag{3.9}$$

$$P_2 \mathrm{diag}(1, \lambda, 1, \lambda^{-1}) P_2^{-1} = H_2. \tag{3.10}$$

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By Lemma 2.8, $\mathrm{diag}(\lambda, 1, \lambda^{-1}, 1)$ and $\mathrm{diag}(1, \lambda, 1, \lambda^{-1})$ are commutators of real J-symmetries. Thus, by Lemma 2.6, it follows that H_1 and H_2 are commutators of real J-symmetries. For $i = 1, 2$, if $H_i = P_i D_i P_i^{-1}$, where P_i and D_i are real symplectic and D_i is diagonal, then $H_i^T = P_i^{-T} D_i P_i^T$, where P_i^T is real symplectic. Hence, H_1^T and H_2^T are commutators of real J-symmetries. □

LEMMA 3.9. *If $\alpha \in \{-1, 0, 1\}$ and $\mu \in \{-1, 1\}$, then* $\mathcal{P}_2(\mu, \alpha) := \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & \alpha\mu & \alpha \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 1 & \mu \end{bmatrix}$ *is a product of 2 commutators of real J-symmetries.*

Proof. If $\mu = 1$, choose $\lambda > 1$ and let

$$K_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & \alpha\lambda \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \text{ and } K_2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} & \lambda^{-1} \end{bmatrix}.$$

By Lemma 3.8, K_1 and K_2 are commutators of real J-symmetries. Now note that $K_1 K_2 = \mathcal{P}_2(1, \alpha)$. Hence, $\mathcal{P}_2(1, \alpha)$ is a product of 2 commutators of real J-symmetries.

If $\mu = -1$, choose $\lambda > 1$ and let

$$K_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & \lambda & 0 & -\alpha\lambda^{-1} \\ 0 & 0 & 1 & -4\lambda^{-1} \\ 0 & 0 & 0 & \lambda^{-1} \end{bmatrix}, \text{ and } K_4 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \lambda & \lambda \end{bmatrix}.$$

Then K_3 and K_4 are commutators of real J-symmetries by Lemma 3.8. Now

$$K_3 K_4 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 4 & -3 & -\alpha & -\alpha \\ 0 & 0 & -3 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Note that if $Q = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \in \mathrm{Sp}(4, \mathbb{R})$, then

$$Q(\mathcal{P}_2(-1, \alpha))Q^{-1} = K_3 K_4. \tag{3.11}$$

Hence, $\mathcal{P}_2(-1, \alpha)$ is also a product of 2 commutators of real J-symmetries by Lemma 2.6. □

LEMMA 3.10. *If $a, b \in \mathbb{R}$ are not both zero such that $a^2 + b^2 \neq 1$, then* $\mathcal{P}_3 := \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \oplus \begin{bmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ -\frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix}$ *is a product of 2 commutators of real J-symmetries.*

Proof. Let $\varphi := \sqrt{a^2 + b^2} \neq 1$. First suppose that $b > 0$. Let $K_1 = \begin{bmatrix} \varphi & 0 \\ -2 & 1 \end{bmatrix} \oplus \begin{bmatrix} \varphi^{-1} & 2\varphi^{-1} \\ 0 & 1 \end{bmatrix}$, and $K_2 = \begin{bmatrix} 1 & \varphi - a \\ 0 & \varphi \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 \\ \varphi^{-1}(a - \varphi) & \varphi^{-1} \end{bmatrix}$. By Lemma 3.8, K_1 and K_2 are commutators of real J-symmetries.

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Observe that

$$K_1K_2 = \begin{bmatrix} \varphi & \varphi(\varphi - a) \\ -2 & 2a - \varphi \end{bmatrix} \oplus \begin{bmatrix} \varphi^{-2}(2a - \varphi) & 2\varphi^{-2} \\ \varphi^{-1}(a - \varphi) & \varphi^{-1} \end{bmatrix}.$$

If we let $w = \sqrt{2b^{-1}}$ and $Q_1 = \begin{bmatrix} w & \frac{\varphi - a}{2}w \\ 0 & \frac{b}{2}w \end{bmatrix} \oplus \begin{bmatrix} \frac{1}{w} & 0 \\ \frac{a-\varphi}{bw} & \frac{2}{bw} \end{bmatrix} \in \mathrm{Sp}(4, \mathbb{R})$, then

$$Q_1(K_1K_2)Q_1^{-1} = \mathcal{P}_3. \tag{3.12}$$

Hence, if $b > 0$, then $\mathcal{P}_3$ is a product of 2 commutators of real J-symmetries.

We proceed similarly for the case when $b < 0$. Consider

$$K_3 = \begin{bmatrix} \varphi & 0 \\ 2 & 1 \end{bmatrix} \oplus \begin{bmatrix} \varphi^{-1} & -2\varphi^{-1} \\ 0 & 1 \end{bmatrix} \text{ and } K_4 = \begin{bmatrix} 1 & a - \varphi \\ 0 & \varphi \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 \\ \varphi^{-1}(\varphi - a) & \varphi^{-1} \end{bmatrix}.$$

By Lemma 3.8, K_3 and K_4 are also commutators of real J-symmetries. Note that

$$K_3K_4 = \begin{bmatrix} \varphi & \varphi(a - \varphi) \\ 2 & 2a - \varphi \end{bmatrix} \oplus \begin{bmatrix} \varphi^{-2}(2a - \varphi) & -2\varphi^{-2} \\ \varphi^{-1}(\varphi - a) & \varphi^{-1} \end{bmatrix}.$$

Let $w = \sqrt{-2b^{-1}}$, and $Q_2 = \begin{bmatrix} w & \frac{a-\varphi}{2}w \\ 0 & -\frac{b}{2}w \end{bmatrix} \oplus \begin{bmatrix} \frac{1}{w} & 0 \\ \frac{a-\varphi}{bw} & -\frac{2}{bw} \end{bmatrix} \in \mathrm{Sp}(4, \mathbb{R})$. Then

$$Q_2(K_3K_4)Q_2^{-1} = \mathcal{P}_3. \tag{3.13}$$

Thus, if $b < 0$, then $\mathcal{P}_3$ is a product of 2 commutators of real J-symmetries.

Suppose now that $b = 0$. Then $a \neq \pm 1$ and $\mathcal{P}_3 = \mathrm{diag}(a, a, a^{-1}, a^{-1})$. If $a > 0$, then

$$\mathcal{P}_3 = (I_2 \boxplus \mathrm{diag}(a, a^{-1}))(\mathrm{diag}(a, a^{-1}) \boxplus I_2).$$

Since each factor on the right-hand side is a commutator of real J-symmetries, $\mathcal{P}_3$ is a product of 2 commutators of real J-symmetries.

If $a < 0$, choose $\lambda > 1$, and let $u = a + a^{-1}$ and $t = u - \lambda - \lambda^{-1}$. Consider

$$K_5 = \begin{bmatrix} \lambda & 0 \\ 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} \lambda^{-1} & -\lambda^{-1} \\ 0 & 1 \end{bmatrix} \text{ and } K_6 = \begin{bmatrix} 1 & t \\ 0 & \lambda^{-1} \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 \\ -t\lambda & \lambda \end{bmatrix},$$

which are commutators of real J-symmetries by Lemma 3.8. Then

$$K_5K_6 = \begin{bmatrix} \lambda & t\lambda \\ 1 & t + \lambda^{-1} \end{bmatrix} \oplus \begin{bmatrix} t + \lambda^{-1} & -1 \\ -t\lambda & \lambda \end{bmatrix}.$$

If $Q_3 = \begin{bmatrix} 1 & a - \lambda \\ \frac{1}{a^{-1}-a} & \frac{a^{-1}-\lambda}{a^{-1}-a} \end{bmatrix} \oplus \begin{bmatrix} \frac{a^{-1}-\lambda}{a^{-1}-a} & \frac{-1}{a^{-1}-a} \\ \lambda - a & 1 \end{bmatrix}$ and $Q_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, then

$$(Q_4Q_3)(K_5K_6)(Q_4Q_3)^{-1} = \mathcal{P}_3. \tag{3.14}$$

Since Q_4Q_3 is symplectic, we have $\mathcal{P}_3$ is a product of 2 commutators of real J-symmetries when $b = 0$. $\square$

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LEMMA 3.11. *Let* $A = B \boxplus \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix}$, *where* $B \in \mathrm{Sp}(2,\mathbb{R})$ *such that* $|\mathrm{tr}\, B| \neq 2$ *and* $\mu \in \{-1, 1\}$. *Then* A *is a product of 2 commutators of real* J*-symmetries.*

Proof. Take a positive real number $\lambda > 1$, and $a, b, c \in \mathbb{R}$ such that $ab \neq 0$ and $ac \neq 0, -4$. Let $K_1 = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b\lambda^{-1} & \lambda^{-1} & 0 \\ b & 0 & 0 & 1 \end{bmatrix}$ and $K_2 = \begin{bmatrix} \lambda^{-1} & 0 & c & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. By Lemma 3.8, K_1 and K_2 are commutators of real J-symmetries. Then

$$K_1 K_2 = \begin{bmatrix} 1 & 0 & \lambda c & 0 \\ 0 & 1 & 0 & 0 \\ a\lambda^{-1} & b\lambda^{-1} & 1 + ac & 0 \\ b\lambda^{-1} & 0 & bc & 1 \end{bmatrix},$$

with characteristic polynomial

$$f_{K_1 K_2}(x) = (1 - x)^2 (1 - (ac + 2)x + x^2). \tag{3.15}$$

It follows that the spectrum of $K_1 K_2$ is

$$\sigma(K_1 K_2) = \{1, \frac{1}{2}\left(ac + 2 + \sqrt{(ac + 2)^2 - 4}\right), \frac{1}{2}\left(ac + 2 - \sqrt{(ac + 2)^2 - 4}\right)\}.$$

If $ac \notin \{0, -4\}$, then the eigenvalues $\frac{1}{2}\left(ac + 2 + \sqrt{(ac + 2)^2 - 4}\right)$ and $\frac{1}{2}\left(ac + 2 - \sqrt{(ac + 2)^2 - 4}\right)$ are distinct and not equal to 1. Furthermore, we have that rank $(K_1 K_2 - I_4) = 3$ and rank $(K_1 K_2 - I_4)^m = 2$ for all $m \geq 2$. By Lemma 2.10 and Lemma 2.12, $K_1 K_2$ is real symplectically similar to $\begin{bmatrix} ac + 2 & \varphi \\ -\varphi & 0 \end{bmatrix} \boxplus \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix}$, for some $\mu, \varphi \in \{-1, 1\}$. Or to prove this directly, we can choose a and c so that $\mu a > 0$ and $\varphi c > 0$, and let

$$S_1 = \sqrt{\lambda^{-1}} \begin{bmatrix} \sqrt{c^{-1}\varphi} & a^{-1}b\sqrt{c^{-1}\varphi} & 0 & 0 \\ 0 & 0 & \lambda\sqrt{a^{-1}\mu} & -ab^{-1}\lambda\sqrt{a^{-1}\mu} \\ -\varphi(1 + ac)\sqrt{c^{-1}\varphi} & -a^{-1}b\varphi(1 + ac)\sqrt{c^{-1}\varphi} & \lambda\sqrt{\varphi c} & 0 \\ 0 & b\mu\sqrt{a^{-1}\mu} & 0 & 0 \end{bmatrix},$$

then

$$K_1 K_2 = S_1^{-1} \left(\begin{bmatrix} ac + 2 & \varphi \\ -\varphi & 0 \end{bmatrix} \boxplus \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix} \right) S_1. \tag{3.16}$$

Suppose $B \in \mathrm{Sp}(2,\mathbb{R})$ such that $|\mathrm{tr}\, B| \neq 2$. Then B is not diagonal, and, by Lemma 2.10(a), B is real symplectically similar to $\begin{bmatrix} \mathrm{tr}\, B & \varphi \\ -\varphi & 0 \end{bmatrix}$ for some $\varphi \in \{-1, 1\}$, and so $B \boxplus \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix}$ is real symplectically similar to $\begin{bmatrix} \mathrm{tr}\, B & \varphi \\ -\varphi & 0 \end{bmatrix} \boxplus \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix}$. Since the real symplectic similarity in (3.16) holds for $ac + 2 \neq \pm 2$, then $B \boxplus \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix}$ is a product of 2 commutators of real J-symmetries. □

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Lemma 3.12. *If $\alpha, \mu \in \{-1, 1\}$, then $\begin{bmatrix} \alpha & -\mu \\ 0 & \alpha \end{bmatrix} \boxplus \begin{bmatrix} 1 & \mu \\ 0 & 1 \end{bmatrix}$ is a product of 2 commutators of real J-symmetries.*

Proof. Suppose $\alpha = 1$. Let $\lambda > 1$ and $a, b \in \mathbb{R}$ such that $ab \neq 0$. If

$$K_1 = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b\lambda^{-1} & \lambda^{-1} & 0 \\ b & 0 & 0 & 1 \end{bmatrix} \text{ and } K_2 = \begin{bmatrix} \lambda^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then $K_1K_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a\lambda^{-1} & b\lambda^{-1} & 1 & 0 \\ b\lambda^{-1} & 0 & 0 & 1 \end{bmatrix}$, and K_1, K_2 are commutators of real J-symmetries by Lemma 3.8. If we let

$$S_1 = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2}b^{-1}(\lambda - a) \\ 0 & 0 & -1 & \frac{1}{2}b^{-1}(\lambda + a) \\ \frac{1}{2}\lambda^{-1}(-\lambda - a) & -b\lambda^{-1} & 0 & 0 \\ \frac{1}{2}\lambda^{-1}(\lambda - a) & -b\lambda^{-1} & 0 & 0 \end{bmatrix} \in \mathrm{Sp}(4, \mathbb{R}),$$

then

$$K_1K_2 = S_1^{-1} \left(\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) S_1. \tag{3.17}$$

By Lemma 2.6, $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is a product of 2 commutators of real J-symmetries. Furthermore, note that $\left(\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. By Lemma 2.6, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ is also a product of 2 commutators of real J-symmetries.

Suppose $\alpha = -1$ and $\mu = 1$. Let $\lambda > 1$, $a > 0$, and $b \neq 0$. If

$$K_1 = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b\lambda^{-1} & \lambda^{-1} & 0 \\ b & 0 & 0 & 1 \end{bmatrix} \text{ and } K_3 = \begin{bmatrix} \lambda^{-1} & 0 & -4a^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then $K_1K_3 = \begin{bmatrix} 1 & 0 & -4a^{-1}\lambda & 0 \\ 0 & 1 & 0 & 0 \\ a\lambda^{-1} & b\lambda^{-1} & -3 & 0 \\ b\lambda^{-1} & 0 & -4a^{-1}b & 1 \end{bmatrix}$.

In view of (3.15), with $c = -4a^{-1}$, we get $f_{K_1K_3}(x) = (1 - x)^2(1 + x)^2$ and $\sigma(K_1K_3) = \{-1, 1\}$. If we let

$$S_2 = \frac{1}{2}\sqrt{\lambda^{-1}a^{-1}} \begin{bmatrix} a & b & 0 & 0 \\ 0 & 0 & 2\lambda & -2ab^{-1}\lambda \\ -2a & -2b & 4\lambda & 0 \\ 0 & 2b & 0 & 0 \end{bmatrix} \in \mathrm{Sp}(4, \mathbb{R}),$$

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then

$$K_1K_3 = S_2^{-1}\left(\begin{bmatrix}-1 & -1\\ 0 & -1\end{bmatrix} \boxplus \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right) S_2. \tag{3.18}$$

By Lemma 2.6, $\begin{bmatrix}-1 & -1\\ 0 & -1\end{bmatrix} \boxplus \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ is a product of 2 commutators of real J-symmetries. Since

$$\left(\begin{bmatrix}-1 & -1\\ 0 & -1\end{bmatrix} \boxplus \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right)^{-1} = \begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix} \boxplus \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix},$$

by Lemma 2.6, $\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix} \boxplus \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$ is also a product of 2 commutators of real J-symmetries. □

LEMMA 3.13. *If $\alpha, \mu \in \{-1, 1\}$, then $\begin{bmatrix}\alpha & \mu\\ 0 & \alpha\end{bmatrix} \boxplus \begin{bmatrix}1 & \mu\\ 0 & 1\end{bmatrix}$ is a product of 3 commutators of real J-symmetries.*

Proof. Suppose that $\alpha = \mu = 1$, and let λ be a positive real number not equal to 1. Note that

$$\left(\begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix} \boxplus \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right)\left(\begin{bmatrix}\lambda^{-1} & \lambda^{-1}\\ 0 & \lambda\end{bmatrix} \boxplus \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \boxplus \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}.$$

By Lemma 3.11, the first factor on the left-hand side is a product of 2 commutators of real J-symmetries, and the second factor is a commutator of real J-symmetries by Lemma 2.8. Hence, $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \boxplus \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ is a product of 3 commutators of real J-symmetries. Since

$$\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \boxplus \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right)^{-1} = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} \boxplus \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix},$$

$\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} \boxplus \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$ is also a product of 3 commutators of real J-symmetries by Lemma 2.6.

Suppose $\alpha = -1$. Choose a positive real number $\lambda \neq 1$ and $a < 0$. Let $K_1 = \begin{bmatrix}\lambda & 0\\ a & \lambda^{-1}\end{bmatrix} \boxplus \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ and $K_2 = \begin{bmatrix}\lambda^{-1} & -4a^{-1}\\ 0 & \lambda\end{bmatrix} \boxplus I_2$. Note that $\lambda + \lambda^{-1} > 2$, so K_1 is a product of 2 commutators of J-symmetries by Lemma 3.11. By Lemma 2.8, K_2 is a commutator of real J-symmetries. Hence,

$$K_1K_2 = \begin{bmatrix}1 & -4a^{-1}\lambda\\ a\lambda^{-1} & -3\end{bmatrix} \boxplus \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},$$

is a product of 3 commutators of real J-symmetries. If we let

$$S = \left(\sqrt{-a^{-1}\lambda}\begin{bmatrix}0 & 1\\ a\lambda^{-1} & -2\end{bmatrix}\right) \boxplus I_2,$$

then

$$K_1K_2 = S\left(\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix} \boxplus \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right) S^{-1}. \tag{3.19}$$

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By Lemma 2.6, it follows that $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is a product of 3 commutators of real J-symmetries. Since $\left(\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, we have that $\begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \boxplus \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ is also a product of 3 commutators of real J-symmetries by Lemma 2.6. ☐

3.3. Back to the proof of Theorem 1. We summarize the technical results into the following lemma.

Lemma 3.14. *Let $A \in \mathrm{Sp}(2n, \mathbb{R})$. If A is a product of $2k$ real J-symmetries, then A is a product of at most $3k$ commutators of real J-symmetries.*

Proof. Let $A \in \mathrm{Sp}(2n, \mathbb{R})$ be a product of $2k$ real J-symmetries. Then we can write $A = \prod_{j=1}^{k} B_j$, where B_j is a product of 2 real J-symmetries for each j. By Lemma 3.1, each B_j can only be real symplectically similar to one of the following

(a) $I_{2n-2} \boxplus \mathrm{diag}(\lambda, \lambda^{-1})$, where $\lambda \neq 0$
(b) $I_{2n-2} \boxplus J_2(1)^{\alpha}$, where $\alpha \in \{-1, 1\}$
(c) $I_{2n-2} \boxplus J_2(-1)^{\alpha}$, where $\alpha \in \{-1, 1\}$
(d) $I_{2n-2} \boxplus R(\theta)$, where $\theta \in (-\pi, \pi) \backslash \{0\}$
(e) $I_{2n-4} \boxplus J_2(1)^{\alpha} \boxplus J_2(1)^{\beta}$, where $\alpha, \beta \in \{-1, 1\}$.
(f) $I_{2n-4} \boxplus \mathcal{P}_2(1, 0)$

The identity I_{2n} is a commutator of real J-symmetries.

If $\lambda > 0$ and $\lambda \neq 1$, then $I_{2n-2} \boxplus \mathrm{diag}(\lambda, \lambda^{-1})$ is a commutator of real J-symmetries by Lemma 2.8. If $\lambda < 0$ and $\lambda \neq -1$, then $I_{2n-2} \boxplus \mathrm{diag}(\lambda, \lambda^{-1})$ is a product of 2 commutators of real J-symmetries by Lemma 3.6. If $n > 1$ and $\lambda = -1$, then $I_{2n-2} \boxplus \mathrm{diag}(\lambda, \lambda^{-1})$ is a product of 3 commutators of real J-symmetries by Lemmas 3.7 and 2.9.

If $\alpha \in \{-1, 1\}$, then $I_{2n-2} \boxplus J_2(1)^{\alpha}$ is a product of 2 commutators of real J-symmetries by Lemma 3.6.

If $\alpha \in \{-1, 1\}$, then $I_{2n-2} \boxplus J_2(-1)^{\alpha}$ is a product of 2 commutators of real J-symmetries by Lemma 3.6.

If $\theta \in (-\pi, \pi) \backslash \{0\}$, then $I_{2n-2} \boxplus R(\theta)$ is a product of 2 commutators of real J-symmetries by Lemma 3.6.

If $\alpha = \beta$, then $I_{2n-4} \boxplus J_2(1)^{\alpha} \boxplus J_2(1)^{\beta}$ is a product of 3 commutators of real J-symmetries by Lemma 3.13.

If $\alpha \neq \beta$, then $I_{2n-4} \boxplus J_2(1)^{\alpha} \boxplus J_2(1)^{\beta}$ is a product of 2 commutators of real J-symmetries by Lemma 3.12.

Finally, $I_{2n-4} \boxplus \mathcal{P}_2(1, 0)$ is a product of 2 commutators of real J-symmetries by Lemmas 3.9 and 2.9.

Since B_j is a product of at most 3 commutators of real J-symmetries for each j, it follows that $A = \prod_{j=1}^{k} B_j$ is a product of at most $3k$ commutators of real J-symmetries. ☐

Every $2n \times 2n$ symplectic matrix P is a product of $\mathrm{rank}(P - I)$ J-symmetries except when P is similar to $-I_{2k} \boxplus I_{2n-2k}$ for some k. We also need to isolate this special case when counting the number of commutator of real J-symmetries factors.

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Lemma 3.15. *$-I_4$ is a product of 4 commutators of real J-symmetries.*

Proof. Let a be a real number such that $a \notin \{-1, 0, 1\}$. Note that

$$-I_4 = \left(\operatorname{diag}(a, a, a^{-1}, a^{-1})\right)\left(\operatorname{diag}(-a^{-1}, -a^{-1}, -a, -a)\right).$$

The first factor on the right-hand side is $\mathcal{P}_3$ with $a \neq 1$ and $b = 0$. By Lemma 3.10, it is a product of 2 commutators of real J-symmetries. The second factor on the right-hand side also has the same form as $\mathcal{P}_3$ with $b = 0$, so it is a product of 2 commutators of real J-symmetries. Therefore, $-I_4$ is a product of 4 commutators of real J-symmetries. □

Lemma 3.16. *Let $A = -I_{2n_1} \boxplus I_{2n_2}$, where n_1 and n_2 are positive integers with $n_1 + n_2 = n$.*

(a) If n_1 is odd, then A is a product of $2n_1 + 1$ commutators of real J-symmetries.
(b) If n_1 is even, then A is a product of $2n_1$ commutators of real J-symmetries.

Proof. (a) By Lemma 2.9, it suffices to show that if n_1 is an odd integer, then $-I_{2n_1}$ is a product of $2n_1 + 1$ commutators of real J-symmetries. Since n_1 is odd, we can write $n_1 = 2m - 1$ where m is a positive integer. We use induction on m. Beginning with $m = 1$, or $n_1 = 1$, we note that $-I_2$ is a product of $3 = 2(1) + 1$ commutators of real J-symmetries by Lemma 3.7.

Suppose that $-I_{4k-2} = -I_{2(2k-1)}$ is a product of $2(2k-1)+1 = 4k-1$ commutators of real J-symmetries. Then

$$-I_{2(2k+1)} = -I_{(4k-2)+4} = \left(-I_{4k-2} \boxplus I_4\right)\left(I_{4k-2} \boxplus -I_4\right).$$

By Lemma 2.9 and the inductive hypothesis, the first factor on the right-hand side is a product of $4k - 1$ commutators of real J-symmetries. By Lemmas 2.9 and 3.15, the second factor on the right-hand side is a product of 4 commutators of real J-symmetries. Therefore, $-I_{4k+2}$ is a product of $4k+3 = 2(2(k+1)-1)+1$ commutators of real J-symmetries. This concludes the induction.

(b) By Lemma 2.9, it suffices to show that if n_1 is an even integer, then $-I_{2n_1}$ is a product of $2n_1$ commutators of real J-symmetries. Suppose $n_1 = 2m$ where m is a positive integer. By Lemma 3.15, $-I_4$ is a product of 4 commutators of real J-symmetries.

Suppose that $-I_{4k} = -I_{2(2k)}$ is a product of $4k$ commutators of real J-symmetries. Then

$$-I_{4(k+1)} = -I_{4k+4} = \left(-I_{4k} \boxplus I_4\right)\left(I_{4k} \boxplus -I_4\right).$$

By Lemma 2.9 and the inductive hypothesis, the first factor on the right-hand side is a product of $4k$ commutators of real J-symmetries. By Lemmas 2.9 and 3.15, the second factor on the right-hand side is a product of 4 commutators of real J-symmetries. Therefore, $-I_{4(k+1)}$ is a product of $4k + 4 = 4(k + 1)$ commutators of real J-symmetries. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $P \in \mathrm{Sp}(2n, \mathbb{R})$, and let $m = \operatorname{rank}(P - I_{2n})$. By [4, Theorem 16], P can be written as a product of m J-symmetries if $J(P - I_{2n})$ is not skew-symmetric, and $m + 1$ J-symmetries if $J(P - I_{2n})$ is skew-symmetric.

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Suppose $J(P - I_{2n})$ is skew-symmetric. Then $(J(P - I_{2n}))^T = -J(P - I_{2n})$ if and only if $(P^T - I_{2n})(-J) = -JP + J$. Hence, $J(P - I_{2n})$ is skew-symmetric if and only if $-P^T J = -JP$, which is equivalent to $P^T JP = JP^2$. Since P is symplectic, $J = JP^2$, which implies that P is a real symplectic involution. It follows from Lemma 2.7 that P is real symplectically similar to $B = -I_{2n_1} \boxplus I_{2n_2}$ for some nonnegative integers n_1 and n_2, with $n_1 + n_2 = n$. By Lemma 3.16, B is a product of $2n_1 + 1 = 3n_1 - 2\lfloor \frac{n_1}{2} \rfloor$ commutators of real J-symmetries if n_1 is odd, and a product of $2n_1 = 3n_1 - 2\lfloor \frac{n_1}{2} \rfloor$ commutators of real J-symmetries of n_1 is even. Since P is real symplectically similar to B, $m = \text{rank}(B - I_{2n}) = 2n_1$. Hence, replacing each instance of n_1 with $\frac{m}{2}$, we have that P is a product of $\frac{3m}{2} - 2\lfloor \frac{m}{4} \rfloor$ commutators of real J-symmetries.

Suppose $J(P - I_{2n})$ is not skew-symmetric. If m is odd, then by [5, Lemma 2(4)], P can be written as a product of $m + 1$ J-symmetries. By Lemma 3.14, P is a product of $3\left(\frac{m+1}{2}\right) = 3\lceil \frac{m}{2} \rceil$ commutators of real J-symmetries. Finally, if m is even, then, by Lemma 3.14, P is a product of $3\left(\frac{m}{2}\right) = 3\lceil \frac{m}{2} \rceil$ commutators of real J-symmetries. □

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