

GODSIL–MCKAY SWITCHINGS FOR GAIN GRAPHS*

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Abstract. We introduce a switching operation, inspired by the Godsil–McKay switching, in order to obtain pairs of G -cospectral gain graphs, that are gain graphs cospectral with respect to every representation of the gain group G . For instance, for two signed graphs, this notion of cospectrality is equivalent to the cospectrality of their signed adjacency matrices together with the cospectrality of their underlying graphs. Moreover, we introduce another more flexible switching in order to obtain pairs of gain graphs cospectral with respect to some fixed unitary representation. Many existing notions of spectrum for graphs and gain graphs are indeed special cases of these spectra associated with particular representations, therefore our construction recovers the classical Godsil–McKay switching and the Godsil–McKay switching for signed and complex unit gain graphs. As in the classical case, not all gain graphs are suitable for these switchings: we analyze the relationships between the properties that make the graph suitable for the one or the other switching. Finally, we apply our construction in order to define a Godsil–McKay switching for the right spectrum of quaternion unit gain graphs.

Key words. Gain graph, Group representation, Godsil–McKay switching, G -cospectrality, π -cospectrality, Quaternionic matrix.

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1. Introduction. Spectral graph theory is the investigation of graphs via linear algebra. It mainly consists in studying invariants such as the spectrum, the spectral radius, the index, and many others, of matrices representing graphs, such as the *adjacency matrix*, the *Laplacian matrix*, and other generalizations. Many geometric and combinatorial properties of a graph can be deduced from these spectral invariants: the theory is rich and well established [13], and it is omnipresent in various fields from the most theoretical to the most applied ones [12].

For example, the first time anyone wondered about the existence of *pairs of nonisomorphic cospectral graphs*, it was in the world of chemistry [20] in the 1950s. In fact, at that time, the question was whether graphs were *determined by their spectrum* (see, for instance, [14, 15] for more historic details and further developments). Cospectral graphs have attracted a lot of attention for many reasons. In particular, they are a useful tool to outline the limits of the information that can be drawn from the spectrum and therefore the limits of spectral graph theory itself. A simple but emblematic example of a pair of cospectral graphs is the one depicted in Fig. 1, first given in [11] and called *Saltire pair* in [14], due to the similarity to the Scottish flag. One of the two graphs is connected and one is not: therefore, connectivity, at least in general, cannot be deduced from the adjacency spectrum.

But the real leap from an anecdotal research to a systematic study in the research of cospectral pairs happened when also a quantitative, probabilistic investigation of the spectral determination began. A remarkable achievement in this direction was the result that *almost all* trees are cospectral [26]. Many methods, operations, and routines have been proposed to obtain cospectrality: one of the most refined results, allowing the consolidation of various known techniques into a single construction, is the so-called *Godsil–McKay switching* [19].

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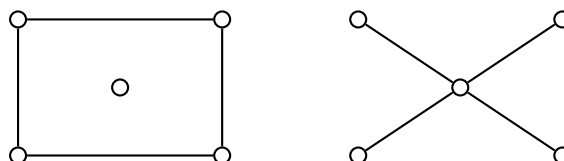


FIGURE 1. Two cospectral graphs.

The ubiquity of graph theory is due to the fact that a graph is a model for a system of interactions. When these interactions may be positive or negative, then the related natural object is instead a *signed graph*, i.e., a graph whose edges may be positive or negative. Formally, a signed graph is a pair (Γ, σ) where Γ is the *underlying graph* and σ is the *signature*, that is, a map assigning a sign to each adjacency. In order to model interactions that are even more varied, there exists a further generalization, given by the notion of *gain graph*. This is a pair (Γ, ψ) where still Γ is the underlying graph and ψ is the *gain function*, assigning to every ordered pair of adjacent vertices u, v of Γ an element $\psi(u, v)$, called *gain*, from a group G , in such a way that $\psi(v, u) = \psi(u, v)^{-1}$. Signed graphs are then particular gain graphs with $G = \mathbb{T}_2 = \{\pm 1\}$. As graphs are studied up to isomorphism, signed and gain graphs are studied up to *switching isomorphism* [28, 29, 30]. When the product of the gains (signs) along any closed walk is the unit element of G , the gain (signed) graph is said to be *balanced* [21]. Equivalently, a gain graph is balanced if it is switching isomorphic to its underlying graph endowed with the trivial gain function. For a regularly updated bibliography on signed and gain graphs, we refer to [32].

Spectral graph theory was generalized in a very natural way to signed graphs, which have $\{0, \pm 1\}$ -valued adjacency matrices [31]. The spectrum of a signed graph is even able to capture characteristics that are specific of signed graphs, such as balance [2]. Moreover, a signed version of the Godsil–McKay switching has been proposed in [4].

The generalization of the spectral theory to gain graphs is less immediate and univocal. Many particular cases have been considered in literature: first of all, the spectral theory of *complex unit gain graphs* [25] that are gain graphs whose gain group is $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, which can be also regarded as particular cases of weighted directed graphs [3]. Clearly, also in this case, complex matrices naturally arise and a spectral characterization of balance was proved [23], as well as an analog of the Godsil–McKay switching [5]. Also the spectral theory for *quaternion unit gain graphs* was recently investigated [6], thanks to the existence of several notions of spectrum for quaternion valued matrices [7, 17, 22, 33]. In [6], many of the classical results for signed and complex unit gain graphs are generalized to quaternion unit gain graphs. One of the by-products of the current paper is a generalization of the Godsil–McKay switching to this class of gain graphs. But our approach is definitely more general, and this is just an example of its possible applications.

For general gain graphs, adjacency matrices are not complex valued, but group algebra valued matrices and then the notion of spectrum is not unique. Anyway, complex matrices naturally arise from the underlying graph or from the cover graphs. In [8], a unifying approach was introduced thanks to the use of the theory of representations. Given a gain graph (Γ, ψ) on a group G together with a unitary representation π of G , there exists a complex adjacency matrix of (Γ, ψ) associated with π , obtained by replacing each occurrence of an element g with the block $\pi(g)$. The spectrum of the resulting matrix is called the π -*spectrum* of the gain graph. With this definition, the analogous spectral characterization of balance was proved in [8]. But in [10] further investigations on π -spectra and π -*cospectrality* highlighted the strong dependence on the choice of the representation π , and the need for a more canonical notion of cospectrality, namely the G -*cospectrality*, which for finite groups is equivalent to cospectrality with respect to all representations.

The main goal of the present paper is the introduction of routines, inspired by the Godsil–McKay switching [19] and its generalizations [4, 5], in order to obtain pairs of G -cospectral gain graphs and/or pairs of π -cospectral gain graphs for some unitary representation π of the group G , as well as an analysis of the relationships between the properties that make a graph suitable for the one or the other switching.

The already known Godsil–McKay switchings can be obtained as particular cases of this more general construction. Our construction significantly extends the scope, and we show this by providing nontrivial examples in the context of gain graphs that, if considered in the classical setting, would have been trivial or not suitable to that approach. In other words, their nontriviality is ensured precisely by the novelty of the construction.

It is worth mentioning that, in the very recent paper [1], within the same framework of spectral theory of abstract gain graphs, the authors provided a generalization of a modified version of the Godsil–McKay switching, originally introduced in [27] and generalized in [4, 5] to signed and complex unit gain graphs.

Section 2 contains an introduction to group representations, a summary on gain graphs and switching isomorphisms, and a summary on the different notions of cospectrality for general gain graphs and their relations.

In Section 3, the notion of G -GM partition is introduced together with its associated switching; in Theorem 3.6, it is proved that this switching produces pairs of G -cospectral gain graphs.

In the first part of Section 4, the notion of π -GM partition, where π is a unitary representation, is introduced, together with its associated switching; in Theorem 4.4 it is proved that this switching produces pairs of π -cospectral gain graphs. Then in Theorem 4.9 it is proved that a partition is G -GM if and only if it is π -GM with respect to all unitary representations of G or, equivalently, if it is λ_G -GM, where λ_G is the left regular representation. In the second part of Section 4, another switching is considered: the assumptions required to be a π -GM partition are somehow weakened, as long as there exists an element of the group whose image via π is $-I$ (when the representation is faithful, this is equivalent to the existence of a *central involution*).

Finally, in Section 5, after giving a brief introduction on quaternions and quaternionic matrices, we describe a Godsil–McKay switching for quaternion unit gain graphs and in Theorem 5.9 we prove, by using the theory of the previous section, that this switching actually produces pairs of right cospectral quaternion unit gain graphs.

2. Preliminaries.

2.1. Group representations and group algebras. Let G be a group with unit element 1_G . Let $M_k(\mathbb{C})$ be the set of all square matrices of size k with entries in \mathbb{C} and let $GL_k(\mathbb{C})$ be the group of all invertible matrices in $M_k(\mathbb{C})$. A *representation* π of *degree* k (we write $\deg \pi = k$) of G is a group homomorphism $\pi: G \rightarrow GL_k(\mathbb{C})$.

Let $U_k(\mathbb{C}) = \{M \in GL_k(\mathbb{C}) : M^{-1} = M^*\}$ be the subgroup of $GL_k(\mathbb{C})$ consisting of unitary matrices, where M^* is the Hermitian transpose of M . The representation π is said to be *unitary* if $\pi(g) \in U_k(\mathbb{C})$, for every $g \in G$. Two representations π and π' of degree k of G are said to be equivalent, and we write $\pi \sim \pi'$, if there exists a matrix $S \in GL_k(\mathbb{C})$ such that $\pi'(g) = S^{-1}\pi(g)S$ for every $g \in G$. This is an equivalence relation and it is known that, for a finite group G , each equivalence class contains a unitary representative: this is the reason why working with unitary representations is not restrictive in the finite case.

We will denote by I_k (or just I when the size is clear from the context) the identity matrix of size k in $M_k(\mathbb{C})$.

The *kernel* of a representation π is $\ker \pi = \{g \in G : \pi(g) = I\}$, and a representation is *faithful* if $\ker \pi = \{1_G\}$.

Given two representations π_1 and π_2 of G , the direct sum representation $\pi_1 \oplus \pi_2$ of G is defined by

$$(\pi_1 \oplus \pi_2)(g) = \left(\begin{array}{c|c} \pi_1(g) & 0 \\ \hline 0 & \pi_2(g) \end{array} \right) \quad \forall g \in G.$$

We will use the notation $\pi^{\oplus i}$ for the i -th iterated direct sum of a representation π with itself.

A representation π of G is said to be *irreducible* if there is no nontrivial invariant subspace for the action of G : that is, G acts on a complex vector space V of dimension k , and there is no nontrivial subspace W of V such that $\pi(g)w \in W$ for any $w \in W$ and for any $g \in G$. It is proven that π is irreducible if and only if it is not equivalent to any direct sum of representations. It is well known that if G has m conjugacy classes, then there exist m irreducible, pairwise inequivalent, representations π_0, \dots, π_{m-1} , which constitute a *complete system of irreducible representations of G* , such that for each representation π of G , there exist $k_0, \dots, k_{m-1} \in \mathbb{N} \cup \{0\}$ such that $\pi \sim \bigoplus_{i=0}^{m-1} \pi_i^{\oplus k_i}$. One says that π contains the irreducible representation π_i with multiplicity k_i if $k_i \neq 0$, and that π_i is a subrepresentation of π .

Now we recall some remarkable representations of a group G . The first one is the *trivial representation* $\pi_0: G \rightarrow \mathbb{C}$, such that $\pi_0(g) = 1$ for every $g \in G$. It is unitary and irreducible, and it always occurs in a complete system of irreducible representations of G .

We also have the *identical representation* $\pi_{id}: G \rightarrow GL_1(\mathbb{C})$, defined on each subgroup G of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ as $\pi_{id}(z) = z$.

A group G naturally acts by left multiplication on itself. This action can be identified with the action of G on the vector space $\mathbb{C}G = \{\sum_{x \in G} c_x x : c_x \in \mathbb{C}\}$. When G is finite, one gets the *left regular representation* λ_G , which is faithful and has degree $|G|$, obtained as the linear extension of the action $\lambda_G(g)h = gh$ for all $g, h \in G$, so that $\lambda_G(g)$ is the $|G| \times |G|$ matrix such that

$$(\lambda_G(g))_{k,h} = \begin{cases} 1 & \text{if } kh^{-1} = g \\ 0 & \text{otherwise} \end{cases} \quad k, h \in G.$$

It is known that, if π_0, \dots, π_{m-1} is a complete system of irreducible representations, then λ_G decomposes (up to equivalence) as:

$$\lambda_G \sim \bigoplus_{i=0}^{m-1} \pi_i^{\oplus \deg \pi_i}.$$

For further information and a more general discussion on group representation theory, we refer to [18].

Even when G is not finite, the space of all finite \mathbb{C} -linear combinations of elements of G is a vector space, denoted by $\mathbb{C}G$. Endowed with the product

$$\left(\sum_{x \in G} f_x x \right) \left(\sum_{y \in G} h_y y \right) := \sum_{x,y \in G} f_x h_y xy, \quad \text{for each } f = \sum_{x \in G} f_x x, h = \sum_{y \in G} h_y y \in \mathbb{C}G,$$

and with the involution

$$f^* := \sum_{x \in G} \overline{f_{x^{-1}}} x,$$

it is called the *group algebra* of G .

A representation π of degree k of G can be naturally extended by linearity to $\mathbb{C}G$. With a little abuse of notation, we denote with π this extension, which is in fact a homomorphism of algebras:

$$(2.1) \quad \begin{aligned} \pi: \mathbb{C}G &\rightarrow M_k(\mathbb{C}) \\ \sum_{x \in G} f_x x &\mapsto \sum_{x \in G} f_x \pi(x). \end{aligned}$$

Moreover, if π is unitary, then $\pi(f^*) = \pi(f)^*$ for any $f \in \mathbb{C}G$.

We denote by $M_{n,m}(\mathbb{C}G)$ the set of the group algebra valued $n \times m$ matrices. The product of $F \in M_{n,m}(\mathbb{C}G)$ and $H \in M_{m,t}(\mathbb{C}G)$ is the matrix $FH \in M_{n,t}(\mathbb{C}G)$ such that $(FH)_{i,j} = \sum_{k=1}^m F_{i,k} H_{k,j}$, where $F_{i,k} H_{k,j}$ is the product in $\mathbb{C}G$. When $n = m$, we use the notation $M_n(\mathbb{C}G)$. Actually, $M_n(\mathbb{C}G)$ is an algebra with the involution $*$ defined by $(F^*)_{i,j} = (F_{j,i})^*$, where the $*$ on the right is the involution in $\mathbb{C}G$. The identity matrix $I_{M_n(\mathbb{C}G)}$ is defined as the diagonal matrix whose diagonal entries are equal to 1_G , that is

$$I_{M_n(\mathbb{C}G)} = \begin{pmatrix} 1_G & & \\ & \ddots & \\ & & 1_G \end{pmatrix} = \text{diag}(1_G, \dots, 1_G).$$

Notice that $I_{M_n(\mathbb{C}G)}A = AI_{M_n(\mathbb{C}G)} = A$, for each $A \in M_n(\mathbb{C}G)$. Moreover, given $g \in G$ and $A \in M_n(\mathbb{C}G)$, we put

$$gA = \text{diag}(g, g, \dots, g)A \quad Ag = A \text{diag}(g, g, \dots, g).$$

In words, the matrix gA (resp. Ag) is obtained from A by multiplying each entry on the left (resp. on the right) by the element $g \in G$. Also observe that $gA = Ag$ when G is an Abelian group.

A representation $\pi: G \rightarrow GL_k(\mathbb{C})$ can be further extended to $M_{n,m}(\mathbb{C}G)$. We still use the notation

$$(2.2) \quad \pi: M_{n,m}(\mathbb{C}G) \rightarrow M_{nk, mk}(\mathbb{C}).$$

For $A \in M_{n,m}(\mathbb{C}G)$, the element $\pi(A) \in M_{nk, mk}(\mathbb{C})$ is called the *Fourier transform of A at π* . The matrix $\pi(A)$ is obtained from A by replacing each entry $a_{i,j} \in \mathbb{C}G$ with the block $\pi(a_{i,j}) \in M_k(\mathbb{C})$, where π is the map defined in Eq. (2.1) (see [9, Lemma 5.1] for the properties of this extension).

In particular this extension $\pi: M_n(\mathbb{C}G) \rightarrow M_{nk}(\mathbb{C})$ for square matrices is a homomorphism of algebras (see, e.g., [8]). Moreover, equivalence relations and subrepresentations are also preserved in their extensions to $M_n(\mathbb{C}G)$ as the next proposition claims.

PROPOSITION 2.1. [8, Proposition 3.4] *Let $A \in M_n(\mathbb{C}G)$ and let π, π' be two representations of G . Then:*

- if $\pi \sim \pi'$ then the matrices $\pi(A)$ and $\pi'(A)$ are similar;
- the matrices $(\pi \oplus \pi')(A)$ and $\pi(A) \oplus \pi'(A)$ are similar;
- if π is unitary, then $\pi(A^*) = \pi(A)^*$.

Observe that the homomorphisms in Eqs. (2.1) and (2.2) may be not injective, even when the representation π is faithful. To see that, let $G = \{1_G, g\}$ be the cyclic group of two elements, and consider the faithful representation $\pi: G \rightarrow GL_1(\mathbb{C})$ such that $\pi(1_G) = 1$ and $\pi(g) = -1$. In particular, the nonzero element $1_G + g$ of $\mathbb{C}G$ satisfies $\pi(1_G + g) = 0$. However, this is not the case if the representation π is (or contains) the regular representation, as the following proposition shows.

PROPOSITION 2.2. *Let G be a group and let λ_G be its left regular representation. Then the map*

$$\lambda_G: \mathbb{C}G \rightarrow M_{|G|}(\mathbb{C}),$$

of Eq. (2.1) and the map

$$\lambda_G: M_n(\mathbb{C}G) \rightarrow M_{n|G|}(\mathbb{C}),$$

of Eq. (2.2) are injective.

Proof. For every $g \in G$ there exist $i, j \in \{1, 2, \dots, |G|\}$ such that $gg_j = g_i$, so that $(\lambda_G(g))_{i,j} = 1$. Clearly, for all $h \in G$ with $h \neq g$, one has $hg_j \neq g_i$ and then $(\lambda_G(h))_{i,j} = 0$. As a consequence, for g_1, g_2, \dots, g_m distinct elements of G , for $c_1, c_2, \dots, c_m \in \mathbb{C}$, the matrix

$$\lambda_G(c_1g_1 + \dots + c_mg_m) = c_1\lambda_G(g_1) + \dots + c_m\lambda_G(g_m),$$

has disjoint entries equals to c_1, c_2, \dots, c_m . In particular if $\lambda_G(c_1g_1 + \dots + c_mg_m)$ is the zero matrix, it must be $c_1 = c_2 = \dots = c_m = 0$. This implies that the map $\lambda_G: \mathbb{C}G \rightarrow M_{|G|}(\mathbb{C})$ is injective.

Moreover if $A \in M_n(\mathbb{C}G)$ and $A \neq 0$, then there exist $i, j \in \{1, 2, \dots, n\}$ such that $A_{i,j} \neq 0$ in $\mathbb{C}G$. By virtue of the injectivity of λ_G on $\mathbb{C}G$ proved above, the matrix $\lambda_G(A_{i,j})$ has nonzero entries and so the same is true for $\lambda_G(A)$. This shows that the map $\lambda_G: M_n(\mathbb{C}G) \rightarrow M_{n|G|}(\mathbb{C})$ is injective. \square

2.2. Gain graphs and switching isomorphism. Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a finite simple graph, with vertex set V_Γ and edge set E_Γ . If $e = \{u, v\} \in E_\Gamma$, we write $u \sim v$ and we say that u and v are adjacent vertices of Γ . Let G be a group and consider a map ψ from the set of ordered pairs of adjacent vertices of Γ to G , such that $\psi(u, v) = \psi(v, u)^{-1}$. The pair (Γ, ψ) is said to be a G -gain graph (or, equivalently, a gain graph on G): the graph Γ is called the *underlying graph* and ψ is said a *gain function* on Γ .

Suppose $|V_\Gamma| = n$ and let us fix an ordering v_1, v_2, \dots, v_n in V_Γ . The adjacency matrix of (Γ, ψ) is the group algebra valued matrix $A_{(\Gamma, \psi)} \in M_n(\mathbb{C}G)$ with entries

$$(A_{(\Gamma, \psi)})_{i,j} = \begin{cases} \psi(v_i, v_j) & \text{if } v_i \sim v_j; \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the definition that $A_{(\Gamma, \psi)}^* = A_{(\Gamma, \psi)}$.

Let W be a *walk* of length h , that is, an ordered sequence of $h + 1$ (not necessarily distinct) vertices of Γ , say w_0, w_1, \dots, w_h such that $w_i \sim w_{i+1}$. One can define the gain $\psi(W) \in G$ of the walk W as

$$\psi(W) := \psi(w_0, w_1) \cdots \psi(w_{h-1}, w_h).$$

A *closed walk* of length h is a walk of length h such that $w_0 = w_h$. In what follows, we denote by $\mathcal{C}^h(\Gamma)$ the set of all closed walks of length h in Γ .

A gain graph (Γ, ψ) is *balanced* if $\psi(W) = 1_G$ for every closed walk W . An example of a balanced gain graph is that endowed with the *trivial gain function*, which is constantly equal to 1_G . When the underlying graph Γ is a tree, the gain graph (Γ, ψ) is trivially balanced.

A fundamental concept in the theory of gain graphs is the *switching equivalence*: two gain functions ψ_1 and ψ_2 on the same underlying graph Γ are switching equivalent if there exists $f: V_\Gamma \rightarrow G$ such that

$$(2.3) \quad \psi_2(v_i, v_j) = f(v_i)^{-1} \psi_1(v_i, v_j) f(v_j),$$

for any pair of adjacent vertices v_i and v_j of Γ . If Eq. (2.3) holds, we also write $\psi_2 = \psi_1^f$.

It turns out that a gain graph (Γ, ψ) is balanced if and only if the gain function ψ is switching equivalent to the trivial gain function (see [30, Lemma 5.3] or [24, Lemma 1.1]). Moreover, in analogy with the complex case (see [25, Lemma 3.2]), the following result holds.

THEOREM 2.3. [8, Theorem 4.1] *Let ψ_1 and ψ_2 be two gain functions on the same underlying graph Γ , with adjacency matrices $A_{(\Gamma, \psi_1)}$ and $A_{(\Gamma, \psi_2)} \in M_n(\mathbb{C}G)$, respectively. Then ψ_1 and ψ_2 are switching equivalent if and only if there exists a diagonal matrix $F \in M_n(\mathbb{C}G)$, with $F_{i,i} \in G$ for each $i = 1, \dots, n$, such that $F^* A_{(\Gamma, \psi_1)} F = A_{(\Gamma, \psi_2)}$.*

Two gain graphs (Γ_1, ψ_1) and (Γ_2, ψ_2) are said to be *switching isomorphic* if there exists a graph isomorphism $\phi: V_{\Gamma_1} \rightarrow V_{\Gamma_2}$ such that ψ_1 is switching equivalent to the gain function $\psi_2 \circ \phi$ on Γ_1 defined as $(\psi_2 \circ \phi)(u, v) = \psi_2(\phi(u), \phi(v))$.

REMARK 2.4. It can be seen that the adjacency matrices of switching isomorphic gain graphs can be obtained from each other by conjugating with a suitable group algebra valued matrix. More precisely, two gain graphs (Γ_1, ψ_1) and (Γ_2, ψ_2) are switching isomorphic if and only if

$$(PF)^* A_{(\Gamma_1, \psi_1)} (PF) = A_{(\Gamma_2, \psi_2)},$$

for some diagonal matrix $F \in M_n(\mathbb{C}G)$, with $F_{i,i} \in G$ for each $i = 1, \dots, n$, and for some matrix $P \in M_n(\mathbb{C}G)$ whose nonzero entries are equal to 1_G and where the positions of the nonzero entries correspond to those of a suitable permutation matrix.

2.3. G -cospectrality, π -spectrum, and π -cospectrality. In this section, we recall some basic definitions and properties from [10] about G -cospectrality and π -cospectrality of a gain graph (Γ, ψ) on a group G , where π is a representation of G .

Let us start by defining a group algebra valued trace map $Tr: M_n(\mathbb{C}G) \rightarrow \mathbb{C}G$ on $M_n(\mathbb{C}G)$ as $Tr(A) = \sum_{i=1}^n A_{i,i}$. Now observe that, if $A \in M_n(\mathbb{C}G)$ is the adjacency matrix of a G -gain graph (Γ, ψ) , by virtue of [8, Lemma 4.1] the entry $(A^h)_{i,j} \in \mathbb{C}G$ is the sum of the gains of all walks of length h from v_i to v_j . In particular, we have

$$Tr(A^h) = \sum_{W \in \mathcal{C}^h(\Gamma)} \psi(W).$$

Let us denote by $[g]$ the conjugacy class of $g \in G$. A *class function* $f: G \rightarrow \mathbb{C}$ is a map such that $g_1, g_2 \in [g] \implies f(g_1) = f(g_2)$. The set $\mathbb{C}_{Class}[G]$ of finitely supported class functions is a \mathbb{C} -vector space.

There exists a natural map $\mu: \mathbb{C}G \rightarrow \mathbb{C}_{Class}[G]$ defined as:

$$\mu \left(\sum_{x \in G} a_x x \right) (g) = \sum_{x \in [g]} a_x.$$

Notice that, if G is Abelian, each conjugacy class contains only one element and μ is nothing but an isomorphism between \mathbb{C} -vector spaces.

DEFINITION 2.5. Two group algebra valued matrices $A, B \in M_n(\mathbb{C}G)$, with $A^* = A$ and $B^* = B$, are *G -cospectral* if

$$\mu(Tr(A^h)) = \mu(Tr(B^h)) \quad \forall h \in \mathbb{N}.$$

Two gain graphs (Γ_1, ψ_1) and (Γ_2, ψ_2) on G are G -cospectral if the adjacency matrices $A_{(\Gamma_1, \psi_1)}$ and $A_{(\Gamma_2, \psi_2)}$ are G -cospectral.

Roughly speaking, we are asking that the sequences $\{Tr(A^h)\}$ and $\{Tr(B^h)\}$ coincide, up to conjugacy. Notice that the equality of the analogous sequences for complex matrices is equivalent to the classical notion of cospectrality.

Moreover, we have shown in [10, Theorem 4.5] that if two gain graphs (Γ_1, ψ_1) and (Γ_2, ψ_2) are switching isomorphic, then they are G -cospectral, whereas the inverse implication is not true. In fact, one of the main goals of the present paper consists in producing pairs of G -cospectral graphs which are not switching isomorphic. The following lemma holds.

LEMMA 2.6. *Let (Γ_1, ψ_1) and (Γ_2, ψ_2) be two G -gain graphs and let $A, B \in M_n(\mathbb{C}G)$ be their adjacency matrices, respectively. Let $C, D \in M_n(\mathbb{C}G)$ such that $CD = DC = I_{M_n(\mathbb{C}G)}$, where each entry of D is a complex multiple of 1_G . If $A = DBC$ then (Γ_1, ψ_1) and (Γ_2, ψ_2) are G -cospectral.*

Proof. Let $S \in M_n(\mathbb{C}G)$. Then

$$Tr(DS) = \sum_{i=1}^n (DS)_{i,i} = \sum_{i,k=1}^n D_{i,k} S_{k,i} = \sum_{k,i=1}^n S_{k,i} D_{i,k} = \sum_{k=1}^n (SD)_{k,k} = Tr(SD),$$

where $D_{i,k} S_{k,i} = S_{k,i} D_{i,k}$ is true for each $i, k = 1, \dots, n$, since the entries of D are all multiple of 1_G . Now using the fact that $CD = I_{M_n(\mathbb{C}G)}$, we obtain $A^h = (DBC)^h = DB^h C$ for each h in \mathbb{N} . Therefore, if we set $S = B^h C$, we get:

$$Tr(A^h) = Tr(DB^h C) = Tr(B^h CD) = Tr(B^h),$$

and so the matrices A and B are G -cospectral, according with Definition 2.5. □

Consider a G -gain graph (Γ, ψ) with adjacency matrix $A \in M_n(\mathbb{C}G)$. The matrix $\pi(A) \in M_{nk}(\mathbb{C})$ is called the *represented adjacency matrix* of (Γ, ψ) with respect to the representation π . Notice that, if π is unitary, then by Proposition 2.1 $\pi(A)$ is a Hermitian matrix and we say that its (real) spectrum is the π -spectrum of (Γ, ψ) , denoted with $\sigma_\pi(\Gamma, \psi)$ or also $\sigma_\pi(A)$. Moreover, again by Proposition 2.1, if $\pi \sim \pi'$, then $\sigma_\pi(A) = \sigma_{\pi'}(A)$. This leads to the following definition.

DEFINITION 2.7. Let (Γ_1, ψ_1) and (Γ_2, ψ_2) be two G -gain graphs. Let π be a unitary representation of G . The graphs (Γ_1, ψ_1) and (Γ_2, ψ_2) are said to be π -cospectral if $\sigma_\pi(\Gamma_1, \psi_1) = \sigma_\pi(\Gamma_2, \psi_2)$.

Although the spectrum with respect to a given representation π was formally introduced only in [8], particular cases of π -spectra have already been considered in the literature:

- If (Γ, ψ) is a G -gain graph and π_0 is the trivial representation of G , then $\pi_0(A_{(\Gamma, \psi)})$ is the adjacency matrix of the underlying graph Γ , and the π_0 -spectrum of (Γ, ψ) is the spectrum of Γ .
- If (Γ, ψ) is a \mathbb{T}_2 -gain graph and π_{id} is the identical representation of \mathbb{T}_2 , then $\pi_{id}(A_{(\Gamma, \psi)})$ is the classical adjacency matrix of the signed graph, and the π_{id} -spectrum of (Γ, ψ) is the classical spectrum of the signed graph.
- If (Γ, ψ) is a \mathbb{T} -gain graph and π_{id} is the identical representation of \mathbb{T} , then $\pi_{id}(A_{(\Gamma, \psi)})$ is the classical adjacency matrix of the complex unit gain graph, and the π_{id} -spectrum of (Γ, ψ) is the classical spectrum of the complex unit gain graph.

- If (Γ, ψ) is a G -gain graph and λ_G is the left regular representation of G , with G finite, then $\lambda_G(A_{(\Gamma, \psi)})$ is the adjacency matrix of the (left) cover graph of (Γ, ψ) , and the λ_G -spectrum of (Γ, ψ) is the spectrum of the cover graph of (Γ, ψ) .

It may happen that (Γ_1, ψ_1) and (Γ_2, ψ_2) are π_1 -cospectral but not π_2 -cospectral, even if π_1 and π_2 are faithful and irreducible [10, Example 5.7]. However, if two G -gain graphs (Γ_1, ψ_1) and (Γ_2, ψ_2) are π_1 -cospectral and π_2 -cospectral, with π_1 and π_2 unitary representations, then (Γ_1, ψ_1) and (Γ_2, ψ_2) are $(\pi_1 \oplus \pi_2)$ -cospectral by Proposition 2.1. The following theorem holds.

THEOREM 2.8. [10, Theorem 4.14] *Let (Γ_1, ψ_1) and (Γ_2, ψ_2) be two G -gain graphs, with G finite. Let π_0, \dots, π_{m-1} be a complete system of irreducible, unitary, representations of G . The following are equivalent.*

1. (Γ_1, ψ_1) and (Γ_2, ψ_2) are G -cospectral;
2. (Γ_1, ψ_1) and (Γ_2, ψ_2) are π -cospectral, for every unitary representation π of G ;
3. (Γ_1, ψ_1) and (Γ_2, ψ_2) are π_i -cospectral, for each $i = 0, \dots, m - 1$.

It follows from Theorem 2.8 that, in some sense, the G -spectrum of a G -gain graph can be defined as the list of its π_i -spectra, with π_i varying in a complete system of irreducible representations.

3. G -Godsil-McKay switching. Inspired by the classical construction given by Godsil and McKay in [19] which was already generalized to signed graphs and complex unit gain graphs [4, 5], we present in this section a construction allowing to produce pairs of G -cospectral gain graphs over an arbitrary group G .

Let (Γ, ψ) be a G -gain graph, and suppose that $\alpha = \{C_0, C_1, \dots, C_k\}$ is a partition of the vertex set V_Γ . Let $|V_\Gamma| = n$. For any $v \in V_\Gamma$, we set

$$(3.4) \quad \Psi_i(v) := \sum_{w \in C_i, w \sim v} \psi(v, w) \in \mathbb{C}G.$$

REMARK 3.1. If π_0 is the trivial representation mapping each element of G to 1, then $\pi_0(\Psi_i(v))$ is the number of vertices of C_i which are adjacent to v .

DEFINITION 3.2. A partition $\alpha = \{C_0, C_1, \dots, C_k\}$ of the vertex set V_Γ of a gain graph (Γ, ψ) is said to be a G -GM partition for (Γ, ψ) if:

- for every $i, j \in \{1, 2, \dots, k\}$ and for every $v, v' \in C_i$, one has

$$\Psi_j(v) = \Psi_j(v');$$

- for every $v \in C_0$, for every $i \in \{1, 2, \dots, k\}$, there exist $g_1, g_2 \in G \cup \{0\} \subset \mathbb{C}G$ such that

$$\Psi_i(v) = \frac{|C_i|}{2} g_1 + \frac{|C_i|}{2} g_2.$$

In words, in a G -GM partition, the number of edges issuing from a vertex v of C_i and arriving to vertices of C_j with gain g , does not depend on the choice of the starting vertex in C_i , and this must be true for every $i, j \in \{1, 2, \dots, k\}$ and every $g \in G$. Moreover, a vertex in the part C_0 must be adjacent to all vertices of a part C_i (half with a gain and half with another, not necessarily distinct, gain), or to exactly half of the vertices (all with the same gain), or it has no adjacent vertex in C_i (case $g_1 = g_2 = 0$).

EXAMPLE 3.3. Let (Γ, ψ) be the \mathbb{T} -gain graph represented in Fig. 2, where the undirected solid lines correspond to edges labeled by 1, whereas the orientation from v_0 to v_j , for $j \in \{2, 3, 7, 8\}$ is due to the

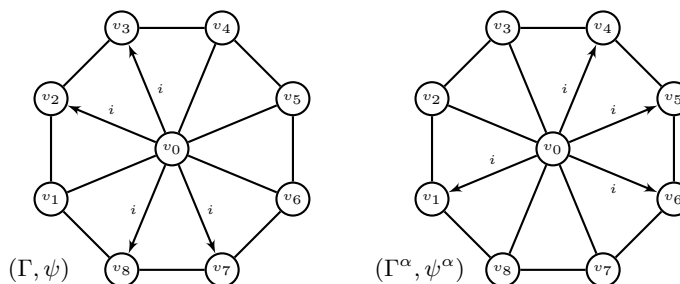


FIGURE 2. The \mathbb{T} -gain graphs (Γ, ψ) and $(\Gamma^\alpha, \psi^\alpha)$ of Examples 3.3 and 3.7.

nontrivial gains $\psi(v_0, v_j) = i$ (and therefore, $\psi(v_j, v_0) = i^{-1} = -i$). If we put $C_0 = \{v_0\}$ and $C_1 = \{v_j : j = 1, \dots, 8\}$, then it is easy to check that $\alpha = \{C_0, C_1\}$ is a \mathbb{T} -GM partition for (Γ, ψ) . In fact

$$\Psi_1(v_0) = 4 \cdot 1 + 4 \cdot i \quad \text{and} \quad \Psi_1(v_j) = 2 \quad \text{for } j = 1, \dots, 8.$$

DEFINITION 3.4. Let α be a G -GM partition for the gain graph (Γ, ψ) . The graph $(\Gamma, \psi)^\alpha = (\Gamma^\alpha, \psi^\alpha)$ is the gain graph such that:

- the adjacency and the gains between pairs of vertices in $\bigcup_{i=1}^k C_i$ are the same as in (Γ, ψ) ;
- for $v \in C_0$ and $i \in \{1, 2, \dots, k\}$ such that $\Psi_i(v) = \frac{|C_i|}{2} g_1 + \frac{|C_i|}{2} g_2$, with $g_1, g_2 \in G \cup \{0\}$, and for every $w \in C_i$, we set:

$$\psi^\alpha(v, w) = \begin{cases} g_1 & \text{if } \psi(v, w) = g_2, \\ g_2 & \text{if } \psi(v, w) = g_1, \end{cases}$$

with the convention that $\psi(v, w) = 0$ (resp. $\psi^\alpha(v, w) = 0$) means that v and w are not adjacent in Γ (resp. in Γ^α).

A G -GM partition α is said to be *nontrivial* if (Γ, ψ) and $(\Gamma, \psi)^\alpha$ are not switching isomorphic.

For each $i = 0, 1, \dots, k$, put $|C_i| = n_i$. Observe that $n_0 + \dots + n_k = |V_\Gamma|$. With a given G -GM partition $\alpha = \{C_0, C_1, \dots, C_k\}$, we associate a group algebra valued matrix $Q_\alpha \in M_{|V_\Gamma|}(\mathbb{C}G)$ in the following way:

$$(Q_\alpha)_{u,v} = \begin{cases} 1_G & \text{if } u = v \in C_0 \\ \frac{2-n_i}{n_i} 1_G & \text{if } u = v \in C_i, i = 1, \dots, k \\ \frac{2}{n_i} 1_G & \text{if } u, v \in C_i, u \neq v, i = 1, \dots, k \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the nonzero entries of Q_α are all real multiples of 1_G and $Q_\alpha^* = Q_\alpha$. Let $J_{M_n(\mathbb{C}G)} \in M_n(\mathbb{C}G)$ be the square matrix of size n whose entries are all equal to $1_G \in \mathbb{C}G$. Notice that, if we put

$$(3.5) \quad Q_n = \frac{2}{n} J_{M_n(\mathbb{C}G)} - I_{M_n(\mathbb{C}G)},$$

then it holds $Q_n^* = Q_n$ and we have

$$Q_\alpha = \begin{pmatrix} I_{M_{n_0}(\mathbb{C}G)} & 0 & \dots & 0 \\ 0 & Q_{n_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q_{n_k} \end{pmatrix}.$$

Observe that the matrix Q_n is inspired by the matrix considered in [19, Theorem 2.2].

LEMMA 3.5. *Let Q_n be the matrix defined in Eq. (3.5). Then the following formulas hold.*

1. $Q_n^2 = I_{M_n(\mathbb{C}G)}$.
2. Let $X \in M_{m,n}(\mathbb{C}G)$ with constant row sums and constant column sums. Then $Q_m X Q_n = X$.
3. Let $\lambda \in \mathbb{C}G$ and let $\mathbf{x} = (\lambda, \dots, \lambda)^T$ be a constant column of length n . Then $Q_n \mathbf{x} = \mathbf{x}$.
4. Let $\mathbf{x} = (x_1, \dots, x_n)^T$ be a column of length n , with $x_i \in \mathbb{C}G$ for each $i = 1, \dots, n$, such that $\sum_{i=1}^n x_i = 0$. Then $Q_n \mathbf{x} = -\mathbf{x}$.
5. Let g_1, g_2 in $G \cup \{0\}$ and let \mathbf{x} be a column of length n , such that half entries of \mathbf{x} are equal to g_1 and half entries are equal to g_2 . Then $Q_n \mathbf{x} = (g_1 + g_2) \cdot (1_G, \dots, 1_G)^T - \mathbf{x}$.

Proof. The proof is analogous to those of Lemma 2.1 and Proposition 2.3 of [5]. To give an idea of how computations work when dealing with elements from $\mathbb{C}G$ instead of complex numbers, we focus here only on the proof of property (2). Observe that the matrix X can be rewritten as

$$X = g_1 X_1 + g_2 X_2 + \dots + g_k X_k,$$

where $g_i \in G$ and the entries of X_i belong to the set $\{0, 1_G\}$, for each $i = 1, \dots, k$. Moreover, by assumption, each matrix X_i has constant row sums and constant column sums. Since the matrices X_i, Q_m, Q_n have entries which are complex multiple of 1_G , they behave as complex matrices in all computations. Therefore, it follows from Lemma 2.1 of [5] that $Q_m X_i Q_n = X_i$ for each $i = 1, \dots, k$. By linearity and using the fact that $g_i 1_G = 1_G g_i$ for each i , the property (2) follows. \square

About property (5), observe that the vector $Q_n \mathbf{x}$ is obtained from \mathbf{x} by switching the entries g_1 and g_2 . In particular, the property (5) recovers the property (3) when $g_1 = g_2 = \lambda$.

THEOREM 3.6. *Let (Γ, ψ) be a G -gain graph and let α be a G -GM partition, with associated matrix $Q_\alpha \in M_n(\mathbb{C}G)$. Then*

$$(3.6) \quad A_{(\Gamma^\alpha, \psi^\alpha)} = Q_\alpha A_{(\Gamma, \psi)} Q_\alpha.$$

In particular, (Γ, ψ) and $(\Gamma, \psi)^\alpha$ are G -cospectral

Proof. We can choose a suitable labeling of V_Γ such that

$$A_{(\Gamma, \psi)} = \begin{pmatrix} C_{0,0} & C_{0,1} & \dots & C_{0,k-1} & C_{0,k} \\ C_{0,1}^* & C_{1,1} & \dots & C_{1,k-1} & C_{1,k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{0,k-1}^* & C_{1,k-1}^* & \dots & C_{k-1,k-1} & C_{k-1,k} \\ C_{0,k}^* & C_{1,k}^* & \dots & C_{k-1,k}^* & C_{k,k} \end{pmatrix},$$

where, for all $i, j \in \{0, 1, \dots, k\}$, the block $C_{i,j} \in M_{n_i, n_j}(\mathbb{C}G)$ describes the adjacencies from vertices in C_i to vertices in C_j , so that:

- the matrix $C_{0,0}$ satisfies $C_{0,0}^* = C_{0,0}$;
- for each $i = 1, \dots, k$, the matrix $C_{i,i}$ satisfies $C_{i,i}^* = C_{i,i}$, and it has constant row sum and constant column sum;
- for each $i, j = 1, \dots, k$, with $i \neq j$, the matrix $C_{i,j}$ has constant row sum and constant column sum;
- for each $i = 1, \dots, k$, the rows of the matrix $C_{0,i}$ (or, equivalently, the columns of the matrix $C_{0,i}^*$) satisfy the property (5) of Lemma 3.5.

By virtue of Lemma 3.5, we obtain:

$$\begin{aligned}
 Q_\alpha A_{(\Gamma, \psi)} Q_\alpha &= \begin{pmatrix} C_{0,0} & C_{0,1} Q_{n_1} & \cdots & C_{0,k-1} Q_{n_{k-1}} & C_{0,k} Q_{n_k} \\ Q_{n_1} C_{0,1}^* & Q_{n_1} C_{1,1} Q_{n_1} & \cdots & Q_{n_1} C_{1,k-1} Q_{n_{k-1}} & Q_{n_1} C_{1,k} Q_{n_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_{n_{k-1}} C_{0,k-1}^* & Q_{n_{k-1}} C_{1,k-1}^* Q_{n_1} & \cdots & Q_{n_{k-1}} C_{k-1,k-1} Q_{n_{k-1}} & Q_{n_{k-1}} C_{k-1,k} Q_{n_k} \\ Q_{n_k} C_{0,k}^* & Q_{n_k} C_{1,k}^* Q_{n_1} & \cdots & Q_{n_k} C_{k-1,k}^* Q_{n_{k-1}} & Q_{n_k} C_{k,k} Q_{n_k} \end{pmatrix} \\
 &= \begin{pmatrix} C_{0,0} & C_{0,1} Q_{n_1} & \cdots & C_{0,k-1} Q_{n_{k-1}} & C_{0,k} Q_{n_k} \\ Q_{n_1} C_{0,1}^* & C_{1,1} & \cdots & C_{1,k-1} & C_{1,k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_{n_{k-1}} C_{0,k-1}^* & C_{1,k-1}^* & \cdots & C_{k-1,k-1} & C_{k-1,k} \\ Q_{n_k} C_{0,k}^* & C_{1,k}^* & \cdots & C_{k-1,k}^* & C_{k,k} \end{pmatrix},
 \end{aligned}$$

where we have used the property (2) of Lemma 3.5 to say that, for each $i, j = 1, \dots, k$, one has $Q_{n_i} C_{i,j} Q_{n_j} = C_{i,j}$. Finally, by using property (5) of Lemma 3.5 and the definition of $(\Gamma, \psi)^\alpha$, we can conclude that $Q_\alpha A_{(\Gamma, \psi)} Q_\alpha = A_{(\Gamma^\alpha, \psi^\alpha)}$, since the matrix $C_{0,i} Q_{n_i}$ (together with its Hermitian transpose $Q_{n_i} C_{0,i}^*$) corresponds to the adjacencies between the part C_0 and the part C_i of $(\Gamma, \psi)^\alpha$, for each $i = 1, \dots, k$. This completes the proof of Eq. (3.6). The last statement is a direct consequence of Lemma 2.6, with $C = D = Q_\alpha$. \square

Observe that, if α is a G -GM partition, then by Remark 3.1 α is also a good partition in the sense of the classical Godsil–McKay switching [19]. In particular, by applying the trivial representation π_0 to the members of Eq. (3.6), one recovers the cospectrality of the two graphs obtained with the classical construction. In some sense, the cospectrality produced by the G -Godsil–McKay construction implies the cospectrality of the underlying graphs according to the classical construction. Moreover, this remark suggests the possibility of investigating Eq. (3.6) with respect to any other representation π of G , by using the adjacency matrix with respect to that representation. This is exactly what we are going to do in Section 4.

The next example shows that the presence of gains makes the situation very rich and complex: in fact, it may happen that the G -Godsil–McKay construction produces gain graphs whose underlying graphs are isomorphic, even if the corresponding G -cospectral gain graphs are not switching isomorphic, when regarded as gain graphs.

EXAMPLE 3.7. In Example 3.3 we showed that $\alpha = \{C_0, C_1\}$ is a G -GM partition for the \mathbb{T} -gain graph (Γ, ψ) represented in Fig. 2. In Fig. 2 also the \mathbb{T} -gain graph $(\Gamma^\alpha, \psi^\alpha)$ is depicted, obtained according to Definition 3.4.

By Theorem 3.6 the gain graphs (Γ, ψ) and $(\Gamma^\alpha, \psi^\alpha)$ are \mathbb{T} -cospectral. Moreover, their underlying graphs Γ and Γ^α are isomorphic. However, (Γ, ψ) and $(\Gamma^\alpha, \psi^\alpha)$ are not switching isomorphic as gain graphs, and so α is nontrivial.

We will prove it by contradiction. Let us assume that there exist a switching function $\xi: V_\Gamma \rightarrow \mathbb{T}$ and a graph automorphism $\phi: V_\Gamma \rightarrow V_{\Gamma^\alpha}$ such that

$$(3.7) \quad \xi(v_i)^{-1} \psi(v_i, v_j) \xi(v_j) = \psi^\alpha(\phi(v_i), \phi(v_j)) \quad \text{for any } v_i \sim v_j.$$

Clearly it must be $\phi(v_0) = v_0$ and both ψ and ψ^α take only value 1 on pair of adjacent vertices in C_1 . As a consequence of Eq. (3.7), we have

$$1 = \xi(v_1)^{-1} \xi(v_2) = \xi(v_2)^{-1} \xi(v_3) = \cdots = \xi(v_8)^{-1} \xi(v_1),$$

that is, ξ must be constant on C_1 . Put $t := \xi(v_1)$. By considering Eq. (3.7) for the pair of adjacent vertices v_0, v_1 , we have

$$\xi(v_0)^{-1} t = \xi(v_0)^{-1} \psi(v_0, v_1) \xi(v_1) = \psi^\alpha(\phi(v_0), \phi(v_1)) = \psi^\alpha(v_0, \phi(v_1)) \in \{1, i\}.$$

Let us focus on the first case: if $\xi(v_0) = t$, then ξ is a constant function and so we obtain that $\psi = \psi^\alpha \circ \phi$. Observe that an automorphism of Γ acts on the set $\{v_1, \dots, v_8\}$ as an element of the dihedral group D_{16} , which is the group of symmetries of an octagon. On the other hand, in order to preserve the gains, such a permutation should send $\{v_1, v_4, v_5, v_6\}$ to $\{v_2, v_3, v_7, v_8\}$. However, it is not difficult to see that no element of D_{16} satisfies this property. This is a contradiction.

Let us study the second case: $\xi(v_0)^{-1}t = i$. This implies that $t = i\xi(v_0)$ and so, by considering Eq. (3.7) for the pair of adjacent vertices v_0, v_2 , we obtain:

$$\psi^\alpha(\phi(v_0), \phi(v_2)) = \xi(v_0)^{-1}\psi(v_0, v_2)t = i^2 = -1.$$

This is impossible, since -1 does not appear as a gain in $(\Gamma^\alpha, \psi^\alpha)$.

4. Godsil–McKay switching with respect to a representation. G -cospectrality is a very strong property, introduced in [10], which implies cospectrality with respect to all representations. In the previous literature, only the notion of cospectrality for gain graphs with respect to some privileged representation of the gain group has been considered, rather than G -cospectrality. For example, for signed graphs and complex unit gain graphs, the classical notion of spectrum and cospectrality is equivalent to that of π_{id} -cospectrality (see Section 2.3), and two cospectral gain graphs are not necessarily G -cospectral. As we will see in the next section, also for quaternion unit gain graphs the spectrum so far considered in the literature is that associated with a particular representation.

For this reason, it is convenient to have a more flexible routine working also for gain graphs with no G -GM partition, producing gain graphs that are cospectral at least with respect to some representations.

We will see two constructions: the first in Section 4.1, which is valid for any group G and any unitary representation π ; the second in Section 4.2, which is applicable only to those representations whose range contains $-I$ (which in the case of faithful representations implies the existence of a central involution in G). This second procedure generalizes Godsil–McKay switching for signed graphs and complex unit graphs [4, 5].

4.1. π -GM switching. Before describing this switching, it is useful to make an observation on what happens when one considers representations that are not faithful.

REMARK 4.1. Let (Γ_1, ψ_1) and (Γ_2, ψ_2) be two G -gain graphs and let π be a unitary representation of G . Suppose that, for any pair of adjacent vertices u and v , one has

$$\psi_1(u, v)\psi_2(u, v)^{-1} \in \ker \pi.$$

Then (Γ_1, ψ_1) and (Γ_2, ψ_2) are π -cospectral. In other words, if we modify (Γ, ψ) by multiplying (on the left and/or on the right) some gains by elements in $\ker \pi$ (consistently for both orientations, in such a way that we still have a gain function), we do not change the π -spectrum: this is obvious because we actually do not change the represented adjacency matrix $\pi(A_{(\Gamma, \psi)})$.

It is important to keep this observation in mind because the switching we are going to describe is not defined up to the operation of multiplying by elements of the kernel. On the contrary, a preliminary multiplication of this kind could even transform a gain graph not suitable for the switching into one that is suitable.

DEFINITION 4.2. A partition $\alpha = \{C_0, C_1, \dots, C_k\}$ of the vertex set V_Γ of a gain graph (Γ, ψ) is said to be a π -GM partition for (Γ, ψ) if:

- for every $i, j \in \{1, 2, \dots, k\}$ and for every $v, v' \in C_i$, one has

$$\pi(\Psi_j(v)) = \pi(\Psi_j(v'));$$

- for every $v \in C_0$, for every $i \in \{1, 2, \dots, k\}$, there exist $g_1, g_2 \in G \cup \{0\} \subset \mathbb{C}G$ such that

$$\Psi_i(v) = \frac{|C_i|}{2} g_1 + \frac{|C_i|}{2} g_2.$$

The analogy with the Definition 3.2 of a G -GM partition is clear and in fact a G -GM partition is a π -GM partition, for every representation π of G . But a π -GM partition is not necessarily a G -GM partition, as Example 4.3 shows. Moreover, a partition α is a π_0 -GM partition, where π_0 is the trivial representation, if and only if it is a suitable partition for the classic Godsil–McKay switching of the underlying graph Γ .

EXAMPLE 4.3. Let (Γ, ψ) be the S_4 -gain graph depicted in Fig. 3, where S_4 is the symmetric group on 4 elements. The usual rules apply: an edge without label represents an edge with trivial gain 1_{S_4} , and when the gain is an involution the orientation of the associated edge is not specified. Let $\pi_p: S_4 \rightarrow U_4(\mathbb{C})$ be the *permutation representation*, associating with each permutation its canonical $\{0, 1\}$ -valued permutation matrix. Let $\alpha = \{C_0, C_1\}$ be the partition with $C_0 = \{v_1\}$ and $C_1 = V_\Gamma \setminus C_0$. At first observe that v_1 is adjacent to exactly half of the vertices of C_1 , all with the same gain 1_{S_4} . Then the second condition of Definition 4.2 holds. Moreover:

$$\begin{aligned} \Psi_1(v_2) = \Psi_1(v_3) = \Psi_1(v_4) = \Psi_1(v_5) &= 1_{S_4} + (12)(34), \\ \Psi_1(v_6) = \Psi_1(v_7) = \Psi_1(v_8) = \Psi_1(v_9) &= (12) + (34). \end{aligned}$$

This implies that α is not a S_4 -GM partition. But

$$\pi_p(1_{S_4} + (12)(34)) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \pi_p((12) + (34)),$$

and then α is π_p -GM. Let π_s be the sign representation associating $\text{sgn}(\sigma) \in \{\pm 1\}$ with any $\sigma \in S_4$ according to its parity: then the partition α is not a π_s -GM partition, since:

$$\pi_s(\Psi_1(v_2)) = \pi_s(1_{S_4} + (12)(34)) = 2, \quad \pi_s(\Psi_1(v_6)) = \pi_s((12) + (34)) = -2.$$

Finally, if π_0 denotes the trivial representation, the partition α is a π_0 -GM partition, since the (underlying) graph induced by C_1 is regular and the unique vertex of C_0 is adjacent to exactly half of the vertices of C_1 .

Even if α is a π -GM partition for a gain graph (Γ, ψ) , and not necessarily a G -GM partition, we can still construct the matrix Q_α and the gain graph $(\Gamma^\alpha, \psi^\alpha)$ exactly as in Definition 3.4, and the following theorem holds.

THEOREM 4.4. *Let (Γ, ψ) be a G -gain graph, let π be a unitary representation of G , and let α be a π -GM partition. Then*

$$\pi(A_{(\Gamma^\alpha, \psi^\alpha)}) = \pi(Q_\alpha)\pi(A_{(\Gamma, \psi)})\pi(Q_\alpha).$$

In particular, (Γ, ψ) and $(\Gamma, \psi)^\alpha$ are π -cospectral.

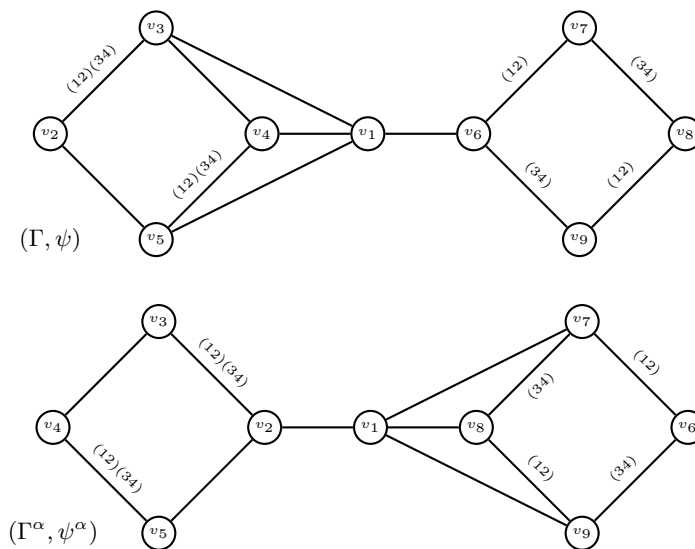


FIGURE 3. The S_4 -gain graphs (Γ, ψ) and $(\Gamma^\alpha, \psi^\alpha)$ of Examples 4.3 and 4.5.

Proof. The extension of π to $M_n(\mathbb{CG})$ is a homomorphism, then one has

$$\pi(Q_\alpha A_{(\Gamma, \psi)} Q_\alpha) = \pi(Q_\alpha) \pi(A_{(\Gamma, \psi)}) \pi(Q_\alpha).$$

By definition of Q_α we have:

$$Q_\alpha A_{(\Gamma, \psi)} Q_\alpha = \begin{pmatrix} C_{0,0} & C_{0,1} Q_{n_1} & \dots & C_{0,k-1} Q_{n_{k-1}} & C_{0,k} Q_{n_k} \\ Q_{n_1} C_{0,1}^* & Q_{n_1} C_{1,1} Q_{n_1} & \dots & Q_{n_1} C_{1,k-1} Q_{n_{k-1}} & Q_{n_1} C_{1,k} Q_{n_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_{n_{k-1}} C_{0,k-1}^* & Q_{n_{k-1}} C_{1,k-1}^* Q_{n_1} & \dots & Q_{n_{k-1}} C_{k-1,k-1} Q_{n_{k-1}} & Q_{n_{k-1}} C_{k-1,k} Q_{n_k} \\ Q_{n_k} C_{0,k}^* & Q_{n_k} C_{1,k}^* Q_{n_1} & \dots & Q_{n_k} C_{k-1,k}^* Q_{n_{k-1}} & Q_{n_k} C_{k,k} Q_{n_k} \end{pmatrix}.$$

Notice that, as in the proof of Theorem 3.6, by virtue of property (5) of Lemma 3.5, the matrix $C_{0,i} Q_{n_i}$ (together with its Hermitian transpose $Q_{n_i} C_{0,i}^*$) corresponds to the adjacencies between the part C_0 and the part C_i of $(\Gamma, \psi)^\alpha$, for each $i = 1, \dots, k$. By construction of $(\Gamma^\alpha, \psi^\alpha)$ the adjacencies and the gains of the parts C_1, \dots, C_k are the same as in (Γ, ψ) and then in order to get the statement we only need to prove that

$$\pi(Q_{n_i} C_{i,j} Q_{n_j}) = \pi(C_{i,j}), \quad \forall i, j \in \{1, 2, \dots, k\}.$$

Let us set $C_i = \{v_1, \dots, v_{n_i}\}$ and $C_j = \{w_1, \dots, w_{n_j}\}$ in such a way that the element in position h, l of the block $C_{i,j}$ is $\psi(v_h, w_l)$ if v_h and w_l are adjacent, and 0 otherwise.

Notice that the sum of the elements of the h -th row of $C_{i,j}$ is $\Psi_j(v_h)$, and the sum along the l -th column is $\Psi_i(w_l)^*$.

Differently from what happens for a G -GM partition, the row sums and the column sums are not constant as elements of \mathbb{CG} , but the first property of Definition 4.2 ensures that the possible differences all live in the kernel of the extension of π to \mathbb{CG} . Therefore, $C_{i,j}$ can be written as the sum of a constant row and column sum matrix with a matrix in the kernel of the extension of π to $M_{n_i, n_j}(\mathbb{CG})$. Let us denote by $S \in \mathbb{CG}$ the sum of all entries of $C_{i,j}$. Let us define $D \in M_{n_i, n_j}(\mathbb{CG})$ by:

$$D_{h,l} = \begin{cases} \frac{S}{n_i} - \Psi_j(v_h) & \text{if } h \neq 1 \text{ and } l = 1 \\ \frac{S}{n_j} - \Psi_i(w_l)^* & \text{if } h = 1 \text{ and } l \neq 1 \\ \frac{S}{n_i} + \frac{S}{n_j} - \Psi_j(v_1) - \Psi_i(w_1)^* & \text{if } h = 1 \text{ and } l = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{pmatrix} \frac{S}{n_i} + \frac{S}{n_j} - \Psi_j(v_1) - \Psi_i(w_1)^* & \frac{S}{n_j} - \Psi_i(w_2)^* & \cdots & \frac{S}{n_j} - \Psi_i(w_{n_j})^* \\ \frac{S}{n_i} - \Psi_j(v_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{S}{n_i} - \Psi_j(v_{n_i}) & 0 & \cdots & 0 \end{pmatrix}.$$

It is easy to check that actually $\pi(D) = 0$, since by the first property of Definition 4.2 one has

$$\pi\left(\frac{S}{n_i}\right) = \pi(\Psi_j(v_h)), \quad \pi\left(\frac{S}{n_j}\right) = \pi(\Psi_i(w_l)^*), \quad \forall h \in \{1, \dots, n_i\}, l \in \{1, \dots, n_j\}.$$

Let us prove that the matrix $C_{i,j} + D$ has constant row sums and constant column sums in $\mathbb{C}G$. Clearly for $h > 1$ and $l > 1$, the sum of the elements of the h -th row of $C_{i,j} + D$ is $\frac{S}{n_i}$ and the sum of the elements of the l -th column of $C_{i,j} + D$ is $\frac{S}{n_j}$. Let us check that the same is true for the first row and the first column:

$$\begin{aligned} \sum_{l=1}^{n_j} (C_{i,j} + D)_{1,l} &= \sum_{l=1}^{n_j} (C_{i,j})_{1,l} + \frac{S}{n_i} + \frac{S}{n_j} - \Psi_j(v_1) - \Psi_i(w_1)^* + \sum_{l=2}^{n_j} \left(\frac{S}{n_j} - \Psi_i(w_l)^*\right) \\ &= \Psi_j(v_1) + \frac{S}{n_i} + \frac{S}{n_j} - \Psi_j(v_1) - \Psi_i(w_1)^* + \frac{n_j - 1}{n_j} S - \sum_{l=2}^{n_j} \Psi_i(w_l)^* \\ &= \frac{S}{n_i} + S - S = \frac{S}{n_i}, \end{aligned}$$

and similarly

$$\begin{aligned} \sum_{h=1}^{n_i} (C_{i,j} + D)_{h,1} &= \sum_{h=1}^{n_i} (C_{i,j})_{h,1} + \frac{S}{n_i} + \frac{S}{n_j} - \Psi_j(v_1) - \Psi_i(w_1)^* + \sum_{h=2}^{n_i} \left(\frac{S}{n_i} - \Psi_j(v_h)\right) \\ &= \Psi_i(w_1)^* + \frac{S}{n_i} + \frac{S}{n_j} - \Psi_j(v_1) - \Psi_i(w_1)^* + \frac{n_i - 1}{n_i} S - \sum_{h=2}^{n_i} \Psi_j(v_h) \\ &= \frac{S}{n_j} + S - S = \frac{S}{n_j}. \end{aligned}$$

Now using property (2) of Lemma 3.5 and the property of the extension of π (see [9, Lemma 5.1]) we can conclude the proof:

$$\begin{aligned} \pi(Q_{n_i} C_{i,j} Q_{n_j}) &= \pi(Q_{n_i} (C_{i,j} + D - D) Q_{n_j}) = \pi(Q_{n_i} (C_{i,j} + D) Q_{n_j} - Q_{n_i} D Q_{n_j}) \\ &= \pi(C_{i,j} + D - Q_{n_i} D Q_{n_j}) = \pi(C_{i,j}) + 0 - \pi(Q_{n_i}) 0 \pi(Q_{n_j}) = \pi(C_{i,j}). \end{aligned} \quad \square$$

EXAMPLE 4.5. In Example 4.3, we presented an S_4 -gain graph (Γ, ψ) and a π_p -GM partition α for (Γ, ψ) . In Fig. 3 the associated gain graph $(\Gamma^\alpha, \psi^\alpha)$ is depicted. By an explicit computation one can check that the 36×36 complex matrices $\pi_p(A_{(\Gamma, \psi)})$ and $\pi_p(A_{(\Gamma^\alpha, \psi^\alpha)})$ are cospectral and so (Γ, ψ) and $(\Gamma^\alpha, \psi^\alpha)$ are π_p -cospectral. However, it is possible to see that $\pi_s(A_{(\Gamma, \psi)})$ and $\pi_s(A_{(\Gamma^\alpha, \psi^\alpha)})$ are not cospectral and so (Γ, ψ) and $(\Gamma^\alpha, \psi^\alpha)$ are not π_s -cospectral. Finally, notice that the underlying graphs Γ and Γ^α are cospectral (actually, they are isomorphic), as we expected since the partition α is also a π_0 -GM partition.

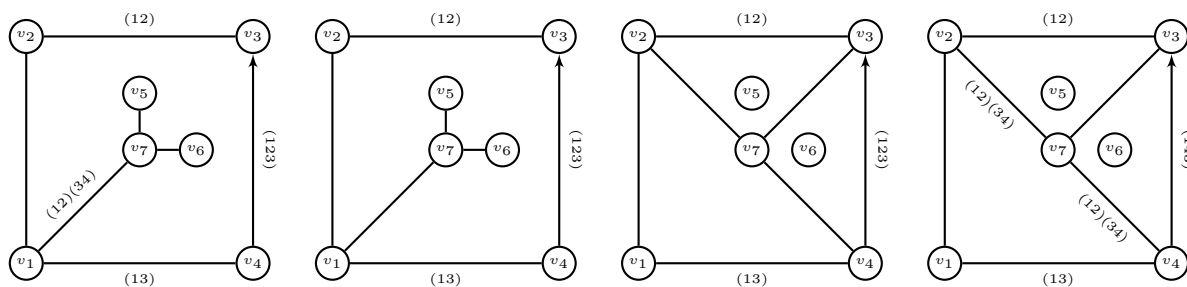


FIGURE 4. From left to right, the S_4 -gain graphs (Γ, ψ_1) , (Γ, ψ_2) , $(\Gamma^\alpha, \psi_2^\alpha)$ and (Γ^α, ψ_3) of Example 4.7.

REMARK 4.6. Notice that for a partition $\alpha = \{C_0, C_1, \dots, C_k\}$ the second condition of Definition 4.2 is the same as in Definition 3.2: for any $v \in C_0$ and any $i \in \{1, \dots, k\}$, if there are edges from v to C_i then half of the vertices of C_i are adjacent to v with a gain $g_1 \in G$ and the other half are adjacent to v with a gain $g_2 \in G$ or are not adjacent at all. Actually, we may weaken this condition by asking that half of the vertices have gain with the same image via π : this would make sense if π were not faithful. Anyway elements in G with the same image via π are equal up to multiplication by elements of $\ker \pi$, this means that this generalization would produce nothing that could not be achieved by combining the switching with the operation described in Remark 4.1 of multiplication of some gains by some elements of $\ker \pi$.

EXAMPLE 4.7. In Fig. 4, four S_4 -gain graphs are depicted. We consider the partition $\alpha = \{C_0, C_1\}$ on Γ , with $C_0 = \{v_7\}$ and $C_1 = V_\Gamma \setminus \{v_7\}$, and the sign representation π_s of S_4 . The partition α is not π_s -GM for (Γ, ψ_1) because the second condition of Definition 4.2 is not satisfied. In the light of Remark 4.6, one can notice that v_7 is adjacent to half of the vertices of C_1 with gains having the same image equal to 1 via π_s . Then one can consider the gain graph (Γ, ψ_2) that is π_s -cospectral with (Γ, ψ_1) by Remark 4.1, since it is obtained from (Γ, ψ_1) by multiplying one edge by an element in $\ker \pi_s$. Now it is easy to check that α is π_s -GM for (Γ, ψ_2) and then (Γ, ψ_2) and $(\Gamma^\alpha, \psi_2^\alpha)$ are π_s -cospectral. Finally, we can again multiply some edges by elements in $\ker \pi_s$ obtaining (Γ^α, ψ_3) , that by Remark 4.1 is π_s -cospectral with $(\Gamma^\alpha, \psi_2^\alpha)$. Finally, by transitivity, the gain graphs (Γ, ψ_1) and (Γ^α, ψ_3) are π_s -cospectral.

If two gain graphs are cospectral with respect to some unitary representations, then they are cospectral with respect to any direct sum of those representations, while the converse is, in general, false ([10, Example 5.4]). On the other hand, for the cospectrality obtained via the by GM-switching, both the implications hold.

PROPOSITION 4.8. Let (Γ, ψ) be a G -gain graph and let π_1, π_2 be unitary representations of G . A partition α is a π_1 -GM partition and a π_2 -GM partition for (Γ, ψ) if and only if it is a $(\pi_1 \oplus \pi_2)$ -GM partition for (Γ, ψ) .

Proof. The second property of Definition 4.2 does not depend on the representation. Therefore, in order to prove the claim, it is enough to show that, for a partition $\alpha = \{C_0, C_1, \dots, C_k\}$, the property

$$\pi_1(\Psi_j(v)) = \pi_1(\Psi_j(v')) \text{ and } \pi_2(\Psi_j(v)) = \pi_2(\Psi_j(v')), \quad \forall i, j \in \{1, 2, \dots, k\}, \forall v, v' \in C_i,$$

is equivalent to

$$(\pi_1 \oplus \pi_2)(\Psi_j(v)) = (\pi_1 \oplus \pi_2)(\Psi_j(v')), \quad \forall i, j \in \{1, 2, \dots, k\}, \forall v, v' \in C_i.$$

This is true since merely at a matrix level one has

$$A_1 = A_2 \text{ and } B_1 = B_2 \iff A_1 \oplus B_1 = A_2 \oplus B_2. \quad \square$$

In the light of Proposition 4.8, it is not reductive to restrict, if necessary, to irreducible representations.

We have already noticed that a G -GM partition is π -GM for every unitary representation. At least when the group G is finite, we know from Theorem 2.8 that G -cospectrality is equivalent to π -cospectrality for all (irreducible) representations of G .

The next theorem implies that for the cospectrality obtained by the GM-switchings the analogous result holds; moreover, it is enough to look at the regular representation λ_G (which is not true for the cospectrality in general instead [10, Remark 4.15]).

THEOREM 4.9. *Let G be a finite group and let π_0, \dots, π_{m-1} be a complete system of irreducible representations of G . Let (Γ, ψ) be a G -gain graph and let α be a partition of the vertex set V_Γ . The following are equivalent.*

1. α is a G -GM partition;
2. α is a π -GM partition for every representation π ;
3. α is a π_i -GM partition for each $i = 0, \dots, m - 1$;
4. α is a λ_G -GM partition.

Proof. Implications (1) \implies (2) \implies (3) are obvious; implication (3) \implies (4) follows from Proposition 4.8.

Let us prove implication (4) \implies (1). Suppose that $\alpha = \{C_0, \dots, C_k\}$ is a λ_G -GM partition. The second condition of Definition 4.2 is equal to that of Definition 3.2, then we only need to prove that

$$\Psi_j(v) = \Psi_j(v'), \quad \forall i, j \in \{1, 2, \dots, k\}, \forall v, v' \in C_i,$$

by assuming that

$$\lambda_G(\Psi_j(v)) = \lambda_G(\Psi_j(v')), \quad \forall i, j \in \{1, 2, \dots, k\}, \forall v, v' \in C_i.$$

By Proposition 2.2 the extension of λ_G to $\mathbb{C}G$ is injective and the claim follows. \square

It follows that G -GM switching and λ_G -GM switching are the same. It is worth noting that this is not true for any faithful representation: we have already seen in the Example 4.3 that a partition may be a π -GM partition, with π faithful, without being a G -GM partition.

4.2. π -GM switching when G has a central involution. In this section, we consider a gain group G and a representation π for which there exists an element $s \in G$ such that $\pi(s) = -I$. Notice that if the representation π is faithful, the element s must be a central involution of G . When such an element exists, we can further weaken the condition for a partition to be a π -GM partition. Therefore, we can update our definition.

DEFINITION 4.10. A partition $\alpha = \{C_0, C_1, \dots, C_k\}$ of the vertex set V_Γ of a gain graph (Γ, ψ) is said to be a π -GM partition for (Γ, ψ) if:

- for every $i, j \in \{1, 2, \dots, k\}$ and for every $v, v' \in C_i$, one has

$$\pi(\Psi_j(v)) = \pi(\Psi_j(v'));$$

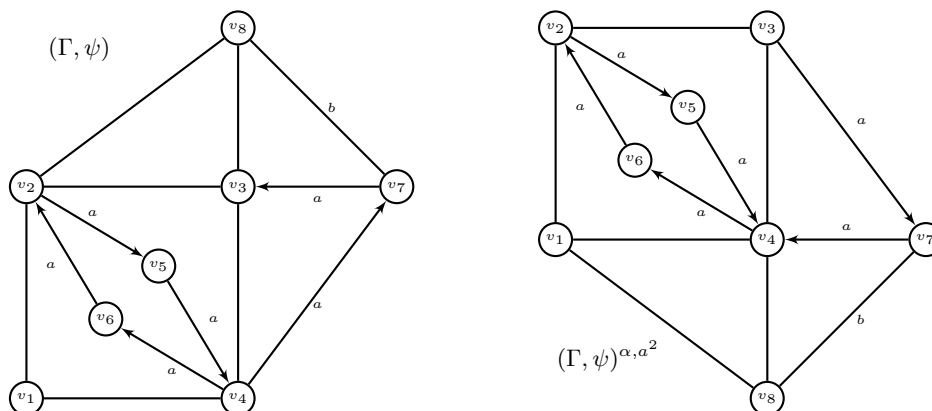


FIGURE 5. The D_8 -gain graphs (Γ, ψ) and $(\Gamma, \psi)^{\alpha, a^2}$ of Examples 4.11, 4.14 and 4.16.

- for every $v \in C_0$, for every $i \in \{1, 2, \dots, k\}$, at least one of the following conditions holds:
 1. there exist $g_1, g_2 \in G \cup \{0\} \subset \mathbb{C}G$ such that $\Psi_i(v) = \frac{|C_i|}{2} g_1 + \frac{|C_i|}{2} g_2$,
 2. there exists $s \in G$ such that $\pi(s) = -I$ and $\pi(\Psi_i(v)) = 0$.

EXAMPLE 4.11. Let

$$D_8 = \langle a, b \mid a^4 = b^2 = 1_{D_8}, bab = a^{-1} \rangle,$$

be the *Dihedral group* of order 8. Recall that the elements of D_8 can be regarded as the symmetries of a square: in particular, a is a rotation of $\pi/2$ and b is a reflection. Let us denote by π_2 the representation of D_8 of degree 2 such that

$$\pi_2(a) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \pi_2(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Observe that π_2 is a faithful representation, and that the element a^2 is a central involution such that $\pi_2(a^2) = -I$. On the left of Fig. 5, the D_8 -gain graph (Γ, ψ) is depicted. Let us consider the partition $\alpha = \{C_0, C_1, C_2\}$ with $C_0 = \{v_7, v_8\}$, $C_1 = \{v_1, v_2, v_3, v_4\}$ and $C_2 = \{v_5, v_6\}$.

One has

$$\Psi_1(v_i) = \begin{cases} 2 \cdot 1_{D_8} & \text{if } i \in \{1, 2, 3, 4, 8\} \\ a + a^3 & \text{if } i \in \{5, 6, 7\} \end{cases} \quad \Psi_2(v_i) = \begin{cases} 0 & \text{if } i \in \{1, 3, 5, 6, 7, 8\} \\ a + a^3 & \text{if } i \in \{2, 4\}. \end{cases}$$

The partition α is not a π_2 -GM partition with respect to the definition given in Section 4.1: one can check that the only obstruction is that the second condition of Definition 4.2 does not hold for v_7 . But we have

$$\pi_2(\Psi_1(v_7)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and then α is a π_2 -GM partition according to Definition 4.10.

REMARK 4.12. Proposition 4.8 is still valid with this updated definition. The first property of Definition 4.10 is in fact exactly the same of Definition 4.2. If the second property in Definition 4.10 holds for $\pi_1 \oplus \pi_2$ then the same is true for both π_1 and π_2 . Vice versa, for a vertex $v \in C_0$ and for $i \in \{1, 2, \dots, k\}$, condition (1) of the second property in Definition 4.10 holds or not independently of the representation: the only thing to prove is that if condition (2) of the second property in Definition 4.10 holds for both π_1 and π_2 , then it

holds for $\pi_1 \oplus \pi_2$. But this is simply the consequence of how the direct sum of representations and matrices is defined. Therefore also Theorem 4.9 can be proved, with this updated definition, with an analogous proof, just taking into account that for the left regular representation λ_G the condition (2) of the second property of Definition 4.10 never occurs.

This time we need to describe the switched graph because it may be different from the one described for G -GM partitions and actually it can depend on the chosen representation.

DEFINITION 4.13. Let α be a π -GM partition for (Γ, ψ) and let $s \in G$ be such that $\pi(s) = -I$. The gain graph $(\Gamma, \psi)^{\alpha, s} = (\Gamma^{\alpha, s}, \psi^{\alpha, s})$ is defined as follows:

- the adjacency and the gains between pairs of vertices in $\bigcup_{i=1}^k C_i$ are as in (Γ, ψ) ;
- for $v \in C_0$ and $i \in \{1, 2, \dots, k\}$, if we are in the case (1) of Definition 4.10, so that $\Psi_i(v) = \frac{|C_i|}{2} g_1 + \frac{|C_i|}{2} g_2$, with $g_1, g_2 \in G \cup \{0\}$, for every $w \in C_i$, then we set:

$$\psi^{\alpha, s}(v, w) = \begin{cases} g_1 & \text{if } \psi(v, w) = g_2, \\ g_2 & \text{if } \psi(v, w) = g_1, \end{cases}$$

with the convention that $\psi(v, w) = 0$ (resp. $\psi^{\alpha, s}(v, w) = 0$) means that v and w are not adjacent in Γ (resp. in $\Gamma^{\alpha, s}$). If we are not in the case (1) and we are in the case (2) of Definition 4.10, so that $\pi(\Psi_i(v)) = 0$, then we set

$$\psi^{\alpha, s}(v, w) = s \psi(v, w).$$

The partition α is said to be *nontrivial* if (Γ, ψ) and $(\Gamma, \psi)^{\alpha, s}$ are not switching isomorphic.

EXAMPLE 4.14. On the right of Fig. 5, the D_8 -gain graph $(\Gamma, \psi)^{\alpha, a^2}$ is depicted, associated to the π_2 -GM partition α for (Γ, ψ) discussed in Example 4.11. In fact in D_8 the element a^2 is a central involution and $\pi_2(a^2) = -I$.

Notice that Γ and Γ^{α, a^2} are not isomorphic, since the vertex v_4 of Γ^{α, a^2} has degree 6, but no vertex in Γ has such a degree. In particular, (Γ, ψ) and $(\Gamma, \psi)^{\alpha, a^2}$ are not switching isomorphic and then α is nontrivial. It is worth noting that the underlying graph Γ^{α, a^2} is not the graph that would have been obtained with a classic Godsil–McKay switching on Γ (which in fact respects all the conditions described in [19]).

Observe that the multiplication by s is the analog of the switch of the sign in the Godsil–McKay switching defined for signed and complex unit gain graphs; more precisely, for subgroups of \mathbb{T} containing $s = -1$ and with respect to the representation π_{id} , the switching described in Definition 4.13 is the same of that defined in [4, 5]. Notice that if both (1) and (2) holds, we have arbitrarily decided that (1) has priority over (2) in the construction of $(\Gamma, \psi)^{\alpha, s}$. This is not completely irrelevant, because when the representation π is not faithful, the switch of g_1 with g_2 described for case (1) may not produce the same gain graph obtained by multiplying by s instead.

THEOREM 4.15. Let (Γ, ψ) be a G -gain graph, let π be a unitary representation of G and α be a π -GM partition. Then

$$\pi(A_{(\Gamma^{\alpha, s}, \psi^{\alpha, s})}) = \pi(Q_\alpha) \pi(A_{(\Gamma, \psi)}) \pi(Q_\alpha).$$

In particular, (Γ, ψ) and $(\Gamma, \psi)^{\alpha, s}$ are π -cospectral.

Proof. A direct computation gives:

$$Q_\alpha A_{(\Gamma, \psi)} Q_\alpha = \begin{pmatrix} C_{0,0} & C_{0,1} Q_{n_1} & \cdots & C_{0,k-1} Q_{n_{k-1}} & C_{0,k} Q_{n_k} \\ Q_{n_1} C_{0,1}^* & Q_{n_1} C_{1,1} Q_{n_1} & \cdots & Q_{n_1} C_{1,k-1} Q_{n_{k-1}} & Q_{n_1} C_{1,k} Q_{n_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_{n_{k-1}} C_{0,k-1}^* & Q_{n_{k-1}} C_{1,k-1}^* Q_{n_1} & \cdots & Q_{n_{k-1}} C_{k-1,k-1} Q_{n_{k-1}} & Q_{n_{k-1}} C_{k-1,k} Q_{n_k} \\ Q_{n_k} C_{0,k}^* & Q_{n_k} C_{1,k}^* Q_{n_1} & \cdots & Q_{n_k} C_{k-1,k}^* Q_{n_{k-1}} & Q_{n_k} C_{k,k} Q_{n_k} \end{pmatrix}.$$

As in the proof of Theorem 4.4 we know that

$$\pi(Q_{n_i} C_{i,j} Q_{n_j}) = \pi(C_{i,j}),$$

for $i, j \in \{1, \dots, k\}$, and then the blocks in position i, j of $\pi(Q_\alpha) \pi(A_{(\Gamma, \psi)}) \pi(Q_\alpha)$ and $\pi(A_{(\Gamma^{\alpha,s}, \psi^{\alpha,s})})$ coincide. We have to prove that for any $i \in \{1, \dots, k\}$ the matrix $Q_{n_i} C_{0,i}^*$ and the associated block in $A_{(\Gamma^{\alpha,s}, \psi^{\alpha,s})}$ have the same image via π . As in the proof of Theorem 3.6, the columns of $Q_{n_i} C_{0,i}^*$ associated with vertices v of C_0 such that there exist $g_1, g_2 \in G \cup \{0\}$ with $\Psi_i(v) = \frac{|C_i|}{2} g_1 + \frac{|C_i|}{2} g_2$, coincide with the corresponding columns in $A_{(\Gamma^{\alpha,s}, \psi^{\alpha,s})}$. In particular, they have the same π -image. Suppose now that $v \in C_0$ is such that

$$\pi(\Psi_i(v)) = 0,$$

and that there exists $s \in G$ such that $\pi(s) = -I$. Let us denote by \mathbf{x} the associated column in $C_{0,i}^*$. More precisely, if $C_i = \{v_1, \dots, v_{n_i}\}$, the entry \mathbf{x}_l is $\psi(v, v_l)^*$ if v and v_l are adjacent and 0 otherwise. We define a constant column \mathbf{y} whose image via π is the zero column as

$$\mathbf{y}_l = \frac{\Psi_i(v)^*}{n_i}, \quad l \in \{1, \dots, n_i\}.$$

In fact, one has $\pi(\Psi_i(v)^*) = 0$ since $\pi(\Psi_i(v)) = 0$. Also notice that

$$\sum_{l=1}^{n_i} (\mathbf{x} - \mathbf{y})_l = 0,$$

and so, by properties (3) and (4) of Lemma 3.5 one has

$$Q_{n_i} \mathbf{x} = Q_{n_i} (\mathbf{x} - \mathbf{y}) + Q_{n_i} \mathbf{y} = 2\mathbf{y} - \mathbf{x},$$

and then

$$\pi(Q_{n_i} \mathbf{x}) = -\pi(\mathbf{x}),$$

since $\pi(\Psi_i(v)) = 0$. On the other hand, by definition of $(\Gamma^{\alpha,s}, \psi^{\alpha,s})$, the column corresponding to the adjacency between the vertex v with the part C_i is exactly $s \mathbf{x}$, whose image via π is $-\pi(\mathbf{x})$, and this concludes the proof. \square

EXAMPLE 4.16. An explicit computation shows that the characteristic polynomial of the π_2 -adjacency matrix of both the D_8 -gain graphs (Γ, ψ) and $(\Gamma, \psi)^{\alpha, a^2}$ of Example 4.14 (depicted in Fig. 5) is

$$x^{16} - 26x^{14} + 263x^{12} - 1306x^{10} + 3297x^8 - 3968x^6 + 1984x^4 - 256x^2.$$

5. Godsil-McKay switching for quaternion unit gain graphs. Let \mathbb{H} be the algebra of real quaternions, that is, the unital associative \mathbb{R} -algebra whose generators are i, j and k , where

$$i^2 = j^2 = k^2 = ijk = -1.$$

Every element $q \in \mathbb{H}$ can be written as

$$(5.8) \quad q = a + bi + cj + dk,$$

where the coefficients $a, b, c, d \in \mathbb{R}$ are uniquely determined. Given $q \in \mathbb{H}$ as in Eq. (5.8), the *imaginary part* of q is $Im(q) = bi + cj + dk$, whereas the *real part* of q is $Re(q) = a$. The *conjugate* of a quaternion q is defined as $\bar{q} := Re(q) - Im(q)$. Notice that \mathbb{H} is not a commutative algebra.

Given $q \in \mathbb{H}$ as in Eq. (5.8), the *norm* of q is defined as

$$|q| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Notice that $|q_1 q_2| = |q_1| |q_2|$. The *inverse* of an element $q \neq 0 \in \mathbb{H}$ is then defined as

$$q^{-1} = \frac{\bar{q}}{|q|^2},$$

and it satisfies the equalities $qq^{-1} = q^{-1}q = 1$. Two quaternions $q_1, q_2 \in \mathbb{H}$ are said to be *similar* if there exists $h \in \mathbb{H}$ such that

$$q_1 = h^{-1} q_2 h.$$

If q_1 and q_2 are similar, then we write $q_1 \sim q_2$. The similarity is clearly an equivalence relation, and we denote by $[q]$ the equivalence class of q with respect to this equivalence. Observe that if $Im(q) = 0$, that is, q is real, then $[q] = \{q\}$, since in this case $qh = hq$ for each h in \mathbb{H} . Moreover, one has $\bar{q} \sim q$ for each $q \in \mathbb{H}$.

By [17, Lemma 2.2], we know that for each $q \in \mathbb{H}$ there is a unique $\lambda_q \in \mathbb{C}$ such that $Im(\lambda_q) \geq 0$ and

$$[q] = [\lambda_q] = [\overline{\lambda_q}].$$

In words, we can say that there exists a 1-to-1 correspondence between the set of similarity classes of quaternions and the set of complex numbers with nonnegative imaginary part. Finally, we recall that the set

$$U(\mathbb{H}) = \{q \in \mathbb{H} : |q| = 1\},$$

of quaternions whose norm is equal to 1 constitutes a multiplicative group. The main object of this section is the investigation of $U(\mathbb{H})$ -gain graphs.

Let us put $\mathbb{H}^n = \{(q_1, \dots, q_n) : q_i \in \mathbb{H} \forall i = 1, \dots, n\}$ and let us denote by $M_n(\mathbb{H})$ the set of $n \times n$ matrices with entries in \mathbb{H} . As in the complex case, given a matrix $A = (a_{ij}) \in M_n(\mathbb{H})$, its *conjugate transpose* is the matrix $A^* = (\overline{a_{ji}}) \in M_n(\mathbb{H})$. A matrix $A \in M_n(\mathbb{H})$ is *Hermitian* if $A = A^*$, it is *normal* if $AA^* = A^*A$ and it is *unitary* if $AA^* = A^*A = I_{M_n(\mathbb{H})}$. Notice that the adjacency matrix of a $U(\mathbb{H})$ -gain graph is a Hermitian matrix in $M_n(\mathbb{H})$.

For a square matrix $A \in M_n(\mathbb{H})$, we say that $q \in \mathbb{H}$ is a *right eigenvalue* if there is a nonzero column vector $\mathbf{x} = (q_1, \dots, q_n)^T$ such that $A\mathbf{x} = \mathbf{x}q$. Notice that if $q \in \mathbb{H}$ is a right eigenvalue, then any $q' \in [q]$ is a right eigenvalue. Indeed, if $q' = h^{-1}qh$ and $A\mathbf{x} = \mathbf{x}q$, then the column vector $\mathbf{x}h$ satisfies the equation $A(\mathbf{x}h) = (\mathbf{x}h)q'$.

Spectral theory of quaternionic matrices is a topic studied in the literature [7, 17, 22, 33], also in relation with gain graphs [6]. It has even been shown that the analogue of the spectral theorem holds.

THEOREM 5.1. [17, Theorem 3.3, Proposition 3.8] *If $A \in M_n(\mathbb{H})$ is normal, then there are matrices $D, U \in M_n(\mathbb{H})$ such that*

- U is a unitary matrix, D is a diagonal matrix, and $U^*AU = D$;
- each diagonal entry of D is a complex number λ such that $\text{Im}(\lambda) \geq 0$;
- $q \in \mathbb{H}$ is a right eigenvalue of A if and only if $[q] = [\lambda]$ for some diagonal element λ of D .

Moreover, if $A = A^*$, the matrix D is real.

It follows from Theorem 5.1 that with a Hermitian matrix $A \in M_n(\mathbb{H})$ a real diagonal matrix D is associated. The *right spectrum* of A , which we denote by $\sigma_r(A)$, is the real multiset of elements on the main diagonal of D . In particular, two Hermitian matrices are *right cospectral* if and only if they are conjugated with each other via a unitary matrix. Clearly, two $U(\mathbb{H})$ -gain graphs are right cospectral if and only if their adjacency matrices are.

Observe that each quaternion $q = a + bi + cj + dk$ can be rewritten as

$$q = (a + bi) + (c + di)j,$$

Similarly, every quaternionic matrix $A \in M_n(\mathbb{H})$ can be decomposed as $A = A_1 + A_2j$, with $A_1, A_2 \in M_n(\mathbb{C})$. Then the *complex adjoint matrix* of A is the matrix $f(A) \in M_{2n}(\mathbb{C})$ defined by

$$f(A) = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix}.$$

The following properties of the complex adjoint matrix hold (see, for instance, [17, 22]).

PROPOSITION 5.2. *For each $A, B \in M_n(\mathbb{H})$ the following properties hold:*

- $f(I_{M_n(\mathbb{H})}) = I_{2n}$;
- $f(A + B) = f(A) + f(B)$;
- $f(AB) = f(A)f(B)$;
- $f(A^*) = f(A)^*$;
- $f(A^{-1}) = f(A)^{-1}$, when A^{-1} exists.

These properties turn out to be very useful in order to investigate the right spectrum of a quaternionic matrix.

PROPOSITION 5.3. *Let $A \in M_n(\mathbb{H})$, with $A^* = A$. Then the spectrum of $f(A)$ consists exactly of two copies of the right spectrum of A . In particular, two Hermitian matrices A and B are right cospectral if and only if $f(A)$ and $f(B)$ are cospectral.*

Proof. Let $A \in M_n(\mathbb{H})$, with $A^* = A$. Then, by Theorem 5.1, there exist a real, diagonal matrix D and a unitary matrix U such that $A = UDU^*$. By Proposition 5.2, we have $f(A) = f(U)f(D)f(U)^*$: but $f(D)$ is a diagonal matrix where each diagonal element of D appears twice: the statement follows. \square

Another interesting consequence of Proposition 5.2 is that the restriction of the map f to $M_1(\mathbb{H})$ induces a unitary representation of $U(\mathbb{H})$, which we are going to introduce in the next definition.

DEFINITION 5.4. We denote by $\pi_{\mathbb{H}}: U(\mathbb{H}) \rightarrow GL_2(\mathbb{C})$ the unitary representation of degree 2 of $U(\mathbb{H})$ such that:

$$\pi_{\mathbb{H}}(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \pi_{\mathbb{H}}(i) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \pi_{\mathbb{H}}(j) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \pi_{\mathbb{H}}(k) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We are now in position to present a Godsil–McKay switching for the right spectrum of quaternion unit gain graphs.

Let (Γ, ψ) be a $U(\mathbb{H})$ -gain graph, and let $\alpha = \{C_0, C_1, \dots, C_k\}$ be a partition of the vertex set V_{Γ} . For any $v \in V_{\Gamma}$ and for each $i \in \{0, 1, \dots, k\}$, we set

$$\Psi_i^{\mathbb{H}}(v) := \sum_{w \in C_i, w \sim v} \psi(v, w),$$

where the sum is in \mathbb{H} . Also, notice that -1 is a central involution in $U(\mathbb{H})$.

DEFINITION 5.5. A partition $\alpha = \{C_0, C_1, \dots, C_k\}$ of the vertex set V_{Γ} of a gain graph (Γ, ψ) is said to be a *quaternionic GM partition* for (Γ, ψ) if:

- for every $i, j \in \{1, 2, \dots, k\}$ and for every $v, v' \in C_i$, one has

$$\Psi_j^{\mathbb{H}}(v) = \Psi_j^{\mathbb{H}}(v');$$

- for every $v \in C_0$, for every $i \in \{1, 2, \dots, k\}$, at least one of the following conditions holds:
 1. there exist $q_1, q_2 \in U(\mathbb{H}) \cup \{0\}$ such that v is adjacent to half of the vertices of C_i with gain q_1 and to the other half with gain q_2 (with the usual convention that the zero gain corresponds to no adjacency);
 2. $\Psi_i^{\mathbb{H}}(v) = 0$.

EXAMPLE 5.6. Let us consider the $U(\mathbb{H})$ -gain graph (Γ, ψ) depicted on the top of Fig. 6, and its partition $\alpha := \{C_0, C_1\}$ with $C_0 = \{v_7, v_8\}$ and $C_1 = \{v_1, \dots, v_6\}$.

We have

$$\begin{aligned} \Psi_1^{\mathbb{H}}(v_i) &= i - i = 0 && \text{for } i = 1, 2, 3, 4; \\ \Psi_1^{\mathbb{H}}(v_i) &= 0 && \text{for } i = 5, 6; \\ \Psi_1^{\mathbb{H}}(v_7) &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}j + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}j - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}k - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}k = 0, \end{aligned}$$

and there is an edge with gain 1 connecting v_8 to exactly half of the vertices of C_1 . As a consequence, α is a quaternionic GM partition, according to Definition 5.5.

DEFINITION 5.7. Let α be a quaternionic GM partition for (Γ, ψ) . Then the gain graph $(\Gamma, \psi)^{\alpha, \mathbb{H}} = (\Gamma^{\alpha, \mathbb{H}}, \psi^{\alpha, \mathbb{H}})$ is defined as follows:

- the adjacency and the gains between pairs of vertices in $\bigcup_{i=1}^k C_i$ are the same as (Γ, ψ) ;
- for $v \in C_0$ and $i \in \{1, 2, \dots, k\}$, if we are in the case (1) of Definition 5.5, then we set:

$$\psi^{\alpha, \mathbb{H}}(v, w) = \begin{cases} q_1 & \text{if } \psi(v, w) = q_2, \\ q_2 & \text{if } \psi(v, w) = q_1, \end{cases}$$

with the convention that $\psi(v, w) = 0$ (resp. $\psi^{\alpha, \mathbb{H}}(v, w) = 0$) means that v and w are not adjacent in Γ (resp. in $\Gamma^{\alpha, \mathbb{H}}$). If we are not in the case (1) and we are in the case (2) of Definition 5.5, so that $\Psi_i^{\mathbb{H}}(v) = 0$, then we set

$$\psi^{\alpha, \mathbb{H}}(v, w) = -\psi(v, w).$$

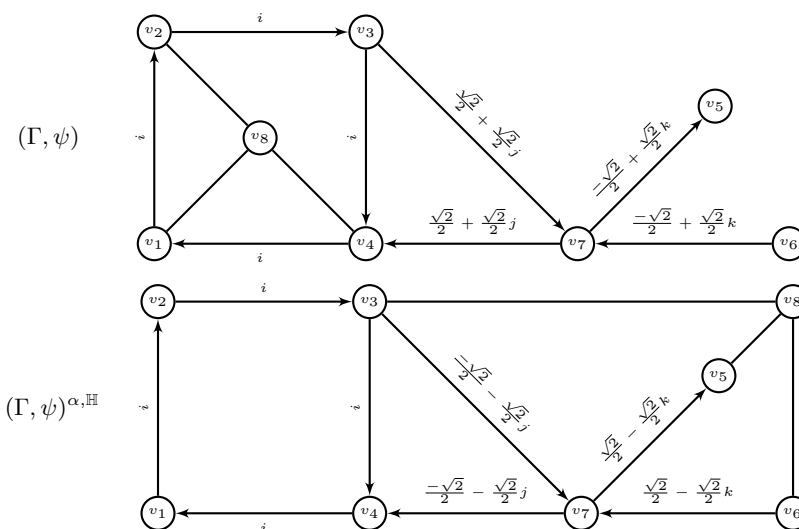


FIGURE 6. The $U(\mathbb{H})$ -gain graphs (Γ, ψ) and $(\Gamma, \psi)^{\alpha, \mathbb{H}}$ from Examples 5.6 and 5.8.

The quaternionic GM partition α is *nontrivial* if (Γ, ψ) and $(\Gamma, \psi)^{\alpha, \mathbb{H}}$ are not switching isomorphic.

EXAMPLE 5.8. In Fig. 6, the $U(\mathbb{H})$ -gain graphs (Γ, ψ) and $(\Gamma, \psi)^{\alpha, \mathbb{H}}$, associated with the partition α described in Example 5.6 are depicted. Notice that (Γ, ψ) and $(\Gamma, \psi)^{\alpha, \mathbb{H}}$ are not switching isomorphic, since the underlying graphs Γ and $\Gamma^{\alpha, \mathbb{H}}$ are clearly nonisomorphic. Then α is a nontrivial quaternionic GM partition for (Γ, ψ) .

THEOREM 5.9. Let (Γ, ψ) be a $U(\mathbb{H})$ -gain graph and let α be a quaternionic GM partition. Then (Γ, ψ) and $(\Gamma, \psi)^{\alpha, \mathbb{H}}$ are right cospectral.

Proof. As a first step, we show that α is a quaternionic GM partition for (Γ, ψ) if and only if it is a $\pi_{\mathbb{H}}$ -GM partition, where $\pi_{\mathbb{H}}$ is the unitary representation of $U(\mathbb{H})$ of Definition 5.4. As in Eq. (3.4), we have $\Psi_j(v) = \sum_{w \in C_j, w \sim v} \psi(v, w) \in \mathbb{C}U(\mathbb{H})$. One can easily check that

$$\pi_{\mathbb{H}}(\Psi_j(v)) = \pi_{\mathbb{H}}(\Psi_j(v')) \iff \Psi_j^{\mathbb{H}}(v) = \Psi_j^{\mathbb{H}}(w),$$

and that

$$\pi_{\mathbb{H}}(\Psi_j(v)) = 0 \iff \Psi_j^{\mathbb{H}}(v) = 0.$$

As a consequence, the properties of Definition 4.10 hold for α if and only if the properties of Definition 5.5 hold, and then a quaternionic GM partition is a $\pi_{\mathbb{H}}$ -GM partition. Moreover, the graph $(\Gamma, \psi)^{\alpha, \mathbb{H}}$ is exactly the $U(\mathbb{H})$ -gain graph $(\Gamma, \psi)^{\alpha, -1}$ of Definition 4.13 with respect to $\pi_{\mathbb{H}}$.

By Theorem 4.15, the gain graphs (Γ, ψ) and $(\Gamma, \psi)^{\alpha, \mathbb{H}}$ are $\pi_{\mathbb{H}}$ -cospectral. As a second step, we prove that the $\pi_{\mathbb{H}}$ -spectrum is nothing but the spectrum of the complex adjoint matrix.

Let $A \in M_n(\mathbb{H})$. We know that for each entry $a_{r,s}$ of A , there are two complex numbers $a_{r,s}^1$ and $a_{r,s}^2$ such that

$$a_{r,s} = a_{r,s}^1 + a_{r,s}^2 j.$$

Now, by applying Definition 5.4, one can check that $\pi_{\mathbb{H}}(a_{r,s}) = \begin{pmatrix} a_{r,s}^1 & a_{r,s}^2 \\ -\bar{a}_{r,s}^2 & \bar{a}_{r,s}^1 \end{pmatrix}$ and so

$$\pi_{\mathbb{H}}(A) = \left(\begin{array}{cc|cc} a_{1,1}^1 & a_{1,1}^2 & \cdots & a_{1,n}^1 & a_{1,n}^2 \\ -\bar{a}_{1,1}^2 & \bar{a}_{1,1}^1 & \cdots & -\bar{a}_{1,n}^2 & \bar{a}_{1,n}^1 \\ \hline \vdots & \vdots & & \vdots & \vdots \\ a_{n,1}^1 & a_{n,1}^2 & \cdots & a_{n,n}^1 & a_{n,n}^2 \\ -\bar{a}_{n,1}^2 & \bar{a}_{n,1}^1 & \cdots & -\bar{a}_{n,n}^2 & \bar{a}_{n,n}^1 \end{array} \right).$$

Let us denote by $\tau : \{1, \dots, 2n\} \rightarrow \{1, \dots, 2n\}$ the permutation defined as

$$\tau(r) = \begin{cases} n+k & \text{if } r = 2k \\ k+1 & \text{if } r = 2k+1. \end{cases}$$

Let e_b be the $2n$ -vector that has 1 in the b -th entry and 0 otherwise. Let us denote by C_{τ} the permutation matrix of size $2n$ whose b -th column is $e_{\tau^{-1}(b)}$, and let R_{τ} be its transpose. Then, by using a shuffling approach as in [16], one obtains:

$$f(A) = R_{\tau} \pi_{\mathbb{H}}(A) C_{\tau}.$$

Indeed

$$\pi_{\mathbb{H}}(A) C_{\tau} = \left(\begin{array}{cccc|cccc} a_{1,1}^1 & a_{1,2}^1 & \cdots & a_{1,n}^1 & a_{1,1}^2 & \cdots & a_{1,n-1}^2 & a_{1,n}^2 \\ -\bar{a}_{1,1}^2 & -\bar{a}_{1,2}^2 & \cdots & -\bar{a}_{1,n}^2 & \bar{a}_{1,1}^1 & \cdots & \bar{a}_{1,n-1}^1 & \bar{a}_{1,n}^1 \\ \hline \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ a_{n,1}^1 & a_{n,2}^1 & \cdots & a_{n,n}^1 & a_{n,1}^2 & \cdots & a_{n,n-1}^2 & a_{n,n}^2 \\ -\bar{a}_{n,1}^2 & -\bar{a}_{n,2}^2 & \cdots & -\bar{a}_{n,n}^2 & \bar{a}_{n,1}^1 & \cdots & \bar{a}_{n,n-1}^1 & \bar{a}_{n,n}^1 \end{array} \right),$$

and

$$R_{\tau} \pi_{\mathbb{H}}(A) C_{\tau} = \left(\begin{array}{cccc|cccc} a_{1,1}^1 & a_{1,2}^1 & \cdots & a_{1,n}^1 & a_{1,1}^2 & \cdots & a_{1,n-1}^2 & a_{1,n}^2 \\ a_{2,1}^1 & a_{2,2}^1 & \cdots & a_{2,n}^1 & a_{2,1}^2 & \cdots & a_{2,n-1}^2 & a_{2,n}^2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ a_{n,1}^1 & a_{n,2}^1 & \cdots & a_{n,n}^1 & a_{n,1}^2 & \cdots & a_{n,n-1}^2 & a_{n,n}^2 \\ \hline -\bar{a}_{1,1}^2 & -\bar{a}_{1,2}^2 & \cdots & -\bar{a}_{1,n}^2 & \bar{a}_{1,1}^1 & \cdots & \bar{a}_{1,n-1}^1 & \bar{a}_{1,n}^1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ -\bar{a}_{n-1,1}^2 & -\bar{a}_{n-1,2}^2 & \cdots & -\bar{a}_{n-1,n}^2 & \bar{a}_{n-1,1}^1 & \cdots & \bar{a}_{n-1,n-1}^1 & \bar{a}_{n-1,n}^1 \\ -\bar{a}_{n,1}^2 & -\bar{a}_{n,2}^2 & \cdots & -\bar{a}_{n,n}^2 & \bar{a}_{n,1}^1 & \cdots & \bar{a}_{n,n-1}^1 & \bar{a}_{n,n}^1 \end{array} \right),$$

which is exactly the matrix $f(A)$. Notice that, since R_{τ} is a permutation matrix, then $R_{\tau}^{-1} = C_{\tau}$ and so $\pi_{\mathbb{H}}(A)$ and $f(A)$ are similar. This implies that they have the same eigenvalues with the same multiplicities.

As a consequence, the fact that (Γ, ψ) and $(\Gamma, \psi)^{\alpha, \mathbb{H}}$ are $\pi_{\mathbb{H}}$ -cospectral implies that the complex adjoint matrices $f(A_{(\Gamma, \psi)})$ and $f(A_{(\Gamma, \psi)^{\alpha, \mathbb{H}}})$ are cospectral: by virtue of Proposition 5.3 it is possible to conclude that (Γ, ψ) and $(\Gamma, \psi)^{\alpha, \mathbb{H}}$ are right cospectral. \square

EXAMPLE 5.10. Consider the $U(\mathbb{H})$ -gain graphs (Γ, ψ) and $(\Gamma, \psi)^{\alpha, \mathbb{H}}$ of Examples 5.6 and 5.8. An explicit computation shows that the characteristic polynomials of the matrices $f(A_{(\Gamma, \psi)})$ and $f(A_{(\Gamma, \psi)^{\alpha, \mathbb{H}}})$, or equivalently, the characteristic polynomials of the matrices $\pi_{\mathbb{H}}(A_{(\Gamma, \psi)})$ and $\pi_{\mathbb{H}}(A_{(\Gamma, \psi)^{\alpha, \mathbb{H}}})$, coincide and are equal to

$$x^{16} - 22x^{14} + 187x^{12} - 776x^{10} + 1639x^8 - 1650x^6 + 625x^4,$$

and so the $U(\mathbb{H})$ -gain graphs (Γ, ψ) and $(\Gamma, \psi)^{\alpha, \mathbb{H}}$ are right cospectral.

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