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EXPRESSING MATRICES IN $\mathrm{SL}_n(F)$ AS PRODUCTS OF COMMUTATORS OF UNIPOTENT MATRICES*

TRUONG HUU DUNG† AND VO THI VAN ANH‡

Abstract. This paper aims to show that for two positive integers $n \geq k$, every nonscalar matrix in the special linear group of degree n over a field can be written as a product of a maximum of two commutators of unipotent matrices of index k. This fact also holds for scalar matrices over a quadratically closed field. Using GAP, some examples are provided to highlight the significance of the field's cardinality and to show that the assumption of quadratically closed fields is essential.

Key words. Unipotent matrix, Commutator, Special linear group, Decomposition of matrix.

AMS subject classifications. 15A23, 20H20.

1. Introduction.

1.1. Notations. Let F be a field and n be a positive integer. We denote by $\mathrm{GL}_n(F)$ and $\mathrm{SL}_n(F)$ the general linear group and the special linear group of degree n over F, respectively. We also use the following notations: $\mathrm{M}_{m\times n}(F)$ the set of all $m \times n$ matrices over a field F; $\mathrm{M}_n(F)$ the set of all $n \times n$ square matrices over a field F; $\mathrm{diag}(a_1, a_2, \dots, a_n)$ the $n \times n$ diagonal matrix with the i-th entry on the main diagonal being a_i; and $A_1 \oplus A_2 \oplus \cdots \oplus A_r$ the direct sum of r square matrices $A_1, A_2, \dots, A_r$. An element A in $\mathrm{GL}_n(F)$ is said to be *unipotent* if there exists a positive integer k such that $(A - I_n)^k = 0$, where I_n is the $n \times n$ identity matrix. Moreover, if k is the smallest integer satisfying such property, then A is called a *unipotent matrix of index k*.

1.2. Unipotent matrices. This paper is mainly concerned with the question of whether every matrix in $\mathrm{SL}_n(F)$ can be written as the product of (two or more) commutators of unipotent matrices of a given index k. Here commutators of unipotent matrices of index k mean elements of the form $[A, B] = ABA^{-1}B^{-1}$ where A and B are unipotent matrices of index k.

The decomposition of a matrix into a product of unipotent matrices has historically attracted significant attention [1, 3, 4, 7, 9, 12, 20]. Writing an element as a product of commutators of unipotent matrices can be seen as a nice description in this case. C. K. Fong and A. R. Sourour [7, Theorem 2] proved that every matrix in $\mathrm{SL}_n(\mathbb{C})$ is a product of three unipotent matrices. Later, J. H. Wang and P. Y. Wu [20, Theorem 3.5] showed that these matrices can be expressed as products of at most four unipotent matrices of index 2 (with respect to the fundamental case $k = 2$). In 2021, X. Hou [12, Theorem 1.1] demonstrated that these four unipotent matrices of index 2 can be rewritten as two commutators of unipotent matrices of index 2.

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†Department of Mathematics, Dong Nai University, 9 Le Quy Don Street, Tan Hiep Ward, Bien Hoa City, Dong Nai Province, Vietnam (dungth0406@gmail.com or thdung@dnpu.edu.vn).

‡Faculty of Mathematics and Computer Science, University of Science, Ho Chi Minh City, Vietnam; Vietnam National University, Ho Chi Minh City, Vietnam; Department of Applied Sciences, Ho Chi Minh City University of Technology and Education, 01 Vo Van Ngan Street, Thu Duc City, Ho Chi Minh City, Vietnam (anhvtv@hcmute.edu.vn).

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While much attention has been given to matrices over fields, considerable investigation has also focused on this decomposition over noncommutative rings, such as division rings [3] and the real quaternion division ring [9, Corollary 1.3].

Our problem can also be placed in the context related to the Sandwich Classification Theorem (SCT) for the lattice of subgroups of a linear group G that are normalized by the elementary subgroup of G. The SCT plays a significant role in the structure theory of linear groups. The decomposition of unipotents was introduced by N. Vavilov, E. Plotkin, and A. Stepanov. It provides a simple proof for the SCT. Recently in [16], A. Stepanov continues a series of publications on the "decomposition of unipotents" method with a new perspective on this decomposition related to the normal structure of Chevalley groups. By decomposing unipotents, one may derive short polynomial expressions representing the conjugates of elementary generators as products of elementaries. We refer the reader to relevant literature [14, 15, 18, 19] on this topic.

The main goal of this paper is to consider the problem on decomposition of matrices in $\mathrm{SL}_n(F)$ into products of commutators of unipotent matrices of index k, where k is a given positive integer greater than 1 and not exceeding n. Precisely, every matrix in $\mathrm{SL}_n(F)$ can be written as a product of at most two commutators of unipotent matrices of index k when the base field F is a quadratically closed field. In particular, for the case when matrices are nonscalar, it is important to note that F may not necessarily be a quadratically closed field. In this paper, the concept of a quadratically closed field is understood as a field that has no quadratic extension. Especially, our results extend that of X. Hou [12] and J. H. Wang and P. Y. Wu [20].

1.3. Outline. The structure of the article is as follows. We present the main results in two separate sections: scalar matrices and nonscalar matrices. Section 2 focuses on the results concerning nonscalar matrices, while Section 3 addresses those related to scalar matrices. The Appendix presents examples that illustrate the main theorems from Sections 2 and 3 with the aid of the computational programs Maple and GAP. Additionally, some open problems are posed for further exploration.

2. Decomposition of nonscalar matrices in $\mathrm{SL}_n(F)$.

Based on our experience, the matrix decomposition is often divided into two cases: scalar matrices and nonscalar matrices. The problems related to nonscalar matrices seem particularly interesting [6, 8, 13]. Therefore, in this section, we consider the problem of decomposing a nonscalar matrix into a product of commutators of unipotent matrices of given index k.

Let A be a unipotent matrix of index k in $\mathrm{M}_n(F)$. It is straightforward to verify that A has eigenvalue 1 only, and its characteristic polynomial is $(x-1)^n$. By the Cayley–Hamilton theorem, $(A-I_n)^n=0$. Since k is the smallest positive integer satisfying $(A-I_n)^k=0$, it follows that $k\le n$. Consequently, throughout this paper, any reference to unipotent matrices assumes that their index does not exceed their size.

In the rest of the paper, the notation $J_n(1)$ denotes the Jordan block of size n corresponding to 1

$$J_n(1)=\begin{pmatrix}1&1&0&\cdots&0\\0&1&1&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\end{pmatrix}\in\mathrm{M}_n(F).$$

It is easy to verify that this matrix is a unipotent matrix of index n.

The following two remarks can be easily verified.

Remark 2.1. *Let $k \geq 2$ be an integer and $A \in \mathrm{M}_n(F)$ be a unipotent matrix of index k. Then, for any matrix $B \in \mathrm{GL}_n(F)$, the matrix $B^{-1}AB$ is a unipotent matrix of index k.*

Remark 2.2. *Let r be a positive integer. Suppose that $A = A_1 \oplus A_2 \oplus \cdots \oplus A_r$, where each $A_i \in \mathrm{M}_{n_i}(F)$ is a unipotent matrix of index k_i, for every $i = 1, \ldots, r$. Then, $A \in \mathrm{M}_n(F)$ is a unipotent matrix of index k, where $k = \max\limits_{i=1,\ldots,r} \{k_i\}$ and $n = n_1 + \cdots + n_r$.*

From Remark 2.1, we derive the following lemma, which will be utilized in the next section.

Lemma 2.3. *If a matrix A can be written as a product of ℓ commutators of unipotent matrices of index k, then any matrix that is similar to A can also be expressed in the same form.*

Proof. Suppose that

$$A = [X_1, Y_1][X_2, Y_2] \cdots [X_\ell, Y_\ell],$$

with $X_1, X_2, \ldots, X_\ell$ and $Y_1, Y_2, \ldots, Y_\ell$ being unipotent matrices of index k. Then, for any matrix $B \in \mathrm{GL}(F)$,

$$B^{-1}AB = (B^{-1}[X_1, Y_1]B)(B^{-1}[X_2, Y_2]B) \cdots (B^{-1}[X_\ell, Y_\ell]B) = [C_1, D_1][C_2, D_2] \cdots [C_\ell, D_\ell],$$

where $C_i = B^{-1}X_iB$ and $D_i = B^{-1}Y_iB$. According to Remark 2.1, C_i and D_i are also unipotent matrices of index k. This completes the proof. ☐

The following lemma describes the direct sum of products of commutators of unipotent matrices.

Lemma 2.4. *If $A_i \in \mathrm{SL}_{n_i}(F)$ is a product of ℓ_i commutators of unipotent matrices of index k_i for $i = 1, 2, \ldots, s$, then the direct sum $A_1 \oplus \cdots \oplus A_s$ is a product of $\ell = \max\{\ell_1, \ldots, \ell_s\}$ commutators of unipotent matrices of index $k = \max\{k_1, \ldots, k_s\}$.*

Proof. We only need to prove the case for $s = 2$. Suppose that $A_i = \prod_{j=1}^{\ell_i} [X_{ij}, Y_{ij}] \in \mathrm{SL}_{m_i}(F)$, where X_{ij}, Y_{ij} are unipotent matrices of index k_i, for $i = 1, 2$. Without loss of generality, we can assume $\ell_1 \geq \ell_2$. Set $X_{2j} = Y_{2j} = I_{m_2}$ for all $j \in \{\ell_2 + 1, \ldots, \ell_1\}$. Then,

$$A_1 \oplus A_2 = \prod_{j=1}^{\ell_1} [X_{1j}, Y_{1j}] \oplus \prod_{j=1}^{\ell_1} [X_{2j}, Y_{2j}] = \prod_{j=1}^{\ell_1} [X_{1j}, Y_{1j}] \oplus [X_{2j}, Y_{2j}] = \prod_{i=j}^{\ell_1} \left[X_{1j} \oplus X_{2j}, Y_{1j} \oplus Y_{2j}\right],$$

is a product of ℓ_1 commutators. Moreover, by Remark 2.2, $X_{1j} \oplus X_{2j}$ and $Y_{1j} \oplus Y_{2j}$ are unipotent matrices of index $k = \max\{k_1, k_2\}$. ☐

The next three results demonstrate the similarity between the direct sum of two matrices, which do not share common eigenvalues, and a quasi-upper triangular matrix formed by them. A similar result for matrices over the complex numbers can be found in [10, Theorem 4.4.22]. We noticed that the Kronecker product presented in [10]. While our proofs differ slightly from those in [10] as they do not involve the use of the Kronecker product, we use a result of Drazin [5, Theorem 1.1] to provide readers with an alternative perspective, which is stated below.

Theorem 2.5. [5, Theorem 1.1] *For any field F, any $m, n \in \mathbb{N}$ and any matrices $A \in \mathrm{M}_n(F)$, $B \in \mathrm{M}_m(F)$ whose eigenvalues all lie in F, the following three properties of the pair A, B are equivalent:*

(i) *for any given polynomials $f, g \in F[t]$ there exists a polynomial $h \in F[t]$ such that $f(A) = h(A)$ and $g(B) = h(B)$;*

(ii) *A and B share no eigenvalue in common;*
(iii) *for $n \times m$ matrices X over F, $AX = XB \Rightarrow X = 0$.*

Using the equivalent conditions of Theorem 2.5, we present a sufficient condition for the existence of solutions to the Sylvester equation $AX - XB = C$ over a field F.

Lemma 2.6. *Let F be a field, and let $A \in \mathrm{M}_n(F)$ and $B \in \mathrm{M}_m(F)$. If A and B have no eigenvalue in common, then for every $C \in \mathrm{M}_{n\times m}(F)$, there exists an $X \in \mathrm{M}_{n\times m}(F)$ such that $AX - XB = C$.*

Proof. Consider the linear transformation

$$T : \mathrm{M}_{n\times m}(F) \to \mathrm{M}_{n\times m}(F),$$

defined by $T(X) = AX - XB$. To prove this lemma, it is sufficient to show that T is an isomorphism. According to Theorem 2.5, T is an injective linear map. Since $\dim_F \mathrm{M}_{n\times m}(F) < \infty$, T is an isomorphism.∎

Next, we show that the direct sum of two matrices without common eigenvalues is similar to a quasi-upper triangular matrix formed by these two matrices.

Corollary 2.7. *If $A \in \mathrm{M}_n(F)$ and $B \in \mathrm{M}_m(F)$ have no eigenvalue in common, then $A \oplus B$ is similar to the matrix*

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

for every matrix $C \in \mathrm{M}_{n\times m}(F)$.

Proof. According to Lemma 2.6, there exists a matrix $X \in \mathrm{M}_{n\times m}(F)$ such that $AX - XB = C$. In this case, it is easy to verify that

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I_n & -X \\ 0 & I_m \end{pmatrix}^{-1} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} I_n & -X \\ 0 & I_m \end{pmatrix}.$$

Thus, the proof is completed. ∎

The next two lemmas present a construction of a unipotent matrix based on two given unipotent matrices, where the index of the new unipotent matrix is higher than the indices of the original unipotent matrices.

Lemma 2.8. *For integers $n \geq k \geq 2$ and $m \in \{1, 2\}$, if $A \in \mathrm{M}_n(F)$ is a unipotent matrix of index k and $B \in \mathrm{M}_m(F)$ is a unipotent matrix of index m, then there exists an $n \times m$ matrix C such that the matrix $D = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a unipotent matrix of index $k + m$.*

Proof. Let $G = A - I_n$ and $H = B - I_m$. Then, $G^k = 0, G^{k-1} \neq 0$, and $H^m = 0, H^{m-1} \neq 0$. Here, if $m = 1$, then H^{m-1} is understood as the identity matrix I_1. Let us denote the entries in the i-th row and j-th column of the matrices G^{k-1} and H^{m-1} by g_{ij} and h_{ij}, respectively.

Since $G^{k-1} \neq 0$ and $H^{m-1} \neq 0$, there exist nonzero entries $g_{i_0 s_0}$ and $h_{r_0 j_0}$. Choose C as an $n \times m$ matrix whose entry in the s_0-th row and r_0-th column is 1, and 0 elsewhere. Then, the entry in the i_0-th row and j_0-th column of $G^{k-1}CH^{m-1}$ is nonzero element $g_{i_0 s_0} \cdot h_{r_0 j_0}$. Therefore, $G^{k-1}CH^{m-1} \neq 0$. Finally, we need to show that with the chosen C, the matrix $D = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a unipotent matrix of index $k + m$. It is easy to compute that for every integer $s \geq 2$,

$$(D - I_{n+m})^s = \begin{pmatrix} G^s & G^{s-1}C + G^{s-2}CH \\ 0 & 0 \end{pmatrix}.$$

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If $m = 1$, then

$$(D - I_{n+1})^{k+1} = \begin{pmatrix} G^{k+1} & G^kC \\ 0 & 0 \end{pmatrix} = 0,$$

and

$$(D - I_{n+1})^{k} = \begin{pmatrix} G^{k+1} & G^{k-1}C \\ 0 & 0 \end{pmatrix} \neq 0.$$

Thus, D is a unipotent matrix of index $k+1$.

If $m = 2$, then

$$(D - I_{n+2})^{k+2} = \begin{pmatrix} G^{k+2} & G^{k+1}C + G^kCH \\ 0 & 0 \end{pmatrix} = 0,$$

and

$$(D - I_{n+2})^{k+1} = \begin{pmatrix} G^{k+1} & G^{k}C + G^{k-1}CH \\ 0 & 0 \end{pmatrix} \neq 0.$$

Thus, D is a unipotent matrix of index $k+2$. □

Lemma 2.9. *Let $n \geq k \geq 2$ be two integers. If $A \in \mathrm{M}_n(F)$ is a unipotent matrix of index k and $B \in \mathrm{M}_2(F)$ is a unipotent matrix of index 2, then there exists an $n \times 2$ matrix Y such that the matrix $D = \begin{pmatrix} A & Y \\ 0 & B \end{pmatrix}$ is a unipotent matrix of index $k+1$.*

Proof. As demonstrated in the proof of Lemma 2.8, a matrix C of size $n \times 2$ can be found such that for every integer $s \geq 2$, the following holds:

$$(D_1 - I_{n+2})^{s} = \begin{pmatrix} G^{s} & G^{s-1}C + G^{s-2}CH \\ 0 & 0 \end{pmatrix},$$

where $D_1 = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, $G = A - I_n$ and $H = B - I_2$. Let $Y = GC$ and $D = \begin{pmatrix} A & Y \\ 0 & B \end{pmatrix}$. Then, for every integer $s \geq 2$, we have:

$$(D - I_{n+2})^{s} = \begin{pmatrix} G^{s} & G^{s}C + G^{s-1}CH \\ 0 & 0 \end{pmatrix}.$$

Therefore, $(D - I_{n+2})^{k+1} = 0$ and $(D - I_{n+2})^{k} \neq 0$. This completes the proof. □

Next, we demonstrate that the direct sum of two commutators sharing no eigenvalue in common is also a commutator.

Lemma 2.10. *Let $A \in \mathrm{M}_n(F)$ and $B \in \mathrm{M}_m(F)$ be two matrices with no eigenvalue in common. If $A = [X, Y]$ and $B = [S, T]$, then $A \oplus B$ is a commutator of two matrices.*

Proof. For any $n \times m$ matrices C and E, let $U = \begin{pmatrix} X & C \\ 0 & S \end{pmatrix}$, $V = \begin{pmatrix} Y & E \\ 0 & T \end{pmatrix}$, and $W = [U, V]$. By calculation, we obtain $W = \begin{pmatrix} A & Z \\ 0 & B \end{pmatrix}$, where

$$Z = -AET^{-1} - AYCS^{-1}T^{-1} + XES^{-1}T^{-1} + CTS^{-1}T^{-1}.$$

Since A and B share no common eigenvalues, by Corollary 2.7, $A \oplus B$ is similar to the matrix W; in other words, there exists an $(n+m) \times (n+m)$ matrix P such that $A \oplus B = P^{-1}WP$. Therefore, $A \oplus B = \left[P^{-1}UP, P^{-1}VP\right]$. The proof is completed. □

The following two corollaries indicate that the direct sum of two commutators of unipotent matrices can be expressed as the commutator of two unipotent matrices of a higher index.

Corollary 2.11. *Let* $n \geq k \geq 2$ *and* $m \in \{1,2\}$ *be integers. Suppose that* $A \in \mathrm{M}_n(F)$ *and* $B \in \mathrm{M}_m(F)$ *are two matrices sharing no eigenvalue in common. If* A *is a commutator of two unipotent matrices of index* k*, and* B *is a commutator of two unipotent matrices of index* m*, then* $A \oplus B$ *is a commutator of two unipotent matrices of index* $k+m$.

Proof. Suppose $A = [X, Y]$ and $B = [S, T]$, where X and Y are unipotent matrices of index k, and S, T are unipotent matrices of index m. According to Lemma 2.8, there exist $n \times m$ matrices C and E such that the $(n+m)\times(n+m)$ matrices $U = \begin{pmatrix} X & C \\ 0 & S \end{pmatrix}$ and $V = \begin{pmatrix} Y & E \\ 0 & T \end{pmatrix}$ are both unipotent matrices of index $k+m$. As proven in Lemma 2.10, there exists an $(n+m)\times(n+m)$ matrix P such that $A \oplus B = \left[P^{-1}UP, P^{-1}VP\right]$. Since U and V are unipotent matrices of index $k+m$, by Remark 2.1, $P^{-1}UP$ and $P^{-1}VP$ are unipotent matrices of index $k+m$ as well. ∎

Note that, according to Corollary 2.11, if the matrix A is a commutator of two unipotent matrices of index k and 1 is not an eigenvalue of A, then the direct sum $A \oplus (1)$ is a commutator of two unipotent matrices of index $k+1$.

In the proof of Corollary 2.11, by utilizing Lemma 2.9 instead of Lemma 2.8, we obtain the following result.

Corollary 2.12. *For integers* $n \geq k \geq 2$*, let* $A \in \mathrm{M}_n(F)$ *and* $B \in \mathrm{M}_2(F)$*. Suppose* A *and* B *share no eigenvalue in common. Then, if* A *is a commutator of unipotent matrices of index* k *and* B *is a commutator of unipotent matrices of index* 2*, then* $A \oplus B$ *is a commutator of unipotent matrices of index* $k+1$.

The following corollary is analogous to a result in X. Hou [12, Lemma 2.3].

Lemma 2.13. *If* a *is a nonzero element in a field* F *satisfying* $a^2 \neq -1$*, then the matrix*

$$A = \begin{pmatrix} a^2 & 0 \\ 0 & (a^2)^{-1} \end{pmatrix},$$

is a commutator of two unipotent matrices of index 2.

Proof. If $a^2 = 1$, then $A = I_2 = [J_2(1), J_2(1)]$, where $J_2(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a unipotent matrix of index 2.

If $a^2 \neq \pm 1$, then $A = [X, Y]$, where

$$X = \begin{pmatrix} 2a^2\left(a^2+1\right)^{-1} & \left(a^4+2a^3-2a-1\right)\left[\left(a^2+1\right)^2\right]^{-1} \\ -\left(a-1\right)\left(a+1\right)^{-1} & 2\left(a^2+1\right)^{-1} \end{pmatrix},$$

and

$$Y = \begin{pmatrix} 2\left(a^2+1\right)^{-1} & -\left(a^4-2a^2+1\right)a^{-1}\left[\left(a^2+1\right)^2\right]^{-1} \\ a & 2a^2\left(a^2+1\right)^{-1} \end{pmatrix},$$

are two unipotent matrices of index 2. ∎

From the above lemma, we can construct a diagonal matrix that can be expressed as a commutator of unipotent matrices of a given index. The following theorem and its corollary present results related to this construction.

Theorem 2.14. *Let F be a field and*

$$a_1^2, (a_1^2)^{-1}, a_2^2, (a_2^2)^{-1}, \dots, a_m^2, (a_m^2)^{-1},$$

be distinct elements in $F \setminus \{0, \pm 1\}$. Then, for an integer

$$k \in \{2, 3, \dots, 2m\},$$

the matrix

$$A = \operatorname{diag}\left(a_1^2, (a_1^2)^{-1}, a_2^2, (a_2^2)^{-1}, \dots, a_m^2, (a_m^2)^{-1}\right),$$

is a commutator of unipotent matrices of index k.

Proof. We can express A as $A = A_1 \oplus A_2 \oplus \cdots \oplus A_m$, where $A_j = \operatorname{diag}({a_j}^2, ({a_j}^2)^{-1})$ for $j = 1, 2, \dots, m$. The matrices $A_1, \dots, A_m$ share no eigenvalue in common, and each of them is a commutator of unipotent matrices of index 2 by Lemma 2.13. Let $k = 2m - r$, with $r \in \{0, 1, \dots, 2m-2\}$.

If r is even, then taking into account Corollary 2.11 successively to the matrices $A_1, A_2, \dots, A_{m-\frac{r}{2}}$, we obtain the matrix $A_1 \oplus A_2 \oplus \ldots \oplus A_{m-\frac{r}{2}}$ as a commutator of unipotent matrices of index $k = 2m - r$. Therefore, by Lemma 2.4, A is a commutator of unipotent matrices of index $k = 2m - r$.

The case that r is odd is similar. Applying Corollary 2.11 successively to the matrices $A_1, A_2, \dots, A_{m-\frac{r+1}{2}}$, we obtain the matrix $A' = A_1 \oplus A_2 \oplus \cdots \oplus A_{m-\frac{r+1}{2}}$ as a commutator of unipotent matrices of index $2m - r - 1$. By Corollary 2.12, the matrix $A' \oplus A_{m-\frac{r-1}{2}}$ is a commutator of unipotent matrices of index $k = 2m - r$. Therefore, by Lemma 2.4, A is a commutator of unipotent matrices of index $k = 2m - r$. Hence, in both cases, A is a commutator of unipotent matrices of index k. ∎

Corollary 2.15. *Let F be a field, and*

$$a_1^2, (a_1^2)^{-1}, a_2^2, (a_2^2)^{-1}, \dots, a_m^2, (a_m^2)^{-1},$$

be distinct elements in $F \setminus \{0, \pm 1\}$. Then, for an integer

$$k \in \{2, 3, \dots, 2m+1\},$$

the matrix

$$A = \operatorname{diag}\left(1, a_1^2, (a_1^2)^{-1}, a_2^2, (a_2^2)^{-1}, \dots, a_m^2, (a_m^2)^{-1}\right),$$

is a commutator of unipotent matrices of index k.

Proof. By Lemma 2.13, each matrix $\operatorname{diag}\left(a_i^2, (a_i^2)^{-1}\right)$ is a commutator of unipotent matrices of index 2 for all $i = 1, 2, \dots, m$. So, applying Lemma 2.4, we obtain that A is a commutator of unipotent matrices of index 2.

Next, we prove this corollary in the case where the index is greater than 2. For $k \in \{3, 4, \dots, 2m+1\}$, according to Theorem 2.14, the matrix

$$\operatorname{diag}\left(a_1^2, (a_1^2)^{-1}, a_2^2, (a_2^2)^{-1}, \dots, a_m^2, (a_m^2)^{-1}\right),$$

is a commutator of unipotent matrices of index $k - 1$. By applying Corollary 2.11, we obtain that A is a commutator of unipotent matrices of index k. ∎

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Next, we present a theorem attributed to Sourour [17, Theorem 1]. The factorization provided by this theorem is useful for the main result discussed in the section.

Theorem 2.16. [17, Theorem 1] *Let A be a nonscalar invertible $n \times n$ matrix over a field F and let b_j and c_j $(1 \leq j \leq n)$ be elements of F such that $\prod_{j=1}^{n} b_j c_j = \det A$. There exist $n \times n$ matrices B and C with eigenvalues $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$, respectively, such that $A = BC$. Furthermore, B and C can be chosen so that B is lower triangularizable, and C is simultaneously upper triangularizable.*

The following theorem presents the decomposition of a nonscalar matrix in $\mathrm{SL}_n(F)$ into a product of commutators of unipotent matrices of a given index. This is one of the main results of this paper.

Theorem 2.17. *Let $n \geq k \geq 2$ be two integers and F be a field having at least $2n+3$ elements. Every nonscalar matrix $A \in \mathrm{SL}_n(F)$ can be decomposed into a product of at most two commutators of unipotent matrices of index k.*

Proof. We consider two possible cases.

Case 1. n is even. Assume $n = 2s$. Choose

$$a_1^2, (a_1^2)^{-1}, a_2^2, (a_2^2)^{-1}, \ldots, a_s^2, (a_s^2)^{-1},$$

as distinct elements in $F \setminus \{0, \pm 1\}$. According to Theorem 2.16, there exists matrices B and C such that both matrices have eigenvalues

$$a_1^2, (a_1^2)^{-1}, a_2^2, (a_2^2)^{-1}, \ldots, a_s^2, (a_s^2)^{-1},$$

and $A = BC$. In this case, both B and C are similar to the matrix

$$D = \operatorname{diag}\left(a_1^2, (a_1^2)^{-1}, a_2^2, (a_2^2)^{-1}, \ldots, a_s^2, (a_s^2)^{-1}\right).$$

Since $2 \leq k \leq 2s$, according to Theorem 2.14, the matrix D is a commutator of unipotent matrices of index k. Therefore, A is a product of two commutators of unipotent matrices of index k.

Case 2. n is odd. Assume $n = 2s+1$. Choose

$$a_1^2, (a_1^2)^{-1}, a_2^2, (a_2^2)^{-1}, \ldots, a_s^2, (a_s^2)^{-1},$$

as distinct elements in $F \setminus \{0, \pm 1\}$. By Theorem 2.16, there exists matrices B and C such that both matrices have eigenvalues

$$1, a_1^2, (a_1^2)^{-1}, a_2^2, (a_2^2)^{-1}, \ldots, a_s^2, (a_s^2)^{-1},$$

and $A = BC$. In this case, both B and C are similar to the matrix

$$D = \operatorname{diag}\left(1, a_1^2, (a_1^2)^{-1}, a_2^2, (a_2^2)^{-1}, \ldots, a_s^2, (a_s^2)^{-1}\right).$$

Since $2 \leq k \leq 2s+1$, by Corollary 2.15, the matrix D is a commutator of unipotent matrices of index k. It implies that A is a product of two commutators of unipotent matrices of index k. $\square$

3. Decomposition of scalar matrices in $\mathrm{SL}_n(F)$.

There are many papers that only decompose nonscalar matrices (see, for example, [6, 8, 13]). Therefore, the case of scalar matrices seems to be more challenging. In this section, we consider the problem of decomposing scalar matrices in $\mathrm{SL}_n(F)$ into a product of commutators of unipotent matrices of a given index k, where F is an quadratically closed field. Because a quadratically closed field is a field that has no quadratic extension, every its element has a square root. Moreover, given that every finite field $\mathbb{F}_q$ has at least one quadratic extension $\mathbb{F}_{q^2}$, it follows that a quadratically closed field must be an infinite field.

First, we state some properties related to a finite-order element in a field. Recall that the order of an element a of a field is the smallest positive integer m such that $a^m = 1$. If m exists, a is said to be finite order.

Lemma 3.1. *Let F be a field and $a \in F \setminus \{1\}$ be an element of order m. Then:*

(i) *$a^i \neq a^j$ for all $i, j \in \{0, 1, \ldots, m-1\}$ and $i \neq j$.*
(ii) *If $a^n = 1$, then m is a divisor of n.*

Proof. First, let us prove (i). Suppose $a^i = a^j$ for some

$$i, j \in \{0, 1, \ldots, m-1\},$$

and $i \geq j$. This implies $a^{i-j} = 1$ and $0 \leq i - j \leq m - 1$. Due to the minimality of order m of a, it must be $i = j$. Hence, (i) is true.

Next, we prove (ii). Assume the division of n by m yields a quotient q and a remainder r, meaning $n = qm + r$ with $0 \leq r < m$. Since $1 = a^n = a^{qm+r} = a^r$, by the minimality of m it follows that $r = 0$. Thus, m is a divisor of n. ◻

Remark 3.2. *Let F be a field containing enough elements and m be the order of an element $a \in F \setminus \{1\}$. By the above lemma, the elements $1, a, a^2, \ldots, a^{m-1}$ are distinct. In this case, we can choose distinct elements $b_1, b_2, \ldots, b_s$ in $F \setminus \{0, \pm 1\}$ such that*

$$b_i, (b_i)^2, b_i b_j, b_i (b_j)^{-1} \notin \{1, a, \ldots, a^{m-1}\},$$

for all $i, j \in \{1, 2, \ldots, s\}$ and $i \neq j$. Indeed, first, we choose b_1 in F such that $b_1, (b_1)^2 \notin \{1, a, \ldots, a^{m-1}\}$. Then, for each $i = 2, 3, \ldots, s$, we choose $b_i \in F$ such that

$$b_i, (b_i)^2, b_i b_j, b_i (b_j)^{-1} \notin \{1, a, \ldots, a^{m-1}\},$$

for all $i, j \in \{1, 2, \ldots, s\}$ and $1 \leq j < i$. With these chosen elements $b_1, \ldots, b_s$, the elements

$$1, a^\ell, b_i, (b_i)^{-1}, b_i a^\ell, (b_i)^{-1} a^\ell,$$

are distinct for every $i \in \{1, 2, \ldots, s\}$ and $\ell \in \{1, 2, \ldots, m-1\}$.

The following lemma presents the problem of decomposing a scalar matrix in $\mathrm{SL}_n(F)$ into a product of commutators of unipotent matrices of index 2. This result is analogous to the one by X. Hou (see [12, Theorem 2.7]).

Lemma 3.3. *Let F be quadratically closed field. Every matrix $A = aI_n$ in $\mathrm{SL}_n(F)$ can be expressed as the product of at most two commutators of unipotent matrices of index 2.*

Proof. Assume n is odd. We can write $A = BC$, where $B = \operatorname{diag}\left(a, a^2, \ldots, a^{n-1}, 1\right)$ and $C = \operatorname{diag}\left(1, a^{n-1}, \ldots, a^2, a\right)$. Both B and C are similar to the matrix

$$(1) \oplus \operatorname{diag}\left(a, a^{n-1}\right) \oplus \operatorname{diag}\left(a^2, a^{n-2}\right) \oplus \cdots \operatorname{diag}\left(a^{\frac{n-1}{2}}, a^{\frac{n+1}{2}}\right).$$

If $a^k = a^{n-k}$, then $(a^k)^2 = 1$ for $k \in \{1, 2, \ldots, (n-1)/2\}$. It implies that $a^k = \pm 1$. In the case where F is a field of characteristic 2, we have $a^k = 1$. If F is a field of characteristic not 2, then it is impossible to have $a^k = -1$ because $1 = (a^n)^k = (a^k)^n$ and n is odd. Thus, for every $k \in \{1, 2, \ldots, (n-1)/2\}$, either $a^k = a^{n-k} = 1$ or $a^k \neq a^{n-k}$. In both cases, each matrix $\operatorname{diag}\left(a^k, a^{n-k}\right)$ is a commutator of two unipotent matrices of index 2 for every $k \in \{1, 2, \ldots, (n-1)/2\}$ by Lemma 2.13. Hence,

$$(1) \oplus \operatorname{diag}\left(a, a^{n-1}\right) \oplus \operatorname{diag}\left(a^2, a^{n-2}\right) \oplus \cdots \operatorname{diag}\left(a^{\frac{n-1}{2}}, a^{\frac{n+1}{2}}\right),$$

is a commutator of two unipotent matrices of index 2 by Lemma 2.4. Therefore, A is a product of two commutators of unipotent matrices of index 2.

Assume n is even. We can write $n = 2m$. Choose $b \in F \setminus \{0, \pm 1\}$ such that $b^2 \neq a^j$ for every $j = 1, 2, \ldots, n$. We have $A = BC$ where

$$B = \operatorname{diag}\left(ab, ab^{-1}\right) \oplus \operatorname{diag}\left(a^3 b, a^3 b^{-1}\right) \oplus \cdots \oplus \operatorname{diag}\left(a^{2m-1} b, a^{2m-1} b^{-1}\right),$$

and

$$C = \operatorname{diag}\left(b^{-1}, b\right) \oplus \operatorname{diag}\left(a^{2m-2} b^{-1}, a^{2m-2} b\right) \oplus \cdots \oplus \operatorname{diag}\left(a^2 b^{-1}, a^2 b\right).$$

The matrix B is similar to the matrix $\left(\bigoplus_{i=1}^{m} \operatorname{diag}\left(a^{2i-1} b, a^{2m-2i+1} b^{-1}\right)\right)$. For each $i = 1, 2, \ldots, m$, the matrix $\operatorname{diag}\left(a^{2i-1} b, a^{2m-2i+1} b^{-1}\right)$ is a commutator of two unipotent matrices of index 2, by Lemma 2.13. Thus, by Lemma 2.3 , B is a commutator of two unipotent matrices of index 2. The matrix C is similar to the matrix $\left(\bigoplus_{i=1}^{m} \operatorname{diag}\left(a^{2i-2} b, a^{2m-2i+2} b^{-1}\right)\right)$. For each $i = 1, 2, \ldots, m$, the matrix $\operatorname{diag}\left(a^{2i-2} b, a^{2m-2i+2} b^{-1}\right)$ is a commutators of two unipotent matrices of index 2, by Lemma 2.13. Thus, by Lemma 2.3, C is a commutator of two unipotent matrices of index 2. Hence, A is a product of at most two commutators of unipotent matrices of index 2. ☐

The decomposition of scalar matrices in $\mathrm{SL}_n(F)$ into a product of commutators of unipotent matrices of a given index is presented in the following theorem. This is the third main result of the paper.

THEOREM 3.4. *Let $n \geq k \geq 2$ be two integers and F be a quadratically closed field. Each scalar matrix in $\mathrm{SL}_n(F)$ can be decomposed into a product of at most two commutators of unipotent matrices of index k.*

Proof. Let $A = aI_n \in \mathrm{SL}_n(F)$ for some $a \in F \setminus \{0\}$. The proof is divided into two cases:

Case 1. $a = 1$.

The result of theorem follows from $A = I_n = [B, B]$, where $B = J_k(1) \oplus I_{n-k}$.

Case 2. $a = -1$.

If F has characteristic 2, then $A = I_n = [B, B]$ where $B = J_k(1) \oplus I_{n-k}$. If F has characteristic not 2, then n must be even. Let $n = 2s$ and choose

$$a_1^2, (a_1^2)^{-1}, a_2^2, (a_2^2)^{-1}, \ldots, a_s^2, (a_s^2)^{-1},$$

as distinct elements in $F \setminus \{0, \pm 1\}$. We can write $A = BC$ where

$$B = \operatorname{diag}\left(a_1^2, (a_1^2)^{-1}, a_2^2, (a_2^2)^{-1}, \ldots, a_s^2, (a_s^2)^{-1}\right),$$

and

$$C = \operatorname{diag}\left((-a_1^2)^{-1}, -a_1^2, (-a_2^2)^{-1}, -a_2^2, \ldots, (-a_s^2)^{-1}, -a_s^2\right).$$

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Since $2 \leq k \leq 2s$, by Theorem 2.14, both B and C are commutators of unipotent matrices of index k. Thus, A is a product of two commutators of unipotent matrices of index k.

Case 3. $a \neq \pm 1$

Let m be the order of a. By Lemma 3.1, we can write $n = qm$ and $A = \left(\bigoplus_{i=1}^{q} aI_m\right)$, where q is some integer. Suppose $k = sm + r$ with $0 \leq r < m$.

Case 3.1. m is even.

Since $k \geq 2$ and $\left(\bigoplus_{i=s+2}^{q} aI_m\right)$ is a product of at most two commutators of unipotent matrices of index 2 by Lemmas 3.3 and 2.4, we only need to prove that $\left(\bigoplus_{i=1}^{s+1} aI_m\right)$ is a product of two commutators of unipotent matrices of index k. By Remark 3.2, we can choose distinct elements $b_1, b_2, \ldots, b_{s+1}$ in $F \setminus \{0, \pm 1\}$ such that

$$a^{2i-1}b_t, a^{m-2i+1}{b_t}^{-1}, a^{2i-2}b_t, a^{m-2i+2}{b_t}^{-1},$$

are distinct for every $i = 1, 2, \ldots, \frac{m}{2}$ and $t = 1, 2, \ldots, s+1$. Then, for each $t = 1, 2, \ldots, s+1$, by the proof of Lemma 3.3, we can write $aI_m = B_tC_t$, where B_t is similar to the matrix

$$B'_t = \left(\bigoplus_{i=1}^{m/2} \operatorname{diag}\left(a^{2i-1}b_t, a^{m-2i+1}{b_t}^{-1}\right)\right),$$

and C_t is similar to the matrix $C'_t = \left(\bigoplus_{i=1}^{m/2} \operatorname{diag}\left(a^{2i-2}b_t, a^{m-2i+2}{b_t}^{-1}\right)\right)$. Thus, it follows that $\left(\bigoplus_{i=1}^{s+1} aI_m\right) = BC$, where $B = \left(\bigoplus_{t=1}^{s+1} B_t\right)$ is similar to $B' = \left(\bigoplus_{t=1}^{s+1} B'_t\right)$, and $C = \left(\bigoplus_{t=1}^{s+1} C_t\right)$ is similar to $C' = \left(\bigoplus_{t=1}^{s+1} C'_t\right)$. Since $2 \leq k \leq (s+1)m$, by Theorem 2.14, both B' and C' are commutators of unipotent matrices of index k. Therefore, $\left(\bigoplus_{i=1}^{s+1} aI_m\right)$ is a product of two commutators of unipotent matrices of index k. Thus, A is a product of two commutators of unipotent matrices of index k.

Case 3.2. m is odd.

Case 3.2.1. $q = 1$.

In this case, it follows that $n = m$ and $2 \leq k \leq m$. According to the proof of Lemma 3.3, we can write $A = BC$, where $B = \operatorname{diag}\left(a, a^2, \ldots, a^{n-1}, 1\right)$ and $C = \operatorname{diag}\left(1, a^{n-1}, \ldots, a^2, a\right)$. By Corollary 2.15, both B and C are commutators of unipotent matrices of index k. Therefore, A is a product of two commutators of unipotent matrices of index k.

Case 3.2.2. $q \geq 2$.

Let

$$\gamma_q = \begin{cases} \frac{q+1}{2}, \text{if } q \text{ is odd} \\ \\ \frac{q}{2}, \text{if } q \text{ is even.} \end{cases}$$

It is easy to see that if q is odd, then $q = 2\gamma_q - 1$, and if q is even, then $q = 2\gamma_q$. By Remark 3.2, we can choose distinct elements $b_1, b_2, \ldots, b_{\gamma_q}$ in $F \setminus \{0, \pm 1\}$ such that elements $a^\ell, b_i, (b_i)^{-1}, b_ia^\ell, (b_i)^{-1}a^\ell$ are

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distinct for all $i \in \{1, 2, \dots, \gamma_q\}$ and $\ell \in \{1, 2, \dots, m-1\}$. For each $i = 1, 2, \dots, \gamma_q$, let

$$\begin{aligned}
B_{i1} &= \operatorname{diag}\left(b_i, ab_i, a^2 b_i, \dots, a^{m-1} b_i\right),\\
B_{i2} &= \operatorname{diag}\left((b_i)^{-1}, a(b_i)^{-1}, a^2(b_i)^{-1}, \dots, a^{m-1}(b_i)^{-1}\right),\\
C_{i1} &= \operatorname{diag}\left(a(b_i)^{-1}, (b_i)^{-1}, a^{m-1}(b_i)^{-1}, \dots, a^2(b_i)^{-1}\right),\\
C_{i2} &= \operatorname{diag}\left(ab_i, b_i, a^{m-1}b_i, \dots, a^2 b_i\right)\\
D_i &= aI_m \oplus aI_m.
\end{aligned}$$

Then, we can write $D_i = B_i.C_i$, where $B_i = B_{i1} \oplus B_{i2}$ and $C_i = C_{i1} \oplus C_{i2.}$ Both B_i and C_i are similar to the matrix

$$E_i = \operatorname{diag}\left(b_i, (b_i)^{-1}\right) \oplus \operatorname{diag}\left(ab_i, a^{m-1}(b_i)^{-1}\right) \oplus \operatorname{diag}\left(a^2 b_i, a^{m-2}(b_i)^{-1}\right) \oplus \cdots \oplus \operatorname{diag}\left(a^{m-1}b_i, a(b_i)^{-1}\right).$$

If q is even, then $A = \left(\bigoplus_{i=1}^{\gamma_q} D_i\right) = \left(\bigoplus_{i=1}^{\gamma_q} B_i\right).\left(\bigoplus_{i=1}^{\gamma_q} C_i\right)$. Both $\left(\bigoplus_{i=1}^{\gamma_q} B_i\right)$ and $\left(\bigoplus_{i=1}^{\gamma_q} C_i\right)$ are similar to the matrix $E = \left(\bigoplus_{i=1}^{\gamma_q} E_i\right)$. Since $2 \le k \le n = 2\gamma_q m$, by Theorem 2.14, E is a commutator of two unipotent matrices of index k. Therefore, A is a product of two commutators of unipotent matrices of index k.

If q is odd, then $A = \left(\bigoplus_{i=1}^{\gamma_q - 1} D_i\right) \oplus aI_m = BC$, where

$$B = \left(\left(\bigoplus_{i=1}^{\gamma_q - 1} B_i\right) \oplus \operatorname{diag}\left(1, a, a^2, \dots, a^{m-1}\right)\right),$$

and

$$C = \left(\left(\bigoplus_{i=1}^{\gamma_q - 1} C_i\right) \oplus \operatorname{diag}\left(a, 1, a^{m-1}, \dots, a^2\right)\right).$$

Both B and C are similar to the matrix $E = (1) \oplus \left(\bigoplus_{i=1}^{\gamma_q - 1} E_i\right) \oplus \left(\bigoplus_{i=1}^{\frac{m-1}{2}} \operatorname{diag}\left(a^i, a^{m-i}\right)\right)$. It is clear that $2 \le k \le n = (2\gamma_q - 1)m$. Therefore, by applying Corollary 2.15, we conclude that E is a commutator of unipotent matrices of index k. Thus, A is a product of two commutators of unipotent matrices of index k.□

As a direct consequence of Theorems 2.17 and 3.4, we obtain the following result.

COROLLARY 3.5. *Let $n \ge k \ge 2$ be two integers and F be a quadratically closed field. Each matrix in $\mathrm{SL}_n(F)$ can be decomposed into a product of at most two commutators of unipotent matrices of index k.*

Appendix and open problems. In the appendix, we use GAP to provide examples involving Theorem 2.17 and Theorem 3.4. The examples presented here consider matrices within the special linear group of order n over the finite field $\mathbb{F}_q$. For specific positive integers n, q, and k, the GAP code to determine if there are any elements in $\mathrm{SL}_n((F)_q)$ that are not the product of at most two commutators of unipotent matrices of index k is as follows:

```
# After input n, q and k, create the special linear group of order n over F_q
gap> G:=SL(n,q);
```

```
gap> I:=Identity(G);
gap> zero:=I-I;

# Construct a function to check if a matrix is an unipotent matrix of index k.
gap> localVars:=List([1..k],x -> 0);
gap> sa_uni:=function(A)
>    local j, tem, F;
>    tem:=I;
>    for j in [1..k] do
>       localVars[j]:=(A-I)*tem;
>       tem:=localVars[j];
>    od;
>    F:=zero;
>    if localVars[k] = zero and localVars[k-1] <> zero then
>       F:=A;
>    fi;
>    return F;
>    end;

# Construct the set uni_set containing all unipotent matrices of index k in G.
gap> uni_set:=Set([]);
>    for A in G do
>       if sa_uni(A) <> zero then
>         AddSet(uni_set,A);
>       fi;
>    od;

# Construct the set 1com_set containing all commutators of unipotent matrices of index k
gap> 1com_set:=Set([]);
>    for B in uni_set do
>      for C in uni_set do
>        AddSet(1com_set,Comm(B,C));
>      od;
>    od;

# Construct the set 2com_set containing all commutators of unipotent matrices of index k
gap> 2com_set:=Set([]);
>    for B in 1com_set do
>      for C in 1com_set do
>        AddSet(2com_set,B*C);
>      od;
>    od;

# Conclusion
gap> if Size(Union(1com_set,2com_set)) = Size(SL(n,q)) then
```

```
>      Print ("Every element of G is the product of at most
two commutators of unipotent matrices of index k.");
>    else
```

```
>      Print("Some elements of G are not the product of at most
two commutators of unipotent matrices of index k. They are:");
>      Print (Difference(SL(n,q), Union(1com_set, 2com_set)));
>    fi;
```

Because the code will take a long time to run in GAP when $\mathrm{SL}(n,q)$ has many elements, we apply this code to small values of n and q.

Using the above code, we present the following three examples to illustrate the importance of the number of elements in F in Theorem 2.17:

Example 1. $n=2, q=8, k=2$ *(nonscalar, enough elements, true).* Since $\mathbb{F}_8$ has more than $2n+3=7$ elements, it satisfies the condition of Theorem 2.17. Moreover, every nonscalar matrices in $\mathrm{SL}_2(\mathbb{F}_8)$ is decomposed into a product of at most two commutators of unipotent matrices of index 2.

Example 2. $n=4, q=2, k\in\{2,3,4\}$*(nonscalar, not enough elements, true).* In this case, $\mathbb{F}_2$ has fewer than $2n+3=7$ elements. However, every nonscalar matrices in $\mathrm{SL}_4(\mathbb{F}_2)$ is decomposed into a product of at most two commutators of unipotent matrices of index k.

Example 3. $n=2, q=2, k=2$ *(nonscalar, not enough elements, false).* The field $\mathbb{F}_2$ has fewer than $2n+3=7$ elements and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a nonscalar matrix in $\mathrm{SL}_2(\mathbb{F}_2)$ that cannot be decomposed into a product of one or two commutators of unipotent matrices of index 2.

The sufficient condition in Theorem 3.4 is that F must be an quadratically closed field. Next, we provide an example that Theorem 3.4 is not true when F is not an quadratically closed field by using the above code.

Example 4. $n=2, q=5, k=2$. The field $\mathbb{F}_5=\{0,1,2,3,4\}$ is non-quadratically closed field. There exists a scalar matrix of $\mathrm{SL}_2(\mathbb{F}_5)$ that cannot be decomposed into a product of one or two commutators of unipotent matrices of index 2. This matrix is $4I_2=\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$.

Additionally, we apply the results from Section 2, combined with the use of Maple, to express a specific matrix $A\in\mathrm{SL}_n(\mathbb{R})$ as a product of two commutators of unipotent matrices of index $k\le n$, considering the cases $n=3$ and $n=4$.

Example 5. Consider $A=\begin{pmatrix} 1 & 6 & 3 \\ -1 & 3 & 1 \\ 1 & -1 & 0 \end{pmatrix}\in\mathrm{SL}_3(\mathbb{R})$. In Case 2 of the proof of Theorem 2.17, by choosing $a_1=\dfrac{1}{\sqrt{3}}$, A can be written as $A=BC$, where $B=\begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & -\frac{7}{3} & \frac{1}{3} \end{pmatrix}$ and $C=\begin{pmatrix} 1 & 6 & 3 \\ 0 & 3 & \frac{4}{3} \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$ are similar to $\operatorname{diag}\left(1,\dfrac{1}{3},3\right)$. By using the proofs in Section 2 (from Lemma 2.8 to Theorem 2.17), we can compute $A=[B_1,C_1]\cdot[B_2,C_2]$, where B_1,C_1,B_2,C_2 are unipotent matrices of index 2, and $A=[B_3,C_3]\cdot[B_4,C_4]$, where B_3,C_3,B_4,C_4 are unipotent matrices of index 3. Specifically,

$$B_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{5}{28} - \frac{3}{14}\sqrt{3} & -\frac{1}{2} + \sqrt{3} & -\frac{16}{7} + \frac{8}{7}\sqrt{3} \\ -\frac{9}{32} + \frac{5}{64}\sqrt{3} & \frac{21}{16} - \frac{21}{32}\sqrt{3} & \frac{5}{2} - \sqrt{3} \end{pmatrix}, C_1 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} + \frac{1}{14}\sqrt{3} & \frac{1}{2} - \frac{1}{3}\sqrt{3} & -\frac{8}{21}\sqrt{3} \\ -\frac{5}{16} - \frac{11}{64}\sqrt{3} & \frac{7}{8} + \frac{49}{96}\sqrt{3} & \frac{3}{2} + \frac{1}{3}\sqrt{3} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 1 & \frac{3}{2} & \frac{27}{4} - 3\sqrt{3} \\ 0 & \frac{7}{4} + \frac{1}{8}\sqrt{3} & \frac{21}{8} - \frac{15}{16}\sqrt{3} \\ 0 & -\frac{1}{2} - \frac{1}{4}\sqrt{3} & \frac{1}{4} - \frac{1}{8}\sqrt{3} \end{pmatrix}, C_2 = \begin{pmatrix} 1 & -\frac{3}{2} & -\frac{3}{4} + \sqrt{3} \\ 0 & \frac{1}{2} + \frac{1}{8}\sqrt{3} & -\frac{1}{2} + \frac{19}{48}\sqrt{3} \\ 0 & -\frac{1}{4}\sqrt{3} & \frac{3}{2} - \frac{1}{8}\sqrt{3} \end{pmatrix},$$

$$B_3 = \begin{pmatrix} \frac{59}{32} + \frac{13}{32}\sqrt{3} & -\frac{35}{16} - \frac{21}{16}\sqrt{3} & -1 - \sqrt{3} \\ \frac{269}{448} - \frac{5}{448}\sqrt{3} & -\frac{51}{32} + \frac{11}{32}\sqrt{3} & -\frac{39}{14} + \frac{9}{14}\sqrt{3} \\ -\frac{63}{128} - \frac{3}{128}\sqrt{3} & \frac{119}{64} - \frac{21}{64}\sqrt{3} & \frac{11}{4} - \frac{3}{4}\sqrt{3} \end{pmatrix},$$

$$C_3 = \begin{pmatrix} \frac{29}{32} - \frac{13}{32}\sqrt{3} & -\frac{21}{16} + \frac{21}{16}\sqrt{3} & -3 + \sqrt{3} \\ \frac{13}{64} - \frac{59}{448}\sqrt{3} & -\frac{5}{32} + \frac{31}{96}\sqrt{3} & -\frac{3}{2} + \frac{5}{42}\sqrt{3} \\ -\frac{37}{128} - \frac{9}{128}\sqrt{3} & \frac{77}{64} + \frac{35}{192}\sqrt{3} & \frac{9}{4} + \frac{1}{12}\sqrt{3} \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 1 & \frac{9}{8} - \frac{1}{8}\sqrt{3} & \frac{109}{16} - \frac{45}{16}\sqrt{3} \\ 0 & \frac{7}{4} + \frac{1}{8}\sqrt{3} & \frac{21}{8} - \frac{15}{16}\sqrt{3} \\ 0 & -\frac{1}{2} - \frac{1}{4}\sqrt{3} & \frac{1}{4} - \frac{1}{8}\sqrt{3} \end{pmatrix}, C_4 = \begin{pmatrix} 1 & -\frac{9}{8} + \frac{1}{8}\sqrt{3} & \frac{3}{16} + \frac{13}{16}\sqrt{3} \\ 0 & \frac{1}{2} + \frac{1}{8}\sqrt{3} & -\frac{1}{2} + \frac{19}{48}\sqrt{3} \\ 0 & -\frac{1}{4}\sqrt{3} & \frac{3}{2} - \frac{1}{8}\sqrt{3} \end{pmatrix}.$$

Example 6. Consider $A = \begin{pmatrix} 1 & 2 & -2 & -2 \\ 2 & 5 & -3 & -4 \\ 1 & 2 & -1 & 4 \\ -1 & 0 & 5 & 9 \end{pmatrix} \in \mathrm{SL}_4(\mathbb{R})$. In Case 1 of the proof of Theorem 2.17, by choosing $a_1 = \dfrac{1}{\sqrt{3}}, a_2 = \dfrac{1}{\sqrt{2}}$, A can be written as $A = BC$, where $B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 4 & \frac{1}{2} & 0 & 0 \\ 2 & 0 & 3 & 0 \\ -2 & 1 & 3 & \frac{1}{3} \end{pmatrix}$ and $C = \begin{pmatrix} \frac{1}{2} & 1 & -1 & -1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & \frac{1}{3} & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ are similar to $\mathrm{diag}\left(\dfrac{1}{2}, 2, \dfrac{1}{3}, 3\right)$. By using the proofs in Section 2 (from Lemma 2.8 to Theorem 2.17), we can compute $A = [B_1, C_1] \cdot [B_2, C_2]$, where B_1, C_1, B_2, C_2 are unipotent matrices of index 2, $A = [B_3, C_3] \cdot [B_4, C_4]$, where B_3, C_3, B_4, C_4 are unipotent matrices of index 3, and $A = [B_5, C_5] \cdot [B_6, C_6]$, where B_5, C_5, B_6, C_6 are unipotent matrices of index 4. Specifically,

$$B_1 = \begin{pmatrix} \frac{49}{3} - 10\sqrt{2} & -\frac{45}{8} + \frac{15}{4}\sqrt{2} & 0 & 0 \\ \frac{1888}{45} - \frac{3584}{135}\sqrt{2} & -\frac{43}{3} + 10\sqrt{2} & 0 & 0 \\ \frac{7}{15} + 20\sqrt{2} - \frac{226}{15}\sqrt{3} & \frac{7}{12} - \frac{15}{2}\sqrt{2} + \frac{16}{3}\sqrt{3} & -\frac{1}{2} + \sqrt{3} & \frac{16}{9} - \frac{8}{9}\sqrt{3} \\ -\frac{2069}{120} + \frac{1472}{45}\sqrt{2} - \frac{1401}{80}\sqrt{3} & 7 - 12\sqrt{2} + 6\sqrt{3} & -\frac{27}{16} + \frac{27}{32}\sqrt{3} & \frac{5}{2} - \sqrt{3} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} \frac{5}{2}\sqrt{2} + \frac{2}{3} & -\frac{15}{16}\sqrt{2} & 0 & 0 \\ -\frac{16}{9} + \frac{908}{135}\sqrt{2} & \frac{4}{3} - \frac{5}{2}\sqrt{2} & 0 & 0 \\ -5\sqrt{2} - \frac{1}{3} + \frac{226}{45}\sqrt{3} & \frac{15}{8}\sqrt{2} - \frac{16}{9}\sqrt{3} & \frac{1}{2} - \frac{1}{3}\sqrt{3} & \frac{8}{27}\sqrt{3} \\ \frac{37}{12} - \frac{344}{45}\sqrt{2} + \frac{407}{80}\sqrt{3} & -1 + 3\sqrt{2} - 2\sqrt{3} & -\frac{9}{8} - \frac{21}{32}\sqrt{3} & \frac{3}{2} + \frac{1}{3}\sqrt{3} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \frac{8}{3} - \frac{4}{3}\sqrt{2} & -\frac{11}{9} + \frac{2}{3}\sqrt{2} & -\frac{454}{15} + \frac{92}{5}\sqrt{2} + \frac{1}{10}\sqrt{3} & \frac{361}{20} - \frac{224}{15}\sqrt{2} - \frac{27}{8}\sqrt{3} \\ 3 - 2\sqrt{2} & -\frac{2}{3} + \frac{4}{3}\sqrt{2} & -38 + 28\sqrt{2} - \frac{3}{2}\sqrt{3} & \frac{653}{20} - \frac{116}{5}\sqrt{2} + \frac{57}{40}\sqrt{3} \\ 0 & 0 & 2 - \frac{3}{4}\sqrt{3} & -\frac{7}{8} + \frac{5}{16}\sqrt{3} \\ 0 & 0 & 2 - \sqrt{3} & \frac{3}{4}\sqrt{3} \end{pmatrix},$$

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$$C_2 = \begin{pmatrix} \frac{4}{3}+\frac{1}{3}\sqrt{2} & -\frac{4}{9}-\frac{1}{3}\sqrt{2} & \frac{5}{3}-\frac{24}{5}\sqrt{2}-\frac{1}{30}\sqrt{3} & -\frac{1}{2}+\frac{62}{15}\sqrt{2}-\frac{131}{40}\sqrt{3} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{3}\sqrt{2}+\frac{2}{3} & -7\sqrt{2}-1+\frac{1}{2}\sqrt{3} & \frac{29}{5}\sqrt{2}+\frac{1}{2}-\frac{3}{40}\sqrt{3} \\ 0 & 0 & \frac{3}{2}+\frac{1}{4}\sqrt{3} & -\frac{3}{4}-\frac{7}{16}\sqrt{3} \\ 0 & 0 & \frac{1}{3}\sqrt{3} & \frac{1}{2}-\frac{1}{4}\sqrt{3} \end{pmatrix},$$

$$B_3 = \begin{pmatrix} \frac{247}{192}-\frac{1311}{128}\sqrt{3}+\frac{17}{256}\sqrt{2}+\frac{495}{64}\sqrt{6} & -\frac{45}{16}\sqrt{6}+\frac{15}{4}\sqrt{3} & \frac{189}{128}+\frac{225}{256}\sqrt{3}-\frac{495}{512}\sqrt{2}-\frac{81}{128}\sqrt{6} & -\frac{15}{16}+\frac{5}{8}\sqrt{2}+\frac{15}{32}\sqrt{6}-\frac{5}{8}\sqrt{3} \\ \frac{3463}{1800}-\frac{871}{32}\sqrt{3}+\frac{109}{540}\sqrt{2}+\frac{12281}{600}\sqrt{6} & \frac{7}{10}-\frac{112}{15}\sqrt{6}+10\sqrt{3} & \frac{323}{80}+\frac{153}{64}\sqrt{3}-\frac{21}{8}\sqrt{2}-\frac{137}{80}\sqrt{6} & -\frac{451}{180}+\frac{5}{3}\sqrt{2}+\frac{56}{45}\sqrt{6}-\frac{5}{3}\sqrt{3} \\ \frac{14669}{480}+\frac{5201}{960}\sqrt{3}-\frac{17}{128}\sqrt{2}-\frac{495}{32}\sqrt{6} & -\frac{32}{3}+\frac{45}{8}\sqrt{6}-\frac{13}{6}\sqrt{3} & -\frac{221}{64}-\frac{97}{128}\sqrt{3}+\frac{495}{256}\sqrt{2}+\frac{81}{64}\sqrt{6} & \frac{263}{72}-\frac{5}{4}\sqrt{2}-\frac{15}{16}\sqrt{6}+\frac{13}{36}\sqrt{3} \\ \frac{18881}{600}+\frac{633}{40}\sqrt{3}-\frac{23}{360}\sqrt{2}-\frac{2569}{100}\sqrt{6} & -\frac{54}{5}+\frac{46}{5}\sqrt{6}-6\sqrt{3} & -\frac{93}{16}-\frac{27}{16}\sqrt{3}+\frac{45}{16}\sqrt{2}+\frac{15}{8}\sqrt{6} & \frac{82}{15}-2\sqrt{2}-\frac{23}{15}\sqrt{6}+\sqrt{3} \end{pmatrix},$$

$$C_3 = \begin{pmatrix} -\frac{377}{256}\sqrt{2}-\frac{9}{512}\sqrt{6}+\frac{431}{768}+\frac{339}{128}\sqrt{3} & \frac{15}{32}\sqrt{2}-\frac{15}{16}\sqrt{3} & \frac{135}{512}\sqrt{2}-\frac{9}{1024}\sqrt{6}-\frac{27}{512}-\frac{45}{256}\sqrt{3} & -\frac{15}{64}\sqrt{2}+\frac{5}{32}\sqrt{3} \\ -\frac{10757}{7200}+\frac{57}{8}\sqrt{3}-\frac{8557}{2160}\sqrt{2}+\frac{227}{4800}\sqrt{6} & \frac{17}{15}+\frac{5}{4}\sqrt{2}-\frac{5}{2}\sqrt{3}-\frac{1}{30}\sqrt{6} & -\frac{57}{320}-\frac{7}{16}\sqrt{3}+\frac{21}{32}\sqrt{2}-\frac{19}{640}\sqrt{6} & \frac{1}{30}-\frac{5}{8}\sqrt{2}+\frac{5}{12}\sqrt{3}+\frac{1}{180}\sqrt{6} \\ \frac{377}{128}\sqrt{2}+\frac{9}{256}\sqrt{6}-\frac{47}{384}-\frac{791}{2880}\sqrt{3} & -\frac{15}{16}\sqrt{2}+\frac{7}{72}\sqrt{3} & -\frac{135}{256}\sqrt{2}+\frac{9}{512}\sqrt{6}+\frac{155}{256}+\frac{7}{384}\sqrt{3} & \frac{15}{32}\sqrt{2}-\frac{7}{432}\sqrt{3} \\ \frac{8173}{1200}-\frac{241}{80}\sqrt{3}+\frac{811}{180}\sqrt{2}+\frac{497}{800}\sqrt{6} & -\frac{11}{5}-\frac{3}{2}\sqrt{2}+\sqrt{3}-\frac{1}{5}\sqrt{6} & -\frac{189}{160}+\frac{3}{32}\sqrt{3}-\frac{9}{8}\sqrt{2}-\frac{3}{320}\sqrt{6} & \frac{17}{10}+\frac{3}{4}\sqrt{2}-\frac{1}{6}\sqrt{3}+\frac{1}{30}\sqrt{6} \end{pmatrix},$$

$$B_4 = \begin{pmatrix} \frac{8}{3}-\frac{4}{3}\sqrt{2} & -\frac{11}{9}+\frac{2}{3}\sqrt{2} & -\frac{283}{10}+\frac{256}{15}\sqrt{2}-\frac{31}{30}\sqrt{6}+\frac{43}{30}\sqrt{3} & \frac{651}{40}-\frac{107}{24}\sqrt{3}-\frac{821}{60}\sqrt{2}+\frac{97}{120}\sqrt{6} \\ 3-2\sqrt{2} & -\frac{2}{3}+\frac{4}{3}\sqrt{2} & -35+26\sqrt{2}-\frac{3}{2}\sqrt{6}+\frac{1}{2}\sqrt{3} & \frac{146}{5}-\frac{23}{40}\sqrt{3}-\frac{419}{20}\sqrt{2}+\frac{57}{40}\sqrt{6} \\ 0 & 0 & 2-\frac{3}{4}\sqrt{3} & -\frac{7}{8}+\frac{5}{16}\sqrt{3} \\ 0 & 0 & 2-\sqrt{3} & \frac{3}{4}\sqrt{3} \end{pmatrix},$$

$$C_4 = \begin{pmatrix} \frac{4}{3}+\frac{1}{3}\sqrt{2} & -\frac{4}{9}-\frac{1}{3}\sqrt{2} & \frac{28}{15}-\frac{43}{10}\sqrt{2}-\frac{11}{30}\sqrt{3}+\frac{1}{30}\sqrt{6} & -\frac{11}{20}-\frac{343}{120}\sqrt{3}+\frac{421}{120}\sqrt{2}-\frac{1}{120}\sqrt{6} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{3}\sqrt{2}+\frac{2}{3} & -\frac{25}{4}\sqrt{2}-1 & \frac{419}{80}\sqrt{2}+\frac{1}{40}\sqrt{6}+\frac{13}{20}+\frac{3}{10}\sqrt{3} \\ 0 & 0 & \frac{3}{2}+\frac{1}{4}\sqrt{3} & -\frac{3}{4}-\frac{7}{16}\sqrt{3} \\ 0 & 0 & \frac{1}{3}\sqrt{3} & \frac{1}{2}-\frac{1}{4}\sqrt{3} \end{pmatrix},$$

$$B_5 = \begin{pmatrix} \frac{6025}{192}+\frac{1329}{128}\sqrt{3}-\frac{2243}{128}\sqrt{2}-\frac{2703}{256}\sqrt{6} & -\frac{45}{4}-\frac{15}{4}\sqrt{3}+\frac{105}{16}\sqrt{2}+\frac{15}{4}\sqrt{6} & -\frac{189}{128}-\frac{207}{256}\sqrt{3}+\frac{189}{256}\sqrt{2}+\frac{369}{512}\sqrt{6} & \frac{15}{16}+\frac{5}{8}\sqrt{3}-\frac{15}{32}\sqrt{2}-\frac{5}{8}\sqrt{6} \\ \frac{148087}{1800}+\frac{33527}{1200}\sqrt{3}-\frac{1010867}{21600}\sqrt{2}-\frac{16781}{600}\sqrt{6} & -\frac{443}{15}-\frac{152}{15}\sqrt{3}+\frac{88}{5}\sqrt{2}+\frac{299}{30}\sqrt{6} & -\frac{333}{80}-\frac{359}{160}\sqrt{3}+\frac{651}{320}\sqrt{2}+\frac{157}{80}\sqrt{6} & \frac{38}{15}+\frac{76}{45}\sqrt{3}-\frac{19}{15}\sqrt{2}-\frac{299}{180}\sqrt{6} \\ -\frac{14221}{480}-\frac{34399}{960}\sqrt{3}+\frac{2243}{64}\sqrt{2}+\frac{2703}{128}\sqrt{6} & \frac{71}{6}+\frac{77}{6}\sqrt{3}-\frac{105}{8}\sqrt{2}-\frac{15}{2}\sqrt{6} & \frac{157}{64}+\frac{335}{128}\sqrt{3}-\frac{189}{128}\sqrt{2}-\frac{369}{256}\sqrt{6} & -\frac{7}{72}-\frac{77}{36}\sqrt{3}+\frac{15}{16}\sqrt{2}+\frac{5}{4}\sqrt{6} \\ -\frac{38551}{600}-\frac{19691}{400}\sqrt{3}+\frac{200353}{3600}\sqrt{2}+\frac{6983}{200}\sqrt{6} & \frac{119}{5}+\frac{86}{5}\sqrt{3}-\frac{102}{5}\sqrt{2}-\frac{61}{5}\sqrt{6} & \frac{27}{16}+\frac{93}{32}\sqrt{3}-\frac{63}{32}\sqrt{2}-\frac{33}{16}\sqrt{6} & -\frac{3}{10}-\frac{43}{15}\sqrt{3}+\frac{7}{5}\sqrt{2}+\frac{61}{30}\sqrt{6} \end{pmatrix},$$

$$C_5 = \begin{pmatrix} -\frac{323}{256}\sqrt{2}-\frac{2703}{512}\sqrt{6}+\frac{3145}{384}+\frac{1329}{256}\sqrt{3} & \frac{15}{32}\sqrt{2}-\frac{45}{16}-\frac{15}{8}\sqrt{3}+\frac{15}{8}\sqrt{6} & \frac{189}{512}\sqrt{2}+\frac{369}{1024}\sqrt{6}-\frac{189}{256}-\frac{207}{512}\sqrt{3} & -\frac{15}{64}\sqrt{2}+\frac{15}{32}+\frac{5}{16}\sqrt{3}-\frac{5}{16}\sqrt{6} \\ \frac{75311}{3600}+\frac{32621}{2400}\sqrt{3}-\frac{33659}{10800}\sqrt{2}-\frac{68477}{4800}\sqrt{6} & -\frac{209}{30}+\frac{23}{20}\sqrt{2}+\frac{151}{30}\sqrt{6}-\frac{73}{15}\sqrt{3} & -\frac{309}{160}-\frac{317}{320}\sqrt{3}+\frac{147}{160}\sqrt{2}+\frac{589}{640}\sqrt{6} & \frac{83}{60}-\frac{73}{120}\sqrt{2}+\frac{73}{90}\sqrt{3}-\frac{151}{180}\sqrt{6} \\ \frac{323}{128}\sqrt{2}+\frac{2703}{256}\sqrt{6}-\frac{2953}{192}-\frac{30877}{5760}\sqrt{3} & -\frac{15}{16}\sqrt{2}+\frac{45}{8}+\frac{71}{36}\sqrt{3}-\frac{15}{4}\sqrt{6} & -\frac{189}{256}\sqrt{2}-\frac{369}{512}\sqrt{6}+\frac{253}{128}+\frac{365}{768}\sqrt{3} & \frac{15}{32}\sqrt{2}-\frac{15}{16}-\frac{71}{216}\sqrt{3}+\frac{5}{8}\sqrt{6} \\ -\frac{3109}{600}-\frac{2607}{200}\sqrt{3}+\frac{4973}{900}\sqrt{2}+\frac{12613}{800}\sqrt{6} & \frac{16}{5}-\frac{21}{10}\sqrt{2}-\frac{29}{5}\sqrt{6}+\frac{24}{5}\sqrt{3} & \frac{117}{80}+\frac{93}{80}\sqrt{3}-\frac{63}{40}\sqrt{2}-\frac{447}{320}\sqrt{6} & \frac{4}{5}+\frac{17}{20}\sqrt{2}-\frac{4}{5}\sqrt{3}+\frac{29}{30}\sqrt{6} \end{pmatrix},$$

$$B_6 = \begin{pmatrix} \frac{8}{3}-\frac{4}{3}\sqrt{2} & -\frac{11}{9}+\frac{2}{3}\sqrt{2} & -\frac{481}{15}-\frac{11}{10}\sqrt{3}+\frac{193}{10}\sqrt{2}+\frac{41}{30}\sqrt{6} & \frac{96}{5}-\frac{313}{120}\sqrt{3}-\frac{1891}{120}\sqrt{2}-\frac{97}{120}\sqrt{6} \\ 3-2\sqrt{2} & -\frac{2}{3}+\frac{4}{3}\sqrt{2} & -41-\frac{7}{2}\sqrt{3}+\frac{59}{2}\sqrt{2}+2\sqrt{6} & \frac{361}{10}+\frac{129}{40}\sqrt{3}-\frac{997}{40}\sqrt{2}-\frac{31}{20}\sqrt{6} \\ 0 & 0 & 2-\frac{3}{4}\sqrt{3} & -\frac{7}{8}+\frac{5}{16}\sqrt{3} \\ 0 & 0 & 2-\sqrt{3} & \frac{3}{4}\sqrt{3} \end{pmatrix},$$

$$C_6 = \begin{pmatrix} \frac{4}{3}+\frac{1}{3}\sqrt{2} & -\frac{4}{9}-\frac{1}{3}\sqrt{2} & \frac{22}{15}-\frac{21}{5}\sqrt{2}+\frac{19}{30}\sqrt{6}-\frac{5}{6}\sqrt{3} & \frac{21}{20}-\frac{269}{120}\sqrt{3}+\frac{97}{30}\sqrt{2}-\frac{89}{120}\sqrt{6} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{3}\sqrt{2}+\frac{2}{3} & -\frac{25}{4}\sqrt{2}-\frac{5}{2}-\frac{1}{2}\sqrt{3}+\sqrt{6} & \frac{79}{16}\sqrt{2}-\frac{31}{40}\sqrt{6}+\frac{89}{40}+\frac{33}{40}\sqrt{3} \\ 0 & 0 & \frac{3}{2}+\frac{1}{4}\sqrt{3} & -\frac{3}{4}-\frac{7}{16}\sqrt{3} \\ 0 & 0 & \frac{1}{3}\sqrt{3} & \frac{1}{2}-\frac{1}{4}\sqrt{3} \end{pmatrix}.$$

We close the work with five intriguing questions. In the proof of Theorem 4.4, we use the following two properties of a quadratically closed field F:
(1) every element in F has a square root,
(2) F is an infinite field.
This raises the following question:

Question 1. Given that F is a finite field with property (1), specifically $F = \mathbb{F}_{2^k}$, does Theorem 4.4 still hold?

The main theorem states that for two positive integers $n \geq k$, every nonscalar matrix in the special linear group of degree n over a field can be written as a product of at most two commutators of unipotent matrices of index k. This fact is also true for scalar matrices over a quadratically closed field. Therefore, we inquire whether two is indeed the smallest such number.

Question 2. For two positive integers $n \geq k > 2$, is there an n by n matrix that can be expressed as a product of two commutators of unipotent matrices of index k, but cannot be expressed as a single commutator of unipotent matrices of index k?

According to [11, Lemma 2.8], if $n = k = 2$, then $-I_2$ can be expressed as a product of two commutators of unipotent matrices of index 2, but it cannot be expressed as a single commutator of unipotent matrices of index 2.

Next, our questions concern further research on matrices over general rings, specifically focusing on division rings and local rings, stemming from recent main results obtained in [2, 3, 9].

Note that our obtained results can be seen as a generalization of Hou's results concerning matrices over the field of complex numbers in the study of decomposition into commutators of unipotent matrices of index 2. On the other hand, it is shown in [9] that every matrix in the special linear group of degree n over the real quaternion division ring can be expressed as a product of at most three commutators of unipotent matrices of index 2. Moreover, three is the smallest such number, illustrated by $-I_n$ for odd n. With these insights, we hope that the answer to the following question is affirmative.

Question 3. For two positive integers $n \geq k$, can every matrix in the special linear group of degree n over the real quaternion division ring be expressed as a product of at most three commutators of unipotent matrices of index k?

Question 4. For two positive integers $n \geq k$, can the matrix $-I_n$ be written as a product exactly three commutators of unipotent matrices of index k if n is odd?

The final question arises from the main tool we use, which is Sourour's theorem [17, Theorem 1]. A recent generalization of Sourour's theorem for matrices over local rings is provided in [2]. Therefore, we now focus our attention on the following challenging problem.

Question 5. For two positive integers $n \geq k$, does the special linear group of degree n over a local ring coincide with the group generated by commutators of unipotent matrices of index k and can matrices decompose if this is the case?

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REFERENCES

[1] M.H. Bien, P.V. Danchev, M. Ramezan-nassab, and T.N. Son. Products of unipotent elements in certain algebras. *Forum Math.*, 35:1655–1666, 2023.

[2] M.H. Bien, P.T. Nhan, and N.H.T. Nhat. Some decompositions of matrices over local rings. *J. Algebra Appl.*, 24: 2550088, 2025.

[3] M.H. Bien, T.N. Son, P.T.T. Thuy, and L.Q. Truong. Products of unipotent matrices of index 2 over division rings. *Acta Math. Hungar*, 173:74–100, 2024.

[4] J.D. Botha. Products of two unipotent matrices of index 2. *Linear Algebra Appl.*, 433:1447–1451, 2010.

[5] M.P. Drazin. On a result of J. J. Sylvester. *Linear Algebra Appl.*, 505:361–366, 2016.

[6] E.A. Egorchenkova and N.L. Gordeev. Products of commutators on a general linear group over a division algebra. *J. Math. Sci. (N.Y.)*, 243:561–572, 2019.

[7] C.K. Fong and A.R. Sourour. The group generated by unipotent operators. *Proc. Amer. Math. Soc.*, 97:453–458, 1986.

[8] P. Gvozdevsky. Commutator lengths in general linear group over a skew-field. *J. Math. Sci. (N.Y.)*, 264:29–38, 2022.

[9] N.T.T. Ha and D.T. Toan. Products of commutators of unipotent matrices of index 2 in $\mathrm{GL}_n(\mathbb{H})$. *Int. Electron. J. Algebra*, 36:121–133, 2024.

[10] R.A. Horn and C.R. Johnson. Topics in matrix analysis. *Cambridge University Press, Cambridge*, 2008.

[11] X. Hou, Z. Xiao, Y. Hao, and Q. Yuan. Decomposition of symplectic matrices into products of symplectic unipotent matrices of index 2. *Electron. J. Linear Algebra*, 35:497–502, 2019.

[12] X. Hou. Decomposition of matrices into commutators of unipotent matrices of index 2. *Electron. J. Linear Algebra*, 37:31–34, 2021.

[13] F. Knppel. $\mathrm{GL}_n^{\pm}(R)$ is 5-reflectional. *Abh. Math. Semin. Univ. Hamburg*, 61:47–51, 1991.

[14] R. Preusser. Sandwich classification for $\mathrm{GL}_n(R), \mathrm{O}_{2n}(R)$ and $\mathrm{U}_{2n}(R, \Lambda)$ revisited. *J. Group Theory*, 21:21–44, 2018.

[15] R. Preusser. Sandwich classification for $\mathrm{O}_{2n+1}(R)$ and $\mathrm{U}_{2n+1}(R, \Delta)$ revisited. *J. Group Theory*, 21:539–571, 2018.

[16] A.V. Stepanov. A new look at the decomposition of unipotents and the normal structure of Chevalley groups. *St. Petersburg. Math. J.*, 28:411–419, 2017; translation from *Algebra i Analiz*, 28:161–173, 2016.

[17] A.R. Sourour. A factorization theorem for matrices. *Linear and Multilinear Algebra*, 19:141–147, 1986.

[18] N.A. Vavilov. Towards the reverse decomposition of unipotents. *J. Math. Sci.*, 243:515-526, 2019; translation from *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 470:21–37, 2018.

[19] N.A. Vavilov. Toward the reverse decomposition of unipotents. II: The relative case. *J. Math. Sci.*, 252: 749–760, 2021; translation from *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 484:5–22, 2019.

[20] J-H. Wang and P.Y. Wu. Products of unipotent matrices of index 2. *Linear Algebra Appl.*, 149:111–123, 1991.