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THE EQUATION $XA + AX^* = 0$ AND THE DIMENSION OF *CONGRUENCE ORBITS[*]

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Abstract. We solve the matrix equation $XA+AX^* = 0$, where $A \in \mathbb{C}^{n\times n}$ is an arbitrary given square matrix, and we compute the dimension of its solution space. This dimension coincides with the codimension of the tangent space of the *congruence orbit of A. Hence, we also obtain the (real) dimension of *congruence orbits in $\mathbb{C}^{n\times n}$. As an application, we determine the generic canonical structure for *congruence in $\mathbb{C}^{n\times n}$ and also the generic Kronecker canonical form of *palindromic pencils $A + \lambda A^*$.

Key words. Canonical forms for *congruence, *Congruence, Codimension, Matrix equations, Orbits, *Palindromic pencils.

AMS subject classifications. 15A24, 15A21.

1. Introduction.

We are interested in the solution of the equation

$$XA + AX^* = 0\,, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a given arbitrary square matrix and X^* denotes the conjugate transpose of the unknown X. This equation is closely related to the equation

$$XA + AX^T = 0 \tag{1.2}$$

(where X^T denotes the transpose of X), which has been solved in the recent work [7], and the present paper can be seen as the continuation of that work. In particular, the main techniques used in [7] for equation (1.2) are still valid for equation (1.1) and, for the sake of brevity, many of the arguments will be referred to that paper. However, there are several important differences between both equations that have led us to address them separately. The main difference is that whereas (1.2) is linear in $\mathbb{C}$, (1.1) is not, though it is linear in $\mathbb{R}$. Then, the solution space of (1.1) is not a complex but a real subspace of $\mathbb{C}^{n\times n}$. To illustrate this fact and the difference between both

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equations, let us consider the scalar case. If A is a complex number and X is a scalar unknown, then the solution of (1.2), provided $A \neq 0$, is $X = 0$. By contrast, the solution of (1.1), for $A \neq 0$, consists of the set of purely imaginary complex numbers. This is connected to the theory of orbits, which is one of the main motivations to address the resolution of (1.1).

The *congruence orbit of A, denoted by $\mathcal{O}(A)$, is the set of matrices which are *congruent to A, that is,

$$\mathcal{O}(A) = \{PAP^* \,:\, P \text{ is nonsingular}\}.$$

The congruence orbit of A is defined in a similar way replacing P^* by P^T. It is known that the congruence orbit of A is a complex manifold [3] in $\mathbb{C}^{n\times n}$, considered as a vector space over the field $\mathbb{C}$. By contrast, the *congruence orbit of A is not a manifold in $\mathbb{C}^{n\times n}$ over $\mathbb{C}$ but over the real field $\mathbb{R}$. The link between the orbits and the solution of the matrix equations above is the tangent space. We will see that the *tangent space* of $\mathcal{O}(A)$ at the point A is the set

$$\mathcal{T}_A = \left\{ XA + AX^* :\, X \in \mathbb{C}^{n\times n} \right\}.$$

As a consequence, the dimension of the solution space of (1.1) is equal to the codimension of this tangent space, which is in turn equal to the codimension of $\mathcal{O}(A)$. Since $\mathcal{O}(A)$ is a real manifold, we have to consider here real dimension instead of the complex dimension considered in [7]. For more information about the theory of orbits and its relationship with (1.1), the interested reader may consult [7] and the references therein.

Another relevant difference between equations (1.1) and (1.2) comes from the reduction to the canonical form, which is the basic step in our resolution procedure. More precisely, to solve (1.1) (respectively, (1.2)) we first transform A into its canonical form for *congruence (resp., congruence), C_A, and then we solve the corresponding equation with C_A instead of A. From the solution of this last equation we are able to recover the solution of the original equation by means of a change of variables involving the *congruency (resp., congruency) matrix leading A to C_A. The canonical forms for congruence and *congruence bear a certain resemblance but they are not equal [9]. Both of them consist of three types of blocks, and these blocks coincide for just one type (Type 0), though the other two types have a similar appearance. These differences in the canonical forms lead in some cases to different solutions, though the techniques used to achieve these solutions are similar. In particular, in both cases the equation for C_A is decomposed into smaller equations involving the canonical blocks. One of the relevant differences related with this is that the hardest cases for equation (1.2) are no longer present in (1.1). The main consequence of this is that, for (1.1), we are able to give an explicit solution of this last equation for all types of blocks (we

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want to stress that though in [7] the complete solution of (1.2) was obtained, there is a particular case for which this solution was not given explicitly, but through an algorithm). Then, provided that the change matrix leading A to its canonical form for *congruence is known, we give a complete explicit solution of (1.1).

Among the applications of equations (1.1) and (1.2) there are two specific contexts where they are of particular interest. The first one is the theory of orbits, mentioned above, where the solution of these equations, and the determination of the dimension of their solution space, may have numerical applications in the computation of the canonical form for congruence (or *congruence) of A (see the introduction of [7] for details). In the second place, the more general equations $XA + BX^T = C$ and $XA + BX^* = C$ arise in the perturbation theory of the generalized palindromic eigenvalue problem [5]. Also, these equations naturally appear in intermediate steps in the design of structure-preserving algorithms for this kind of eigenvalue problems [11]. Related with these two contexts, in [7] the dimension of the solution space of (1.2) has been used to determine the generic Kronecker canonical form of T-palindromic pencils. In a similar way, the dimension of the solution space of (1.1) allow us to determine the generic Kronecker canonical form of *palindromic pencils. The same approach was followed in [6] to derive the generic Kronecker canonical form of singular matrix pencils (though the notion of genericity in [6] is different to the one considered in the present paper). We also want to remark that other equations somewhat related to (1.1) and (1.2) have been considered in [4, 12] in relation with Hamiltonian systems.

It is worth to point out that in the recent work [11, Lemma 8] mentioned above, the authors have provided necessary and sufficient conditions for the existence of a unique solution of $XA + BX^* = C$ (and the same was done in [5, Lemma 5.10] for the equation $XA + BX^T = C$). Forty years earlier, Ballantine considered in [2, Theorem 2] the non-homogeneous equation $XA + AX^* = C$, with C Hermitian and A Hermitian positive definite, and he gave necessary and sufficient conditions for the existence of a positive stable solution. In the present paper we are mainly interested in the case where the homogeneous equation (1.1) for general A has a multiple solution. Actually, as a consequence of our codimension count, we will see that (1.1) never has a unique solution (see Theorem 5.1). This is coherent with the characterization given in [11] when specialized to (1.1).

The paper is organized as follows. In Section 2, we recall the canonical form for *congruence and we also establish the relationship between (1.1) and the theory of orbits using the tangent space. In Section 3, we summarize the dimension count of the solution space of (1.1) in terms of the canonical form for *congruence of A. In Section 4, we solve (1.1) and, as a consequence, we prove the result stated in Section 3. In Section 5, we provide the generic canonical structure for *congruence in $\mathbb{C}^{n\times n}$, and also the generic Kronecker canonical form of *palindromic pencils. Finally, in

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Section 6, we summarize the most relevant contributions of the paper and we present some lines of future research.

2. The canonical form for *congruence. To calculate the real dimension of the solution space of (1.1) we will make use of the following result, whose proof mimics the one of Lemma 1 in [7] and is omitted.

LEMMA 2.1. *Let $A, B \in \mathbb{C}^{n\times n}$ be two *congruent matrices such that $B = PAP^*$. Let $Y \in \mathbb{C}^{n\times n}$ and $X := P^{-1}YP$. Then Y is a solution of $YB+BY^* = 0$ if and only if X is a solution of $XA + AX^* = 0$. Therefore, the linear mapping $Y \mapsto P^{-1}YP$ is an isomorphism between the solution space (over $\mathbb{R}$) of $YB + BY^* = 0$ and the solution space (over $\mathbb{R}$) of $XA + AX^* = 0$, and, as a consequence, both spaces have the same real dimension.*

Lemma 2.1 indicates that the solution of (1.1) can be recovered from the solution of the equation obtained by replacing A with another *congruent matrix $B = PAP^*$. This suggests a natural procedure to solve (1.1), namely, to reduce A by *congruence to a simpler form and then solve the equation with this new matrix as a coefficient matrix instead of A. We will use as this simple form the canonical form for *congruence introduced by Horn and Sergeichuk [9] (see also [10, 14]).

In order to recall the canonical form for *congruence, let us define the following $k \times k$ matrices as in [9],

$$\Gamma_k = \begin{bmatrix} 0 & & & & & (-1)^{k+1} \\ & & & & \iddots & (-1)^k \\ & & & -1 & \iddots & \\ & & 1 & 1 & & \\ & -1 & -1 & & & \\ 1 & 1 & & & & 0 \end{bmatrix} \quad (\Gamma_1 = [1]),$$

and the $k \times k$ Jordan block with eigenvalue $\lambda \in \mathbb{C}$,

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \quad (J_1(\lambda) = [\lambda]).$$

Also, we define, for each $\mu \in \mathbb{C}$, the $2k \times 2k$ matrix

$$H_{2k}(\mu) = \begin{bmatrix} 0 & I_k \\ J_k(\mu) & 0 \end{bmatrix} \quad \left(H_2(\mu) = \begin{bmatrix} 0 & 1 \\ \mu & 0 \end{bmatrix} \right).$$

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THEOREM 2.2. **(Canonical form for *congruence)** [9, Theorem 1.1 (b)] *Each square complex matrix is *congruent to a direct sum, uniquely determined up to permutation of summands, of canonical matrices of the three types:*

Type 0	$J_k(0)$
Type I	$\alpha\Gamma_k, \qquad \lvert\alpha\rvert = 1$
Type II	$H_{2k}(\mu), \quad \lvert\mu\rvert > 1$

One of our main motivations to address the resolution of (1.1) is because of its relation with the theory of orbits. In this context, our interest focuses not on the explicit solution of (1.1) but on the dimension of its solution space over $\mathbb{R}$. The connection between (1.1) and the *congruence orbits of A is shown in the following result. We omit the proof because it is similar to the one of Lemma 2 in [7].

LEMMA 2.3. *Let $A \in \mathbb{C}^{n\times n}$ be given and let $\mathcal{O}(A)$ be the *congruence orbit of A. Then the tangent space of $\mathcal{O}(A)$ at A is*

$$\mathcal{T}_A = \left\{ XA + AX^* \;:\; X \in \mathbb{C}^{n\times n} \right\}.$$

As a consequence of Lemma 2.3, the real dimension of the solution space of (1.1) over $\mathbb{R}$ is the codimension of $\mathcal{O}(A)$. This motivates the following definition.

DEFINITION 2.4. Given $A \in \mathbb{C}^{n\times n}$, the codimension of A is the codimension of its *congruence orbit $\mathcal{O}(A)$ (this codimension coincides with the dimension of the solution space of $XA + AX^* = 0$).

We want to stress again that, since $\mathcal{O}(A)$ is not a manifold over $\mathbb{C}$ but over $\mathbb{R}$, we are considering real dimension in Definition 2.4.

3. Main results. As a consequence of Lemma 2.1, the codimension of a given matrix A is equal to the codimension of its canonical form for *congruence. Then, we may restrict ourselves to A being a direct sum

$$A = \operatorname{diag}(D_1, \ldots, D_p)\,, \tag{3.1}$$

where $D_1, \ldots, D_p$ are canonical blocks of Type 0, I and II as in Theorem 2.2. Following an analogous procedure to the one in [7, §3], we partition the unknown $X = [X_{ij}]_{i,j=1}^p$ in (1.1) conformally with the partition of A in (3.1), where X_{ij} is a block with the appropriate size. Then (1.1) is equivalent to the system of p^2 equations obtained by equating to zero all the (i, j) blocks in the left hand side of (1.1). In particular, for the diagonal (i, i) block, we get

$$X_{ii}D_i + D_iX_{ii}^* = 0, \tag{3.2}$$

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and, for the (i, j) and (j, i) blocks together, we get

$$\begin{aligned} X_{ij}D_j + D_i X_{ji}^* &= 0 \\ X_{ji}D_i + D_j X_{ij}^* &= 0. \end{aligned} \tag{3.3}$$

Then (1.1) is decoupled into p smaller independent equations (3.2), for $i = 1, \dots, p$, together with $p(p-1)/2$ independent systems of equations (3.3), for $1 \le i < j \le p$. As a consequence, the dimension of the solution space of (1.1) is the sum of the dimension of the solution spaces of all equations (3.2), that is, the sum of the codimensions of the blocks D_i, and all systems of equations (3.3). This motivates the following definition.

Definition 3.1. Let $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$. Then the real interaction between M and N, denoted by $\operatorname{inter}(M, N)$, is the real dimension of the solution space (X, Y) over $\mathbb{R}$ of the linear system

$$\begin{aligned} XM + NY^* &= 0 \\ YN + MX^* &= 0, \end{aligned}$$

for the unknowns $X \in \mathbb{C}^{n\times m}$ and $Y \in \mathbb{C}^{m\times n}$.

Then we have the following result.

Lemma 3.2. *The real codimension of the block diagonal matrix $D = \operatorname{diag}(D_1, D_2, \dots, D_p)$ is the sum of the real codimensions of the diagonal blocks D_i for all $i = 1, \dots, p$, and the sum of the real interactions between D_i and D_j for all $1 \le i < j \le p$.*

The main result of the paper is stated in Theorem 3.3 below. It shows the real codimension of the *congruence orbit of an arbitrary matrix $A \in \mathbb{C}^{n\times n}$ in terms of the canonical form for *congruence of A. We stress that, instead of the complex dimension considered in [7], here we deal with real dimension, which implies that some factors of 2 appear in the statement of the theorem. Before stating Theorem 3.3 we want to remark an important difference with respect to [7, Theorem 2] for congruence orbits: observe that in Theorem 3.3 there is no contribution from interactions between Type I and Type II blocks. We will prove that these interactions are always zero.

From now on, we will use the standard notation $\lfloor q\rfloor$ (respectively, $\lceil q\rceil$) for the largest (resp., smallest) integer that is less (resp., greater) than or equal to q. We will use also the notation $\mathrm{i} := \sqrt{-1}$.

Theorem 3.3. **(Breakdown of the codimension count)** *Let $A \in \mathbb{C}^{n\times n}$ be a matrix with canonical form for *congruence,*

$$\begin{aligned} C_A = &J_{p_1}(0) \oplus J_{p_2}(0) \oplus \cdots \oplus J_{p_a}(0) \\ &\oplus \alpha_1\Gamma_{q_1} \oplus \alpha_2\Gamma_{q_2} \oplus \cdots \oplus \alpha_b\Gamma_{q_b} \\ &\oplus H_{2r_1}(\mu_1) \oplus H_{2r_2}(\mu_2) \oplus \cdots \oplus H_{2r_c}(\mu_c), \end{aligned}$$

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where $p_1 \geq p_2 \geq \cdots \geq p_a$. *Then the real codimension of the orbit of* A *for the action of* **congruence, i.e., the dimension of the solution space of* (1.1) *over* $\mathbb{R}$, *depends only on* C_A. *It can be computed as the sum*

$$c_{\text{Total}} = c_0 + c_1 + c_2 + c_{00} + c_{11} + c_{22} + c_{01} + c_{02}$$

whose components are given by:

1. *The codimension of the Type 0 blocks*
$$c_0 = \sum_{i=1}^{a} 2 \left\lceil \frac{p_i}{2} \right\rceil .$$
2. *The codimension of the Type I blocks*
$$c_1 = \sum_{i=1}^{b} q_i.$$
3. *The codimension of the Type II blocks*
$$c_2 = \sum_{i=1}^{c} 2r_i \, .$$
4. *The codimension due to interactions between Type 0 blocks*
$$c_{00} = \sum_{\substack{i,j=1\\ i<j}}^{a} \operatorname{inter}(J_{p_i}(0), J_{p_j}(0)) \, ,$$
where
$$\operatorname{inter}(J_{p_i}(0), J_{p_j}(0)) = \begin{cases} 2p_j, & \text{if } p_j \text{ is even,} \\ 2p_i, & \text{if } p_j \text{ is odd and } p_i \neq p_j, \\ 2(p_i+1), & \text{if } p_j \text{ is odd and } p_i = p_j. \end{cases}$$
5. *The codimension due to interactions between Type I blocks*
$$c_{11} = \sum 2 \min\{q_i, q_j\} \, ,$$
where the sum runs over all pairs of blocks $(\alpha_i \Gamma_{q_i}, \alpha_j \Gamma_{q_j})$, $i<j$, *in* C_A *such that: (a)* q_i *and* q_j *have the same parity (both odd or both even) and* $\alpha_i = \pm \alpha_j$, *and (b)* q_i *and* q_j *have different parity and* $\alpha_i = \pm \mathrm{i} \alpha_j$.
6. *The codimension due to interactions between Type II blocks*
$$c_{22} = \sum 4 \min\{r_i, r_j\} \, ,$$
where the sum runs over all pairs $(H_{2r_i}(\mu_i), H_{2r_j}(\mu_j))$, $i<j$, *of blocks in* C_A *such that* $\mu_i = \mu_j$.

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7. *The codimension due to interactions between Type 0 and Type I blocks*

$$c_{01} = N_{\text{odd}} \cdot \sum_{i=1}^{b} 2q_i \,,$$

where N_{odd} is the number of Type 0 blocks with odd size in C_A.

8. *The codimension due to interactions between Type 0 and Type II blocks*

$$c_{02} = N_{\text{odd}} \cdot \sum_{i=1}^{c} 4r_i \,,$$

where N_{odd} is the number of Type 0 blocks with odd size in C_A.

We want to stress several remarkable differences between Theorem 3.3 and the corresponding result in [7, Theorem 2] for the congruence orbits. The first one is the absence of any interaction between Type I and Type II blocks in Theorem 3.3, as noticed before the statement. Also, the codimension of Type I blocks in Theorem 3.3 is not exactly twice the codimension of Type I blocks in [7, Theorem 2], and for both the codimension and the interaction of Type II blocks, there is a summand in [7, Theorem 2] that is no longer present in Theorem 3.3.

Theorem 3.3 provides the dimension of the solution space of (1.1). In order to get the solution, we can follow the procedure below:

1. Transform A into its canonical form for *congruence: $C_A = PAP^*$.
2. Solve the equation $YC_A + C_AY^* = 0$. For this, decompose this equation into the smaller equations (3.2) and (3.3) and solve these equations independently.
3. Compute $X = P^{-1}YP$ as stated in Lemma 2.1.

In the following section, we will show how to solve (3.2) and (3.3) for the Type 0, I and II blocks of C_A. As a consequence, we completely solve (1.1) up to the knowledge of the *congruency matrix P leading A to its canonical form for *congruence C_A.

4. Codimension of individual blocks and interactions. In this section, we compute all quantities appearing in Theorem 3.3, namely the codimension of the Type 0, Type I and Type II blocks in the canonical form for *congruence of A, and the interaction between pairs of blocks. We want to stress that our procedure does not only give the dimension of the solution space of (1.1), but also the explicit solution of this equation.

Most of the arguments employed in [7] for equation (1.2) are still valid for equation (1.1), and also the conclusions are the same. Nonetheless, in certain cases (1.1) requires additional arguments. We will focus mainly on these cases and refer to [7] for the remaining ones. For the sake of brevity, we will only state three lemmas. In

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Lemma 4.1 we will group the results regarding the codimension of individual blocks. Lemma 4.2 contains the interactions due to pairs of blocks with the same type and in Lemma 4.3 we deal with the interaction between blocks of different type. We recall that we are considering real dimension instead of the complex one considered in [7].

Lemma 4.1. **(Codimension of individual blocks)**

(i) *The real codimension of an individual $k \times k$ Type 0 block is*

$$\operatorname{codim}(J_k(0)) = 2\left\lceil \frac{k}{2} \right\rceil .$$

(ii) *The real codimension of an individual $k \times k$ Type I block is*

$$\operatorname{codim}(\alpha\Gamma_k) = k.$$

(iii) *The real codimension of an individual $k \times k$ Type II block is*

$$\operatorname{codim}(H_{2k}(\mu)) = 2k.$$

Proof. Following similar arguments to the ones in [7, §4] for the congruence we find that the solution of $XJ_k(0) + J_k(0)X^* = 0$ is

$$X = \begin{bmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -\overline{x}_1 & 0 & x_2 & 0 & x_3 & \dots & x_{\frac{k}{2}} \\ -\overline{x}_2 & 0 & x_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & -\overline{x}_1 & 0 & x_2 & \dots & x_{\frac{k}{2}-1} \\ -\overline{x}_3 & 0 & -\overline{x}_2 & 0 & x_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\overline{x}_{\frac{k}{2}} & 0 & -\overline{x}_{\frac{k}{2}-1} & 0 & -\overline{x}_{\frac{k}{2}-2} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & -\overline{x}_1 \end{bmatrix} \quad (k \text{ even}),$$

$$X = \begin{bmatrix} x_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ -\overline{x}_2 & -\overline{x}_1 & x_2 & 0 & x_3 & \dots & x_{\frac{k+1}{2}} \\ 0 & 0 & x_1 & 0 & 0 & \dots & 0 \\ -\overline{x}_3 & 0 & -\overline{x}_2 & -\overline{x}_1 & x_2 & \dots & x_{\frac{k-1}{2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\overline{x}_{\frac{k+1}{2}} & 0 & -\overline{x}_{\frac{k-1}{2}} & 0 & -\overline{x}_{\frac{k-3}{2}} & \dots & x_2 \\ 0 & 0 & 0 & 0 & 0 & \dots & x_1 \end{bmatrix} \quad (k \text{ odd}),$$

where $x_1, \dots, x_{\lceil k/2 \rceil}$ are arbitrary (complex) parameters. From this, the statement on the codimension immediately follows.

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For Type I blocks, we have the equation

$$X\Gamma_k + \Gamma_k X^* = 0\,. \tag{4.1}$$

Let us consider separately the cases k even and k odd.

► k **even:** Similar reasonings as the ones for solving (1.2) in [7, Lemma 5] give rise to the following conclusions:

a) X is Toeplitz.
b) X is lower triangular.
c) $(-1)^i x_{k,k-i+1} = \overline{x}_{i1}$.

We want to stress that conditions a) and b) are the same as the ones obtained in the proof of Lemma 5 in [7], but c) is different to equation (17) in [7] due to the presence of the conjugate in the right hand side. This leads to a solution which is different than the corresponding one in [7], as we will see below.

For odd i, conditions a) and c) above imply

$$\begin{cases} x_{i1} = x_{k,k-i+1} \\ \overline{x}_{i1} = -x_{k,k-i+1}, \end{cases}$$

and this in turn implies

$$\overline{x}_{i1} = -x_{i1}.$$

Hence, x_{i1} is purely imaginary.

By analogous reasonings we conclude that x_{i1} is real for even i. Then X must be of the form

$$X = \begin{bmatrix} b_1\mathfrak{i} & & & & & 0 \\ a_1 & b_1\mathfrak{i} & & & & \\ b_2\mathfrak{i} & a_1 & b_1\mathfrak{i} & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ b_{\frac{k}{2}}\mathfrak{i} & \dots & b_2\mathfrak{i} & a_1 & b_1\mathfrak{i} & \\ a_{\frac{k}{2}} & b_{\frac{k}{2}}\mathfrak{i} & \dots & b_2\mathfrak{i} & a_1 & b_1\mathfrak{i} \end{bmatrix},$$

for some arbitrary real values $a_1, \dots, a_{k/2}$ and $b_1, \dots, b_{k/2}$. A direct computation shows that X above is indeed a solution of (4.1) for all real values $a's$ and $b's$, so this is the general solution of (4.1).

► k **odd:** Reasoning as above we conclude that the general solution of (4.1) in this

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case is

$$X = \begin{bmatrix} b_1\mathrm{i} & & & & & 0 \\ a_1 & b_1\mathrm{i} & & & & \\ b_2\mathrm{i} & a_1 & b_1\mathrm{i} & & & \\ \vdots & \ddots & \ddots & \ddots & & \\ a_{\frac{k-1}{2}} & \dots & b_2\mathrm{i} & a_1 & b_1\mathrm{i} & \\ b_{\frac{k+1}{2}}\mathrm{i} & a_{\frac{k-1}{2}} & \dots & b_2\mathrm{i} & a_1 & b_1\mathrm{i} \end{bmatrix}.$$

Now the statement on the codimension is immediate.

Finally, for Type II blocks, we have the system of equations

$$\left\{ \begin{array}{c} X_{12}J_k(\mu) = -X_{12}^* \\ X_{21} = -J_k(\mu)X_{21}^* \\ X_{11} = -X_{22}^* \\ X_{22}J_k(\mu) = -J_k(\mu)X_{11}^*, \end{array} \right.$$

obtained by partitioning the original equation

$$XH_{2k}(\mu) + H_{2k}(\mu)X^* = 0$$

conformally with the partition of $H_{2k}(\mu)$. Notice that the first and the second equations are decoupled from the other two equations. Hence, we will solve separately the first equation, then the second equation, and finally the third and the fourth ones together.

- For the first equation notice that, since $\mu \neq 0$, $J_k(\mu)$ is invertible. Then the first equation is equivalent to

$$X_{12}^* = -J_k(\mu)^{-*}X_{12},$$

where A^{-*} stands for the inverse of the conjugate transpose of A. Now, by replacing this in the initial equation we achieve

$$X_{12}J_k(\mu) = J_k(\mu)^{-*}X_{12},$$

which is a Sylvester equation. The reader should notice that there is a remarkable difference with the proof of Lemma 6 in [7]. In [7], $\mu = \pm 1$ were allowed, and these values required a special examination. Now we have $|\mu| > 1$, so these values are excluded, and this considerably simplifies the solution. More precisely, the matrices $J_k(\mu)$ and $J_k(\mu)^{-*}$ have no common eigenvalues, and then the solution of the previous equation is $X_{12} = 0$ [8, Ch. XII §1]. Similar reasonings lead to $X_{21} = 0$.

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- Replacing the third equation in the fourth one we obtain

$$X_{22}J_k(\mu) = J_k(\mu)X_{22},$$

which is again a Sylvester equation. The solution of this equation is an arbitrary upper triangular Toeplitz matrix [8, Ch. XII §1]. From this, the statement on the codimension follows. □

LEMMA 4.2. **(Interaction between blocks of the same type)** *The real interaction between two blocks of the same type is:*

(i) *For two Type 0 blocks $J_k(0), J_\ell(0)$, with $k \geq \ell$,*

$$\operatorname{inter}(J_k(0), J_\ell(0)) = \begin{cases} 2\ell\,, & \text{if } \ell \text{ is even,} \\ 2k\,, & \text{if } \ell \text{ is odd and } k \neq \ell, \\ 2(k+1)\,, & \text{if } \ell \text{ is odd and } k = \ell. \end{cases}$$

(ii) *For two Type I blocks,*

$$\operatorname{inter}(\alpha\Gamma_k, \beta\Gamma_\ell) = \begin{cases} 0\,, & \text{if } k,\ell \text{ have the same parity and } \alpha \neq \pm\beta, \\ 0\,, & \text{if } k,\ell \text{ have different parity and } \alpha \neq \pm \mathrm{i}\beta, \\ 2\min\{k,\ell\}, & \text{otherwise.} \end{cases}$$

(iii) *For two Type II blocks,*

$$\operatorname{inter}(H_{2k}(\mu), H_{2\ell}(\widetilde{\mu})) = \begin{cases} 4\min\{k,\ell\}\,, & \text{if } \mu = \widetilde{\mu}, \\ 0\,, & \text{if } \mu \neq \widetilde{\mu}. \end{cases}$$

Proof. The arguments in all three cases are similar to the corresponding ones for congruence employed in [7]. The solutions of the associated equations are also the same as the corresponding solutions there, but some cases deserve more detailed comments. More precisely, let us begin with the case of two type 0 blocks. In this case, the solution of

$$\begin{aligned} XJ_k(0) &= -J_\ell(0)Y^* \\ YJ_\ell(0) &= -J_k(0)X^* \end{aligned}$$

can be obtained from the solution of the system of equations with T instead of $*$ (see [7, Lemma 7]) just by replacing Y with $\overline{Y}$.

For two Type I blocks, we have the system of equations

$$\begin{aligned} \alpha X\Gamma_k &= -\beta\Gamma_\ell Y^* \\ \beta Y\Gamma_\ell &= -\alpha\Gamma_k X^*. \end{aligned} \tag{4.2}$$

Though the procedure that we follow to solve (4.2) is similar to the one used to solve the system of equations (40) in [7], the presence of the factors α and β in

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(4.2) introduces a difference that requires slight changes in the reasonings. Since Γ_ℓ is invertible, we find $Y = -(\alpha/\beta)\Gamma_k X^* \Gamma_\ell^{-1}$ and, taking conjugate transposes, $Y^* = -(\overline{\alpha}/\overline{\beta})\Gamma_\ell^{-T} X \Gamma_k^T$ (note that $\Gamma_m^* = \Gamma_m^T$). Replacing this expression for Y^* in the first equation of (4.2) we get the system of equations

$$X\left(\alpha\overline{\beta}\,\Gamma_k\Gamma_k^{-T}\right) = \left(\overline{\alpha}\beta\,\Gamma_\ell\Gamma_\ell^{-T}\right)X \tag{4.3}$$

$$Y = -(\alpha/\beta)\Gamma_k X^* \Gamma_\ell^{-1}, \tag{4.4}$$

which is equivalent to (4.2). To solve (4.3)-(4.4), we just have to solve (4.3) for X and then to obtain Y from (4.4). Note that (4.3) is a Sylvester equation. To solve it, we recall that $\Gamma_s\Gamma_s^{-T}$ is similar to $J_s((-1)^{s+1})$ [9, p. 1016]. As a consequence, and since $|\alpha| = |\beta| = 1$, the matrices $\alpha\overline{\beta}\,\Gamma_k\Gamma_k^{-T}$ and $\overline{\alpha}\beta\,\Gamma_\ell\Gamma_\ell^{-T}$ have different eigenvalues if and only if: (a) k, ℓ have the same parity and $\alpha \neq \pm\beta$, or (b) k, ℓ have different parity and $\alpha \neq \pm\mathrm{i}\beta$. In both cases the solution of (4.3) is $X = 0$ [8, Ch.VIII, §1], and this implies $Y = 0$ by (4.4). For the remaining cases, we may follow similar arguments as the ones in the proof of Lemma 8 in [7] to obtain

$$X = Q\left[\begin{array}{ccc|cccc} 0 & \dots & 0 & x_1 & x_2 & \dots & x_\ell \\ 0 & \dots & 0 & 0 & x_1 & \ddots & \vdots \\ \vdots & & \vdots & \vdots & & \ddots & x_2 \\ 0 & \dots & 0 & 0 & \dots & 0 & x_1 \end{array}\right]P^{-1},$$

where $x_1, x_2, \dots, x_\ell$ are free parameters, and P, Q are the nonsingular matrices leading $\alpha\overline{\beta}\Gamma_k\Gamma_k^{-T}$ and $\overline{\alpha}\beta\Gamma_\ell\Gamma_\ell^{-T}$, respectively, to their Jordan canonical form, that is, $(\alpha\overline{\beta})\,\Gamma_k\Gamma_k^{-T} = PJ_k((-1)^{k+1}\alpha\overline{\beta})P^{-1}$ and $(\overline{\alpha}\beta)\,\Gamma_\ell\Gamma_\ell^{-T} = QJ_\ell((-1)^{\ell+1}\overline{\alpha}\beta)Q^{-1}$.

Finally, for two Type II blocks, the solution of

$$\left\{\begin{array}{l} XH_{2k}(\mu) = -H_{2\ell}(\widetilde{\mu})Y^* \\ YH_{2\ell}(\widetilde{\mu}) = -X_{2k}(\mu)X^* \end{array}\right.$$

can be obtained using similar arguments as the ones in the proof of Lemma 9 in [7] for the congruence case. Nonetheless, the casuistry here is slightly different. More precisely, the solution depends on whether $\mu = \widetilde{\mu}$, $\overline{\mu}\widetilde{\mu} = 1$ or $\mu \neq \widetilde{\mu}$, $\overline{\mu}\widetilde{\mu} \neq 1$. Since $|\mu|, |\widetilde{\mu}| > 1$, the case $\overline{\mu}\widetilde{\mu} = 1$ is impossible. $\square$

LEMMA 4.3. **(Interaction between blocks of different type)** *The real interaction between two blocks of different type is:*

(i) *For one Type 0 block and one Type I block,*

$$\operatorname{inter}(J_k(0), \alpha\Gamma_\ell) = \left\{\begin{array}{ll} 0\,, & \text{if } k \text{ is even,} \\ 2\ell\,, & \text{if } k \text{ is odd.} \end{array}\right.$$

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(ii) *For one Type 0 block and one Type II block,*

$$\operatorname{inter}(J_k(0), H_{2\ell}(\mu)) = \begin{cases} 0\,, & \text{if } k \text{ is even,} \\ 4\ell\,, & \text{if } k \text{ is odd.} \end{cases}$$

(iii) *For one Type I block and one Type II block,*

$$\operatorname{inter}(\alpha\Gamma_k, H_{2\ell}(\mu)) = 0.$$

Proof. In this case, there are no relevant differences with the congruence case described in [7, Lemma 10]. The solution of the corresponding equations can be obtained in a similar way. For claim (iii) we just notice that $|\mu| > 1$ and, in particular $\mu \neq (-1)^{k+1}$. □

Lemmas 4.1, 4.2 and 4.3 immediately imply the codimension count stated in Theorem 3.3.

5. Minimal codimension and generic structure. The goal of this section is to answer the following question: which is the typical canonical structure by *congruence in $\mathbb{C}^{n\times n}$? In other words, we want to determine the generic canonical structure for *congruence of matrices in $\mathbb{C}^{n\times n}$. We understand by *generic canonical structure* the canonical structure for *congruence of a certain set of matrices that has codimension zero. The first step in this direction is Theorem 5.1 below.

THEOREM 5.1. *The minimal (real) codimension for a *congruence orbit in $\mathbb{C}^{n\times n}$ is n.*

Proof. Given $A \in \mathbb{C}^{n\times n}$ the real codimension c_{Total} of its *congruence orbit is given by Theorem 3.3. We will first show that $c_{\text{Total}} \geq n$. For this, we will show that $c_0 + c_1 + c_2 \geq n$ (following the notation in Theorem 3.3). Let k_0, k_1 and $2k_2$ be the total size corresponding to, respectively, Type 0 blocks, Type I blocks and Type II blocks in the canonical form for *congruence of A. Notice that $k_0 + k_1 + 2k_2 = n$. Now, using the basic inequality $\lceil x\rceil + \lceil y\rceil \geq \lceil x + y\rceil$, we have that

$$c_0 \geq 2\left\lceil \frac{k_0}{2} \right\rceil, \qquad c_1 = k_1\,, \qquad c_2 = 2k_2\,.$$

Then

$$c_0 + c_1 + c_2 \geq 2\left\lceil \frac{k_0}{2} \right\rceil + k_1 + 2k_2 \geq n\,.$$

Now, notice that the real codimension of

$$H_n(\mu) \quad \text{if } n \text{ is even} \qquad \text{and} \qquad H_{n-1}(\mu) \oplus \Gamma_1 \quad \text{if } n \text{ is odd}$$

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is n. This shows that the bound is sharp and the result is proved. $\square$

As a consequence of Theorem 5.1, there are no *congruence orbits of codimension zero. Therefore, the generic canonical structure must include more than one orbit. This fact should not be surprising because it is the same situation as in the case of similarity and congruence [1, 7]. In order to find a generic structure, we should be unconcerned with the particular values α, μ in Type I and Type II blocks. On the contrary, when we consider a particular orbit, these values are fixed. This leads us to the following notion, which is motivated by the notion of *bundle* introduced by Arnold [1] for the action of similarity and already extended in [7] for the action of congruence.

Definition 5.2. Let $A \in \mathbb{C}^{n\times n}$ with canonical form for *congruence

$$C_A = \bigoplus_{i=1}^{a} J_{p_i}(0) \;\oplus\; \bigoplus_{i=1}^{b} \alpha_i \mathcal{G}_i \;\oplus\; \bigoplus_{i=1}^{t} \mathcal{H}(\mu_i),$$

with $|\alpha_i| = 1$, for $i = 1, \dots, b$, $|\mu_i| > 1$, for $i = 1, \dots, t$, and $\mu_i \neq \mu_j$, $\alpha_i \neq \alpha_j$ if $i \neq j$, where

$$\mathcal{G}_i = \Gamma_{s_{i,1}} \oplus \cdots \oplus \Gamma_{s_{i,q_i}} \qquad \text{for } i = 1, \dots, b,$$

and

$$\mathcal{H}(\mu_i) = H_{2r_{i,1}}(\mu_i) \oplus H_{2r_{i,2}}(\mu_i) \oplus \cdots \oplus H_{2r_{i,g_i}}(\mu_i), \qquad \text{for } i = 1, \dots, t.$$

Then the bundle $\mathcal{B}(A)$ of A for the action of *congruence is defined by the following union of *congruence orbits

$$\mathcal{B}(A) = \bigcup_{\substack{|\alpha_i'|=1, i=1,\dots,b \\ |\mu_i'|>1,\ i=1,\dots,t \\ \mu_i'\neq\mu_j',\, \alpha_i'\neq\alpha_j', i\neq j}} \mathcal{O}\left(\bigoplus_{i=1}^{a} J_{p_i}(0) \;\oplus\; \bigoplus_{i=1}^{b} \alpha_i' \mathcal{G}_i \;\oplus\; \bigoplus_{i=1}^{t} \mathcal{H}(\mu_i') \right). \tag{5.1}$$

The real codimension of the bundle (5.1) is the number

$$\operatorname{codim} \mathcal{B}(A) = c_{\text{Total}}(A) - 2t - b,$$

where $c_{\text{Total}}(A)$ is the real codimension of A.

Notice that whereas in the real codimension of the bundle we subtract twice the number of different values μ_i associated to Type II blocks, we only subtract once the number of different values α_i corresponding to Type I blocks. The reason for this difference is that whereas μ_i are arbitrary complex numbers with absolute value greater than one (they vary upon a set of real dimension 2), the α_i numbers are

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unitary complex numbers, so they vary upon a one-dimensional real set. Note also that a bundle consists of matrices with the same canonical structure for *congruence, that is, with the same number of different blocks of the same type and with the same sizes, but with arbitrary μ_i' and α_i' parameters ($|\mu_i'| > 1$, $|\alpha_i'| = 1$).

Now we can state the following result, which is an immediate consequence of Theorem 5.1, Theorem 3.3, and the definition of codimension of a bundle for the action of *congruence. It gives us the generic canonical structure for *congruence in $\mathbb{C}^{n\times n}$.

THEOREM 5.3. **(Generic canonical form for *congruence)**

1. *Let n be even and $A \in \mathbb{C}^{n\times n}$ be a matrix whose canonical form for *congruence is*
$$G_A = H_2(\mu_1) \oplus H_2(\mu_2) \oplus \cdots \oplus H_2(\mu_{n/2}), \tag{5.2}$$
with $|\mu_i| > 1$, $i = 1, \ldots, n/2$ and $\mu_i \neq \mu_j$ if $i \neq j$. Then $\operatorname{codim}(\mathcal{B}(A)) = 0$. *Therefore, we can say that the generic canonical form for *congruence of a matrix in $\mathbb{C}^{n\times n}$ is the one in* (5.2) *with unspecified values $\mu_1, \mu_2, \ldots, \mu_{n/2}$.*
2. *Let n be odd and $A \in \mathbb{C}^{n\times n}$ be a matrix whose canonical form for *congruence is*
$$G_A = H_2(\mu_1) \oplus H_2(\mu_2) \oplus \cdots \oplus H_2(\mu_{(n-1)/2}) \oplus (\alpha\Gamma_1), \tag{5.3}$$
with $|\mu_i| > 1$, $i = 1, \ldots, (n-1)/2$, $\mu_i \neq \mu_j$ if $i \neq j$ and $|\alpha| = 1$. Then $\operatorname{codim}(\mathcal{B}(A)) = 0$. *Therefore, we can say that the generic canonical form for *congruence of a matrix in $\mathbb{C}^{n\times n}$ is the one in* (5.3) *with unspecified values $\mu_1, \mu_2, \ldots, \mu_{(n-1)/2}$, α.*

Let us justify the presence of the Type I block $\alpha\Gamma_1$ in the generic structure (5.3). When A is nonsingular, this kind of blocks in the canonical form for *congruence of A are associated with eigenvalues γ with $|\gamma| = 1$ in the Jordan canonical form of $A^{-*}A$ [9]. If n is odd and A is nonsingular, then $A^{-*}A$ has always an eigenvalue γ with $|\gamma| = 1$. To see this, notice that λ is an eigenvalue of $A^{-*}A$ if and only if $-\lambda$ is an eigenvalue of the *palindromic pencil $A + \lambda A^*$. It is known that the nonzero eigenvalues of this kind of pencils are paired up in the form $(\lambda, 1/\overline{\lambda})$, in such a way that λ is an eigenvalue of $A + \lambda A^*$ if and only if $1/\overline{\lambda}$ is, and both eigenvalues have the same algebraic multiplicity [13, Theorem 2.2]. Then, if n is odd, there is at least one eigenvalue λ with $\lambda = 1/\overline{\lambda}$, so $|\lambda| = 1$.

As an application of Theorem 5.3, we are able to determine also the generic canonical structure of *palindromic matrix pencils $A + \lambda A^*$. Note that, as a consequence of Theorem 5.3, the generic canonical form for *congruence of *palindromic pencils is $G_A + \lambda G_A^*$, where G_A is given by (5.2) if n is even and by (5.3) if n is odd. From this

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we can obtain the generic Kronecker canonical form for strict equivalence [8, Chapter XII] of *palindromic pencils by taking into account that $G_A + \lambda G_A^*$ is strictly equivalent to $G_A^{-*} G_A + \lambda I_n$ and the Jordan canonical form of $H_{2k}(\mu)^{-*} H_{2k}(\mu)$ is $J_k(\mu) \oplus J_k(1/\overline{\mu})$. Then we get the following theorem.

THEOREM 5.4. *The generic Kronecker canonical form of* **palindromic pencils in* $\mathbb{C}^{n\times n}$ *is:*

1. *If* n *is even:*
$$(\lambda+\mu_1)\oplus(\lambda+1/\overline{\mu}_1)\oplus(\lambda+\mu_2)\oplus(\lambda+1/\overline{\mu}_2)\oplus\cdots\oplus(\lambda+\mu_{n/2})\oplus(\lambda+1/\overline{\mu}_{n/2}),$$
where $\mu_1,\ldots,\mu_{n/2}$ *are unspecified complex numbers such that* $|\mu_i|>1$, $i=1,\ldots,n/2$, *and* $\mu_i\neq\mu_j$ *if* $i\neq j$.
2. *If* n *is odd:*
$$(\lambda+\mu_1)\oplus(\lambda+1/\overline{\mu}_1)\oplus(\lambda+\mu_2)\oplus(\lambda+1/\overline{\mu}_2)\oplus\cdots\oplus$$
$$(\lambda+\mu_{(n-1)/2})\oplus(\lambda+1/\overline{\mu}_{(n-1)/2})\oplus(\lambda+\gamma),$$
where γ *is an unspecified complex number with* $|\gamma|=1$ *and* $\mu_1,\ldots,\mu_{(n-1)/2}$ *are unspecified complex numbers such that* $|\mu_i|>1$, $i=1,\ldots,(n-1)/2$ *and* $\mu_i\neq\mu_j$ *if* $i\neq j$.

6. Conclusions. In this paper, we have solved the equation $XA+AX^*=0$, for a given matrix $A\in\mathbb{C}^{n\times n}$. As a consequence, we have computed the dimension of the *congruence orbit of A and we have determined the generic canonical structure for *congruence in $\mathbb{C}^{n\times n}$. As an application, we have also obtained the generic Kronecker structure of *palindromic pencils in $\mathbb{C}^{n\times n}$. This work completely closes the study of the solution space of the related equations $XA+AX^T=0$ and $XA+AX^*=0$ which, despite their similar appearance, present relevant differences. At the same time, this work can be seen as a first step in describing the space of *congruence orbits in $\mathbb{C}^{n\times n}$. One of the major goals in this context is to completely describe the inclusion relationships between the orbit closures.

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