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ON SOLUTIONS OF MATRIX EQUATION $AX = B$ OVER A BEZOUT DOMAIN*

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Abstract. Let $\mathrm{R}_{m,n}$ be the set of $m\times n$ matrices over a Bezout domain R with identity $e \neq 0$ and let $0_{m,k}$ be the zero $m\times k$ matrix. Further, let $d_i(A) \in \mathrm{R}$ be an ideal generated by the i-th order minors of the matrix $A \in \mathrm{R}_{m,n}$, $i = 1, 2, \ldots, \min\{m, n\}$. In this article, we investigate a structure of solutions of a matrix equation $AX = B$, where $A \in \mathrm{R}_{m,n}$ and $B \in \mathrm{R}_{m,k}$ are known matrices and X is unknown matrix over R. It is known that matrix equation $AX = B$ is solvable over a Bezout domain R if and only if $\operatorname{rank} A = \operatorname{rank} A_B = r$ and $d_i(A) = d_i(A_B)$ for all $i = 1, 2, \ldots, r$, where $A_B = \begin{bmatrix} A & B \end{bmatrix}$. On the other hand, $AX = B$ is solvable over R if and only if matrices $\begin{bmatrix} A & 0_{m,k} \end{bmatrix}$ and A_B are right-equivalent, that is, the Hermitian normal forms of these matrices coincide. In this article, we give alternative necessary and sufficient conditions for the solvability of equation $AX = B$ over a Bezout domain R. If a solution of this equation exists, we also give an algorithm for its construction. We prove also that the matrix equation $AX = B$ over R has a symmetric solution if and only if $AX = B$ has a solution over R and the matrix AB^T is symmetric. If symmetric solution exists, we propose the method for its construction.

Key words. Bezout domain, Elementary divisor domain, Matrix equation, Symmetric solution.

AMS subject classifications. 13A05, 15A06, 15A24, 15A36.

1. Introduction. Let R be a Bezout domain with identity $e \neq 0$, that is, R is an integral domain with the property that every finitely generated ideal over R is principal (see [6], [10]). Denote by $\mathrm{R}_{m,n}$ the set of $m \times n$ matrices over R and by $GL(n, \mathrm{R})$ the set of $n \times n$ invertible matrices over R. In what follows, I_n is the identity $n \times n$ matrix, $0_{m,k}$ is the zero $m \times k$ matrix and $d_i(A) \in \mathrm{R}$ is an ideal generated by the i-th order minors of the matrix $A \in \mathrm{R}_{m,n}$, $i = 1, 2, \ldots, \min\{m, n\}$.

The rank of a matrix A, denoted by $\operatorname{rank} A$, is the highest order of a nonzero minor of the matrix A. (The rank of the zero matrix is 0.) The transpose matrix of a matrix $A \in \mathrm{R}_{m,n}$ will be denoted by A^T. A matrix $A \in \mathrm{R}_{n,n}$ is called a symmetric matrix if $A = A^T$. In what follows $C^* = \operatorname{Adj}(C)$ means the classical adjoint matrix for a nonsingular matrix $C \in \mathrm{R}_{n,n}$, i.e., $C^*C = I_n \det C$.

Let $A \in \mathrm{R}_{m,n}$, $B \in \mathrm{R}_{m,k}$ and $\operatorname{rank} A \geq 1$. Consider the matrix equation

$$AX = B, \tag{1.1}$$

where X is unknown matrix in $\mathrm{R}_{n,k}$. Solving linear matrix equations over fields and commutative rings is one of the most important research problems, which has applications in many areas of mathematics and its applications. The generalization of solvability of the matrix equation $AX = B$ over a field to the equation (1.1) over rings is nontrivial. Furthermore, this is a very complicated problem, but its solution will have many applications in theory of differential equations, system theory, control theory, stability theory, engineering and scientific computations. It may be noted if $B = 0_{m,k}$, then homogeneous matrix equation $AX = 0_{m,k}$ is solvable if and only if the $\operatorname{rank} A < n$ (see [1], [16]).

The problem of solvability of the equation (1.1) over rings has drawn the attention of many mathematicians. Theoretical aspects of solving linear matrix equations can be found in [1], [2], [10], [11] and [16]. On

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the other hand, matrix equations have applications in many applied problems (see [3], [9], [17], [24] and references therein).

There are many papers, where the basic aim is to find necessary and sufficient conditions for the existence of a solution X of the matrix equation $AX = B$, such that X belongs to some special class of matrices. A problem of existence of symmetric solutions of the equation (1.1) was studied by many authors. For example, the existence of symmetric solutions of equation (1.1) over a field was investigated in [7] (see also papers [4], [5], [12], [14], [20] and references therein). A structure symmetric solutions of equation (1.1) over a principal ideal domain are studied in [21]. We note that the results of paper [21] are true for the equation (1.1) over domains of elementary divisors. Ranks of symmetric solutions of linear matrix equations have been considered in [23] and [25]. Solutions with prescribed characteristic polynomials of matrix equation $AX = B$ considered in [19].

Denote by $A_0 = \begin{bmatrix} A & 0_{m,k} \end{bmatrix} \in \mathrm{R}_{m,n+k}$ and by $A_B = \begin{bmatrix} A & B \end{bmatrix} \in \mathrm{R}_{m,n+k}$ the extended matrix of equations (1.1). If R is an elementary divisor domain, then equation (1.1) is solvable if and only if the Smith normal forms of the matrices A_0 and A_B coincide. It shall be noted that the study of conditions under which a Bezout domain is an elementary divisor domain is an open problem. It is known (see [10], [11], [16]) that equation (1.1) over a Bezout domain R is solvable if and only if $\operatorname{rank} A = \operatorname{rank} A_B = r$ and $d_i(A) = d_i(A_B)$ for all $i = 1, 2, \ldots, r$. According to [18], equation (1.1) is solvable over a Bezout domain R if and only if the matrices A_0 and A_B are right equivalent, that is, the Hermitian normal forms of these matrices coincide. Cramer's rule to residue class rings of Bezout domains was given in [24].

In this work we investigate the solvability of equation (1.1) over a Bezout domain R. In Section 2, we give alternative necessary and sufficient conditions for solvability of equation (1.1) over R. If equation (1.1) is solvable, then we give an algorithm for constructing of a general solution of equation (1.1). In Section 3, we prove that the equation $AX = B$ over a Bezout domain R has a symmetric solution if and only if $AX = B$ has a solution over R and the matrix AB^T is symmetric. In Section 4, we represent constructive conditions under which for matrix equation $AX = B$ over a Bezout domain there exists a symmetric solution. If a symmetric solution exists, we propose the method for its construction. A numerical example is also given in Section 5.

2. On solvability of the matrix equation $AX = B$ over a Bezout domain. Currently, there are many direct methods based on matrix factorization and iteration methods for solving linear matrix equations (see [15], [18] [24], [26] and references therein). The idea of representing a matrix in a product of structured matrices, such as triangular, orthogonal, and diagonal matrices, is among the important methods in the theory of numerical computation [26], [27]. In this chapter, we will use the matrix decomposition as a product of invertible matrices and block diagonal matrices to solve the equation (1.1) in the Bezout domain.

In a Bezout domain R all finitely generated ideals are principal. So, for two given elements $a, b \in \mathrm{R}$, such that the row $[a,\ b] \neq [0,\ 0]$, there exists a greatest common divisor $d = (a,\ b)$ and elements $x, y \in \mathrm{R}$, satisfying the Bezout identity $ax + by = d$. This means that for the row $[a,\ b]$ there exists an invertible matrix $U \in GL(2, \mathrm{R})$ such that $[a,\ b]U = [d,\ 0]$. Thus, for a matrix $A \in \mathrm{R}_{m,n}$ of $\operatorname{rank} A = r \geq 1$, there exist matrices $U \in GL(m, \mathrm{R})$ and $V \in GL(n, \mathrm{R})$ such that

$$UAV = \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix},$$

where $C \in \mathrm{R}_{r,r}$ (see [6], [10], [13]). It is clear that $\det C \neq 0$.

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Suppose for the matrix $A \in \mathrm{R}_{m,n}$ there exist matrices $U_1 \in GL(m, \mathrm{R})$ and $V_1 \in GL(n, \mathrm{R})$, so that either $U_1 \neq U$ or $V_1 \neq V$, such that

$$U_1 A V_1 = \begin{bmatrix} C_1 & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix},$$

where $C_1 \in \mathrm{R}_{r,r}$ and $\det C_1 \neq 0$.

So, for matrices $U_0 = U_1 U^{-1} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \in GL(m, \mathrm{R})$ and $V_0 = V_1^{-1} V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \in GL(n, \mathrm{R})$, where $U_{11}, V_{11} \in \mathrm{R}_{r,r}$, $U_{22} \in \mathrm{R}_{m-r,m-r}$ and $V_{22} \in \mathrm{R}_{n-r,n-r}$, we have

$$U_0 \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} = \begin{bmatrix} C_1 & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} V_0. \tag{2.2}$$

Hence, from equation (2.2), we get $U_{21} = 0_{m-r,r}$ and $V_{12} \in \mathrm{R}_{r,n-r}$. This means that $U_{11}, V_{11} \in GL(r, \mathrm{R})$. Thus, matrices C and C_1 are equivalent. This means that the matrix C is unique defined with respect to the equivalence for the matrix A.

The next theorem gives a necessary and sufficient condition for solvability of equation (1.1) over a Bezout domain.

THEOREM 2.1. *Let* $A \in \mathrm{R}_{m,n}$, $B \in \mathrm{R}_{m,k}$ *and* $\operatorname{rank} A = r \geq 1$. *Further, let* $U \in GL(m, \mathrm{R})$ *and* $V \in GL(n, \mathrm{R})$ *such that* $UAV = \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}$, *where* $C \in \mathrm{R}_{r,r}$. *The matrix equation* $AX = B$ *is solvable over a Bezout domain* R *if and only if the following conditions hold:*

(i) $UB = \begin{bmatrix} B_1 \\ 0_{m-r,k} \end{bmatrix}$, *where* $B_1 \in \mathrm{R}_{r,k}$;

(ii) *if* $r = 1$, *then* $B_1 \equiv 0_{1,k} \pmod{C}$ *and if* $r \geq 2$, *then* $C^* B_1 \equiv 0_{r,k} \pmod{\det C}$, *i.e.,* $B_1 = CD$, *where* $D \in \mathrm{R}_{r,k}$.

If equation $AX = B$ *is solvable, then for arbitrary matrix* $P \in \mathrm{R}_{m-r,k}$ *the set of matrices* $X_P = \begin{bmatrix} D \\ P \end{bmatrix}$ *is a general solution of equation* $AX = B$.

Proof. Let $X_1 \in \mathrm{R}_{n,k}$ be a solution of matrix equation $AX = B$. Further, let $U \in GL(m, \mathrm{R})$ and $V \in GL(n, \mathrm{R})$ such that $UAV = \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}$, where $C \in \mathrm{R}$ if $r = 1$ and if $r \geq 2$, then $C \in \mathrm{R}_{r,r}$. From equality $AX_1 = B$ we have

$$UAVV^{-1} X_1 = UB. \tag{2.3}$$

Put $V^{-1} X_1 = \begin{bmatrix} X_{11} \\ X_{12} \end{bmatrix}$ and $UB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, where $X_{11} \in \mathrm{R}_{r,k}$ and $B_1 \in \mathrm{R}_{r,k}$. We rewrite the equality (2.3) in the form

$$\begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{12} \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

From this we have $CX_{11} = B_1$ and $B_2 = 0_{m-r,k}$.

If $\operatorname{rank} A = 1$, then $C \in \mathrm{R}$ and $CX_{11} = B_1$, i.e., $B_1 \equiv 0_{1,k} \pmod{C}$. If $\operatorname{rank} A \geq 2$, then $CX_{11} = B_1$. From this it follows that $C^* B_1 = X_{11}(\det C)$. So in the case, when $r \geq 2$, we have $C^* B_1 \equiv 0_{r,k} \pmod{\det C}$.

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Conversely, let $A \in \mathrm{R}_{m,n}$, $B \in \mathrm{R}_{m,k}$ and $\operatorname{rank} A = r \geq 1$. Further, let matrices $U \in GL(m,\mathrm{R})$ and $V \in GL(n,\mathrm{R})$ such that $UAV = \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}$ and $UB = \begin{bmatrix} B_1 \\ 0_{m-r,k} \end{bmatrix}$, where $C \in \mathrm{R}_{r,r}$ and $B_1 \in \mathrm{R}_{r,k}$. We note that the matrix C is unique defined with respect to the equivalence for the matrix A.

If $\operatorname{rank} A = 1$, then $B_1 \equiv 0_{1,k} \pmod{C}$, i.e., $CD = B_1$, where $D \in \mathrm{R}_{1,k}$. If $\operatorname{rank} A \geq 2$, then $C^* B_1 \equiv 0_{r,k}$ $(\mathrm{mod}\ \det C)$. Thus, $C^* B_1 = D(\det C)$, where $D \in \mathrm{R}_{r,k}$. From this it follows that $CD = B_1$.

So, we have $CD = B_1$ and the matrix $X_0 = V \begin{bmatrix} D \\ 0_{n-r,k} \end{bmatrix}$ is the solution of matrix equation $AX = B$. The solution X_0 we will call a minimal solution of $AX = B$. It is clear if $r = n$, then solution X_0 is unique for equation $AX = B$. If $r < n$, then for arbitrary matrix $P \in \mathrm{R}_{n-r,k}$, we obtain

$$\begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} \begin{bmatrix} D \\ P \end{bmatrix} = \begin{bmatrix} B_1 \\ 0_{m-r,k} \end{bmatrix}.$$

From the above, it follows that for arbitrary matrix $P \in \mathrm{R}_{n-r,k}$ the set of matrices $X_P = V \begin{bmatrix} D \\ P \end{bmatrix} \in \mathrm{R}_{n,k}$ gives all solutions of the equation $AX = B$.

Let $r < n$ and let the matrix equation $AX = B$ be solvable. Below we will describe some properties of solutions of equation $AX = B$. Let $X_1 = V \begin{bmatrix} D \\ P_1 \end{bmatrix} \in \mathrm{R}_{n,k}$ be a solution of $AX = B$ and $P_1 \in \mathrm{R}_{n-r,k}$ be nonzero matrix. If $P_1 = TP_2$, where $T \in \mathrm{R}_{n-r,n-r}$, then for matrix X_1 there exists a representation in the form

$$X_1 = V \begin{bmatrix} D \\ TP_2 \end{bmatrix} = V \begin{bmatrix} I_r & 0_{r,n-r} \\ 0_{n-r,r} & T \end{bmatrix} V^{-1} V \begin{bmatrix} D \\ P_2 \end{bmatrix}.$$

It is obvious that $X_2 = V \begin{bmatrix} D \\ P_2 \end{bmatrix}$ is the solution of equation $AX = B$ and the right divisor of the solution X_1.

Let T be an arbitrary $(n-r)$-by-$(n-r)$ matrix over R. Consider the equality

$$V \begin{bmatrix} I_r & 0_{r,n-r} \\ 0_{n-r,r} & T \end{bmatrix} V^{-1} X_1 = V \begin{bmatrix} G \\ TP_1 \end{bmatrix} = X_T.$$

Thus, the matrix X_T is the solution of matrix equation $AX = B$. So, if $T = 0_{n-r,n-r}$, then every solution of equation $AX = B$ is the right divisor of the minimal solution X_0 of this equation. This completes the proof of Theorem 2.1. ∎

From the proof of the sufficiency of Theorem 2.1, we obtain the following statements.

Corollary 2.2. *Let a matrix equation $AX = B$ be solvable. Then all solutions of the equation $AX = B$ are the right divisors of minimal solution X_0 of this equation.*

Corollary 2.3. *Let $A \in \mathrm{R}_{m,n}$, $B \in \mathrm{R}_{m,k}$ and let the matrix equation $AX = B$ be solvable. If $n \leq k$, then equation $AX = B$ has a solution $X_Q \in \mathrm{R}_{n,k}$ such that X_Q is the left divisor of all solutions of this equation.*

Proof. Let $n \leq k$ and let the matrix equation $AX = B$ be solvable. If $\operatorname{rank} A = n$, then $n \leq m$ and $AX = B$ has a unique solution. Thus, in this case, corollary 2.3 is trivial.

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Let $\operatorname{rank} A = r < n$ and $X_1 \in \mathrm{R}_{n,k}$ be a solution of equation $AX = B$. Thus, $r \le m$ and $r \le k$. For matrices A and B there exist matrices $U \in GL(m, \mathrm{R})$, $V \in GL(n, \mathrm{R})$ and $W \in GL(k, \mathrm{R})$ such that

$$UAV = \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} \text{ and } UBW = \begin{bmatrix} B_{11} & 0_{r,k-r} \\ 0_{m-r,r} & 0_{m-r,k-r} \end{bmatrix},$$

where $C, B_{11} \in \mathrm{R}_{r,r}$. From equality $AX_1 = B$, we get

$$UAVV^{-1}X_1W = \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} V^{-1}X_1W = UBW = \begin{bmatrix} B_{11} & 0_{r,k-r} \\ 0_{m-r,r} & 0_{m-r,k-r} \end{bmatrix}. \tag{2.4}$$

Put $V^{-1}X_1W = \begin{bmatrix} X_{11} & X_{12} \\ \multicolumn{2}{c}{X_2} \end{bmatrix}$, where $X_{11} \in \mathrm{R}_{r,r}$, $X_{12} \in \mathrm{R}_{r,k-r}$, and $X_2 \in \mathrm{R}_{n-r,k}$. From equality (2.4) we get $X_{12} = 0_{r,k-r}$.

Thus, for arbitrary matrix $P \in \mathrm{R}_{n-r,k}$ the matrix $X_P = V \begin{bmatrix} X_{11} & 0_{r,k-r} \\ \multicolumn{2}{c}{P} \end{bmatrix} W^{-1}$ is the solution of equation $AX = B$. So, for arbitrary matrix $Q \in GL(r, \mathrm{R})$, the matrix $X_Q = V \begin{bmatrix} X_{11} & 0_{r,k-r} \\ 0_{n-r,k-r} & Q \end{bmatrix} W^{-1}$ is a solution of equation $AX = B$. It is easy to see that $X_P = X_Q \begin{bmatrix} I_r & 0_{r,k-r} \\ \multicolumn{2}{c}{Q^{-1}P} \end{bmatrix} W^{-1}$. The proof of Corollary 2.3 is complete. $\square$

We note that Corollary 2.2 and Corollary 2.3 generalize the main result of [22]. On the other hand, the results of [21] are true for a matrix equation $AX = B$, where A and B are m-by-n matrices over an elementary divisor domain. So, Theorem [22] is easily obtained as a consequence from Lemma 1 [21].

3. Symmetric solutions of the matrix equation $AX{=}B$. In this section, we investigate symmetric solutions of the matrix equation $AX = B$, where A and B are m-by-n matrices over a Bezout domain R and X is unknown n-by-n matrix over R. Let R = D be a principal ideal domain. The equation $AX = B$ over a principal ideal domain D has a symmetric solution if and only if equation $AX = B$ has a solution over D and $AB^T = BA^T$ [21]. In this section, we affirm that this criteria is true for equation $AX = B$ over a Bezout domain R.

Theorem 3.1. *Let $A, B \in \mathrm{R}_{m,n}$. The matrix equation $AX = B$ has a symmetric solution over a Bezout domain* R *if and only if equation $AX = B$ is solvable over* R *and $AB^T = BA^T$.*

Proof. Necessity. Let $X_s \in \mathrm{R}_{n,n}$ be a symmetric solution of the matrix equation $AX = B$, that is, $AX_s = B$ and $X_s = X_s^T$. Thus, $X_sA^T = B^T$. From the last equality, we have $AX_sA^T = BA^T = AB^T$. The necessity is proved.

Sufficiency. Let $X_p \in \mathrm{R}_{n,n}$ be the solution of the equation $AX = B$, that is, $AX_p = B$. This implies that $X_p^TA^T = B^T$. Further, let $\operatorname{rank} A = r \ge 1$. For the matrix A there exist matrices $U \in GL(m, \mathrm{R})$ and $V \in GL(n, \mathrm{R})$ such that $UAV = \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}$, where $C \in \mathrm{R}_{r,r}$. We note that the matrix C is unique defined with respect to the equivalence for the matrix A.

We introduce for matrix $V \in GL(n, \mathrm{R})$ the following notation $(V^{-1})^T = V^{-T}$. From equalities $AX_p = B$

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and $X_p^T A^T = B^T$ we have

$$\left\{ \begin{array}{lcl} UAVV^{-1}X_pV^{-T} & = & UBV^{-T}, \\ V^{-1}X_p^TV^{-T}V^TA^TU^T & = & V^{-1}B^TU^T, \end{array} \right. \tag{3.5}$$

Put $V^{-1}X_pV^{-T} = G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ and $UBV^{-T} = B_1 = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, where $G_{11} \in \mathrm{R}_{r,r}$, $G_{12} \in \mathrm{R}_{r,n-r}$, $G_{21} \in \mathrm{R}_{n-r,r}$, $B_{11} \in \mathrm{R}_{r,r}$, $B_{12} \in \mathrm{R}_{r,n-r}$ and $B_{21} \in \mathrm{R}_{m-r,r}$. Thus, $V^{-1}X_p^TV^{-T} = G^T$ and $V^{-1}B^TU^T = B_1^T$. From (3.5) we obtain

$$\begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{3.6}$$

and

$$\begin{bmatrix} G_{11}^T & G_{21}^T \\ G_{12}^T & G_{22}^T \end{bmatrix} \begin{bmatrix} C^T & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix} = \begin{bmatrix} B_{11}^T & B_{21}^T \\ B_{12}^T & B_{22}^T \end{bmatrix}. \tag{3.7}$$

By Theorem 2.1, from (3.6) it follows that $B_{21} = 0_{m-r,r}$ and $B_{22} = 0_{m-r,n-r}$. Similarly, from equality (3.7) it follows that $B_{21}^T = 0_{r,m-r}$ and $B_{22}^T = 0_{n-r,m-r}$. Thus,

$$\begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix},$$

and

$$\begin{bmatrix} G_{11}^T & G_{21}^T \\ G_{12}^T & G_{22}^T \end{bmatrix} \begin{bmatrix} C^T & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix} = \begin{bmatrix} B_{11}^T & 0_{r,m-r} \\ B_{12}^T & 0_{n-r,m-r} \end{bmatrix}.$$

From equality $AB^T = BA^T$ we have $UAB^TU^T = UBA^TU^T$. Hence,

$$UAB^TU^T = UAVV^{-1}B^TU^T = \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} G^T \begin{bmatrix} C^T & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix} = \begin{bmatrix} CG_{11}^TC^T & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix},$$

and

$$UBA^TU^T = UBV^{-T}V^TA^TU^T = \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} G \begin{bmatrix} C^T & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix} = \begin{bmatrix} CG_{11}C^T & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}.$$

From the last two equalities we get $CG_{11}C^T = CG_{11}^TC^T$. This implies that $G_{11} = G_{11}^T$. Thus, the matrix G_{11} is symmetric.

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It is easily verified that the symmetric matrix $X_g = V \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^T & 0_{n-r,n-r} \end{bmatrix} V^T$ is the common solution of equation $AX = B$ and $XA^T = B^T$. So, for the equations $AX = B$ there exists a symmetric solution. Now we prove that for arbitrary matrix $P \in \mathrm{R}_{n-r,n-r}$ the set of matrices $X_p = V \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^T & P \end{bmatrix} V^T$ give a general solution of the equation $AX = B$. So, we have the following equality

$$AX_p = AV \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^T & P \end{bmatrix} V^T = U^{-1} \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} V^{-1} V \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^T & P \end{bmatrix} V^T =$$

$$U^{-1} \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & P \end{bmatrix} V^T = U^{-1} \begin{bmatrix} B_{11} & B_{12} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} V^T = U^{-1} U B V^{-T} V^T = B.$$

Let $P = S \in \mathrm{R}_{n-r,n-r}$ be a symmetric matrix. In this case, we have that the set of symmetric matrices $X_s = V \begin{bmatrix} G_{11} & G_{12} \\ G_{12}^T & S \end{bmatrix} V^T$ gives a general symmetric solution of matrix equations $AX = B$, i.e., $AX_s = B$. Based on Sylvester's inequality for the rank of the product of two rectangular matrices A and X_s, we obtain $\operatorname{rank} B \le \operatorname{rank} X_s \le n + \operatorname{rank} B - \operatorname{rank} A$ (see also [23] and [25]). The proof of Theorem 3.1 is complete. ☐

Corollary 3.2. *Let $A \in \mathrm{R}_{m,n}$. If $\operatorname{rank} A < n$, then homogenous equation $AX = 0_{m,n}$ has a symmetric solution.*

4. A structure of symmetric solutions of equation AX=B **.** Conditions of Theorem 3.1 are very general, and they represent just the starting point in the search for more constructive conditions in the particular cases. Below we give an alternative conditions under which for the matrix equation $AX = B$ over a Bezout domain there exists a symmetric solution. If a symmetric solution exists, we propose the method for its construction.

Theorem 4.1. *Let $A, B \in \mathrm{R}_{m,n}$ and $\operatorname{rank} A = r$. Further, let $U \in GL(m, \mathrm{R})$ and $V \in GL(n, \mathrm{R})$ such that $UAV = \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}$, where $C \in \mathrm{R}_{r,r}$. The equation $AX = B$ has a symmetric solution if and only if the following conditions hold:*

(i) $UBV^{-T} = \begin{bmatrix} B_0 \\ 0_{m-r,n} \end{bmatrix}$, *where* $B_0 \in \mathrm{R}_{r,n}$*;*

(ii) $C^* B_0 = (\det C) \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}$, *where $D_{11} \in \mathrm{R}_{r,r}$ is a symmetric matrix and $D_{12} \in \mathrm{R}_{r,n-r}$.*

If a symmetric solution of the equation $AX = B$ exists, then for arbitrary symmetric matrix $S \in \mathrm{R}_{n-r,n-r}$ the set of matrices $X_S = V \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^T & S \end{bmatrix} V^T$ gives a general symmetric solution of $AX = B$.

Proof. Necessity. Let a symmetric matrix $X_s \in \mathrm{R}_{n,n}$ be a solution of matrix equation $AX = B$. Further, let $U \in GL(m, \mathrm{R})$ and $V \in GL(n, \mathrm{R})$ such that $UAV = \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}$, where $C \in \mathrm{R}_{r,r}$. By Theorem

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2.1, from equality $AX_s = B$ we have

$$\begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} V^{-1}X_0 = UB = \begin{bmatrix} CB_1 \\ 0_{m-r,n} \end{bmatrix},$$

where $B_1 \in \mathrm{R}_{r,n}$. Put $V^{-T} = (V^{-1})^T$. Multiplying the last equality from the right by V^{-T}, we obtain

$$\begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} V^{-1}X_sV^{-T} = \begin{bmatrix} CB_1 \\ 0_{m-r,n} \end{bmatrix} V^{-T} = \begin{bmatrix} B_0 \\ 0_{m-r,n} \end{bmatrix}, \tag{4.8}$$

where $B_0 \in \mathrm{R}_{r,n}$.

We write the symmetric matrix $V^{-1}X_sV^{-T}$ and matrix $UBV^{-T} = \begin{bmatrix} B_0 \\ 0_{m-r,n} \end{bmatrix}$ in block forms as

$$V^{-1}X_sV^{-T} = Y_s = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \quad \text{and} \quad UBV^{-T} = \begin{bmatrix} B_0 \\ 0_{m-r,n} \end{bmatrix} = \begin{bmatrix} CD_{11} & CD_{12} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix},$$

where $Y_{11}, D_{11} \in \mathrm{R}_{r,r}$, $Y_{22} \in \mathrm{R}_{n-r,n-r}$, and $D_{12} \in \mathrm{R}_{r,n-r}$. From the last equality we get $B_0 = C\begin{bmatrix} D_{11} & D_{12} \end{bmatrix}$. So, we have $C^*B_0 = (\det C)\begin{bmatrix} D_{11} & D_{12} \end{bmatrix}$.

Since Y_s is a symmetric matrix, then Y_{11} and Y_{22} are symmetric matrices and $Y_{21} = Y_{12}^T$. So, we can write equality (4.8) in the form

$$\begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} Y_s = \begin{bmatrix} CD_{11} & CD_{12} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}.$$

From this equality we obtain that $Y_{11} = D_{11}$, i.e., D_{11} is the symmetric matrix and $Y_{12} = D_{12}$, $Y_{21} = D_{12}^T$.

It is clear that for arbitrary matrix $P \in \mathrm{R}_{n-r,n-r}$ the following equality holds

$$\begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^T & P \end{bmatrix} = \begin{bmatrix} CD_{11} & CD_{12} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}.$$

So, for arbitrary symmetric matrix $S \in \mathrm{R}_{n-r,n-r}$ the matrix $X_S = V\begin{bmatrix} D_{11} & D_{12} \\ D_{12}^T & S \end{bmatrix}V^T$ is a symmetric solution of equation $AX = B$. The necessity is proved.

Sufficiency. Let $U \in GL(m,\mathrm{R})$, $V \in GL(n,\mathrm{R})$ and let $UAV = \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}$, where $C \in \mathrm{R}_{r,r}$. Further, let $UBV^{-T} = \begin{bmatrix} B_0 \\ 0_{m-r,n} \end{bmatrix}$, where $B_0 \in \mathrm{R}_{r,n}$ and $C^*B_0 = (\det C)\begin{bmatrix} D_{11} & D_{12} \end{bmatrix}$, where $D_{11} \in \mathrm{R}_{r,r}$ is a symmetric matrix and $D_{12} \in \mathrm{R}_{r,n-r}$. From the last equality we get $B_0 = C\begin{bmatrix} D_{11} & D_{12} \end{bmatrix}$.

So, $UBV^{-T} = \begin{bmatrix} CD_{11} & CD_{12} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}$ and for the symmetric matrix $Y_0 = \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^T & 0_{n-r,n-r} \end{bmatrix} \in \mathrm{R}_{n,n}$ we have

$$\begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} Y_0 = \begin{bmatrix} CD_{11} & CD_{12} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix}, \tag{4.9}$$

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and

$$Y_0 \begin{bmatrix} C^T & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix} = \begin{bmatrix} D_{11}C^T & 0_{r,m-r} \\ D_{12}^T C^T & 0_{n-r,m-r} \end{bmatrix}. \tag{4.10}$$

From the equality (4.9) it follows

$$U^{-1} \begin{bmatrix} C & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} V^{-1} V Y_0 V^T = AX_0 = U^{-1} \begin{bmatrix} CD_{11} & CD_{12} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} V^T = B.$$

Since Y_0 is the symmetric matrix, then $X_0 = VY_0V^T$ is also the symmetric matrix. Thus, the symmetric matrix X_0 is a solution of equation $AX = B$. Similarly, from equality (4.10) we have $X_0A^T = B^T$. From the above it follows that the equation $AX = B$ has a symmetric solution.

It is easily verified that for arbitrary symmetric matrix $S \in \mathrm{R}_{n-r,n-r}$ the set of matrices

$$\widetilde{X}_s = V \begin{bmatrix} 0_{r,r} & 0_{r,n-r} \\ 0_{m-r,r} & S \end{bmatrix} V^T \in \mathrm{R}_{n,n},$$

is a set of common symmetric solutions of homogeneous matrix equations $AX = 0_{m,n}$ and $XA^T = 0_{n,m}$. Hence, the set of matrices $X_s = V \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^T & S \end{bmatrix} V^T$ is a general symmetric solution of matrix equation $AX = B$, and the proof is complete. ∎

Corollary 4.2. *Let $A, B \in \mathrm{R}_{m,n}$ and let the equation $AX = B$ be solvable. If $\operatorname{rank} A = 1$, then equation $AX = B$ has a symmetric solution.*

Let R be an elementary divisor domain. From Theorem 4.1, we obtain the following statement.

Corollary 4.3. *Let $A, B \in \mathrm{R}_{m,n}$ and $\operatorname{rank} A = r$. Further, let $U \in GL(m, \mathrm{R})$ and $V \in GL(n, \mathrm{R})$ such that*

$$UAV = S_A = \begin{bmatrix} \operatorname{diag}(a_1, a_2, \dots, a_r) & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix},$$

be the Smith normal form of the matrix A. The matrix equation $AX = B$ has a symmetric solution if and only if the following conditions hold:

(i) $UBV^{-T} = \begin{bmatrix} B_0 \\ 0_{m-r,n} \end{bmatrix}$, *where* $B_0 \in \mathrm{R}_{r,n}$;

(ii) $B_0 = (\operatorname{diag}(a_1, a_2, \dots, a_r)) \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}$, *where $D_{11} \in \mathrm{R}_{r,r}$ is a symmetric matrix and $D_{12} \in \mathrm{R}_{r,n-r}$.*

If a symmetric solution of matrix equation $AX = B$ exists, then for arbitrary symmetric matrix $S \in \mathrm{R}_{n-r,n-r}$ the set of matrices $X_S = V \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^T & S \end{bmatrix} V^T$ is a general symmetric solution of $AX = B$.

5. Example. Let $\mathrm{R} = \mathbb{Z} + \lambda\mathbb{Q}[\lambda]$ be the ring of all polynomials in an indeterminate λ with rational coefficients and integral constant term. The ring R is a Bezout domain, but it is not a principal ideal domain and it is not a unique factorization domain. Moreover, ring R is an elementary divisor domain (see [6], [8]).

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Consider the following example. Let

$$A = \begin{bmatrix} 5,5\lambda^2+3,5\lambda & -11\lambda^2-7\lambda & 2,75\lambda^3+5,25\lambda^2+6\lambda+2 \\ 5\lambda+1 & -10\lambda-2 & 2,5\lambda^2+3,5\lambda+4 \\ -8\lambda-1 & 16\lambda+2 & -4\lambda^2-5,5\lambda-6 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 5,5\lambda^4+10,5\lambda^3+\lambda^2-3\lambda & -11\lambda^3+4\lambda^2+7\lambda & 22\lambda^3+28\lambda^2+24\lambda+8 \\ 5\lambda^3+7\lambda^2-2\lambda-2 & -10\lambda^2+8\lambda+2 & 20\lambda^2+16\lambda+16 \\ -8\lambda^3-11\lambda^2+4\lambda+2 & 16\lambda^2-14\lambda-2 & -32\lambda^2-24\lambda-24 \end{bmatrix},$$

be 3×3 matrices with entries from the domain R. Consider the matrix equation $AX=B$.

For invertible matrices

$$U = \begin{bmatrix} 1 & 0,5\lambda-2 & \lambda-1 \\ \lambda-1 & 0,5\lambda^2-2,5\lambda+4 & \lambda^2-2\lambda+2 \\ 1 & 0,5\lambda+1 & \lambda+1 \end{bmatrix} \text{ and } V = \begin{bmatrix} 0,5\lambda-3 & -0,5\lambda^2+3,5\lambda & 2 \\ 0,5\lambda-1 & -0,5\lambda^2+1,5\lambda & 1 \\ 1 & -\lambda+1 & 0 \end{bmatrix},$$

we have

$$UAV = \begin{bmatrix} C & 0_{2,1} \\ 0_{1,2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda^2+2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ where } \quad C = \operatorname{diag}(1,\ \lambda^2+2),$$

and

$$UB = \begin{bmatrix} \lambda^3+\lambda^2-2\lambda+2 & -2\lambda^2+4\lambda-2 & 4\lambda^2 \\ \lambda^4+2\lambda^3+4\lambda-4 & -2\lambda^3+2\lambda^2-4\lambda+4 & 4\lambda^3+4\lambda^2+8\lambda+8 \\ 0 & 0 & 0 \end{bmatrix} =$$

$$\operatorname{diag}(1,\ \lambda^2+2,\ 0) \left[\begin{array}{cc|c} \lambda^3+\lambda^2-2\lambda+2 & -2\lambda^2+4\lambda-2 & 4\lambda^2 \\ \lambda^2+2\lambda-2 & -2\lambda+2 & 4\lambda+4 \\ \hline 0 & 0 & 0 \end{array}\right].$$

In accordance with Theorem 2.1, matrix equation $AX=B$ is solvable. Hence, for arbitrary $p_1, p_2, p_3 \in$ R the set of matrices

$$X_p = V \left[\begin{array}{cc|c} \lambda^3+\lambda^2-2\lambda+2 & -2\lambda^2+4\lambda-2 & 4\lambda^2 \\ \lambda^2+2\lambda-2 & -2\lambda+2 & 4\lambda+4 \\ \hline p_1 & p_2 & p_3 \end{array}\right] =$$

$$\begin{bmatrix} 4\lambda^2-6+2p_1 & -6\lambda+6+2p_2 & 14\lambda+2p_3 \\ 2\lambda^2-2+p_1 & -2\lambda+2+p_2 & 6\lambda+p_3 \\ 2\lambda & 0 & 4 \end{bmatrix},$$

is a general solution of equation $AX=B$.

It is easily verified that AB^t is the symmetric matrix. Thus, the equation $AX=B$ has a symmetric solution. So, by Theorem 4.1 and Corollary 4.3, we get

$$UBV^{-t} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda^2+2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left[\begin{array}{cc|c} 3\lambda^4+6\lambda^3-15\lambda^2+16\lambda-6 & 3\lambda^3+9\lambda^2-10\lambda+6 & -5\lambda^3-7\lambda^2+14\lambda-8 \\ 3\lambda^3+9\lambda^2-10\lambda+6 & 3\lambda^2+12\lambda-2 & -5\lambda^2-12\lambda+8 \\ \hline 0 & 0 & 0 \end{array}\right].$$

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Based on this, we obtain that for arbitrary $s \in \mathrm{R}$ the set of matrices

$$X_s = V \left[\begin{array}{cc|c} 3\lambda^4+6\lambda^3-15\lambda^2+16\lambda-6 & 3\lambda^3+9\lambda^2-10\lambda+6 & -5\lambda^3-7\lambda^2+14\lambda-8 \\ 3\lambda^3+9\lambda^2-10\lambda+6 & 3\lambda^2+12\lambda-2 & -5\lambda^2-12\lambda+8 \\ \hline -5\lambda^3-7\lambda^2+14\lambda-8 & -5\lambda^2-12\lambda+8 & s \end{array}\right] V^T =$$

$$\begin{bmatrix} -32\lambda^2-36\lambda+42+4s & -16\lambda^2-18\lambda+22+2s & 2\lambda \\ -16\lambda^2-18\lambda+22+2s & -8\lambda^2-8\lambda+10+s & 0 \\ 2\lambda & 0 & 4 \end{bmatrix},$$

is a general symmetric solution of equation $AX = B$.

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