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DERIVATIONS ON RANK K TRIANGULAR MATRICES*

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Abstract. Let $n \geqslant 2$ be an integer and let $T_n(\mathcal{R})$ be the algebra of $n \times n$ upper triangular matrices over a unital ring $\mathcal{R}$. In this paper, we characterize derivations $\psi : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ on strictly upper triangular matrices, i.e., additive maps ψ satisfying $\psi(AB) = A\psi(B) + \psi(A)B$ for all strictly upper triangular matrices $A, B \in T_n(\mathcal{R})$. We then deduce this result a complete structural characterization of derivations $\psi : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ on rank k upper triangular matrices, where $1 \leqslant k \leqslant n$ is a fixed integer and $\mathcal{R}$ is a division ring.

Key words. Derivation, Upper triangular matrix, Rank, Functional identity.

AMS subject classifications. 15A03, 15A04, 16R60, 16W25.

1. Introduction. Let $\mathcal{R}$ be a ring and let $\mathcal{S}$ be a nonempty subset of $\mathcal{R}$. An additive map $\psi : \mathcal{R} \to \mathcal{R}$ is called a *derivation on* $\mathcal{S}$ if $\psi(AB) = A\psi(B) + \psi(A)B$ for all $A, B \in \mathcal{S}$, or simply a *derivation* when $\mathcal{S} = \mathcal{R}$. Let $n \geqslant 2$ be an integer. We denote by $T_n(\mathcal{R})$ the algebra of $n \times n$ upper triangular matrices over $\mathcal{R}$. The study of derivations of rings, as well as its relation to the structure of rings, has been an active and continuing subject for a long history due to the initial works by Herstein [12] and Posner [19] date back to 1957. The first structural characterization of derivations on upper triangular matrix rings was obtained in 1987 by Jøndrup [14]. He studied derivations $\psi : T_n(\mathcal{R}) \to T_n(\mathcal{R})$, with $\mathcal{R}$ being a simple algebra finite dimensional over its center and showed that every derivation ψ is inner, i.e., $\psi = \mathrm{ad}_X$ for some $X \in T_n(\mathcal{R})$, where ad_X is given in (1.2). Subsequently, Coelho and Miles [10] generalized this result to arbitrary unital rings $\mathcal{R}$ and showed that all derivations $\psi : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ are precisely maps of the form

$$\psi = \mathrm{ad}_X + \overline{d}, \tag{1.1}$$

for some derivation $d : \mathcal{R} \to \mathcal{R}$ and $X \in T_n(\mathcal{R})$, where $\overline{d} : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ is the derivation induced by d as given in (1.3). Later, an alternative proof of this result via induction was given in Ref. [15]. Since then, derivations on numerous rings and algebras of upper triangular matrices have intensively studied and generalized in various directions, see, for example [1, 3, 6, 7, 13, 17, 20, 21, 22, 24].

Motivated by these works, as well as by the studies of functional identities on upper triangular matrices [2, 4, 5], linear preserver problems on subsets of matrices [16, 18], and commuting maps on rank k matrices [8, 9, 11, 23], it is natural to address a question of what is the structural form of derivations $\phi : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ on subsets of upper triangular matrices, in particular, on rank k upper triangular matrices. Surprisingly, the structure of ϕ is extremely simple which is of the form (1.1) when $\mathcal{R}$ is a division ring with at least three elements (see Lemma 4.1). Nevertheless, when $|\mathcal{R}| = 2$ and $k = n$, the problem is more involved, and it is related to the study of derivations on strictly upper triangular matrices (see Lemma 4.2). In this regard, we first study derivations $\phi : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ on strictly upper triangular matrices over a unital ring $\mathcal{R}$ in Theorem 3.2. Our main strategy in Theorem 3.2 is to reduce derivations $\phi|_{N_n(\mathcal{R})} : N_n(\mathcal{R}) \to T_n(\mathcal{R})$ to

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derivations $\varphi : N_n(\mathcal{R}) \to N_n(\mathcal{R})$ via X_α (see (1.5) and Example 1.4), where $N_n(\mathcal{R})$ denotes the subset of $T_n(\mathcal{R})$ consisting of strictly upper triangular matrices. Then Theorem 3.2 follows from the characterization of derivations $\varphi : N_n(\mathcal{R}) \to N_n(\mathcal{R})$ (see Lemma 3.1). We notice that a similar attempt in linear Lie derivations of $N_n(\mathcal{R})$ over commutative rings $\mathcal{R}$ has been considered by Ou et al. in Ref. [17]. However, our approach and result are different from Ou et al. Using Theorem 3.2, we deduce a complete structural characterization of derivations $\phi : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ on rank $1 \leqslant k \leqslant n$ upper triangular matrices over a division ring $\mathcal{R}$ in Theorem 4.3. We remark, in Theorem 4.3, that such additive maps ϕ are of the form (1.1), but the structure of ϕ is much more complex and fertile when $|\mathcal{R}| = 2$ and $k = n$.

We begin our discussion with an overview of some examples of derivations on certain subsets of upper triangular matrices, which play a fundamental role in describing the structural results of derivations in Theorems 3.2 and 4.3. Throughout these examples, unless stated otherwise, $\mathcal{R}$ denotes a unital ring and $n \geqslant 2$ is an integer. Let $N_n(\mathcal{R})$ and $D_n(\mathcal{R})$ denote the subsets of $T_n(\mathcal{R})$ consisting of strictly upper triangular matrices and diagonal matrices, respectively. Notice that $T_n(\mathcal{R}) = D_n(\mathcal{R}) \oplus N_n(\mathcal{R})$ is the direct sum of $D_n(\mathcal{R})$ and $N_n(\mathcal{R})$. We denote by $\pi_N : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ and $\pi_D : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ the projections onto $N_n(\mathcal{R})$ and $D_n(\mathcal{R})$, respectively, which are the additive maps such that $\pi_N(A) = A$ for all $A \in N_n(\mathcal{R})$, $\ker \pi_N = D_n(\mathcal{R})$, and $\pi_D + \pi_N = I$, where $I : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ is the identity map. We adopt $\mathcal{R} = \mathbb{F}_2$, the Galois field of two elements, when $|\mathcal{R}| = 2$ and denote $[A, B] = AB - BA$ for $A, B \in T_n(\mathcal{R})$.

Example 1.1. Let $X \in T_n(\mathcal{R})$ and let $\mathrm{ad}_X : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ be the additive map defined by

$$\mathrm{ad}_X(A) = [A, X], \tag{1.2}$$

for all $A \in T_n(\mathcal{R})$. One usually calls ad_X an *inner derivation* of $T_n(\mathcal{R})$.

Example 1.2. Let $d : \mathcal{R} \to \mathcal{R}$ be a derivation. Let $\overline{d} : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ be the additive map defined by

$$\overline{d}(A) = A^d, \tag{1.3}$$

for all $A \in T_n(\mathcal{R})$, where $A^d = (d(a_{ij}))$ is the matrix obtained from $A = (a_{ij})$ by applying d entrywise. Since $(AB)^d = AB^d + A^dB$ for all $A, B \in T_n(\mathcal{R})$, $\overline{d}$ is a derivation induced by d.

For each pair of integers $1 \leqslant i, j \leqslant n$, we denote by E_{ij} the $n \times n$ matrix unit over $\mathcal{R}$ whose (i, j)th entry is one and zero elsewhere.

Example 1.3. Let $f : T_n(\mathcal{R}) \to \mathcal{R}$ be an additive map. Let $\mu_f : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ be the additive map defined by

$$\mu_f(A) = f(A)E_{1n}, \tag{1.4}$$

for all $A \in T_n(\mathcal{R})$. Then μ_f is a derivation on $N_n(\mathcal{R})$ if and only if $aE_{ij} \in \ker f$ for all $a \in \mathcal{R}$ and integers $1 \leqslant i < j - 1 \leqslant n - 1$. To see this, the sufficiency follows from Lemma 2.1 (ii). Consider the necessity. Let $A, B \in N_n(\mathcal{R})$. Then $Af(B)E_{1n} = 0 = f(A)E_{1n}B$ implies that $f(AB) = 0$. It follows that $f(aE_{ij}) = 0$ for all $a \in \mathcal{R}$ and integers $1 \leqslant i < j - 1 \leqslant n - 1$.

Example 1.4. Let $\alpha_1, \ldots, \alpha_{n-1} \in \mathcal{R}$. We denote

$$X_\alpha = E_{1n} + \sum_{i=1}^{n-1} \alpha_i E_{i+1,i}, \tag{1.5}$$

where $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$. Note that

$$[aE_{st}, X_\alpha] = a\alpha_{t-1}E_{s,t-1} - \alpha_s a E_{s+1,t} \in T_n(\mathcal{R}),$$

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for all $a \in \mathcal{R}$ and integers $1 \leqslant s < t \leqslant n$. We thus have $[A, X_\alpha] \in T_n(\mathcal{R})$ for all $A \in N_n(\mathcal{R})$. Let $\eta_{X_\alpha} : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ be the additive map defined by

$$\eta_{X_\alpha}(A) = [\pi_N(A), X_\alpha], \tag{1.6}$$

for all $A \in T_n(\mathcal{R})$. Then η_{X_α} is a derivation on $N_n(\mathcal{R})$.

Example 1.5. Let $X_1, \ldots, X_n \in T_n(\mathcal{R})$ and let $\Phi_{\mathbf{X}} : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ be the additive map defined by

$$\Phi_{\mathbf{X}}(A) = \sum_{i=1}^{n} a_{ii} X_i, \tag{1.7}$$

for all $A = (a_{ij}) \in T_n(\mathcal{R})$, where $\mathbf{X} = \{X_1, \ldots, X_n\}$. Clearly, $\Phi_{\mathbf{X}}$ is a left $\mathcal{R}$-linear map. Since $N_n(\mathcal{R}) \subseteq \ker \Phi_{\mathbf{X}}$ and $AB \in \ker \Phi_{\mathbf{X}}$ for all $A, B \in N_n(\mathcal{R})$, it follows from Lemma 2.1 (i) that $\Phi_{\mathbf{X}}$ is a derivation on $N_n(\mathcal{R})$. Moreover, when $\mathcal{R} = \mathbb{F}_2$ is the Galois field of two elements and $X_1 + \cdots + X_n = 0$, we have $\Phi_{\mathbf{X}}(A) = 0$ for invertible matrices $A \in T_n(\mathbb{F}_2)$. Then $\Phi_{\mathbf{X}}$ is a derivation on invertible upper triangular matrices.

2. Preliminaries.

Lemma 2.1. *Let $\mathcal{R}$ be a ring and let $\mathcal{S}$ be a subset of $\mathcal{R}$. If $\psi : \mathcal{R} \to \mathcal{R}$ is an additive map satisfying one of the following conditions:*

- **(i)** $\mathcal{S} \subseteq \ker \psi$ *and* $AB \in \ker \psi$ *for all* $A, B \in \mathcal{S}$,
- **(ii)** $\mathcal{S}\psi(\mathcal{S}) = \psi(\mathcal{S})\mathcal{S} = \{0\}$ *and* $AB \in \ker \psi$ *for all* $A, B \in \mathcal{S}$,

then ψ is a derivation on $\mathcal{S}$.

Proof. The lemma follows immediately from the observation that $\psi(AB) = \psi(A)B = A\psi(B) = 0$ for all $A, B \in \mathcal{S}$. □

Lemma 2.2. *Let $\mathcal{R}$ be a unital ring and let $\psi : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ be an additive map such that $N_n(\mathcal{R}) \subseteq \ker \psi$. Then ψ is a derivation on $N_n(\mathcal{R})$. Moreover, if ψ is a left $\mathcal{R}$-linear map, then there exist $X_1, \ldots, X_n \in T_n(\mathcal{R})$ such that*

$$\psi(A) = \sum_{i=1}^{n} a_{ii} X_i,$$

for all $A = (a_{ij}) \in T_n(\mathcal{R})$.

Proof. Since $N_n(\mathcal{R})$ is closed under multiplication, it follows from Lemma 2.1 (i) that ψ is a derivation on $N_n(\mathcal{R})$. Suppose that ψ is left $\mathcal{R}$-linear. We set $X_i = \psi(E_{ii}) \in T_n(\mathcal{R})$ for $i = 1, \ldots, n$. By the left linearity of ψ, we have

$$\psi(A) = \sum_{i=1}^{n} a_{ii}\psi(E_{ii}) = \sum_{i=1}^{n} a_{ii} X_i,$$

for all $A = (a_{ij}) \in T_n(\mathcal{R})$ as required. □

Lemma 2.3. *Let $\mathbb{D}$ be a division ring. Let $n \geqslant 2$ and $1 \leqslant k \leqslant n$ be integers such that $k \neq n$ when $|\mathbb{D}| = 2$. Then $T_n(\mathbb{D})$ is spanned by rank k matrices.*

Proof. The result is trivial when $k = 1$. Consider $1 < k \leqslant n$. We now claim that each E_{st}, $1 \leqslant s \leqslant t \leqslant n$, can be expressed as a sum of rank k matrices. We argue in two cases.

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Case I: $1 < k < n$. Let $1 \leqslant s < t \leqslant n$ be integers. We see that

$$E_{ss} = \sum_{i\in\mathcal{J}}(E_{ii}+E_{i,i+1}) - \left(E_{s,s+1} + \sum_{i\in\mathcal{J}\setminus\{s\}}(E_{ii}+E_{i,i+1})\right), \tag{2.8}$$

$$E_{st} = \sum_{i\in\mathcal{J}} E_{ii} + \left(E_{st} - \sum_{i\in\mathcal{J}} E_{ii}\right),$$

are represented as a sum of two rank k triangular matrices, where $\mathcal{J}$ is a subset of $\{1,\ldots,n-1\}$ such that $|\mathcal{J}| = k$ and $s \in \mathcal{J}$. Next, we have

$$E_{nn} = \left(E_{nn} - \sum_{i=1}^{k-1} E_{ii}\right) + \sum_{i=1}^{k-1} E_{ii},$$

is a sum of rank k triangular matrices, because each E_{ii}, $i = 1,\ldots,k-1$, can be expressed as a sum of rank k matrices as given in (2.8).

Case II: $k = n$. Then $|\mathbb{D}| \geqslant 3$. Let $\alpha \in \mathbb{D}\setminus\{0,1\}$. For each integer $1 \leqslant s \leqslant n$, we see that

$$E_{ss} = (I_n - \alpha E_{ss}) + ((\alpha+1)E_{ss} - I_n),$$

and for each pair of integers $1 \leqslant s < t \leqslant n$,

$$E_{st} = (E_{st} - I_n) + I_n,$$

are represented as a sum of two invertible triangular matrices. ∎

3. Derivations on strictly upper triangular matrices. Let $n \geqslant 2$ be an integer. We write $\Lambda_n = \{(i,j) \mid 1 \leqslant i < j \leqslant n\}$ throughout this section.

Lemma 3.1. *Let $n \geqslant 2$ be an integer and let $\mathcal{R}$ be a unital ring. Then $\psi : N_n(\mathcal{R}) \to N_n(\mathcal{R})$ is an additive map satisfying*

$$\psi(AB) = A\psi(B) + \psi(A)B, \tag{3.9}$$

for all $A, B \in N_n(\mathcal{R})$ if and only if

- **(a)** *when $n = 2$, ψ is any additive map.*
- **(b)** *when $n \geqslant 3$, there exists a derivation $d : \mathcal{R} \to \mathcal{R}$, a matrix $X \in T_n(\mathcal{R})$ and an additive map $f : N_n(\mathcal{R}) \to \mathcal{R}$, with $aE_{ij} \in \ker f$ for all $a \in \mathcal{R}$ and integers $1 \leqslant i < j-1 \leqslant n-1$, such that*

$$\psi(A) = [A,X] + A^d + f(A)E_{1n},$$

for all $A \in N_n(\mathcal{R})$.

Proof. The result is obvious when $n = 2$. Consider $n \geqslant 3$. Examples 1.1–1.3 prove the sufficiency. We turn to the necessity. The proof is by induction on n. First suppose $n = 3$. For each $(i,j) \in \Lambda_3$, there are additive maps $f_{ij}, g_{ij}, h_{ij} : N_3(\mathcal{R}) \to N_3(\mathcal{R})$ such that

$$\psi(aE_{ij}) = f_{ij}(a)E_{12} + g_{ij}(a)E_{23} + h_{ij}(a)E_{13}, \tag{3.10}$$

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Volume 41, pp. 86-99, January 2025.

for all $a \in \mathcal{R}$. Let $a \in \mathcal{R}$. Setting $A = E_{12}$ and $B = aE_{12}$ in (3.9), we have $0 = E_{12}\psi(aE_{12}) + \psi(E_{12})aE_{12}$, and hence $g_{12}(a)E_{13} = 0$ by (3.10). Then $g_{12} = 0$. Likewise, setting $A = aE_{23}$ and $B = E_{23}$ in (3.9), we obtain $f_{23} = 0$, and taking $A = aE_{13}$ and $B = E_{13}$ in (3.9), we get $f_{13} = g_{13} = 0$. Let $\phi : N_3(\mathcal{R}) \to N_3(\mathcal{R})$ be the additive map defined by

$$\phi(A) = \psi(A) - f(A)E_{13}, \tag{3.11}$$

for all $A \in N_3(\mathcal{R})$, where $f : N_3(\mathcal{R}) \to \mathcal{R}$ is the additive map defined by

$$f(A) = h_{12}(a_{12}) + h_{23}(a_{23}),$$

for all $A = (a_{ij}) \in N_3(\mathcal{R})$. Clearly, $aE_{13} \in \ker f$ for all $a \in \mathcal{R}$. Then $\phi(AB) = A\phi(B) + \phi(A)B$ for all $A, B \in N_3(\mathcal{R})$ and

$$\phi(aE_{ij}) = k_{ij}(a)E_{ij},$$

for all $a \in \mathcal{R}$ and $(i,j) \in \Lambda_3$, where $k_{12} = f_{12}$, $k_{23} = g_{23}$ and $k_{13} = h_{13}$. Since $\phi((aE_{12})E_{23}) = \phi(aE_{13}) = \phi(E_{12}(aE_{23}))$, it follows that $aE_{12}(k_{23}(1)E_{23}) + (k_{12}(a)E_{12})E_{23} = k_{13}(a)E_{13} = E_{12}(k_{23}(a)E_{23}) + (k_{12}(1)E_{12})aE_{23}$ for all $a \in \mathcal{R}$. We thus obtain

$$k_{12}(a) + ak_{23}(1) = k_{13}(a) = k_{12}(1)a + k_{23}(a), \tag{3.12}$$

for all $a \in \mathcal{R}$. Let $X = -k_{12}(1)E_{11} + k_{23}(1)E_{33} \in T_3(\mathcal{R})$ and let $d : \mathcal{R} \to \mathcal{R}$ be the additive map defined by

$$d(a) = k_{12}(a) - k_{12}(1)a, \tag{3.13}$$

for all $a \in \mathcal{R}$. Then $\phi(aE_{12}) - [aE_{12}, X] = k_{12}(a)E_{12} - k_{12}(1)aE_{12} = d(a)E_{12}$ for all $a \in \mathcal{R}$. By (3.12) and (3.13), we have $d(a) = k_{23}(a) - ak_{23}(1)$ for all $a \in \mathcal{R}$. Then $\phi(aE_{23}) - [aE_{23}, X] = d(a)E_{23}$ for all $a \in \mathcal{R}$. Moreover, by (3.12) and (3.13), we see that $d(a) = k_{13}(a) - ak_{23}(1) - k_{12}(1)a$ for $a \in \mathcal{R}$. So $\phi(aE_{13}) - [aE_{13}, X] = d(a)E_{13}$ for all $a \in \mathcal{R}$. By the additivity of ϕ and $[\,\cdot\,, X]$,

$$\phi(A) - [A, X] = d(a_{12})E_{12} + d(a_{23})E_{23} + d(a_{13})E_{13} = A^d, \tag{3.14}$$

for all $A = (a_{ij}) \in N_3(\mathcal{R})$. We now show that d is a derivation. Let $a, b \in \mathcal{R}$. By (3.14),

$$\begin{aligned} d(ab)E_{13} &= \phi(aE_{12}(bE_{23})) - [aE_{12}(bE_{23}), X] \\ &= aE_{12}(\phi(bE_{23}) - [bE_{23}, X]) + (\phi(aE_{12} - [aE_{12}, X])bE_{23} \\ &= aE_{12}(d(b)E_{23}) + (d(a)E_{12})bE_{23} \\ &= (ad(b) + d(a)b)E_{13}. \end{aligned}$$

Hence, d is a derivation. Together with (3.11) and (3.14), this establishes the base case $n = 3$.

Assuming the result holds for $n \geqslant 3$, we prove it for $n+1$. Let us first claim that there exist additive maps $\varphi_1 : N_n(\mathcal{R}) \to M_{n,1}(\mathcal{R})$ and $\varphi : M_{n,1}(\mathcal{R}) \to M_{n,1}(\mathcal{R})$, a derivation $d : \mathcal{R} \to \mathcal{R}$, a matrix $H \in T_n(\mathcal{R})$ and an additive map $g : N_n(\mathcal{R}) \to \mathcal{R}$, with $aE_{ij} \in \ker g$ for all $a \in \mathcal{R}$ and integers $1 \leqslant i < j - 1 \leqslant n - 1$, such that

$$\psi\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^d + [A, H] + g(A)E_{1n} & \varphi_1(A) + \varphi(B) \\ 0 & 0 \end{pmatrix}, \tag{3.15}$$

for all $A \in N_n(\mathcal{R})$ and $B \in M_{n,1}(\mathcal{R})$, where $M_{n,1}(\mathcal{R})$ is the additive group of $n \times 1$ matrices over $\mathcal{R}$. Let $B_1, B_2 \in M_{n,1}(\mathcal{R})$. For each $i = 1, 2$, we write

Electronic Journal of Linear Algebra, ISSN 1081-3810
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$$\psi\begin{pmatrix} 0_n & B_i \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Y_i & Z_i \\ 0 & 0 \end{pmatrix},$$

for some $Y_i \in N_n(\mathcal{R})$ and $Z_i \in M_{n,1}(\mathcal{R})$, where 0_n is the $n \times n$ zero matrix. By (3.9),

$$\begin{aligned} 0 &= \psi\left(\begin{pmatrix} 0_n & B_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0_n & B_2 \\ 0 & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} 0_n & B_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Y_2 & Z_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} Y_1 & Z_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0_n & B_2 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0_n & Y_1B_2 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence, $Y_1B_2 = 0$ for all $B_2 \in M_{n,1}(\mathcal{R})$, and so $Y_1 = 0$. Then there exists an additive map $\varphi : M_{n,1}(\mathcal{R}) \to M_{n,1}(\mathcal{R})$ such that

$$\psi\begin{pmatrix} 0_n & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0_n & \varphi(B) \\ 0 & 0 \end{pmatrix},$$

for all $B \in M_{n,1}(\mathcal{R})$. By the additivity of ψ, we thus deduce that

$$\psi\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = \psi\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \psi\begin{pmatrix} 0_n & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \zeta(A) & \varphi_1(A) + \varphi(B) \\ 0 & 0 \end{pmatrix}, \tag{3.16}$$

for all $A \in N_n(\mathcal{R})$ and $B \in M_{n,1}(\mathcal{R})$, where $\zeta : N_n(\mathcal{R}) \to N_n(\mathcal{R})$ and $\varphi_1 : N_n(\mathcal{R}) \to M_{n,1}(\mathcal{R})$ are additive maps. Let $A_1, A_2 \in N_n(\mathcal{R})$. By (3.9) and (3.16), we see that

$$\begin{aligned} \begin{pmatrix} \zeta(A_1A_2) & \varphi_1(A_1A_2) \\ 0 & 0 \end{pmatrix} &= \psi\begin{pmatrix} A_1A_2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \psi\left(\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \zeta(A_2) & \varphi_1(A_2) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \zeta(A_1) & \varphi_1(A_1) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A_1\zeta(A_2) + \zeta(A_1)A_2 & A_1\varphi_1(A_2) \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Then

$$\zeta(A_1A_2) = A_1\zeta(A_2) + \zeta(A_1)A_2 \quad \text{and} \quad \varphi_1(A_1A_2) = A_1\varphi_1(A_2), \tag{3.17}$$

for all $A_1, A_2 \in N_n(\mathcal{R})$. The induction assumption deduces that there is a derivation $d : \mathcal{R} \to \mathcal{R}$ and $H \in T_n(\mathcal{R})$ such that

$$\zeta(A) = [A, H] + A^d + g(A)E_{1n}, \tag{3.18}$$

for all $A \in N_n(\mathcal{R})$, where $g : N_n(\mathcal{R}) \to \mathcal{R}$ is an additive map satisfying

$$aE_{ij} \in \ker g \quad \text{for all } a \in \mathcal{R} \text{ and } 1 \leqslant i < j - 1 \leqslant n - 1. \tag{3.19}$$

It follows from (3.16) and (3.18) that (3.15) is claimed. Let $K_1 = H \oplus 1 \in T_{n+1}(\mathcal{R})$ and let $\phi_1 : N_{n+1}(\mathcal{R}) \to N_{n+1}(\mathcal{R})$ be the additive map defined by

$$\phi_1(X) = \psi(X) - [X, K_1] - X^d, \tag{3.20}$$

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for all $X \in N_{n+1}(\mathcal{R})$. Then $\phi_1(XY) = X\phi_1(Y) + \phi_1(X)Y$ for all $X, Y \in N_{n+1}(\mathcal{R})$, and by (3.15), we obtain

$$\phi_1\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} g(A)E_{1n} & \varphi_1(A) + \widetilde{\varphi}_2(B) \\ 0 & 0 \end{pmatrix}, \tag{3.21}$$

for all $A \in N_n(\mathcal{R})$ and $B \in M_{n,1}(\mathcal{R})$, where $\widetilde{\varphi}_2 : M_{n,1}(\mathcal{R}) \to M_{n,1}(\mathcal{R})$ is the additive map given by $\widetilde{\varphi}_2(B) = \varphi(B) - (B - HB + B^d)$ for all $B \in M_{n,1}(\mathcal{R})$. We claim that for each $(i,j) \in \Lambda_n$,

$$g(aE_{ij}) = \begin{cases} ag(E_{12}) & \text{when } (i,j) = (1,2), \\ g(E_{n-1,n})a & \text{when } (i,j) = (n-1,n), \\ 0 & \text{otherwise} \end{cases} \tag{3.22}$$

for all $a \in \mathcal{R}$. By (3.19), we have $g(aE_{ij}) = 0$ for all $a \in \mathcal{R}$ and integers $1 \leqslant i < j-1 \leqslant n-1$. Let $1 < i < n-1$ and let $a \in \mathcal{R}$. Since $i+1 \neq n$, it follows from (3.21) that

$$0 = \phi_1((aF_{i,i+1})F_{n,n+1}) = aF_{i,i+1}\phi_1(F_{n,n+1}) + \phi_1(aF_{i,i+1})F_{n,n+1} = g(aE_{i,i+1})F_{1,n+1}.$$

Here, F_{ij} denotes the $(n+1) \times (n+1)$ matrix unit over $\mathcal{R}$ whose (i,j)th entry is one and zero elsewhere. Hence, $g(aE_{i,i+1}) = 0$ for all $a \in \mathcal{R}$ and $i = 2, \ldots, n-2$. Next, in view of (3.21), we write $\phi_1(F_{n,n+1}) = \alpha_1 F_{1,n+1} + \cdots + \alpha_n F_{n,n+1}$ for some $\alpha_1, \ldots, \alpha_n \in \mathcal{R}$. By (3.21), we have

$$0 = \phi_1((aF_{12})F_{n,n+1}) = aF_{12}\phi_1(F_{n,n+1}) + \phi_1(aF_{12})F_{n,n+1} = (a\alpha_2 + g(aE_{12}))F_{1,n+1}.$$

So $g(aE_{12}) = ag(E_{12})$ for all $a \in \mathcal{R}$. Let $a \in \mathcal{R}$. Write $\phi_1(aF_{n,n+1}) = \beta_1 F_{1,n+1} + \cdots + \beta_n F_{n,n+1}$ for some $\beta_1, \ldots, \beta_n \in \mathcal{R}$. Considering $\phi_1((aF_{n-1,n})F_{n,n+1}) = \phi_1(F_{n-1,n}(aF_{n,n+1}))$, we get

$$a\alpha_n F_{n-1,n+1} + g(aE_{n-1,n})F_{1,n+1} = \beta_n F_{n-1,n+1} + g(E_{n-1,n})aF_{1,n+1}.$$

Hence, $g(aE_{n-1,n}) = g(E_{n-1,n})a$ for all $a \in \mathcal{R}$, which completes the proof of (3.22).

Let $K_2 = -g(E_{n-1,n})F_{1,n-1} + g(E_{12})F_{2n} \in N_{n+1}(\mathcal{R})$. By the additivity of g and (3.22), there exists an additive map $\varphi_3 : M_{n,1}(\mathcal{R}) \to M_{n,1}(\mathcal{R})$ such that

$$XK_2 - K_2X = \begin{pmatrix} g(A)E_{1n} & \varphi_3(B) \\ 0 & 0 \end{pmatrix} \quad \text{for all } X = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \in N_{n+1}(\mathcal{R}),$$

with $A \in N_n(\mathcal{R})$. Let $\phi_2 : N_{n+1}(\mathcal{R}) \to N_{n+1}(\mathcal{R})$ be the additive map defined by

$$\phi_2(X) = \phi_1(X) - [X, K_2], \tag{3.23}$$

for all $X \in N_{n+1}(\mathcal{R})$. Clearly, $\phi_2(XY) = X\phi_2(Y) + \phi_2(X)Y$ for all $X, Y \in N_{n+1}(\mathcal{R})$, and by (3.21), we obtain

$$\phi_2\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0_n & \varphi_1(A) + \varphi_2(B) \\ 0 & 0 \end{pmatrix}, \tag{3.24}$$

for all $A \in N_n(\mathcal{R})$ and $B \in M_{n,1}(\mathcal{R})$, where $\varphi_2(B) = \widetilde{\varphi}_2(B) - \varphi_3(B)$ for all $B \in M_{n,1}(\mathcal{R})$. We continue to show that

$$\varphi_2(AB) = A\varphi_2(B), \tag{3.25}$$

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for all $A \in N_n(\mathcal{R})$ and $B \in M_{n,1}(\mathcal{R})$. To see this, by (3.24), we have

$$\begin{aligned}\begin{pmatrix}0_n & \varphi_2(AB)\\ 0 & 0\end{pmatrix} &= \phi_2\left(\begin{pmatrix}A & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0_n & B\\ 0 & 0\end{pmatrix}\right)\\ &= \begin{pmatrix}A & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0_n & \varphi_2(B)\\ 0 & 0\end{pmatrix}+\begin{pmatrix}0_n & \varphi_1(A)\\ 0 & 0\end{pmatrix}\begin{pmatrix}0_n & B\\ 0 & 0\end{pmatrix}\\ &= \begin{pmatrix}0_n & A\varphi_2(B)\\ 0 & 0\end{pmatrix},\end{aligned}$$

for all $A \in N_n(\mathcal{R})$ and $B \in M_{n,1}(\mathcal{R})$, which establishes (3.25).

Our next claim is that there exists an additive map $\mu_1 : N_n(\mathcal{R}) \to \mathcal{R}$ and $c_3, \dots, c_n \in \mathcal{R}$ such that

$$\varphi_1(A) = \mu_1(A)\mathrm{e}_1 + \sum_{2\leqslant i<j\leqslant n} a_{ij}c_j\mathrm{e}_i, \tag{3.26}$$

for all $A = (a_{ij}) \in N_n(\mathcal{R})$. Here, $\mathrm{e}_i \in M_{n,1}(\mathcal{R})$ denotes the column matrix whose ith entry is one and zero elsewhere. Since φ_1 is additive, there are additive maps $\mu_i : N_n(\mathcal{R}) \to \mathcal{R}$, $i = 1, \dots, n$, such that $\varphi_1(A) = \mu_1(A)\mathrm{e}_1 + \cdots + \mu_n(A)\mathrm{e}_n$ for all $A \in N_n(\mathcal{R})$. Let $a \in \mathcal{R}$ and $(i, j) \in \Lambda_n$. By (3.17), we see that

$$E_{1\ell}\,\varphi_1(aE_{ij}) = \varphi_1(E_{1\ell}(aE_{ij})) = 0,$$

for all $2 \leqslant \ell \leqslant n$ with $\ell \neq i$. It follows that

$$\varphi_1(aE_{ij}) = \begin{cases}\mu_1(aE_{1j})\mathrm{e}_1 & \text{for } 1 = i < j \leqslant n,\\ \mu_1(aE_{ij})\mathrm{e}_1 + \mu_i(aE_{ij})\mathrm{e}_i & \text{for } 2 \leqslant i < j \leqslant n\end{cases},$$

for all $a \in \mathcal{R}$. Set $\mu_1(E_{1j}) = c_j \in \mathcal{R}$ for $j = 3, \dots, n$. Let $2 \leqslant i < j \leqslant n$. By (3.17), we have

$$\mu_i(aE_{ij})\mathrm{e}_1 = E_{1i}\,\varphi_1(aE_{ij}) = \varphi_1((aE_{1i})E_{ij}) = aE_{1i}\,\varphi_1(E_{ij}) = a\varphi_1(E_{1j}) = ac_j\mathrm{e}_1,$$

and, in consequence, $\mu_i(aE_{ij}) = ac_j$ for all $a \in \mathcal{R}$. Then for any $2 \leqslant i < j \leqslant n$,

$$\varphi_1(aE_{ij}) = \mu_1(aE_{ij})\mathrm{e}_1 + ac_j\mathrm{e}_i,$$

for all $a \in \mathcal{R}$. By the additivity of φ_1 and μ_1, we prove (3.26). Write $K_3 = \sum_{i=3}^{n} c_iF_{i,n+1} \in N_{n+1}(\mathcal{R})$. One sees immediately that

$$[X, K_3] = \sum_{i=3}^{n} x_{1i}c_iF_{1,n+1} + \sum_{2\leqslant i<j\leqslant n} x_{ij}c_jF_{i,n+1},$$

for all $X = (x_{ij}) \in N_{n+1}(\mathcal{R})$. Let $\phi_3 : N_{n+1}(\mathcal{R}) \to N_{n+1}(\mathcal{R})$ be the additive map defined by

$$\phi_3(X) = \phi_2(X) - [X, K_3], \tag{3.27}$$

for all $X \in N_{n+1}(\mathcal{R})$. Then $\phi_3(XY) = X\phi_3(Y) + \phi_3(X)Y$ for all $X, Y \in N_{n+1}(\mathcal{R})$. By (3.24), (3.26), and (3.27),

$$\phi_3\begin{pmatrix}A & B\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0_n & \varphi_2(B)\\ 0 & 0\end{pmatrix} + \mu(A)F_{1,n+1}, \tag{3.28}$$

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for all $A \in N_n(\mathcal{R})$ and $B \in M_{n,1}(\mathcal{R})$, where $\mu : N_n(\mathcal{R}) \to \mathcal{R}$ is the additive map given by $\mu(A) = \mu_1(A) - \sum_{i=3}^{n} a_{1i}c_i$ for all $A = (a_{ij}) \in N_n(\mathcal{R})$.

We now continue to show that there exists an additive map $\eta_1 : M_{n,1}(\mathcal{R}) \to \mathcal{R}$ and $c \in \mathcal{R}$ such that

$$\varphi_2(B) = \eta_1(B)\mathrm{e}_1 + Bc, \tag{3.29}$$

for all $B \in M_{n,1}(\mathcal{R})$. By the additivity of φ_2, there exist additive maps $\eta_i : M_{n,1}(\mathcal{R}) \to \mathcal{R}$, $i = 1, \ldots, n$, such that $\varphi_2(B) = \eta_1(B)\mathrm{e}_1 + \cdots + \eta_n(B)\mathrm{e}_n$ for all $B \in M_{n,1}(\mathcal{R})$. Let $a \in \mathcal{R}$ and let $1 \leqslant i \leqslant n$. By (3.25),

$$E_{1\ell}\,\varphi_2(a\mathrm{e}_i) = \varphi_2(E_{1\ell}(a\mathrm{e}_i)) = 0,$$

for all $2 \leqslant \ell \leqslant n$ with $\ell \neq i$. Then

$$\varphi_2(a\mathrm{e}_i) \quad = \quad \begin{cases} \eta_1(a\mathrm{e}_1)\mathrm{e}_1 & \text{for } i = 1, \\ \eta_1(a\mathrm{e}_i)\mathrm{e}_1 + \eta_i(a\mathrm{e}_i)\mathrm{e}_i & \text{for } 2 \leqslant i \leqslant n \end{cases},$$

for all $a \in \mathcal{R}$. Set $\eta_1(\mathrm{e}_1) = c \in \mathcal{R}$. For each integer $2 \leqslant i \leqslant n$, we see from (3.25) that

$$\eta_i(a\mathrm{e}_i)\mathrm{e}_1 = E_{1i}\,\varphi_2(a\mathrm{e}_i) = \varphi_2((aE_{1i})\mathrm{e}_i)) = aE_{1i}\,\varphi_2(\mathrm{e}_i) = a\,\varphi_2(E_{1i}\mathrm{e}_i) = a\varphi_2(\mathrm{e}_1) = ac\mathrm{e}_1,$$

and thus, $\eta_i(a\mathrm{e}_i) = ac$ for all $a \in \mathcal{R}$. Then for each $2 \leqslant i \leqslant n$,

$$\varphi_2(a\mathrm{e}_i) = \eta_1(a\mathrm{e}_i)\mathrm{e}_1 + a\mathrm{e}_i c,$$

for all $a \in \mathcal{R}$, which completes the proof of (3.29). It follows from (3.28) and (3.29) that

$$\begin{aligned} \phi_3\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0_n & Bc \\ 0 & 0 \end{pmatrix} + (\mu(A) + \eta_1(B))F_{1,n+1} \\ &= \left[\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, cF_{n+1,n+1}\right] + (\mu(A) + \eta_1(B))F_{1,n+1}, \end{aligned}$$

for all $A \in N_n(\mathcal{R})$ and $B \in M_{n,1}(\mathcal{R})$. Write $K_4 = cF_{n+1,n+1} \in T_{n+1}(\mathcal{R})$. Let $f : N_{n+1}(\mathcal{R}) \to \mathcal{R}$ and let $\phi_4 : N_{n+1}(\mathcal{R}) \to N_{n+1}(\mathcal{R})$ be the additive maps defined by

$$f\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} = \mu(A) + \eta_1(B),$$

for all $A \in N_n(\mathcal{R})$ and $B \in M_{n,1}(\mathcal{R})$, and

$$\phi_4(X) = \phi_3(X) - [X, K_4], \tag{3.30}$$

for all $X \in N_{n+1}(\mathcal{R})$. It is clear that $\phi_4(XY) = X\phi_4(Y) + \phi_4(X)Y$ for all $X, Y \in N_{n+1}(\mathcal{R})$ and $\phi_4(X) = f(X)F_{1,n+1}$ for all $X \in N_{n+1}(\mathcal{R})$. It follows from Example 1.3 that $aF_{ij} \in \ker f$ for all $a \in \mathcal{R}$ and integers $1 \leqslant i < j - 1 \leqslant n$. Consequently, in view of (3.20), (3.23), (3.27), and (3.30), we thus obtain

$$\psi(X) = [X, K] + X^d + f(X)F_{1,n+1},$$

for all $X \in T_{n+1}(\mathcal{R})$, where $K = \sum_{i=1}^{4} K_i \in T_{n+1}(\mathcal{R})$. This completes the proof. ◻

We are now ready for the main result of this section.

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THEOREM 3.2. *Let $n \geqslant 2$ be an integer and let $\mathcal{R}$ be a unital ring. Then $\psi : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ is a derivation on $N_n(\mathcal{R})$ if and only if there exists an additive map $\Phi : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ with $N_n(\mathcal{R}) \subseteq \ker \Phi$, a derivation $d : \mathcal{R} \to \mathcal{R}$, a matrix $X \in T_n(\mathcal{R})$, scalars $\alpha_1, \ldots, \alpha_{n-1} \in \mathcal{R}$, and an additive map $f : T_n(\mathcal{R}) \to \mathcal{R}$, with $aE_{ij} \in \ker f$ for all $a \in \mathcal{R}$ and $1 \leqslant i < j - 1 \leqslant n - 1$, such that*

$$\psi = \Phi + \mathrm{ad}_X + \overline{d} + \eta_{X_\alpha} + \mu_f,$$

where $\mathrm{ad}_X : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ is the inner derivation as given in (1.2), $\overline{d} : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ is the derivation induced by d as given in (1.3), $\mu_f : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ is the derivation as given in (1.4), and $\eta_{X_\alpha} : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ is the derivation as given in (1.6) with $X_\alpha = E_{1n} + \sum_{i=1}^{n-1} \alpha_i E_{i+1,i}$ and $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$.

Proof. The sufficiency follows easily from Lemma 2.2 and Examples 1.2–1.4. Consider now the necessity. Let $\psi_1, \psi_2 : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ be the composite of additive maps defined by

$$\psi_1 = \psi \circ \pi_N \quad \text{and} \quad \psi_2 = \psi \circ \pi_D,$$

where π_N and π_D are projections onto $N_n(\mathcal{R})$ and $D_n(\mathcal{R})$, respectively. Clearly, $\psi = \psi_1 + \psi_2$ and ψ_1 is a derivation on $N_n(\mathcal{R})$ with $D_n(\mathcal{R}) \subseteq \ker \psi_1$. Moreover, since $N_n(\mathcal{R}) \subseteq \ker \psi_2$, we have ψ_2 is a derivation on $N_n(\mathcal{R})$ by Lemma 2.2. We first claim that there exist $\alpha_1, \ldots, \alpha_{n-1} \in \mathcal{R}$ such that

$$\psi_1(A) - [A, X_\alpha] \in N_n(\mathcal{R}), \tag{3.31}$$

for all $A \in N_n(\mathcal{R})$, where $X_\alpha = E_{1n} + \sum_{i=1}^{n-1} \alpha_i E_{i+1,i}$ with $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$. Since ψ_1 is additive, there are additive maps $f_{ij} : N_n(\mathcal{R}) \to \mathcal{R}$, $1 \leqslant i \leqslant j \leqslant n$, such that

$$\psi_1(A) = \sum_{1 \leqslant i \leqslant j \leqslant n} f_{ij}(A) E_{ij}, \tag{3.32}$$

for all $A \in N_n(\mathcal{R})$. Let $(p, q), (s, t) \in \Lambda_n$ and $a \in \mathcal{R}$. As ψ_1 is a derivation on $N_n(\mathcal{R})$, we have $\psi_1((aE_{pq})E_{st}) = aE_{pq}\psi_1(E_{st}) + \psi_1(aE_{pq})E_{st}$ and $\psi_1(E_{pq}(aE_{st})) = E_{pq}\psi_1(aE_{st}) + \psi_1(E_{pq})aE_{st}$. Using (3.32), we obtain

$$\psi_1((aE_{pq})E_{st}) = \sum_{j=q}^{n} a f_{qj}(E_{st}) E_{pj} + \sum_{i=1}^{s} f_{is}(aE_{pq}) E_{it}, \tag{3.33}$$

$$\psi_1(E_{pq}(aE_{st})) = \sum_{j=q}^{n} f_{qj}(aE_{st}) E_{pj} + \sum_{i=1}^{s} f_{is}(E_{pq}) aE_{it}, \tag{3.34}$$

for any $(p, q), (s, t) \in \Lambda_n$ and $a \in \mathcal{R}$. Let $1 < s+1 < t \leqslant n$ be integers and $a \in \mathcal{R}$. Since $aE_{st} = aE_{s,s+1}E_{s+1,t}$, it follows from (3.32) and (3.33) that

$$\sum_{1 \leqslant i \leqslant j \leqslant n} f_{ij}(aE_{st}) E_{ij} = \sum_{j=s+1}^{n} a f_{s+1,j}(E_{s+1,t}) E_{sj} + \sum_{i=1}^{s+1} f_{i,s+1}(aE_{s,s+1}) E_{it}.$$

We thus deduce that for each $(s, t) \in \Lambda_n$ with $s + 1 < t$,

$$f_{ii}(aE_{st}) = 0, \quad i = 1, \ldots, n, \tag{3.35}$$

for all $a \in \mathcal{R}$. Let $1 \leqslant s < t < n$ be integers. Note that

$$\{(i, s+1) \mid i = 1, \ldots, s\} \cap \{(t, j) \mid j = t+1, \ldots, n\} = \emptyset. \tag{3.36}$$

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Since $\psi_1((aE_{t,t+1})E_{s,s+1}) = 0 = \psi_1(E_{t,t+1}(aE_{s,s+1}))$ for all $a \in \mathcal{R}$, it follows from (3.33), (3.34), and (3.36) that $f_{t+1,t+1}(aE_{s,s+1}) = 0 = f_{ss}(aE_{t,t+1})$ for all $a \in \mathcal{R}$ and $1 \leqslant s < t < n$. Consequently, for each integer $1 \leqslant s < n$, we have

$$f_{ii}(aE_{s,s+1}) = 0, \quad i = 1, \ldots, n, \ \text{ with } i \neq s, s+1, \tag{3.37}$$

for all $a \in \mathcal{R}$. Next, let $1 \leqslant s < n$ be an integer and $a \in \mathcal{R}$. Since $\psi_1((aE_{s,s+1})E_{s,s+1}) = 0 = \psi_1(E_{s,s+1}(aE_{s,s+1}))$, it follows from (3.33) and (3.34) that

$$\begin{array}{rcl} 0 & = & \sum_{j=s+1}^{n} a f_{s+1,j}(E_{s,s+1})E_{sj} + \sum_{i=1}^{s} f_{is}(aE_{s,s+1})E_{i,s+1} \\ & = & \sum_{j=s+1}^{n} f_{s+1,j}(aE_{s,s+1})E_{sj} + \sum_{i=1}^{s} f_{is}(E_{s,s+1})aE_{i,s+1}. \end{array}$$

This leads to

$$f_{ss}(aE_{s,s+1}) + a f_{s+1,s+1}(E_{s,s+1}) = 0 \ \text{ and } \ f_{s+1,s+1}(aE_{s,s+1}) + f_{ss}(E_{s,s+1})a = 0, \tag{3.38}$$

for all $a \in \mathcal{R}$ and $s = 1, \ldots, n-1$. Hence, $f_{ss}(E_{s,s+1}) = -f_{s+1,s+1}(E_{s,s+1})$ for $s = 1, \ldots, n-1$. This result, together with (3.38), gives

$$f_{ss}(aE_{s,s+1}) = -a f_{s+1,s+1}(E_{s,s+1}) = a f_{ss}(E_{s,s+1}), \tag{3.39}$$

$$f_{s+1,s+1}(aE_{s,s+1}) = -f_{ss}(E_{s,s+1})a = f_{s+1,s+1}(E_{s,s+1})a, \tag{3.40}$$

for all $a \in \mathcal{R}$ and $s = 1, \ldots, n-1$. Let $\alpha_i = f_{ii}(E_{i,i+1}) \in \mathcal{R}$ for $i = 1, \ldots, n-1$. By (3.39) and (3.40), we deduce that

$$\begin{array}{rcl} [aE_{s,s+1}, X_\alpha] & = & aE_{s,s+1}\Big(E_{1n} + \sum_{i=1}^{n-1} \alpha_i E_{i+1,i}\Big) - \Big(E_{1n} + \sum_{i=1}^{n-1} \alpha_i E_{i+1,i}\Big) aE_{s,s+1} \\ & = & a\alpha_s E_{ss} - \alpha_s a E_{s+1,s+1} \\ & = & a f_{ss}(E_{s,s+1})E_{ss} - f_{ss}(E_{s,s+1})aE_{s+1,s+1} \\ & = & f_{ss}(aE_{s,s+1})E_{ss} + f_{s+1,s+1}(aE_{s,s+1})E_{s+1,s+1} \end{array},$$

for $a \in \mathcal{R}$ and $s = 1, \ldots, n-1$. This observation, together with (3.32), (3.35), (3.37), (3.39), and (3.40), deduces that $\psi_1(A) - [A, X_\alpha] \in N_n(\mathcal{R})$ for all $A \in N_n(\mathcal{R})$. Hence, (3.31) is proved.

Let $\phi : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ be the additive map defined by

$$\phi(A) = \psi_1(A) - [\pi_N(A), X_\alpha], \tag{3.41}$$

for all $A \in T_n(\mathcal{R})$. One sees immediately that $D_n(\mathcal{R}) \subseteq \ker \phi$. It follows from (3.31) that ϕ maps from $N_n(\mathcal{R})$ into $N_n(\mathcal{R})$ satisfying $\phi(AB) = A\phi(B) + \phi(A)B$ for all $A, B \in N_n(\mathcal{R})$. By Lemma 3.1, there exists a derivation $d : \mathcal{R} \to \mathcal{R}$, a matrix $X \in T_n(\mathcal{R})$ and an additive map $g : N_n(\mathcal{R}) \to \mathcal{R}$, with $aE_{ij} \in \ker g$ for all $a \in \mathcal{R}$ and $1 \leqslant i < j-1 \leqslant n-1$, such that

$$\phi(A) = [A, X] + A^d + g(A)E_{1n}, \tag{3.42}$$

for all $A \in N_n(\mathcal{R})$. Let $\Phi : T_n(\mathcal{R}) \to T_n(\mathcal{R})$ be the additive map defined by

$$\Phi(A) = \psi_2(A) - \pi_D(A)^d - [\pi_D(A), X] - f(\pi_D(A))E_{1n}, \tag{3.43}$$

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for all $A \in T_n(\mathcal{R})$, where $f : T_n(\mathcal{R}) \to \mathcal{R}$ is an additive map such that $f(A) = g(A)$ for all $A \in N_n(\mathcal{R})$. Thus, $aE_{ij} \in \ker f$ for all $a \in \mathcal{R}$ and $1 \leqslant i < j-1 \leqslant n-1$. Since $N_n(\mathcal{R}) \subseteq \ker \Phi$, it follows from Lemma 2.2 that Φ is a derivation on $N_n(\mathcal{R})$. In view of (3.41)$-$(3.43), we obtain

$$\begin{aligned} \psi(A) &= \psi_1(A) + \psi_2(A) \\ &= \phi(A) + [\pi_N(A), X_\alpha] + \Phi(A) + \pi_D(A)^d + [\pi_D(A), X] + f(\pi_D(A))E_{1n} \ , \\ &= \Phi(A) + [A, X] + A^d + \eta_{X_\alpha}(A) + f(A)E_{1n} \end{aligned}$$

for all $A \in T_n(\mathcal{R})$, and hence $\psi = \Phi + \overline{d} + \mathrm{ad}_X + \eta_{X_\alpha} + \mu_f$. This completes the proof. ◻

4. Derivations on rank k upper triangular matrices. This section is devoted to characterize derivations $\psi : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ on rank k upper triangular matrices over division rings $\mathbb{D}$.

Lemma 4.1. *Let $n \geqslant 2$ and $1 \leqslant k \leqslant n$ be integers such that $k \neq n$ when $|\mathbb{D}| = 2$. Then the following are equivalent:*

- **(i)** *$\psi : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ is a derivation on rank k matrices.*
- **(ii)** *$\psi : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ is a derivation.*
- **(iii)** *There exists a derivation $d : \mathbb{D} \to \mathbb{D}$ and $X \in T_n(\mathbb{D})$ such that*

$$\psi = \mathrm{ad}_X + \overline{d},$$

where $\overline{d} : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ is the derivation induced by d as defined in (1.3) and $\mathrm{ad}_X : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ is the inner induced by X as defined in (1.2)

Proof. (iii) $\Rightarrow$ (i) By Examples 1.2 and 1.1.

(ii) $\Rightarrow$ (iii) By [10] and [15, Theorem 2].

(i) $\Rightarrow$ (ii) Let $A, B \in T_n(\mathbb{D})$. By Lemma 2.3, we have $A = \sum_{i=1}^p A_i$ and $B = \sum_{j=1}^q B_j$, where p, q are positive integers and matrices $A_i, B_j \in T_n(\mathbb{D})$ are of rank k. Then

$$\begin{aligned} \psi(AB) = \psi\left(\sum_{i=1}^p \sum_{j=1}^q A_i B_j\right) &= \sum_{i=1}^p \sum_{j=1}^q \psi(A_i B_j) \\ &= \sum_{i=1}^p \sum_{j=1}^q (A_i \psi(B_j) + \psi(A_j) B_j) \\ &= A\psi(B) + \psi(A)B. \end{aligned}$$

Hence, ψ is a derivation. This completes the proof. ◻

When $\mathbb{D} = \mathbb{F}_2$ is the Galois field of two elements, we have the following result.

Lemma 4.2. *Let $n \geqslant 2$ be an integer and let $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ be an additive map. Then the following hold.*

- **(i)** *If ψ is a derivation on invertible matrices, then ψ is a derivation on $N_n(\mathbb{F}_2)$.*
- **(ii)** *If $\psi(I_n) = 0$ and ψ is a derivation on $N_n(\mathbb{F}_2)$, then ψ is a derivation on invertible matrices.*

Proof. We first claim that if $\psi(I_n) = 0$, then

$$\psi \text{ is a derivation on invertible matrices} \quad \Leftrightarrow \quad \psi \text{ is a derivation on } N_n(\mathbb{F}_2). \tag{4.44}$$

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To show this, we note easily that $A, B \in N_n(\mathbb{F}_2)$ if and only if $I_n + A$ and $I_n + B$ are invertible matrices in $T_n(\mathbb{F}_2)$. By the additivity of ψ and $\psi(I_n) = 0$, one sees immediately that

$$\psi((I_n + A)(I_n + B)) = \psi(A) + \psi(B) + \psi(AB),$$

$$(I_n + A)\psi(I_n + B) + \psi(I_n + A)(I_n + B) = \psi(B) + A\psi(B) + \psi(A) + \psi(A)B.$$

Consequently, (4.44) is proved.

(i) It suffices to show that $\psi(I_n) = 0$. To see this, $\psi(I_n) = \psi(I_n I_n) = I_n\psi(I_n) + \psi(I_n)I_n = 2\psi(I_n)$, and so $\psi(I_n) = 0$. The result follows from (4.44).

(ii) The result follows immediately from (4.44). ∎

We now establish our main theorem of this section.

THEOREM 4.3. *Let $n \geqslant 2$ and $1 \leqslant k \leqslant n$ be integers and let $\mathbb{D}$ be a division ring. Then $\psi : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ is a derivation on rank k matrices if and only if*

(a) *when $1 \leqslant k \leqslant n$ with $k \neq n$ if $\mathbb{D} = \mathbb{F}_2$, there exists a derivation $d : \mathbb{D} \to \mathbb{D}$ and $X \in T_n(\mathbb{D})$ such that*

$$\psi = \mathrm{ad}_X + \overline{d},$$

where $\mathrm{ad}_X : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ is the inner derivation as given in (1.2) and $\overline{d} : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ is the derivation induced by d as given in (1.3).

(b) *when $k = n$ and $\mathbb{D} = \mathbb{F}_2$, there exist scalars $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{F}_2$, matrices $X, X_1, \ldots, X_n \in T_n(\mathbb{F}_2)$ satisfying $X_1 + \cdots + X_n = 0$, and an additive map $f : T_n(\mathbb{F}_2) \to \mathbb{F}_2$ with $I_n, E_{ij} \in \ker f$ for all $1 \leqslant i < j - 1 \leqslant n - 1$, such that*

$$\psi = \mathrm{ad}_X + \eta_{X_\alpha} + \mu_f + \Phi_{\mathbf{X}},$$

where $\mathrm{ad}_X : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is the inner derivation as given in (1.2), $\mu_f : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is the derivation as given in (1.4), $\eta_{X_\alpha} : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is the derivation as given in (1.6) with $X_\alpha = E_{1n} + \sum_{i=1}^{n-1} \alpha_i E_{i+1,i}$ and $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$, and $\Phi_{\mathbf{X}} : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is the derivation as given in (1.7) with $\mathbf{X} = \{X_1, \ldots, X_n\}$.

Proof. Since η_{X_α}, μ_f, and $\Phi_{\mathbf{X}}$ are derivations on $N_n(\mathbb{F}_2)$ such that $\eta_{X_\alpha}(I_n) = \mu_f(I_n) = \Phi_{\mathbf{X}}(I_n) = 0$, the sufficiency follows immediately Examples 1.2−1.5 and Lemma 4.2 (ii).

For the necessity, we note that the result follows immediately from Lemma 4.1 if $1 \leqslant k \leqslant n$ with $k \neq n$ when $\mathbb{D} = \mathbb{F}_2$. Consider now $k = n$ and $\mathbb{D} = \mathbb{F}_2$. By Lemma 4.2 (i), ψ is a derivation on $N_n(\mathbb{F}_2)$. It follows from Theorem 3.2 that there exists an additive map $\Psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ with $N_n(\mathbb{F}_2) \subseteq \ker \Psi$, a derivation $d : \mathbb{F}_2 \to \mathbb{F}_2$, a matrix $X \in T_n(\mathbb{F}_2)$, scalars $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{F}_2$, and an additive map $g : T_n(\mathbb{F}_2) \to \mathbb{F}_2$, with $E_{ij} \in \ker g$ for all $1 \leqslant i < j - 1 \leqslant n - 1$, such that

$$\psi = \Psi + \mathrm{ad}_X + \overline{d} + \eta_{X_\alpha} + \mu_g, \tag{4.45}$$

where $X_\alpha = E_{1n} + \sum_{i=1}^{n-1} \alpha_i E_{i+1,i}$ with $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$. We note that Ψ is linear, and it is a derivation on $N_n(\mathbb{F}_2)$ by Lemma 2.2. Then there exist $Y_1, \ldots, Y_n \in T_n(\mathbb{F}_2)$ such that

$$\Psi(A) = \sum_{i=1}^{n} a_{ii} Y_i,$$

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for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$. In view of 4.45, together with $\psi(I_n) = \overline{d}(I_n) = \mathrm{ad}_X(I_n) = \eta_{X_\alpha}(I_n) = 0$, we deduce that $\Psi(I_n) + g(I_n)E_{1n} = 0$. We set $X_i = Y_i + g(E_{ii})E_{1n}$ for $i = 1, \ldots, n$. Let $\Phi_{\mathbf{X}} : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ be the derivation as defined in (1.7) with $\mathbf{X} = \{X_1, \ldots, X_n\}$ and $X_1 + \cdots + X_n = 0$. Let $\mu_f : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ be the derivation as defined in (1.4) with $f(A) = g(A) - \sum_{i=1}^n a_{ii} g(E_{ii})$ and $I_n, E_{ij} \in \ker f$ for all $1 \leqslant i < j - 1 \leqslant n - 1$. Since g is linear, we deduce that $\Phi_{\mathbf{X}} + \mu_f = \Psi + \mu_g$. Since $d : \mathbb{F} \to \mathbb{F}$ is a linear derivation and $d(1_{\mathbb{F}}) = 0$, we deduce that $\overline{d} = 0$. Hence, we can write (4.45) as

$$\psi = \Phi_{\mathbf{X}} + \mathrm{ad}_X + \eta_{X_\alpha} + \mu_f.$$

This completes our proof. ◻

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