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THE NUMERICAL RANGE OF LINEAR OPERATORS ON THE 2-DIMENTIONAL KREIN SPACE*

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Abstract. The aim of this note is to provide the complete characterization of the numerical range of linear operators on the 2-dimensional Krein space $\mathbb{C}^2$.

Key words. Krein space, Numerical range, Rank one operator.

AMS subject classifications. 46C20, 47A12.

1. Introduction. The concept of numerical range of linear operators on a Hilbert space was introduced by Toeplitz [16] and has been generalized in several directions. The theory of numerical ranges of linear operators on a Krein space has also been considered by some authors (see [2, 3, 10, 11, 13, 14, 15] and the references therein). There are many motivations for the study of the numerical range of linear operators on Hilbert spaces or Krein spaces. We enumerate some of them: the localization of the spectrum of an operator, related inequalities, control theory and applications to physics (cf. [8]). Recently, an application of a generalized numerical range to NMR spectroscopy has been discussed (cf. [7, 12]). The aim of this paper is the complete determination of the numerical range of linear operators on the 2-dimensional Krein space $\mathbb{C}^2$. By addition of scalar operators, the study of the numerical range of operators on $\mathbb{C}^2$ is reduced to that of rank one operators. The numerical range of such rank one operators has been already investigated in [14] for non-neutral vectors.

Let C, A be (non-zero) rank one operators on the 2-dimensional Krein space $\mathbb{C}^2$, endowed with the indefinite inner product space $[\cdot,\cdot]$ defined by $[\xi,\nu] = (J\xi,\nu) = \nu^* J\xi$ for $J = I_1 \oplus (-I_1)$. We refer [1, 4, 6] for general reference on Krein spaces or Krein spaces operators. For the rank one operators C, A, there exist non-zero vectors

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$\eta, \zeta, \kappa, \tau$ such that

$$C\xi = [\xi, \eta]\zeta, \ A\xi = [\xi, \kappa]\tau, \ \xi \in \mathbb{C}^2. \tag{1.1}$$

Denote by $SU(1,1)$ the group of 2×2 complex matrices U with determinant 1 such that $U^*JU = J$. We consider the indefinite *C-numerical range of A* denoted and defined by

$$W_C^J(A) = \{[U\zeta, \kappa]\overline{[U\eta, \tau]} : U \in SU(1,1)\}, \tag{1.2}$$

which has been characterized in [14] for $\eta, \zeta, \kappa, \tau$, *non-neutral vectors*, that is, $[\eta, \eta]$, $[\zeta, \zeta]$, $[\kappa, \kappa]$, $[\tau, \tau]$ do not vanish. An analogous object for a 2-dimensional Hilbert space is the *C-numerical range* of A defined as

$$W_C(A) = \{\kappa^* U\zeta \ \overline{\tau^* U\eta} : U \in SU(2)\},$$

for rank one operators C, A. The range $W_C(A)$ is a (possible degenerate) closed elliptical disc (cf. [9]). This paper treats the analogous object for Krein spaces.

The main aim of this note is to complete the characterization of $W_C^J(A)$ considering the case of the vectors $\eta, \zeta, \kappa, \tau$ being neutral. For the range $W_C(A)$ of any dimensional matrices A, C , the numerical method to draw the boundary is given in [7] based on a result in [5]. A numerical algorithm is not known for Krein spaces numerical ranges except for some special cases. We give a complete characterization for operators on 2-dimensional spaces.

The following classification takes place, being the different cases treated in the next five sections.

First Case: All the vectors η, ζ, κ, τ are neutral.
Second Case: One of the vectors η, ζ, κ, τ is non-neutral and the other three are neutral.
Third Case: The vectors κ, τ are neutral and η, ζ are non-neutral.
Fourth Case: The vectors κ, ζ are neutral and η, τ are non-neutral.
Fifth Case: The vectors κ, η are neutral and ζ, τ are non-neutral.
Sixth Case: One of the vectors η, ζ, κ, τ is neutral and the other three are non-neutral.

Using the concrete description of $W_C^J(A)$ given in Sections 2–6 and in [14], we prove the following results in Section 7.

THEOREM 1.1. *Let C, A be arbitrary linear operators on a 2-dimensional Krein space, and $J = I_1 \oplus -I_1$. If the boundary of $W_C^J(A)$ is non-empty, then it is a singleton or it lies on a possibly degenerate conic.*

THEOREM 1.2. *Let C, A be arbitrary linear operators on the Krein space $\mathbb{C}^2$, and $J = I_1 \oplus -I_1$. Then the fundamental group $\pi_1(W_C^J(A))$ of $W_C^J(A)$ is a trivial group,*

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or an abelian group isomorphic to the additive group of the integers, $\mathbb{Z}$*. The number of the connected components of the complement* $\mathbb{C}\backslash W_C^J(A)$ *is* 1 *or* 2*.*

2. The first case. In the sequel, we identify the complex plane with $\mathbb{R}^2$ and we denote by E_{ij} the 2×2 matrix with the (i,j)th entry equal to 1 and all the others 0. For $V, W \in U(1,1)$, we have

$$(2.1)\quad VAV^{-1}\xi = [V^{-1}\xi, \kappa]V\tau = [\xi, V\kappa]V\tau,\ WCW^{-1}\xi = [W^{-1}\xi, \eta]\zeta = [\xi, W\eta]W\zeta,$$

and so we may assume that $\zeta = (1,1)^T$, $\tau = (1,-1)^T$. We also may consider that $\kappa = (\overline{k_1}, -\overline{k_2})$, $\eta = (\overline{q_1}, \overline{q_2})$, with $|k_1| = |k_2| = |q_1| = |q_2| = 1$. Under these assumptions, we find

$$W_C^J(A) = \{(k_1\alpha + k_1\overline{\beta} + k_2\overline{\alpha} + k_2\beta)(q_1\overline{\alpha} + q_2\beta + q_1\overline{\beta} + q_2\alpha) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}.$$

Writing $k_1 = \exp(i\,s)\exp(i\theta)$, $k_2 = \exp(i\,s)\exp(-i\theta)$, $q_1 = \exp(i\,t)\exp(i\phi)$, $q_2 = \exp(i\,t)\exp(-i\,\phi)$, $s, t, \theta, \phi \in \mathbb{R}$, we obtain

$$(2.2)\quad \begin{aligned} W_C^J(A) = \{&4\exp(i\,s)\,\Re(\exp(i\theta)(\alpha + \overline{\beta}))\exp(i\,t)\Re(\exp(-i\phi)(\alpha+\beta)) : \\ &\alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}. \end{aligned}$$

Thus, $W_C^J(A)$ is contained in a straight line passing through the origin. We may assume that $\kappa = (\exp(-i\theta), -\exp(i\theta))^T$, $\eta = (\exp(-i\phi), \exp(i\phi))^T$, $\zeta = (1,1)^T$ and $\tau = (1,-1)^T$. Under these assumptions, we prove the following.

Proposition 2.1. *Let* $C = \exp(i\phi)(E_{11} + E_{21}) - \exp(-i\phi)(E_{12} + E_{22})$*,* $A = \exp(i\theta)(E_{11} - E_{21}) + \exp(-i\theta)(E_{12} - E_{22})$*,* $0 \le \theta, \phi \le 2\pi$*. If either* θ *or* ϕ *is not congruent to* 0 *modulo* π*, then* $W_C^J(A)$ *is the real line. Otherwise,* $W_C^J(A) = [0, +\infty)$ *for* $\theta = \phi = 0$ *or* $\theta = \phi = \pi$*, and* $W_C^J(A) = (-\infty, 0]$ *for* $\theta = 0$*,* $\phi = \pi$ *or* $\theta = \pi$*,* $\phi = 0$*.*

Proof. Having in mind (2.2), we easily find for $0 \le \theta$, $\phi \le 2\pi$,

$$\begin{aligned} &W_C^J(A) = \{4\Re(\exp(i\theta)(\alpha + \overline{\beta}))\,\Re(\exp(-i\phi)(\alpha+\beta))\} = \{4(\cosh t\cos(u+\theta) \\ &+ \sinh t\cos(v-\theta))(\cosh t\cos(u-\phi) + \sinh t\cos(v-\phi)) : t \ge 0, 0 \le u, v \le 2\pi\}. \end{aligned}$$

For fixed $0 \le u, v, \theta, \phi \le 2\pi$, we study the behavior of the real valued function

$$\psi(t) = 4(\cosh t\cos(u+\theta) + \sinh t\cos(v-\theta))(\cosh t\cos(u-\phi) + \sinh t\cos(v-\phi))$$

as $t \to +\infty$. For this purpose, we consider the sign of the function

$$\begin{aligned} f(u,v) &= (\cos(u+\theta) + \cos(v-\theta))(\cos(u-\phi) + \cos(v-\phi)) \\ &= 4\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u+v}{2} - \phi\right)\cos\left(\frac{u-v}{2} + \theta\right)\cos\left(\frac{u-v}{2}\right), \quad u, v \in \mathbb{R}. \end{aligned}$$

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Hence, it is sufficient to determine the sign of the function $\tilde{f}(x,y) = 4\cos x\,\cos(x-\phi)\,\cos y\,\cos(y+\theta)$ for the variables ranging over the reals. We observe that function $\cos x\,\cos(x-\phi)$ takes positive and negative values except for $\phi \equiv 0$ modulo π, while $\cos y\,\cos(y+\theta)$ takes positive and negative values except for $\theta \equiv 0$ modulo π. Therefore, if either ϕ or θ is not congruent to 0 modulo π, then $f(u,v)$ takes positive and negative values, and so $W_C^J(A)$ coincides with the real line.

To finish the proof we consider the following cases: (1) $\theta = \phi = 0$; (2) $\theta = 0$ and $\phi = \pi$; (3) $\theta = \pi$ and $\phi = 0$; (4) $\theta = \phi = \pi$. We concentrate on the case (1). Considering $t = 0$, $u = \pi/2$, we produce the origin. Taking $u = v = 0$, we get $[0, +\infty) \subset W_C^J(A)$ and the reverse inclusion is clear.

The treatment of the remaining cases is similar and the proposition follows. $\square$

3. The fourth and fifth cases. In the fourth case, we may consider $\zeta = (1,1)^T$, $\kappa = (1,-1)$, $\tau = (1,0)^T$, and $\eta = (1,0)^T$ or $\eta = (0,1)^T$. In the fifth case, we may take $\eta = (1,1)^T$, $\kappa = (1,1)^T$, $\zeta = (1,0)^T$, and $\tau = (1,0)^T$ or $\tau = (0,1)^T$.

Proposition 3.1. *If $C = E_{11} + E_{21}$ and $A = E_{11} + E_{12}$, then $W_C^J(A) = \mathbb{C}\backslash(-\infty, 1] \cup \{0\}$.*

Proof. Some computations yield

$$\begin{aligned} W_C^J(A) &= \{(\alpha + \overline{\alpha} + \beta + \overline{\beta})\overline{\alpha} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\} \\ &= \{2(\cosh t \cos\theta + \sinh t \cos\phi)\cosh t \exp(i\theta) : t \geq 0, 0 \leq \theta, \phi \leq 2\pi\}. \end{aligned} \tag{3.1}$$

By choosing in (3.1) $\theta = \pi/2$ and $\phi = 0$ or $\phi = \pi$, we conclude that the imaginary axis is contained in $W_C^J(A)$. It can be easily seen that a non-zero real number λ belongs to $W_C^J(A)$ if and only if it is of the form

$$\lambda(t,\phi) = 2(\cosh t + \sinh t \cos\phi)\cosh t, \;\; t \geq 0, \;\; 0 \leq \phi \leq 2\pi.$$

For a fixed t, λ attains its minimum value when $\cos\phi = -1$ and this minimum equals $1 + \exp(-2t)$. Letting t vary on its domain, we find $\lambda > 1$. Thus,

$$W_C^J(A) \cap \mathbb{R} = \{0\} \cup (1, +\infty). \tag{3.2}$$

Considering $0 < \theta < \pi$ and $\phi = \pi$ in (3.1), we find

$$\begin{aligned} \{2(\cosh t\cos\theta - \sinh t)\cosh t \exp(i\theta) : t \geq 0\} &= \{\lambda \exp(i\theta) : \lambda \leq 2\cos\theta\} \\ &\subset W_C^J(A), \end{aligned} \tag{3.3}$$

while for $\phi = \pi/2$ we get the inclusion

$$\{2\cosh t \cos\theta\; \cosh t \exp(i\theta) : t \geq 0\} = \{\lambda \exp(i\theta) : \lambda \geq 2\cos\theta\} \subset W_C^J(A). \tag{3.4}$$

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Therefore the proposition follows from (3.2), (3.3), (3.4). □

Proposition 3.2. *If $C = -E_{12} - E_{22}$ and $A = E_{11} + E_{12}$, then $W_C^J(A) = \mathbb{C}\backslash(-\infty, -1]$.*

Proof. We have

$$\begin{aligned} W_C^J(A) &= \{(\alpha + \overline{\alpha} + \beta + \overline{\beta})\beta : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\} \\ &= \{2(\cosh t \cos\phi + \sinh t \cos\theta) \sinh t \exp(i\theta) : t \geq 0, 0 \leq \theta, \phi \leq 2\pi\}. \end{aligned}$$

Choosing $\phi = \pi/2$ and letting $\theta \neq 0$ vary in $[-\pi/2, \pi/2]$, we conclude that

$$\{2\sinh t \cos\theta \sinh t \exp(i\theta) : t \geq 0\} = \{\lambda \exp(i\theta) : 0 \leq \lambda\} \subset W_C^J(A), \tag{3.5}$$

and considering $\phi = \pi$, we find

$$\{2(-\cosh t + \sinh t \cos\theta) \sinh t \exp(i\theta) : t \geq 0\} = \{\lambda \exp(i\theta) : \lambda \leq 0\} \subset W_C^J(A). \tag{3.6}$$

It follows from (3.5), (3.6) that every complex number with nonvanishing imaginary part belongs to $W_C^J(A)$.

Taking $\theta = \phi = 0$ and $t \to +\infty$, we conclude that $[0, +\infty) \subset W_C^J(A)$, because $2\sinh^2 t \in W_C^J(A)$. For a fixed ϕ different from 0 or π, we consider the real valued function of real variable $f(t) = 2(\cosh t \cos\phi + \sinh t)\sinh t$. Its derivative $f'(t) = (\exp(2t) + \exp(-2t))\cos\phi + \exp(2t) - \exp(-2t)$ vanishes at $\exp(2t) = \sqrt{(1 - \cos\phi)/(1 + \cos\phi)}$, and the function attains here the minimum value $-1 + \sqrt{1 - \cos^2\phi}$. Thus, the proposition follows. □

Proposition 3.3. *If $A = C = E_{11} - E_{12}$, then $W_C^J(A) = \mathbb{C}\backslash(-\infty, 0]$.*

Proof. We easily find

$$\begin{aligned} W_C^J(A) &= \{(\alpha - \beta)(\overline{\alpha} + \beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\} \\ &= \{(\cosh t + \sinh t \exp(i\theta))(\cosh t + \sinh t \exp(i\phi)) : 0 \leq t, \ \ 0 \leq \theta, \phi \leq 2\pi\}. \end{aligned}$$

Setting

$$G = \{(\cosh t + \sinh t \exp(i\theta)) : 0 \leq t, \ \ 0 \leq \theta \leq 2\pi\},$$

we clearly have

$$\{z^2 : z \in G\} \subset W_C^J(A) = \{z_1 z_2 : z_1, z_2 \in G\}.$$

We show that G is the family of circles $(x - r)^2 + y^2 = r^2 - 1$, $r \geq 1$. In fact, if $(x, y) \in G$, then $2rx = x^2 + y^2 + 1$ and so $x > 0$. Conversely, if $x > 0$ and $y \in \mathbb{R}$, then the real number $r = \frac{1}{2}\left(x + \frac{1}{x}\right) + \frac{y^2}{2x}$ satisfies $(x - r)^2 + y^2 = r^2 - 1$. Thus, $G = \{z \in \mathbb{C} : \Re(z) > 0\}$. □

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PROPOSITION 3.4. *If* $C = E_{11} - E_{12}$ *and* $A = E_{21} - E_{22}$*, then* $W_C^J(A) = \mathbb{C}\backslash(-\infty, 0] \cup \{-1\}$.

Proof. We obtain

$$W_C^J(A) = \{(\sinh t + \cosh t \exp(i\theta))(\sinh t + \cosh t \exp(i\phi)) : t \in \mathbb{R}, 0 \leq \theta, \phi \leq 2\pi\}.$$

By similar arguments to those used in Proposition 3.3, it can be easily seen that $\sinh t + \cosh t \exp(i\theta)$, $t \in \mathbb{R}$, $0 \leq \theta \leq 2\pi$, ranges over the complementary of the imaginary axis, taken in the complex plane, with i and $-i$ deleted. ◻

4. The second case. In this case, we may assume that $\tau = (1,0)^T$, $\zeta = (1,1)^T$, $\kappa = (\overline{k_1}, -\overline{k_2})^T$, $\eta = (1, \overline{q})^T$, $k_1, k_2, q \in \mathbb{C}$, $|k_1| = |k_2| = |q| = 1$, and we have

$$W_C^J(A) = \{(k_1\alpha + k_2\overline{\alpha} + k_1\overline{\beta} + k_2\beta)(\overline{\alpha} + q\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}.$$

Writing $k_1 = \exp(i\, s)\exp(i\theta)$, $k_2 = \exp(i\, s)\exp(-i\theta)$, $s, \theta \in \mathbb{R}$, it follows that

$$\begin{aligned} W_C^J(A) &= \{2\exp(i\, s)\Re(\exp(-i\theta)(\overline{\alpha} + \beta))(\overline{\alpha} + q\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\} \\ &= \{2k_1\, \Re(\alpha + \beta)\,(\alpha + q\,\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}, \end{aligned} \tag{4.1}$$

and so we may assume that $k_1 = 1$.

PROPOSITION 4.1. *Let* $C = (E_{11} + E_{21}) - \overline{q}(E_{21} + E_{22})$, $|q| = 1$*, and let* $A = E_{11} + E_{12}$*. The following hold:*

1) If $q = 1$*, then* $W_C^J(A) = \{0\} \cup \{z \in \mathbb{C} : \Re(z) > 0\}$.

2) If $q \neq 1$*, then* $W_C^J(A) = \mathbb{C}\backslash\{\lambda(1-q) : \lambda \in \mathbb{R}\} \cup \{1-q\}$.

Proof. 1)It may be easily seen that the origin belongs to $W_C^J(A)$. We have

$$\Re(2\Re(\alpha + \beta)\,(\alpha + \beta)) = 2\Re(\alpha + \beta)^2 \geq 0$$

with occurrence of equality if and only if $\Re(\alpha + \beta) = 0$. Thus,

$$W_C^J(A) \subset \{0\} \cup \{z \in \mathbb{C} : \Re(z) > 0\}.$$

We prove that the reverse inclusion holds. For every $z \in \mathbb{C}$ with $\theta = \operatorname{Arg}(z) \in (-\pi/2, \pi/2)$, we may find $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 - |\beta|^2 = 1$ and $z = 2\Re(\alpha+\beta)\,(\alpha+\beta)$. In fact, let $\alpha = \cosh t \exp(i\theta)$ and $\beta = \sinh t \exp(i\theta)$, $t \in \mathbb{R}$, so that $2\exp(2t)\cos\theta = |z|$. Since $\cos\theta > 0$, we have $2\exp(2t)\exp(i\theta)\cos\theta = z$.

2) Let $q = q_1 + i\, q_2 \neq 1$. From (4.1) we easily get

$$W_C^J(A) = \{2\Re(z)\, w : z, w \in \mathbb{C}, 2\Re(z\overline{w}) - 2\Re(q\, z\, \overline{w}) = |1-q|^2\},$$

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that is, the elements of $W_C^J(A)$ are the complex numbers of the form $2x_0\,(x + i\,y)$ such that

$$(4.2)\quad (q_2\,x + (1 - q_1)\,y)\,y_0 = (1/2)|1 - q|^2 - (1 - q_1)\,x_0\,x + q_2\,x_0\,y,\ x_0, y_0, x, y \in \mathbb{R}.$$

If $q_2\,x + (1 - q_1)\,y \neq 0$, then y_0 may always be found such that (4.2) is satisfied, and thus

$$S = \{(x, y) \in \mathbb{R}^2 : q_2\,x + (1 - q_1)\,y \neq 0\} \subset W_C^J(A).$$

Moreover, $\Re(-i\,(x+iy)\overline{1 - q}) = q_2\,x+(1-q_1)\,y = 0$ if and only if $z = x+i\,y = \lambda\,(1-q)$, $\lambda \in \mathbb{R}$. Clearly,

$$S = \mathbb{C}\backslash\{\lambda(1 - q) : \lambda \in \mathbb{R}\}.$$

Some calculations show that there exist complex numbers α, β satisfying $\lambda(1 - q) = 2\Re(\alpha + \beta)\,(\alpha + q\,\beta)$ and $|\alpha|^2 - |\beta|^2 = 1$ if and only if $\lambda = 1$. Finally, it can easily be seen that $1 - q$ belongs to $W_C^J(A)$. $\square$

5. The sixth case. In this case, we may assume that κ is neutral. By replacing the inner product $[\xi_1, \xi_2]$ by $-[\xi_1, \xi_2]$, we may consider (1): $\zeta = \tau = (1, 0)^T$ or (2): $\zeta = (0, 1)^T$, $\tau = (1, 0)^T$. In either case, we may suppose that $\kappa = (1, 1)^T$, by replacing A by $\mathrm{diag}(1, \exp(i\theta))\,A\,\mathrm{diag}(1, \exp(-i\theta))$ for some $\theta \in \mathbb{R}$. By performing a transformation of this type, the components of the vector $\eta = (\eta_1, \eta_2)$ may be assumed to be real and such that $|\eta_1| \neq |\eta_2|$. Thus, one of the following situations occurs:

$1st$ *Subcase.* $A\xi = [\xi, (1, 1)^T]\,(1, 0)^T$, $C\xi = [\xi, (1, q)^T]\,(1, 0)^T$, $\quad -1 < q < 1$.
$2nd$ *Subcase.* $A\xi = [\xi, (1, 1)^T]\,(1, 0)^T$, $C\xi = [\xi, (q, 1)^T]\,(1, 0)^T$, $\quad -1 < q < 1$.
$3rd$ *Subcase.* $A\xi = [\xi, (1, 1)^T]\,(1, 0)^T$, $C\xi = [\xi, (1, q)^T]\,(0, 1)^T$, $\quad -1 < q < 1$.
$4rd$ *Subcase.* $A\xi = [\xi, (1, 1)^T]\,(1, 0)^T$, $C\xi = [\xi, (q, 1)^T]\,(0, 1)^T$, $\quad -1 < q < 1$.

Firstly, we treat the $1st$ *Subcase.*

PROPOSITION 5.1. *If $C = E_{11} - qE_{12}$ and $A = E_{11} - E_{12}$, then*

$$W_C^J(A) = \{(x, y) \in \mathbb{R}^2 : x > 1/2 - |q|/2\,\cosh t,\ y = \sqrt{1 - q^2}/2\,\sinh t,\ t \in \mathbb{R}\}.$$

Proof. By some computations we get

$$\begin{aligned} W_C^J(A) &= \{(\alpha - \beta)(\overline{\alpha} + q\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\} \\ &= \{(\cosh t + \sinh t \exp(i\theta))(\cosh t + q \sinh t \exp(i\phi)) : t \geq 0, 0 \leq \theta, \phi \leq 2\pi\}. \end{aligned}$$

The case $q = 0$ is easily treated, so we may assume that $0 < q < 1$. The above set is the union of the family of circles centered at $z = z(t, \theta) = x_0 + i\,y_0 = \cosh^2 t +$

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$\sinh t\, \cosh t\, \exp(i\theta)$ whose radii $r = r(t, \theta)$ satisfy $R = r^2 = q^2 \sinh^2 t\,(1 + 2\sinh^2 t + 2\sinh t \cosh t \cos\theta)$, and $R = q^2\,((x_0 - 1)^2 + y_0^2)$. It may be shown that the centers of these circles describe the open half-plane $x > 1/2$. Thus,

$$W_C^J(A) = \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 = q^2((x_0 - 1)^2 + y_0^2), x_0 > 1/2,\ y_0 \in \mathbb{R}\ \}.$$

The boundary of $W_C^J(A)$ is the envelope of the family of circles when their centers run over $x = 1/2$ (cf. Proposition 2.3 in [14]) and the result follows. □

Next, the 2*nd Subcase* is studied.

PROPOSITION 5.2. *If $C = qE_{11} - E_{12}$, $-1 < q < 1$, $A = E_{11} - E_{12}$, then $W_C^J(A) = \mathbb{C}$.*

Proof. Some computations yield

$$\begin{aligned} W_C^J(A) &= \{(\alpha - \beta)(q\overline{\alpha} + \beta) : \alpha, \beta \in \mathbb{C},\ \ |\alpha|^2 - |\beta|^2 = 1\} \\ &= \{(\cosh t + \sinh t \exp(i\theta))(q \cosh t + \sinh t \exp(i\phi)) : t \geq 0, 0 \leq \theta, \phi \leq 2\pi\}, \end{aligned}$$

where we may assume $q > 0$. Therefore, $W_C^J(A)$ is the union of the family of circles centered at $z(t, \theta) = x_0 + i\, y_0 = q\,(\cosh t + \sinh t \exp(i\theta)) \cosh t$, whose radii $r = r(t, \theta)$ satisfy $R = r^2 = \sinh^2 t\,(1 + 2\sinh^2 t + 2\sinh t \cosh t\, \cos\theta)$, and the following relation holds

$$R = \frac{1}{q^2}((x_0 - q)^2 + y_0^2).$$

The centers of the circles describe the open half-plane $x > q/2$. Thus,

$$W_C^J(A) = \left\{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 = \frac{1}{q^2}((x_0 - q)^2 + y_0^2) x_0 > q/2,\ y_0 \in \mathbb{R}\right\}.$$

Clearly, we have

$$(x - x_0)^2 + (y - y_0)^2 = \frac{1}{q^2}((x_0 - q)^2 + y_0^2).$$

If $(x - x_0)^2 = \frac{1}{q^2}(x_0 - q)^2$ and $(y - y_0)^2 = \frac{1}{q^2} y_0^2$, then (x, y) belongs to $W_C^J(A)$. It is always possible to find a real y_0 such that the first equation holds, while the second equation is satisfied if $x_0 = \frac{q}{1 \pm q}(1 \pm x)$. Since $x_0 > q/2$, this equation has a real solution x_0 for $x > (q - 1)/2$ and also for $x < (1 + q)/2$. □

The 3*rd Subcase* is investigated in the following.

PROPOSITION 5.3. *If $C = E_{12} - qE_{22}$, $-1 < q < 1$, and $A = E_{11} - E_{12}$, then*

$$W_C^J(A) = \left\{(x, y) \in \mathbb{R}^2 : (x + q/2)^2 + \frac{y^2}{(1 - q^2)} > \frac{1}{4}\right\}.$$

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Proof. We easily find

$$\begin{aligned} W_C^J(A) &= \{(\overline{\beta}-\overline{\alpha})(q\overline{\alpha}+\beta) : \alpha,\beta\in\mathbb{C}, |\alpha|^2-|\beta|^2=1\} \\ &= \{(\sinh t+\cosh t\exp(i\theta))(q\,\sinh t+\cosh t\exp(i\phi)) : t\geq 0, 0\leq\theta,\phi\leq 2\pi\}, \end{aligned}$$

and we may consider $q \geq 0$. Since $W_C^J(A)$ is invariant under rotations around the origin in the case $q=0$, we may concentrate on the case $q>0$. Thus, $W_C^J(A)$ is the union of the family of circles centered at $z = x_0 + i\,y_0 = q\,\sinh t\,(\sinh t + \cosh t\exp(i\,\theta))$ and whose radii $r = r(t,\theta)$ satisfy $R = r^2 = \cosh^2 t\,(1+2\sinh^2 t + 2\sinh t\,\cosh t\cos\theta)$. The following relation holds

$$R = \frac{1}{q^2}((x+q)^2+y^2).$$

The centers $z = z(t,\theta)$ of the circles describe the half-plane $x > -q/2$. Thus,

$$\begin{aligned} W_C^J(A) = \{(x,y)\in\mathbb{R}^2 : (x-x_0)^2+(y-y_0)^2 = 1/q^2\,((x_0+q)^2+y_0^2), \\ x_0 > -q/2,\ y_0\in\mathbb{R}\}. \end{aligned}$$

For a fixed complex number $x+i\,y$, we consider the Apolonius circle

$$\{(X,Y)\in\mathbb{R}^2 : (X+q)^2+Y^2 = q^2((X-x)^2+(Y-y)^2)$$

with center $(-\frac{q(1+q\,x)}{1-q^2}, -\frac{q^2\,y}{1-q^2})$, and radius $\frac{q}{1-q^2}\sqrt{(x+q)^2+y^2}$. Hence,

$$\begin{aligned} M(x,y) &= \max\{x_0 : |(x_0+i\,y_0)+\frac{q(1+q\,x)}{1-q^2}+i\,\frac{q^2\,y}{1-q^2}| = \frac{q}{1-q^2}\sqrt{(x+q)^2+y^2}\} \\ &= -\frac{q(1+q\,x)}{1-q^2}+\frac{q\sqrt{(x+q)^2+y^2}}{1-q^2}. \end{aligned}$$

The point $x+i\,y$ belongs to $W_C^J(A)$ if and only if $M(x,y) > -q/2$. Thus, we conclude

$$W_C^J(A) = \{(x,y)\in\mathbb{R}^2 : 4((x+q)^2+y^2)-(1+q^2+2q\,x)^2 > 0\}. \quad \square$$

The $4th$ *Subcase* is analysed in the following.

PROPOSITION 5.4. *If* $C = qE_{21}-E_{22}$, $-1<q<1$, *and* $A = E_{11}-E_{12}$, *then*

$$W_C^J(A) = \{(x,y)\in\mathbb{R}^2 : y = \sqrt{1-q^2}/2\sinh t,\ x > -1/2-|q|/2\cosh t,\ \ t\in\mathbb{R}\}.$$

Proof. We find

$$\begin{aligned} W_C^J(A) &= \{(\overline{\beta}-\overline{\alpha})(q\overline{\alpha}+\beta) : \alpha,\beta\in\mathbb{C}, |\alpha|^2-|\beta|^2=1\} \\ &= \{(\sinh t+\cosh t\exp(i\theta))(\sinh t+q\cosh t\exp(i\phi)) : t\geq 0, 0\leq\theta,\phi\leq 2\pi\}. \end{aligned}$$

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The case $q = 0$ is trivial, so we assume $0 < q < 1$. Under this assumption, $W_C^J(A)$ is the union of the family of circles centered at $z = z(t,\theta) = x + i\,y = \sinh t\,(\sinh t + \cosh t \exp(i\,\theta))$, whose radii $r = r(t,\theta)$ satisfy $R = r^2 = q^2 \cosh^2(1 + 2\sinh^2 t + 2\sinh t\,\cosh t\,\cos\theta)$, and also $R = q^2\,((x+1)^2 + y^2)$. The centers of the circles range over the half-plane $x > -1/2$ and the proposition easily follows from Proposition 5.1. ∎

6. The third case. In this case, we are assuming that κ, τ are neutral and η, ζ are non- neutral. We may consider $\zeta = (1,0)^T$, $\tau = (1,-1)^T$, $\kappa = (1,-k)$, $k \in \mathbb{C}$ with $|k| = 1$ and $\eta = (\eta_1, \eta_2)$, $\eta_1, \eta_2 \in \mathbb{R}$, $|\eta_1| \neq |\eta_2|$. Under these assumptions, we obtain

$$W_C^J(A) = \{(\overline{\alpha} + \overline{k\beta})(\eta_1(\alpha+\beta) + \eta_2(\overline{\alpha} + \overline{\beta})) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}.$$

Firstly, we consider the three special Subcases: (1) $\eta_2 = 0$, $\eta_1 = 1$; (2) $\eta_1 = 0$, $\eta_2 = 1$; (3) $k = 1$. Finally, in Proposition 6.4 we treat the case $k \neq 1$ and $\eta_1 \neq \eta_2 \neq 0$.

If the Subcase (1) occurs, then

$$W_C^J(A) = \{(\alpha+\beta)\overline{(\alpha + k\,\beta)} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}.$$

If $k = 1$, then $W_C^J(A)$ is the positive real line. Let $k \neq 1$ and let k_1 be a complex number such that $k_1^2 = k$. Thus, $|k_1| = 1$, $k_1 \neq 1$, $k_1 \neq -1$ and

$$\begin{aligned} W_C^J(A) &= \overline{k_1}\{(\alpha+\beta)(k_1\overline{\alpha} + \overline{k_1\beta}) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\} \\ &= \overline{k_1}\{k_1 + (k_1 + \overline{k_1})|\beta|^2 t + 2\Re(k_1\overline{\alpha}\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\}. \end{aligned}$$

Multiplying $W_C^J(A)$ by a complex number k_1, and performing some calculations, the next proposition follows.

Proposition 6.1. *Let $C = E_{11}, A = k_1(E_{11} - E_{21}) + \overline{k_1}(E_{21} - E_{22})$, $k_1 = \exp(i\phi)$, $-\pi < \phi < \pi$, $\phi \neq 0$. Then $W_C^J(A) = \{i\,\sin\phi + t : t \in \mathbb{R}\}$.*

In the Subcase (2), we replace $(1,-k)^T$ by $(1,-\overline{k})^T$ and we prove.

Proposition 6.2. *Let $C = -E_{12}$, $A = (E_{11} - E_{21}) + k(E_{12} - E_{22})$, $k = \exp(i\phi)$, $-\pi < \phi \leq \pi$. Then $W_C^J(A) = \{z \in \mathbb{C} : |z| \geq \sin(|\phi|/2)\}$ for $\phi \neq 0$, and $W_C^J(A) = \mathbb{C}\backslash\{0\}$ for $\phi = 0$.*

Proof. We get

$$W_C^J(A) = \{(\alpha + k\beta)(\alpha+\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\},$$

being this set invariant under the rotation $z \mapsto \exp(i\theta)\,z$, $\theta \in \mathbb{R}$. Therefore, to conclude the proof it is sufficient to show that the function defined on $[0,+\infty) \times [0, 2\pi]$ by $f(t,\theta) = |\cosh t + \exp(i\,\phi)\,\exp(i\,\theta)\sinh t|^2\,|\cosh t + \exp(i\,\theta)\sinh t|^2$ ranges over

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$[\sin^2(|\phi|/2), +\infty)$ for $\phi \neq 0$, and $(0, +\infty)$ in the case $\phi = 0$. This can be done by simple standard arguments. □

Next we consider the Subcase (3).

PROPOSITION 6.3. *Let* $C = E_{11} - qE_{12}$, $q \in \mathbb{R}$, $q \neq 1$, $q \neq -1$, *and let* $A = E_{11} + E_{12} - E_{21} - E_{22}$. *The following hold:*

1) If $|q| > 1$, *then* $W_C^J(A) = \mathbb{C}\backslash\{0\}$.

2) If $-1 < q < 1$, *then* $W_C^J(A) = \{(x, y) \in \mathbb{R}^2 : x > 0,\ |y| \leq |q|\, x/\sqrt{1-q^2}\}$.

Proof. Performing some computations, we obtain

$$\begin{aligned} W_C^J(A) &= \{(\alpha+\beta)(\overline{\alpha} + \overline{\beta} + q\,\alpha + q\,\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\} \\ &= \{|z| + |q|\, z : z \in \mathbb{C}\backslash\{0\}\}. \end{aligned}$$

1) Let $|q| > 1$. The equation $|z| + |q|\, z = 0$ holds if and only if $z \neq 0$. Given a non-zero complex number $z_0 = x_0 + iy_0$ it is always possible to find a complex z satisfying $z_0 = |z| + |q|\, z$, since the real system $x_0 = \sqrt{x^2 + y^2} + |q|\, x, \quad y_0 = |q| y$ is possible.

2) Let $-1 < q < 1$. We may assume $q \neq 0$. It can be easily checked that the equations of the above system are satisfied for real numbers x_0, y_0 if and only if $x_0 > 0$ and $\frac{y_0^2}{x_0^2} < \frac{q^2}{1-q^2}$. □

PROPOSITION 6.4. *Let* $C = E_{11} - qE_{12}$, $q \in \mathbb{R}$, $q \neq 1$, $q \neq -1$, $A = k_1(E_{11} - E_{21}) + \overline{k_1}(E_{12} - E_{22})$, $k_1 = \exp(i\,\psi)$, $-\pi < \psi < \pi$, $\psi \neq 0$. *The following hold:*

1) If $|q| > 1$, *then*

$$W_C^J(A) = \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{q^2 - 1} + \frac{(y - \sin\psi)^2}{q^2} \geq \sin^2(\psi)\right\}. \tag{6.1}$$

2) If $0 < |q| < 1$, *then*

$$W_C^J(A) = \left\{(x, y) \in \mathbb{R}^2 : \frac{(y - \sin\psi)^2}{q^2} - \frac{x^2}{1 - q^2} \leq \sin^2(\psi)\right\}.$$

Proof. Some computations yield

$$\begin{aligned} W_C^J(A) &= \{(k_1\alpha + \overline{k_1}\beta)(\overline{\alpha} + \overline{\beta} + q\alpha + q\beta) : \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 = 1\} \\ &= \{[(k_1 \cosh t + \overline{k_1} \sinh t \exp(i\phi)][\cosh t + \sinh t \exp(-i\phi)] + q\,[k_1 \cosh t \\ &\quad + \overline{k_1} \sinh \exp(i\phi)][\cosh t + \sinh t \exp(i\phi)] \exp(i\theta) : t \geq 0, 0 \leq \phi, \theta \leq 2\pi\}. \end{aligned}$$

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Then, $W_C^J(A)$ is the union of the family of circles centered at

$$\begin{aligned} z = z(t,\phi,\psi) = x + i\,y &= [(k_1 \cosh t + \overline{k_1} \sinh t \exp(i\phi)][\cosh t + \sinh t \exp(-i\phi)] \\ &= k_1 + 2\cos(\psi)\sinh^2 t + 2\sinh t \cosh t \cos(\phi - \psi), \end{aligned}$$

where $\psi = \arg k_1$. The radii of the above family, $r = r(t,\theta)$, satisfy $R = r^2 = q^2\,[\cosh^2 t + \sinh^2 t + 2\sinh t \cosh t \cos(\phi - 2\psi)][\cosh^2 t + \sinh^2 t + 2\sinh t \cosh t \cos\phi]$. Thus, $R = q^2\,(x^2 + \sin^2(\psi))$, $\;y = \sin\psi$, and we easily get

$$W_C^J(A) = \{(x,y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - \sin\psi)^2 = q^2\,(x_0^2 + \sin^2\psi),\ x \in \mathbb{R}\}. \tag{6.2}$$

1) Let $|q| > 1$. If $x + i\,y$ is a boundary point of $W_C^J(A)$, then it satisfies

$$F(x,y;x_0) = (x - x_0)^2 + (y - \sin\psi)^2 - q^2(x_0^2 + \sin^2\psi) = 0,$$

and $F_x(x,y;x_0) = -2\,(x + (q^2-1)x_0) = 0$. By eliminating x_0 in the above equations, we get

$$\frac{x^2}{q^2 - 1} + \frac{(y - \sin\psi)^2}{q^2} = \sin^2\psi,$$

and so the boundary of $W_C^J(A)$ is contained in this ellipse. Conversely, we *claim* that every point of this ellipse is contained in the boundary of $W_C^J(A)$. Indeed, every point (x,y) of the ellipse is parametrically represented as $x = \sqrt{(q^2-1)}\sin\psi\cos\theta$, $y = \sin\psi + q\sin\psi\sin\theta$, $\theta \in \mathbb{R}$, and satisfies

$$\left(x + \frac{\sin\psi\cos\theta}{\sqrt{q^2-1}}\right)^2 + (y - \sin\psi)^2 - q^2\left(\frac{\sin^2\psi\cos^2\theta}{q^2-1} + \sin^2\psi\right) = 0.$$

Thus, (x,y) is an element of (6.2). To finish the proof, we observe that if $x_0 \to \infty$, then the circle $F(x,y;x_0) = 0$ contains points $x + i\,0$ with $x \to \infty$. Let $0 < |q| < 1$. By similar arguments to the above, we find that the boundary of $W_C^J(A)$ is the hyperbola:

$$\frac{(y - \sin\psi)^2}{q^2} - \frac{x^2}{1 - q^2} = \sin^2(\psi).$$

If $x_0 \to +\infty$, then the circle $F(x,y;x_0) = 0$ contains points x of the real line with $x \to +\infty$. $\square$

7. Proof of main theorems and a concluding remark. As a consequence of the descriptions of the shape of $W_C^J(A)$ in Sections 2–6 and [14], we obtain Theorem 1.1. We remark that $W_C^J(A)$, or each connected component of its complement $\mathbb{C}\backslash W_C^J(A)$, is convex. This property does not hold for 3-dimensional Krein spaces. From Theorem 1.2 it follows that $W_C^J(A)$ has at most 1 hole.

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