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IMPROVING THE SUPERADDITIVITY OF SOME DETERMINANTAL MATRIX MAPS*

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Abstract. In this note, a generalization by Yuan and Leng of Minkowski's determinant inequality is improved. An interpolation of the Yuan and Leng's inequality is shown by using the negativity of some related functional. Some refined versions of Minkowski's inequality and of Ky Fan's inequality are presented.

Key words. Positive definite matrix, Minkowski's determinant inequality, Fan's determinant inequality, Yuan and Leng's inequality.

AMS subject classifications. 15A15, 15A24, 15A45.

1. Introduction.

Throughout the notation, $\mathbb{P}_n$ stands for the set of all positive definite matrices of order n. The symbol I_n denotes the $n \times n$ identity matrix.

Minkowski's determinant inequality [3, Theorem 7.8.8, p. 482] asserts that

$$(\det(A+B))^{1/n} \geq (\det A)^{1/n} + (\det B)^{1/n} \quad \text{for } A, B \in \mathbb{P}_n, \tag{1.1}$$

(see [11, p. 125], [12, p. 104], [1, Cor. II.3.21, p. 47]).

A more general result due to Ky Fan [2] (see [8, p. 687], [12, p. 104]) is as follows:

$$\left(\frac{\det(A+B)}{\det(A+B)_k}\right)^{\frac{1}{n-k}} \geq \left(\frac{\det A}{\det A_k}\right)^{\frac{1}{n-k}} + \left(\frac{\det B}{\det B_k}\right)^{\frac{1}{n-k}} \quad \text{for } A, B \in \mathbb{P}_n, \tag{1.2}$$

where k is a positive integer satisfying $1 \leq k < n$, and A_k, B_k and $(A+B)_k$ are the kth leading principal submatrices of A, B and $A+B$, respectively.

Observe that inequality (1.1) can be viewed as a case of (1.2) for $k = 0$ with the convention $\det A_0 = 1$, $\det B_0 = 1$ and $\det(A+B)_0 = 1$.

Yuan and Leng [11, Theorem 1.1] (see also [12, p. 104]) proved that if $A, B \in \mathbb{P}_n$, $1 \leq k < n$, and $a \geq 0$ and $b \geq 0$ are two real numbers satisfying $A > aI_n$ and $B > bI_n$ then

$$\left(\frac{\det(A+B)}{\det(A+B)_k} - \det((a+b)I_{n-k})\right)^{\frac{1}{n-k}}$$

$$\geq \left(\frac{\det A}{\det A_k} - \det(aI_{n-k})\right)^{\frac{1}{n-k}} + \left(\frac{\det B}{\det B_k} - \det(bI_{n-k})\right)^{\frac{1}{n-k}}. \tag{1.3}$$

Self-improving is a permanent feature of sub-additivity and super-additivity inequalities (see e.g., [9, 10]).

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The aim of the present paper is to refine Yuan and Leng's inequality (1.3) by using some properties of the functional $\Delta_{n,k}$ defined by (1.4) (see Theorem 1). By employing some additional two auxiliary matrices in Theorem 2, satisfying automatically the requirements of Theorem 1, we interpolate inequality (1.3) in the manner as Hudzik and Landes [4] and Maligranda [6, 7] used for norm inequalities (see also [9, 10]). Finally, we derive corresponding refinements of the inequalities (1.1) and (1.2), due to Minkowski and Fan, respectively (see Section 3).

To do so, we define the functional

$$\Delta_{n,k}((A,a),(B,b)) := \left(\frac{\det A}{\det A_k} - \det(aI_{n-k})\right)^{\frac{1}{n-k}} + \left(\frac{\det B}{\det B_k} - \det(bI_{n-k})\right)^{\frac{1}{n-k}}$$

$$- \left(\frac{\det(A+B)}{\det(A+B)_k} - \det((a+b)I_{n-k})\right)^{\frac{1}{n-k}}, \tag{1.4}$$

for $A, B \in \mathbb{P}_n$, $a, b \geq 0$, $A > aI_n$, $B > bI_n$, $1 \leq k < n$.

On the Cartesian product $\mathbb{P}_n \times [0,\infty)$, we introduce the standard algebraic operations

$$(A,a) + (B,b) := (A+B, a+b), \tag{1.5}$$

and

$$t(A,a) := (tA, ta) \quad \text{for } t \in [0,\infty). \tag{1.6}$$

For convenience, we shall use the following notation

$$\mathcal{D}_n := \{(A,a) \in \mathbb{P}_n \times [0,\infty) : A > aI_n\},$$

$$F_{n,k}(A,a) := \left(\frac{\det A}{\det A_k} - \det(aI_{n-k})\right)^{\frac{1}{n-k}} \quad \text{for } (A,a) \in \mathcal{D}_n. \tag{1.7}$$

It follows that $\frac{\det A}{\det A_k} - \det(aI_{n-k}) > 0$ (see [11, p. 129]), so (1.7) is well-defined.

Thus statement (1.4) can be shorten as

$$\Delta_{n,k}((A,a),(B,b)) = F_{n,k}(A,a) + F_{n,k}(B,b) - F_{n,k}(A+B, a+b), \tag{1.8}$$

for $(A,a), (B,b) \in \mathcal{D}_n$.

An equivalent form of Yuan and Leng's result (1.3) is the basic inequality

$$\Delta_{n,k}((A,a),(B,b)) \leq 0 \quad \text{for } (A,a) \in \mathcal{D}_n \text{ and } (B,b) \in \mathcal{D}_n. \tag{1.9}$$

It says that the determinantal matrix map $F_{n,k}(\cdot,\cdot)$ is superadditive on its domain $\mathcal{D}_n$.

2. Refining the inequality of Yuan and Leng. The following is a complement to Yuan and Leng's inequality (1.9).

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Theorem 1. *Let $A, B \in \mathbb{P}_n$, $0 \leq a, b \in \mathbb{R}$, $A > aI_n$, $B > bI_n$, $1 \leq k < n$, $k \in \mathbb{N}$. Suppose that there exist $(C, c), (D, d) \in \mathcal{D}_n$ such that $(B, b) = (C, c) + (D, d)$ and $\Delta_{n,k}((C, c), (D, d)) = 0$.*

Then

$$\Delta_{n,k}((A, a), (B, b)) \leq \min\{\Delta_{n,k}((A, a), (C, c)), \Delta_{n,k}((D, d), (A + C, a + c))\} \leq 0. \tag{2.10}$$

Proof. It follows from (1.8) via the equality $(B, b) = (C, c) + (D, d)$ that

$$\begin{aligned} &\Delta_{n,k}((A, a), (B, b)) + \Delta_{n,k}((C, c), (D, d)) \\ &= F_{n,k}(A, a) + F_{n,k}(B, b) - F_{n,k}(A + B, a + b) \\ &+F_{n,k}(C, c) + F_{n,k}(D, d) - F_{n,k}(C + D, c + d) \\ &= F_{n,k}(A, a) + F_{n,k}(C, c) + F_{n,k}(D, d) - F_{n,k}(A + C + D, a + c + d), \end{aligned}$$

and

$$\begin{aligned} &\Delta_{n,k}((A, a), (C, c)) + \Delta_{n,k}((D, d), (A + C, a + c)) \\ &= F_{n,k}(A, a) + F_{n,k}(C, c) - F_{n,k}(A + C, a + c) \\ &+F_{n,k}(D, d) + F_{n,k}(A + C, a + c) - F_{n,k}(A + C + D, a + c + d) \\ &= F_{n,k}(A, a) + F_{n,k}(C, c) + F_{n,k}(D, d) - F_{n,k}(A + C + D, a + c + d). \end{aligned}$$

Hence,

$$\begin{aligned} &\Delta_{n,k}((A, a), (B, b)) + \Delta_{n,k}((C, c), (D, d)) \\ &= \Delta_{n,k}((A, a), (C, c)) + \Delta_{n,k}((D, d), (A + C, a + c)). \end{aligned} \tag{2.11}$$

Now, since $\Delta_{n,k}((C, c), (D, d)) = 0$, from (2.11) we get the equality

$$\begin{aligned} &\Delta_{n,k}((A, a), (B, b)) \\ &= \Delta_{n,k}((A, a), (C, c)) + \Delta_{n,k}((D, d), (A + C, a + c)). \end{aligned} \tag{2.12}$$

□

Moreover, by (1.9), we have

$$\Delta_{n,k}((A, a), (C, c)) \leq 0 \quad \text{and} \quad \Delta_{n,k}((D, d), (A + C, a + c)) \leq 0,$$

so (2.12) gives

$$\Delta_{n,k}((A, a), (B, b)) \leq \Delta_{n,k}((A, a), (C, c)) \leq 0$$

and

$$\Delta_{n,k}((A, a), (B, b)) \leq \Delta_{n,k}((D, d), (A + C, a + c)) \leq 0.$$

In consequence, inequalities (2.10) are satisfied, as claimed.

We now present and prove a matrix analog of the Maligranda norm inequality [6, 7] (see also Hudzik and Landes [4]). It is a special case of inequalities (2.10) in Theorem 1.

THEOREM 2. *Let $A, B \in \mathbb{P}_n$, $0 \le a, b \in \mathbb{R}$, $A > aI_n$, $B > bI_n$, $1 \le k < n$, $k \in \mathbb{N}$. Assume $F_{n,k}(A,a) \neq 0$ and $F_{n,k}(B,b) \neq 0$, where $F_{n,k}(\cdot,\cdot)$ is given by (1.7).*

Then

$$\begin{aligned} F_{n,k}(A,a) + F_{n,k}(B,b) &\le F_{n,k}(A,a) + F_{n,k}(B,b) \\ &- \left[2 - F_{n,k}\left(\frac{(A,a)}{F_{n,k}(A,a)} + \frac{(B,b)}{F_{n,k}(B,b)}\right)\right] \min\{F_{n,k}(A,a), F_{n,k}(B,b)\} \\ &\le F_{n,k}(A+B, a+b). \end{aligned} \tag{2.13}$$

Proof. The function $F_{n,k}(\cdot,\cdot)$ is positively homogeneous. That is, for any $E \in \mathbb{P}_n$, $e \ge 0$ with $E > eI_n$, and $0 < t \in \mathbb{R}$, the following equality holds

$$F_{n,k}(t(E,e)) = tF_{n,k}(E,e). \tag{2.14}$$

In fact, by (1.7) we have

$$\begin{aligned} F_{n,k}(t(E,e)) &= \left(\frac{\det(tE)}{\det(tE)_k} - \det(teI_{n-k})\right)^{\frac{1}{n-k}} \\ &= \left(\frac{t^n \det E}{t^k \det E_k} - t^{n-k}\det(eI_{n-k})\right)^{\frac{1}{n-k}} \\ &= t\left(\frac{\det E}{\det E_k} - \det(eI_{n-k})\right)^{\frac{1}{n-k}} = tF_{n,k}(E,e). \end{aligned}$$

(I). To prove the left-hand side inequality of the double inequality (2.13), it is sufficient to observe by (2.14), (1.8) and (1.9) that

$$\begin{aligned} &\left[2 - F_{n,k}\left(\frac{(A,a)}{F_{n,k}(A,a)} + \frac{(B,b)}{F_{n,k}(B,b)}\right)\right] \min\{F_{n,k}(A,a), F_{n,k}(B,b)\} \\ &= \left[F_{n,k}\left(\frac{(A,a)}{F_{n,k}(A,a)}\right) + F_{n,k}\left(\frac{(B,b)}{F_{n,k}(B,b)}\right)\right. \\ &\left. -F_{n,k}\left(\frac{(A,a)}{F_{n,k}(A,a)} + \frac{(B,b)}{F_{n,k}(B,b)}\right)\right] \min\{F_{n,k}(A,a), F_{n,k}(B,b)\} \\ &= (\Delta_{n,k}\left(\frac{(A,a)}{F_{n,k}(A,a)}, \frac{(B,b)}{F_{n,k}(B,b)}\right)) \min\{F_{n,k}(A,a), F_{n,k}(B,b)\} \le 0. \end{aligned}$$

(II). In order to show the right-hand side inequality of the double inequality (2.13), we observe that the assertion (2.13) is symmetric in (A,a) and (B,b). Therefore, we are allowed to assume without loss of generality that $F_{n,k}(A,a) \le F_{n,k}(B,b)$. Moreover, in the case $F_{n,k}(A,a) = F_{n,k}(B,b)$ inequality (2.13) holds trivially. So, it remains to consider the case $F_{n,k}(A,a) < F_{n,k}(B,b)$. From this we have $0 < \frac{F_{n,k}(A,a)}{F_{n,k}(B,b)} < 1$.

For any $t \in (0,1)$, we introduce pairs (C,c) and (D,d) by

$$(C,c) := t(B,b) \quad \text{and} \quad (D,d) := (1-t)(B,b). \tag{2.15}$$ □

It is clear by (1.5)–(1.6) that $(C,c), (D,d) \in \mathcal{D}_n$ and

$$(B,b) = (C,c) + (D,d).$$

Furthermore, the condition $\Delta_{n,k}((C,c),(D,d))=0$ holds valid, because by (1.8), (2.15) and (2.14), we have

$$\begin{aligned}\Delta_{n,k}((C,c),(D,d)) &= F_{n,k}(C,c)+F_{n,k}(D,d)-F_{n,k}(C+D,c+d)\\ &= F_{n,k}(t(B,b))+F_{n,k}((1-t)(B,b))-F_{n,k}(B,b)\\ &= tF_{n,k}(B,b)+(1-t)F_{n,k}(B,b)-F_{n,k}(B,b)=0.\end{aligned}$$

In consequence, we are permitted to make use of Theorem 1. So, it follows from inequalities (2.10) that

$$\Delta_{n,k}((A,a),(B,b))\leq\Delta_{n,k}((A,a),(C,c)).$$

Hence, by (1.8), we get

$$\begin{aligned}&F_{n,k}(A,a)+F_{n,k}(B,b)-F_{n,k}(A+B,a+b)\\ &\leq F_{n,k}(A,a)+F_{n,k}(C,c)-F_{n,k}(A+C,a+c).\end{aligned}$$

However, $(C,c)=t(B,b)$ with $t\in(0,1)$, so we obtain

$$\begin{aligned}&F_{n,k}(A,a)+F_{n,k}(B,b)-(F_{n,k}(A,a)+F_{n,k}(tB,tb)-F_{n,k}(A+tB,a+tb))\\ &\leq F_{n,k}(A+B,a+b),\end{aligned}$$

and next, by (2.14),

$$\begin{aligned}&F_{n,k}(A,a)+F_{n,k}(B,b)-[F_{n,k}(A,a)+tF_{n,k}(B,b)-F_{n,k}(A+tB,a+tb)]\\ &\leq F_{n,k}(A+B,a+b).\end{aligned}$$

Now, by putting $t:=\frac{F_{n,k}(A,a)}{F_{n,k}(B,b)}$, we conclude that $0<t<1$ and

$$\begin{aligned}&F_{n,k}(A,a)+F_{n,k}(B,b)\\ &-\left[F_{n,k}(A,a)+\frac{F_{n,k}(A,a)}{F_{n,k}(B,b)}F_{n,k}(B,b)-F_{n,k}\left((A,a)+\frac{F_{n,k}(A,a)}{F_{n,k}(B,b)}(B,b)\right)\right]\\ &\leq F_{n,k}(A+B,a+b),\end{aligned}$$

which means that

$$\begin{aligned}&F_{n,k}(A,a)+F_{n,k}(B,b)-\left[2-F_{n,k}\left(\frac{(A,a)}{F_{n,k}(A,a)}+\frac{(B,b)}{F_{n,k}(B,b)}\right)\right]F_{n,k}(A,a)\\ &\leq F_{n,k}(A+B,a+b).\end{aligned}$$

Thus, we obtain the required right-hand inequality of (2.13) because $F_{n,k}(A,a)\leq F_{n,k}(B,b)$. This completes the proof.

3. Improvements of Yuan–Leng, Minkowski, and Fan's inequalities. Let $A, B \in \mathbb{P}_n$, $a, b \geq 0$, $A > aI_n$, $B > bI_n$, $1 \leq k < n$ with $F_{n,k}(A,a) \neq 0$ and $F_{n,k}(B,b) \neq 0$. Note that

$$\frac{(A,a)}{F_{n,k}(A,a)} + \frac{(B,b)}{F_{n,k}(B,b)} = \left(\frac{A}{F_{n,k}(A,a)} + \frac{B}{F_{n,k}(B,b)}, \frac{a}{F_{n,k}(A,a)} + \frac{b}{F_{n,k}(B,b)} \right).$$

For this reason, Theorem 2 gives the following refinement of Yuan and Leng's inequality (1.3):

$$\begin{aligned} &\left(\frac{\det A}{\det A_k} - \det(aI_{n-k}) \right)^{\frac{1}{n-k}} + \left(\frac{\det B}{\det B_k} - \det(bI_{n-k}) \right)^{\frac{1}{n-k}} \\ &\leq \left(\frac{\det A}{\det A_k} - \det(aI_{n-k}) \right)^{\frac{1}{n-k}} + \left(\frac{\det B}{\det B_k} - \det(bI_{n-k}) \right)^{\frac{1}{n-k}} - r \qquad (3.16) \\ &\leq \left(\frac{\det(A+B)}{\det(A+B)_k} - \det((a+b)I_{n-k}) \right)^{\frac{1}{n-k}}, \end{aligned}$$

where

$$\begin{aligned} r := &\left[2 - \left(\frac{\det(\frac{A}{F_{n,k}(A,a)} + \frac{B}{F_{n,k}(B,b)})}{\det(\frac{A}{F_{n,k}(A,a)} + \frac{B}{F_{n,k}(B,b)})_k} - \det((\frac{a}{F_{n,k}(A,a)} + \frac{b}{F_{n,k}(B,b)})I_{n-k}) \right)^{\frac{1}{n-k}} \right] \\ &\times \min\{F_{n,k}(A,a), F_{n,k}(B,b)\} \leq 0. \qquad (3.17) \end{aligned}$$

In the special case $a = b = 0$, the above statements (3.16)-(3.17) reduce to the following refinement of Fan's inequality (1.2):

$$\begin{aligned} &\left(\frac{\det A}{\det A_k} \right)^{\frac{1}{n-k}} + \left(\frac{\det B}{\det B_k} \right)^{\frac{1}{n-k}} \\ &\leq \left(\frac{\det A}{\det A_k} \right)^{\frac{1}{n-k}} + \left(\frac{\det B}{\det B_k} \right)^{\frac{1}{n-k}} - r \leq \left(\frac{\det(A+B)}{\det(A+B)_k} \right)^{\frac{1}{n-k}}, \qquad (3.18) \end{aligned}$$

where

$$r := \left[2 - \left(\frac{\det(\frac{A}{F_{n,k}(A,0)} + \frac{B}{F_{n,k}(B,0)})}{\det(\frac{A}{F_{n,k}(A,0)} + \frac{B}{F_{n,k}(B,0)})_k} \right)^{\frac{1}{n-k}} \right] \min\{F_{n,k}(A,0), F_{n,k}(B,0)\} \leq 0, \qquad (3.19)$$

with $F_{n,k}(A,0) = \left(\frac{\det A}{\det A_k} \right)^{\frac{1}{n-k}}$ and $F_{n,k}(B,0) = \left(\frac{\det B}{\det B_k} \right)^{\frac{1}{n-k}}$.

In particular, if $k = 0$, then from (3.18)–(3.19) we obtain a refinement of Minkowski's inequality (1.1), as follows:

$$(\det A)^{\frac{1}{n}} + (\det B)^{\frac{1}{n}} \leq (\det A)^{\frac{1}{n}} + (\det B)^{\frac{1}{n}} - r \leq (\det(A+B))^{\frac{1}{n}}, \qquad (3.20)$$

where

$$r := \left[2 - \left(\det(\frac{A}{(\det A)^{\frac{1}{n}}} + \frac{B}{(\det B)^{\frac{1}{n}}}) \right)^{\frac{1}{n}} \right] \min\{(\det A)^{\frac{1}{n}}, (\det B)^{\frac{1}{n}}\} \leq 0, \qquad (3.21)$$

with $F_{n,0}(A,0) = (\det A)^{\frac{1}{n}}$ and $F_{n,0}(B,0) = (\det B)^{\frac{1}{n}}$.

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