

GENERALIZED PASCAL k -ELIMINATED FUNCTIONAL MATRIX WITH $2N$ VARIABLES*

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Abstract. In this paper, we introduce the Pascal k -eliminated functional matrix and the Pascal symmetric functional matrix with $2n$ variables. Some algebraic properties of these matrices are presented and proved. In addition, we demonstrate a direct application of these properties for LU decompositions of some well-known matrices (such as symmetric Pascal matrices).

Key words. Pascal matrix, Pascal k -eliminated functional matrix, Pascal symmetric functional matrix, LU decompositions, Cholesky factorization.

AMS subject classifications. 15A06, 34A30.

1. Introduction. In [1], the Pascal matrix $P_n[x]$ is defined by

$$(1.1) \quad (P_n[x])_{ij} = \begin{cases} \binom{i}{j} x^{i-j}, & i \geq j \\ 0, & i < j \end{cases} \quad (i, j = 0, 1, \dots, n).$$

Call and Velleman [1] discussed the inverse and exponential property of $P_n[x]$, i.e., $P_n[x+y] = P_n[x]P_n[y]$ for all x, y , and a few basic properties of this matrix. In [2], the symmetric Pascal matrix Q_n is defined by

$$(1.2) \quad (Q_n)_{ij} = \binom{i+j}{j} \quad (i, j = 0, 1, \dots, n),$$

and it has been shown that Q_n can be expressed as the product of a lower triangular Pascal matrix, $P_n[1]$, and an upper triangular Pascal matrix, $P_n^T[1]$.

In [3], the extended generalized lower triangular Pascal matrix for two variables $\Phi_n[x, y]$ is defined by

$$(1.3) \quad (\Phi_n[x, y])_{ij} = \begin{cases} \binom{i}{j} x^{i-j} y^{i+j}, & i \geq j \\ 0, & i < j \end{cases} \quad (i, j = 0, 1, \dots, n),$$

and the extended generalized rectangular Pascal matrix $\Psi_n[x, y]$ is defined by

$$(1.4) \quad (\Psi_n[x, y])_{ij} = x^{i-j} y^{i+j} \binom{i+j}{j} \quad (i, j = 0, 1, \dots, n).$$

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In [3], the authors also demonstrated that $\Psi_n[x, y]$ has the LU decomposition $\Psi_n[x, y] = \Phi_n[x, y]P_n^T \left[\frac{y}{x} \right]$. The Pascal functional matrix was introduced in [4], by

$$(1.5) \quad (P_{n,\lambda}[x])_{ij} = \begin{cases} \binom{i}{j} x^{(i-j)\lambda}, & i \geq j \\ 0, & i < j \end{cases} \quad (i, j = 0, 1, \dots, n),$$

where $x^{n|\lambda}$ is the generalized upper factorial, which is defined as follows:

$$x^{n|\lambda} = \begin{cases} x(x+\lambda)(x+2\lambda) \cdots (x+(n-1)\lambda), & n \geq 1 \\ 0, & n = 0 \end{cases}.$$

In [5], the notion of Pascal functional matrix was extended to a more general Pascal functional matrix, $G_n[x]$, defined by

$$(1.6) \quad (G_n[x])_{ij} = \begin{cases} \binom{i}{j} g_{i-j}(x), & i \geq j \\ 0, & i < j \end{cases} \quad (i, j = 0, 1, \dots, n),$$

where $\{g_n(x)\}$ is an arbitrary sequence of binomial-type polynomials, i.e., for all n , $g_n(x+y) = \sum_{k=0}^n \binom{n}{k} g_k(x)g_{n-k}(y)$ for any x and y .

In [6], the Pascal k -eliminated functional matrix with two variables, is defined by

$$(1.7) \quad (P_{n,k}[x, y])_{ij} = \begin{cases} \binom{i+k}{j+k} x^{i-j} y^j, & i \geq j \\ 0, & i < j \end{cases} \quad (i, j = 0, 1, \dots, n, k \in \mathbb{N} \cup \{0\}).$$

Another variant of Pascal functional matrix was introduced in [8], defined by

$$(1.8) \quad (\mathcal{P}_n[f(t, x)])_{ij} = \begin{cases} \binom{i}{j} f^{(i-j)}(t, x), & i \geq j \\ 0, & i < j \end{cases} \quad (i, j = 0, 1, \dots, n),$$

where $f^{(k)}(t, x)$ is k th order derivative of f with respect to t . In [8], it was shown that all well-known variants of Pascal matrices in [4,5] are special cases of this generalization of Pascal functional matrix. In [3-8], the authors proved some algebraic properties of such matrices and derived combinatorial identities from these properties.

2. The Pascal k -eliminated functional matrix with $2n$ variables.

DEFINITION 2.1. Using the sequence $t_0 = 1, t_1, t_2, \dots$, we define the sequence $t^{[0]} = 1, t^{[1]}, t^{[2]}, \dots$ by the relation

$$t^{[n]} = t_n t^{[n-1]}.$$

Notation. For convenience, we accept the following notation:

$$t^{[i]+[j]} := t^{[i]}t^{[j]} \quad \text{and} \quad t^{[i]-[j]} := \frac{t^{[i]}}{t^{[j]}}.$$

DEFINITION 2.2. Let $x_1, \dots, x_n, y_1, \dots, y_n$ be any $2n$ variables. The Pascal k -eliminated functional matrix with $2n$ variables $\Phi_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n]$ for $k \in \mathbb{N} \cup \{0\}$ is defined by

$$(2.1) \quad (\Phi_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n])_{ij} = \binom{i+k}{j+k} x^{[i]-[j]} y^{[i]+[j]} \quad (i, j = 0, 1, \dots, n),$$

where it is assumed that

$$\binom{i}{j} = 0 \quad \text{if} \quad j > i.$$

It can be shown that all well-known variants of Pascal matrices in [1,3,4,6,7] are special cases of this new generalization of Pascal functional matrix with $2n$ variables. At first we study a special case of this matrix where $x_1 = \dots = x_n = x$, $y_1 = \dots = y_n = y$ and $k \in \mathbb{N} \cup \{0\}$:

$$\Phi_{n,k}[x; y] = \Phi_{n,k}[x, \dots, x; y, \dots, y],$$

called the Pascal k -eliminated functional matrix with two variables [4]. It is easily seen that for $k = 0$, $\Phi_{n,0}[x, y]$ is the Pascal matrix $\Phi_n[x, y]$ in [3], and for $k = 0$ and $y = 1$, $\Phi_{n,0}[x, 1]$ is the Pascal matrix $P_n[x]$ in [1]. Also taking $x_1 = x$, $x_2 = x + \lambda$, $\dots$, $x_n = x + (n-1)\lambda$, $y_1 = y$, $y_2 = y + \lambda$, $\dots$, $y_n = y + (n-1)\lambda$ and $k = 0$ in (2.1), it yields (1.5).

THEOREM 2.3. For any four real numbers x_1, x_2, x_3, x_4 and $k \in \mathbb{N} \cup \{0\}$, we have

$$(i) \quad \Phi_{n,k}[x_1; y_1] \cdot \Phi_{n,k}[x_2; y_2] = \Phi_{n,k} \left[\frac{x_1}{y_2} + x_2 y_1; y_1 y_2 \right].$$

Also for any two real numbers x and y , we have

$$\begin{aligned} (ii) \quad & \Phi_{n,k}[-x; y] = \Phi_{n,k}[x; -y], \\ (iii) \quad & \Phi_{n,k}^{-1}[x; y] = \Phi_{n,k} \left[-x; \frac{1}{y} \right] = \Phi_{n,k} \left[x, -\frac{1}{y} \right] \quad (y \neq 0), \\ (iv) \quad & \prod_{i=1}^m \Phi_{n,k}[x_i; y_i] = \Phi_{n,k} \left[\left(x_1 + \sum_{i=2}^k x_i y_1 (y_2 \cdots y_{i-1})^2 y_i \right) \prod_{j=2}^m \frac{1}{y_j}, \prod_{i=1}^m y_i \right]. \end{aligned}$$

Proof. Let $(\Phi_{n,k}[x_1; y_1] \cdot \Phi_{n,k}[x_2; y_2])_{ij} = a_{ij}$. Then

$$a_{ij} = \sum_{t=0}^n \binom{i+k}{t+k} x_1^{i-t} y_1^{i+t} \binom{t+k}{j+k} x_2^{t-j} y_2^{t+j}$$

$$\begin{aligned}
 &= \sum_{t=0}^n \binom{i+k}{j+k} \binom{i-j}{t-j} x_1^{i-t} y_1^{i+t} x_2^{t-j} y_2^{t+j} \\
 &= \binom{i+k}{j+k} (y_1 y_2)^{i+j} \sum_{t=0}^n \binom{i-j}{t-j} \left(\frac{x_1}{y_2}\right)^{i-t} (x_2 y_1)^{t-j} \\
 &= \binom{i+k}{j+k} (y_1 y_2)^{i+j} \left(\frac{x_1}{y_2} + x_2 y_1\right)^{i-j} \\
 &= \left(\Phi_{n,k} \left[\frac{x_1}{y_2} + x_2 y_1; y_1 y_2\right]\right)_{ij}.
 \end{aligned}$$

Now, using (i) and $\Phi_{n,k}[0;1] = I_{n+1}$, considering $y_1 y_2 = 1$ and $\frac{x_1}{y_2} + x_2 y_1 = 0$, we have $x_2 = -x_1$ and $y_2 = \frac{1}{y_1}$. For the proof of (iv), we apply induction on m . This completes the proof. $\square$

COROLLARY 2.4. *It holds that*

$$\begin{aligned}
 \Phi_{n,k}^m[x; y] &= \Phi_{n,k} \left[xy^{-(m-1)}(1 + y^2 + y^4 + \cdots + y^{2(m-1)}); y^m \right] \\
 &= \Phi_{n,k} \left[\frac{xy^{-(m-1)}(1 - y^{2(m)})}{1 - y^2}; y^m \right].
 \end{aligned}$$

EXAMPLE 2.5.

$$\begin{aligned}
 &\Phi_{3,2}[x; y] \cdot \Phi_{3,2} \left[-x; \frac{1}{y} \right] \\
 &= \Phi_{3,2}[x; y] \cdot \Phi_{3,2} \left[x; -\frac{1}{y} \right] \\
 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3xy & y^2 & 0 & 0 \\ 6x^2y^2 & 4xy^3 & y^4 & 0 \\ 10x^3y^3 & 10x^2y^4 & 5xy^5 & y^6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{3x}{y} & \frac{1}{y^2} & 0 & 0 \\ \frac{6x^2}{y^2} & -\frac{4x}{y^3} & \frac{1}{y^4} & 0 \\ -\frac{10x^3}{y^3} & \frac{10x^2}{y^4} & -\frac{5x}{y^5} & \frac{1}{y^6} \end{bmatrix} = I_4.
 \end{aligned}$$

The matrices $\Phi_{n,0}[x; y]$ and $\Phi_{n,0}[x; 1]$ are the same as $\Phi_n[x; y]$ and $P_n[x]$, respectively, which are defined in [1,3]. We also consider the $(n+1) \times (n+1)$ matrices

$$(W_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n])_{ij} = \begin{cases} \binom{k+i-j}{k} x^{[i]-[j]} y^{[i]+[j]}, & i \geq j \\ 0, & i < j \end{cases}$$

and

$$(U_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n])_{ij} = \begin{cases} (-1)^m \binom{k+1}{m} \frac{x^{[i]-[j]}}{y^{[i]+[j]}}, & i = j + m, m = 0, 1, \dots, k+1 \\ 0, & \text{otherwise} \end{cases},$$

where $i, j = 0, 1, \dots, n$.

THEOREM 2.6. For $k \in \mathbb{N} \cup \{0\}$, we have

- (i) $\Phi_{n,k}[-x_1, \dots, -x_n; y_1, \dots, y_n] = \Phi_{n,k}[x_1, \dots, x_n; -y_1, \dots, -y_n]$,
- (ii) $\Phi_{n,k}^{-1}[x_1, \dots, x_n; y_1, \dots, y_n] = \Phi_{n,k}\left[-x_1, \dots, -x_n; \frac{1}{y_1}, \dots, \frac{1}{y_n}\right]$
 $= \Phi_{n,k}\left[x_1, \dots, x_n; -\frac{1}{y_1}, \dots, -\frac{1}{y_n}\right]$,
- (iii) $W_{n,k}^{-1}[x_1, \dots, x_n; y_1, \dots, y_n] = U_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n]$.

Proof. Here we prove (iii). Let

$$(W_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n] U_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n])_{ij} = a_{ij}.$$

Obviously, $a_{ii} = 1$ ($i = 0, 1, \dots, n$) and $a_{ij} = 0$ ($i < j$). When $i > j$ we have

$$\begin{aligned} a_{ij} &= \sum_{t=0}^i \binom{k+i-t}{k} x^{[i]-[t]} y^{[i]+[t]} (-1)^{t-j} \binom{k+1}{t-j} \frac{x^{[t]-[j]}}{y^{[t]+[j]}} \\ &= x^{[i]-[j]} y^{[i]-[j]} \sum_{t=j}^i (-1)^{t-j} \binom{k+i-t}{k} \binom{k+1}{t-j} \\ &= x^{[i]-[j]} y^{[i]-[j]} \sum_{s=0}^{i-j} (-1)^s \binom{k+i-j-s}{k} \binom{k+1}{s} = 0. \quad \square \end{aligned}$$

EXAMPLE 2.7.

$$\begin{aligned} &W_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3] U_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3] \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2x_1y_1 & y_1^2 & 0 & 0 \\ 3x_1x_2y_1y_2 & 2x_2y_1^2y_2 & y_1^2y_2^2 & 0 \\ 4x_1x_2x_3y_1y_2y_3 & 3x_2x_3y_1^2y_2y_3 & 2x_3y_1^2y_2^2y_3 & y_1^2y_2^2y_3^2 \end{bmatrix} \\ &\quad \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{2x_1}{y_1} & \frac{1}{y_1^2} & 0 & 0 \\ \frac{x_1x_2}{y_1y_2} & -\frac{2x_2}{y_1^2y_2} & \frac{1}{y_1^2y_2^2} & 0 \\ 0 & \frac{x_2x_3}{y_1^2y_2y_3} & -\frac{2x_3}{y_1^2y_2^2y_3} & \frac{1}{y_1^2y_2^2y_3^2} \end{bmatrix} = I_4. \end{aligned}$$

Again, we need the matrices $P_n[x_1, \dots, x_n]$, $S_n[x_1, \dots, x_n]$, $\overline{P}_m[x_1, \dots, x_m]$ and $G_n[x_1, \dots, x_n]$:

$$P_n[x_1, \dots, x_n] := \Phi_{n,0}[x_1, \dots, x_n; 1, \dots, 1],$$

$$S_n[x_1, \dots, x_n] := W_{n,0}[x_1, \dots, x_n; 1, \dots, 1],$$

$$\overline{P}_m[x_1, \dots, x_m] = \begin{bmatrix} 1 & O^T \\ O & P_m[x_1, \dots, x_m] \end{bmatrix} \in \mathbb{R}^{(m+2) \times (m+2)} \quad (m \geq 0),$$

$$G_m[x_1, \dots, x_m] = \begin{bmatrix} I_{n-m} & O^T \\ O & S_m[x_1, \dots, x_m] \end{bmatrix} \in \mathbb{R}^{(n+1) \times (n+1)} \quad (m = 0, 1, \dots, n-1),$$

$$G_n[x_1, \dots, x_n] := S_n[x_1, \dots, x_n].$$

LEMMA 2.8. For $m \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$, we have

$$W_{m,k}[x_1, \dots, x_m; y_1, \dots, y_m] \bar{P}_{m-1} \begin{bmatrix} \frac{x_2}{y_2}, \dots, \frac{x_m}{y_m} \end{bmatrix} = P_{m,k}[x_1, \dots, x_m; y_1, \dots, y_m].$$

Proof. Let

$$\left(W_{m,k}[x_1, \dots, x_m; y_1, \dots, y_m] \bar{P}_{m-1} \begin{bmatrix} \frac{x_2}{y_2}, \dots, \frac{x_m}{y_m} \end{bmatrix} \right)_{ij} = a_{ij}.$$

Obviously, $a_{ii} = (y^{[i]})^2$ ($i = 0, 1, \dots, m$) and $a_{ij} = 0$ ($i < j$). When $i > j$ we have

$$\begin{aligned} a_{ij} &= \sum_{t=0}^i \binom{k+i-t}{k} x^{[i]-[t]} y^{[i]+[t]} \binom{t-1}{j-1} \frac{x^{[t]-[j]}}{y^{[t]-[j]}} \\ &= x^{[i]-[j]} y^{[i]+[j]} \sum_{t=j}^i \binom{k+i-t}{k} \binom{t-1}{j-1} \\ &= x^{[i]-[j]} y^{[i]+[j]} \sum_{s=j-1}^{i-1} \binom{k+i-1-s}{k} \binom{s}{j-1} \\ &= x^{[i]-[j]} y^{[i]+[j]} \binom{i+k}{j+k}. \quad \square \end{aligned}$$

EXAMPLE 2.9.

$$\begin{aligned} & W_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3] \bar{P}_2 \begin{bmatrix} \frac{x_2}{y_2}, \frac{x_3}{y_3} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2x_1y_1 & y_1^2 & 0 & 0 \\ 3x_1x_2y_1y_2 & 2x_2y_1^2y_2 & y_1^2y_2^2 & 0 \\ 4x_1x_2x_3y_1y_2y_3 & 3x_2x_3y_1^2y_2y_3 & 2x_3y_1^2y_2^2y_3 & y_1^2y_2^2y_3^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{x_2}{y_2} & 1 & 0 \\ 0 & \frac{x_2x_3}{y_2y_3} & \frac{2x_3}{y_3} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2x_1y_1 & y_1^2 & 0 & 0 \\ 3x_1x_2y_1y_2 & 3x_2y_1^2y_2 & y_1^2y_2^2 & 0 \\ 4x_1x_2x_3y_1y_2y_3 & 6x_2x_3y_1^2y_2y_3 & 4x_3y_1^2y_2^2y_3 & y_1^2y_2^2y_3^2 \end{bmatrix} \\ &= P_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3]. \end{aligned}$$

By Lemma 2.8 and the definition of $G_m[x_1, \dots, x_m]$, we get the following result:

THEOREM 2.10. *The extended generalized Pascal matrix $\Phi_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n]$ can be factorized by $W_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n]$ and $G_m[x_1, \dots, x_m]$ ($m = 0, 1, \dots, n-1$):*

$$\begin{aligned} & \Phi_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n] \\ &= W_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n] G_{n-1} \left[\frac{x_2}{y_2}, \dots, \frac{x_n}{y_n} \right] G_{n-2} \left[\frac{x_3}{y_3}, \dots, \frac{x_n}{y_n} \right] \cdots G_1 \left[\frac{x_n}{y_n} \right]. \end{aligned}$$

EXAMPLE 2.11.

$$\begin{aligned} & \Phi_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3] \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2x_1y_1 & y_1^2 & 0 & 0 \\ 3x_1x_2y_1y_2 & 3x_2y_1^2y_2 & y_1^2y_2^2 & 0 \\ 4x_1x_2x_3y_1y_2y_3 & 6x_2x_3y_1^2y_2y_3 & 4x_3y_1^2y_2^2y_3 & y_1^2y_2^2y_3^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{x_2}{y_2} & 1 & 0 \\ 0 & \frac{x_2x_3}{y_2y_3} & \frac{2x_3}{y_3} & 1 \end{bmatrix} \\ & \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{x_3}{y_3} & 1 \end{bmatrix} = W_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3] G_2 \left[\frac{x_2}{y_2}, \frac{x_3}{y_3} \right] G_1 \left[\frac{x_3}{y_3} \right]. \end{aligned}$$

If we take $x_1 = \cdots = x_n = x$, $y_1 = \cdots = y_n = y$ and $k = 0$ (or $x_1 = \cdots = x_n = x$, $y_1 = \cdots = y_n = 1$) in Theorem 2.10, then we get the following results [3,4]:

COROLLARY 2.12. *It holds that*

$$\Phi_n[x; y] = W_{n,k}[x; y] G_{n-1} \left[\frac{x}{y} \right] G_{n-2} \left[\frac{x}{y} \right] \cdots G_1 \left[\frac{x}{y} \right],$$

where

$$\Phi_n[x; y] = P_{n,0}[x, \dots, x; y, \dots, y] \quad \text{and} \quad W_n[x; y] = W_{n,0}[x, \dots, x; y, \dots, y].$$

COROLLARY 2.13. *It holds that*

$$P_{n,k}[x] = (G_n[x])^k G_{n-1}[x] G_{n-2}[x] \cdots G_1[x],$$

where

$$P_{n,k}[x] = P_{n,k}[x, \dots, x] \quad \text{and} \quad (G_n[x])^k = W_{n,k}[x, \dots, x; x, \dots, x].$$

For the inverse of the extended generalized Pascal matrix $\Phi_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n]$, applying Theorems 2.6 and 2.10 yields the following.

THEOREM 2.14. *It holds that*

$$\begin{aligned} & \Phi_{n,k}^{-1}[x_1, \dots, x_n; y_1, \dots, y_n] \\ &= \Phi_{n,k} \begin{bmatrix} -x_1, \dots, -x_n; \frac{1}{y_1}, \dots, \frac{1}{y_n} \end{bmatrix} \\ &= \Phi_{n,k} \begin{bmatrix} x_1, \dots, x_n; -\frac{1}{y_1}, \dots, -\frac{1}{y_n} \end{bmatrix} \\ &= G_1^{-1} \begin{bmatrix} \frac{x_n}{y_n} \end{bmatrix} G_2^{-1} \begin{bmatrix} \frac{x_{n-1}}{y_{n-1}}, \frac{x_n}{y_n} \end{bmatrix} \cdots G_{n-1}^{-1} \begin{bmatrix} \frac{x_2}{y_2}, \dots, \frac{x_n}{y_n} \end{bmatrix} W_{n,k}^{-1}[x_1, \dots, x_n; y_1, \dots, y_n] \\ &= F_1 \begin{bmatrix} \frac{x_n}{y_n} \end{bmatrix} F_2 \begin{bmatrix} \frac{x_{n-1}}{y_{n-1}}, \frac{x_n}{y_n} \end{bmatrix} \cdots F_{n-1} \begin{bmatrix} \frac{x_2}{y_2}, \dots, \frac{x_n}{y_n} \end{bmatrix} U_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n], \end{aligned}$$

where

$$\begin{aligned} F_m \begin{bmatrix} \frac{x_{n-m+1}}{y_{n-m+1}}, \dots, \frac{x_n}{y_n} \end{bmatrix} &= G_m^{-1} \begin{bmatrix} \frac{x_{n-m+1}}{y_{n-m+1}}, \dots, \frac{x_n}{y_n} \end{bmatrix} \\ &= \begin{bmatrix} I_{n-k} & O^T \\ O & D_m \begin{bmatrix} \frac{x_{n-m+1}}{y_{n-m+1}}, \dots, \frac{x_n}{y_n} \end{bmatrix} \end{bmatrix} \end{aligned}$$

for $m = 1, \dots, n-1$, and $D_n[x_1, \dots, x_n] = U_{n,0}[x_1, \dots, x_n; 1, \dots, 1]$.

In particular,

$$P_{n,k}^{-1}[x_1, \dots, x_n; y_1, \dots, y_n] = J_n[y_1, \dots, y_n] P_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n] J_n[y_1, \dots, y_n],$$

where

$$J_n[y_1, \dots, y_n] = \text{diag} \left(1, -\frac{1}{y_1^2}, \frac{1}{y_1^2 y_2^2}, \dots, (-1)^n \frac{1}{y_1^2 \cdots y_n^2} \right).$$

3. The symmetric Pascal matrix with $2n$ variables. In this section, we define the extended generalized symmetric Pascal matrix $\Psi_n[x_1, \dots, x_n; y_1, \dots, y_n]$ by

$$(\Psi_n[x_1, \dots, x_n; y_1, \dots, y_n])_{ij} = \binom{i+j}{i} x^{[i]-[j]} y^{[i]+[j]},$$

where $i, j = 0, 1, \dots, n$.

THEOREM 3.1. *It holds that*

$$\begin{aligned} P_n^T \begin{bmatrix} \frac{y_1}{x_1}, \dots, \frac{y_n}{x_n} \end{bmatrix} &= F_1 \begin{bmatrix} \frac{x_n}{y_n} \end{bmatrix} F_2 \begin{bmatrix} \frac{x_{n-1}}{y_{n-1}}, \frac{x_n}{y_n} \end{bmatrix} \cdots F_{n-1} \begin{bmatrix} \frac{x_2}{y_2}, \dots, \frac{x_n}{y_n} \end{bmatrix} \\ &\quad \times U_{n,0}[x_1, \dots, x_n; y_1, \dots, y_n] \Psi_n[x_1, \dots, x_n; y_1, \dots, y_n], \end{aligned}$$

$$P_{n,0}^T[x_1, \dots, x_n; y_1, \dots, y_n] = F_1 \left[\frac{x_n}{y_n} \right] F_2 \left[\frac{x_{n-1}}{y_{n-1}}, \frac{x_n}{y_n} \right] \cdots F_{n-1} \left[\frac{x_2}{y_2}, \dots, \frac{x_n}{y_n} \right] \\ \times U_{n,0}[1, \dots, 1; y_1, \dots, y_n] \Psi_n[x_1, \dots, x_n; y_1, \dots, y_n],$$

and the Cholesky factorization of $\Psi_n[x_1, \dots, x_n; y_1, \dots, y_n]$ is given by

$$(3.1) \quad \Psi_n[x_1, \dots, x_n; y_1, \dots, y_n] = P_{n,0}^T[1, \dots, 1; x_1 y_1, \dots, x_n y_n] \\ \times P_{n,0}^T \left[y_1, \dots, y_n; \frac{1}{x_1}, \dots, \frac{1}{x_n} \right],$$

$$(3.2) \quad \Psi_n[x_1, \dots, x_n; y_1, \dots, y_n] = P_{n,0}^T[x_1, \dots, x_n; y_1, \dots, y_n] P_n^T \left[\frac{y_1}{x_1}, \dots, \frac{y_n}{x_n} \right].$$

Proof. Let $\left(P_{n,0}^T[1, \dots, 1; x_1 y_1, \dots, x_n y_n] P_{n,0}^T \left[y_1, \dots, y_n; \frac{1}{x_1}, \dots, \frac{1}{x_n} \right] \right)_{ij} = a_{ij}$.
 Then

$$a_{ij} = \begin{cases} \sum_{k=0}^j \binom{i}{k} \binom{j}{k} x^{[i]-[j]} y^{[i]+[j]}, & i \geq j \\ \sum_{k=0}^i \binom{i}{k} \binom{j}{k} x^{[i]-[j]} y^{[i]+[j]}, & i < j \end{cases},$$

which by using the Vandermonde identities

$$\sum_{k=0}^i \binom{i}{k} \binom{j}{k} = \sum_{k=0}^i \binom{i}{k} \binom{j}{j-k} = \binom{i+j}{j}$$

and

$$\sum_{k=0}^j \binom{i}{k} \binom{j}{k} = \sum_{k=0}^j \binom{i}{i-k} \binom{j}{k} = \binom{i+j}{j},$$

implies (3.1). Similarly we get (3.2). $\square$

EXAMPLE 3.2.

$$W_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3] \overline{P}_2 \left[\frac{x_2}{y_2}, \frac{x_3}{y_3} \right] \\ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2x_1 y_1 & y_1^2 & 0 & 0 \\ 3x_1 x_2 y_1 y_2 & 2x_2 y_1^2 y_2 & y_1^2 y_2^2 & 0 \\ 4x_1 x_2 x_3 y_1 y_2 y_3 & 3x_2 x_3 y_1^2 y_2 y_3 & 2x_3 y_1^2 y_2^2 y_3 & y_1^2 y_2^2 y_3^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{x_2}{y_2} & 1 & 0 \\ 0 & \frac{x_2 x_3}{y_2 y_3} & \frac{2x_3}{y_3} & 1 \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2x_1 y_1 & y_1^2 & 0 & 0 \\ 3x_1 x_2 y_1 y_2 & 3x_2 y_1^2 y_2 & y_1^2 y_2^2 & 0 \\ 4x_1 x_2 x_3 y_1 y_2 y_3 & 6x_2 x_3 y_1^2 y_2 y_3 & 4x_3 y_1^2 y_2^2 y_3 & y_1^2 y_2^2 y_3^2 \end{bmatrix}, \\ = P_{3,1}[x_1, x_2, x_3; y_1, y_2, y_3].$$

By using Theorems 2.6 and 3.1, we have:

COROLLARY 3.3. *It holds that*

$$\begin{aligned}\Psi_n^{-1}[x_1, \dots, x_n; y_1, \dots, y_n] &= P_n^T \left[-\frac{y_1}{x_1}, \dots, -\frac{y_n}{x_n} \right] P_{n,0} \left[x_1, \dots, x_n; -\frac{1}{y_1}, \dots, -\frac{1}{y_n} \right] \\ &= P_{n,0}^T [y_1, \dots, y_n; -x_1, \dots, -x_n] P_{n,0} \left[1, \dots, 1; -\frac{1}{x_1 y_1}, \dots, -\frac{1}{x_n y_n} \right].\end{aligned}$$

Applying Theorems 2.14 and 3.1, we get the following.

COROLLARY 3.4. *It holds that*

$$\begin{aligned}\Psi_n^{-1}[x_1, \dots, x_n; y_1, \dots, y_n] &= J_n[1, \dots, 1] P_n^T \left[\frac{y_1}{x_1}, \dots, \frac{y_n}{x_n} \right] J_n[1, \dots, 1] \\ &\quad \times J_n[y_1, \dots, y_n] P_{n,0}[x_1, \dots, x_n; y_1, \dots, y_n] J_n[y_1, \dots, y_n] \\ &= J_n \left[\frac{1}{x_1}, \dots, \frac{1}{x_n} \right] P_{n,0}^T \left[y_1, \dots, y_n; \frac{1}{x_1}, \dots, \frac{1}{x_n} \right] J_n \left[\frac{1}{x_1}, \dots, \frac{1}{x_n} \right] \\ &\quad \times J_n[y_1, \dots, y_n] P_{n,0}^T[1, \dots, 1; x_1 y_1, \dots, x_n y_n] J_n[y_1, \dots, y_n].\end{aligned}$$

For the previous two kinds of extended generalized Pascal matrices, we can get:

COROLLARY 3.5. *It holds that*

$$\begin{aligned}\det \Phi_{n,k}[x_1, \dots, x_n; y_1, \dots, y_n] &= y_1^{2n} y_2^{2(n-1)} \dots y_n^2, \\ \det \Phi_{n,k}^{-1}[x_1, \dots, x_n; y_1, \dots, y_n] &= y_1^{-2n} y_2^{-2(n-1)} \dots y_n^{-2}, \\ \det \Psi_n[x_1, \dots, x_n; y_1, \dots, y_n] &= y_1^{2n} y_2^{2(n-1)} \dots y_n^2, \\ \det \Psi_n^{-1}[x_1, \dots, x_n; y_1, \dots, y_n] &= y_1^{-2n} y_2^{-2(n-1)} \dots y_n^{-2}.\end{aligned}$$

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