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GLT SEQUENCES AND NORMAL MATRICES*

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Abstract. The theory of generalized locally Toeplitz (GLT) sequences is an apparatus for computing the asymptotic spectral distribution of matrices A_n arising from numerical discretizations of differential equations. Indeed, when the mesh fineness parameter n tends to infinity, these matrices A_n give rise to a sequence $\{A_n\}_n$, which often turns out to be a GLT sequence. In this paper, we extend the theory of GLT sequences in several directions: we show that every GLT sequence enjoys a normal form, we identify the spectral symbol of every GLT sequence formed by normal matrices, and we prove that, for every GLT sequence $\{A_n\}_n$ formed by normal matrices and every continuous function $f : \mathbb{C} \to \mathbb{C}$, the sequence $\{f(A_n)\}_n$ is again a GLT sequence whose spectral symbol is $f(\kappa)$, where κ is the spectral symbol of $\{A_n\}_n$. In addition, using the theory of GLT sequences, we prove a spectral distribution result for perturbed normal matrices.

Key words. Generalized locally Toeplitz sequences, Spectral distribution, Normal form, Normal matrices and perturbed normal matrices, Matrix functions.

AMS subject classifications. 15B05, 15A18, 47B06, 47B15, 15A16.

1. Introduction.

When a linear differential equation (DE) is discretized by a linear numerical method, the computation of the numerical solution reduces to solving a linear system $A_n\mathbf{u}_n = \mathbf{f}_n$, whose size d_n increases with the mesh fineness parameter n. What is often observed in practice is that A_n enjoys an asymptotic spectral distribution in the limit of mesh refinement $n \to \infty$. More precisely, it often turns out that, for a large class of test functions F,

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{i=1}^{d_n} F(\lambda_i(A_n)) = \frac{1}{\mu_k(D)} \int_D F(f(\mathbf{x}))\mathrm{d}\mathbf{x},$$

where $\lambda_i(A_n)$, $i = 1, \ldots, d_n$, are the eigenvalues of A_n, μ_k is the Lebesgue measure in $\mathbb{R}^k$, and $f : D \subset \mathbb{R}^k \to \mathbb{C}$. In this scenario, the function f is referred to as the spectral symbol of the sequence $\{A_n\}_n$, and we write $\{A_n\}_n \sim_\lambda f$. We refer the reader to Remark 2.1 for the informal meaning behind the spectral distribution $\{A_n\}_n \sim_\lambda f$ and to [19, Chapter 1] for a list of practical uses of the spectral symbol f.

The theory of generalized locally Toeplitz (GLT) sequences is an apparatus for computing the spectral symbol f. Indeed, the sequence of discretization matrices $\{A_n\}_n$ turns out to be a GLT sequence for many DEs and numerical methods. Nowadays, the main references for the theory of GLT sequences and related applications are the books [19, 20] and the papers [3, 6, 7, 8]. We, therefore, refer the reader to these works for a comprehensive treatment of the topic. For a more concise introduction to the subject, we recommend [14, 18].

In this paper, we extend the theory of GLT sequences in several directions. In particular,

- we show that every GLT sequence enjoys a normal form in a sense specified later on (see Theorem 3.1);

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- we identify the spectral symbol of every GLT sequence formed by normal matrices (see Theorem 3.3);
- we prove that, for every GLT sequence $\{A_n\}_n$ formed by normal matrices and every continuous function $f : \mathbb{C} \to \mathbb{C}$, the sequence $\{f(A_n)\}_n$ is again a GLT sequence whose spectral symbol is given by the composite function $f(\kappa)$, where κ is the spectral symbol of $\{A_n\}_n$ (see Theorem 3.4).

In addition, using the theory of GLT sequences, we prove a spectral distribution result for perturbed normal matrices (see Theorem 3.2). In this regard, it is curious to note how a pure matrix analysis result apparently unrelated to GLT sequences can be proved using GLT sequences. It is also worth pointing out that the proofs of the main results (Theorems 3.1–3.4) require some auxiliary results that are of interest also in themselves. We expressly refer to Theorems 4.1 and 4.4, which we have labeled as "theorems" in order to emphasize their importance over the other auxiliary results.

The paper is organized as follows. In Section 2, we overview the basics of the theory of GLT sequences. In Section 3, we state the main results. In Section 4, we prove the main results.

2. Overview of the theory of GLT sequences. In this section, we overview the basics of the theory of GLT sequences.

2.1. Singular value and spectral distribution of a sequence of matrices. Let μ_k be the Lebesgue measure in $\mathbb{R}^k$. Throughout this paper, all terminology from measure theory (such as "measurable set", "measurable function", and "a.e.") always refers to the Lebesgue measure. Let $C_c(\mathbb{R})$ (resp., $C_c(\mathbb{C})$) be the space of continuous complex-valued functions with bounded support defined on $\mathbb{R}$ (resp., $\mathbb{C}$). The singular values and eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$ are denoted by $\sigma_1(A), \ldots, \sigma_n(A)$ and $\lambda_1(A), \ldots, \lambda_n(A)$, respectively.

Definition 2.1 (singular value and spectral distribution of a sequence of matrices). *Let $\{A_n\}_n$ be a sequence of matrices with A_n of size $d_n \to \infty$, and let $f : D \subset \mathbb{R}^k \to \mathbb{C}$ be a measurable function defined on a set D with $0 < \mu_k(D) < \infty$.*

- *We say that $\{A_n\}_n$ has a spectral (or eigenvalue) distribution described by f, and we write $\{A_n\}_n \sim_\lambda f$, if*

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{i=1}^{d_n} F(\lambda_i(A_n)) = \frac{1}{\mu_k(D)} \int_D F(f(\mathbf{x}))\mathrm{d}\mathbf{x}, \qquad \forall\, F \in C_c(\mathbb{C}). \tag{2.1}$$

  In this case, f is called the spectral (or eigenvalue) symbol of $\{A_n\}_n$.
- *We say that $\{A_n\}_n$ has a singular value distribution described by f, and we write $\{A_n\}_n \sim_\sigma f$, if*

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{i=1}^{d_n} F(\sigma_i(A_n)) = \frac{1}{\mu_k(D)} \int_D F(|f(\mathbf{x})|)\mathrm{d}\mathbf{x}, \qquad \forall\, F \in C_c(\mathbb{R}). \tag{2.2}$$

  In this case, f is called the singular value symbol of $\{A_n\}_n$.

Remark 2.1. The informal meaning behind the spectral distribution (2.1) is the following [19, p. 46]: assuming that f is continuous a.e., the eigenvalues of A_n, except possibly for $o(d_n)$ outliers, are approximately equal to the samples of f over a uniform grid in the domain D (for n large enough). A completely analogous meaning can be given for the singular value distribution (2.2). We refer the reader to [4, Theorems 2.1–2.2], [10, Corollary 3.3], [11, Theorems 4.2 and 4.4], [12, Theorem 1.5] and [13, Theorem 1.3] for a mathematical proof under certain assumptions of the informal meaning given here.

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2.2. Special matrix sequences. In what follows, a matrix sequence is a sequence of the form $\{A_n\}_n$, where $A_n \in \mathbb{C}^{n\times n}$ for every n. If $A \in \mathbb{C}^{n\times n}$ and $1 \le p \le \infty$, we denote by $\|A\|_p$ the Schatten p-norm of A, i.e., the p-norm of the vector $(\sigma_1(A), \ldots, \sigma_n(A))$ formed by the singular values of A. The Schatten 2-norm $\|A\|_2$ coincides with the Frobenius norm. The Schatten ∞-norm $\|A\|_\infty$ coincides with the classical spectral (Euclidean) norm and is preferably denoted by $\|A\|$. For more on Schatten p-norms, see [9].

2.2.1. Zero-distributed sequences. A matrix sequence $\{Z_n\}_n$ such that $Z_n = R_n + N_n$ with

$$\lim_{n\to\infty} n^{-1}\mathrm{rank}(R_n) = \lim_{n\to\infty} \|N_n\| = 0$$

is referred to as a zero-distributed sequence. It can be shown that a matrix sequence $\{Z_n\}_n$ is zero-distributed if and only if $\{Z_n\}_n \sim_\sigma 0$, i.e.,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} F(\sigma_i(Z_n)) = F(0), \qquad \forall\, F \in C_c(\mathbb{R});$$

see [19, Chapter 3].

2.2.2. Diagonal sampling sequences. If $n \in \mathbb{N}$ and $a : [0,1] \to \mathbb{C}$, the nth diagonal sampling matrix generated by a is the $n \times n$ diagonal matrix given by

$$D_n(a) = \operatorname*{diag}_{i=1,\ldots,n} a\Big(\frac{i}{n}\Big).$$

$\{D_n(a)\}_n$ is called the diagonal sampling sequence generated by a.

2.2.3. Toeplitz sequences. If $n \in \mathbb{N}$ and $f : [-\pi,\pi] \to \mathbb{C}$ is a function in $L^1([-\pi,\pi])$, the nth Toeplitz matrix generated by f is the $n \times n$ matrix

$$T_n(f) = [f_{i-j}]_{i,j=1}^n,$$

where the numbers f_k are the Fourier coefficients of f:

$$f_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\mathrm{e}^{-\mathrm{i}k\theta}\mathrm{d}\theta, \qquad k \in \mathbb{Z}. \tag{2.3}$$

$\{T_n(f)\}_n$ is called the Toeplitz sequence generated by f. For more details on Toeplitz matrices generated by a function f, see [15] and [19, Chapter 6].

2.3. Approximating classes of sequences.

Definition 2.2 (approximating class of sequences). *Let $\{A_n\}_n$ be a matrix sequence and let $\{\{B_{n,m}\}_n\}_m$ be a sequence of matrix sequences. We say that $\{\{B_{n,m}\}_n\}_m$ is an approximating class of sequences (a.c.s.) for $\{A_n\}_n$, and we write $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$, if the following condition is met: for every m there exists n_m such that, for $n \ge n_m$,*

$$A_n = B_{n,m} + R_{n,m} + N_{n,m}, \qquad \mathrm{rank}(R_{n,m}) \le c(m)n, \qquad \|N_{n,m}\| \le \omega(m),$$

where n_m, $c(m)$, $\omega(m)$ depend only on m and

$$\lim_{m\to\infty} c(m) = \lim_{m\to\infty} \omega(m) = 0.$$

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Roughly speaking, $\{\{B_{n,m}\}_n\}_m$ is an a.c.s. for $\{A_n\}_n$ if, for all sufficiently large m, the sequence $\{B_{n,m}\}_n$ approximates $\{A_n\}_n$ in the sense that A_n is eventually equal to $B_{n,m}$ plus a small-rank matrix (with respect to the matrix size n) plus a small-norm matrix. It turns out that the notion of a.c.s. is a notion of convergence in the space of matrix sequences

$$\mathscr{E} = \{\{A_n\}_n : \{A_n\}_n \text{ is a matrix sequence}\}.$$

More precisely, for every $A \in \mathbb{C}^{n\times n}$ let

$$p(A) = \inf\left\{\frac{\text{rank}(R)}{n} + \|N\| : \quad R, N \in \mathbb{C}^{n\times n}, \quad R + N = A\right\} = \min_{i=0,\ldots,n}\left(\frac{i}{n} + \sigma_{i+1}(A)\right), \tag{2.4}$$

where $\sigma_1(A) \geq \ldots \geq \sigma_n(A)$ and $\sigma_{n+1}(A) = 0$ by convention. For the second equation in (2.4), see [19, Section 5.2.2]. Set

$$p_{\text{a.c.s.}}(\{A_n\}_n) = \limsup_{n\to\infty} p(A_n), \qquad \{A_n\}_n \in \mathscr{E},$$
$$d_{\text{a.c.s.}}(\{A_n\}_n, \{B_n\}_n) = p_{\text{a.c.s.}}(\{A_n - B_n\}_n), \qquad \{A_n\}_n, \{B_n\}_n \in \mathscr{E}.$$

It is proved in [19, Chapter 5] that $d_{\text{a.c.s.}}$ is a distance on $\mathscr{E}$ which turns $\mathscr{E}$ into a pseudometric space $(\mathscr{E}, d_{\text{a.c.s.}})$ where the statement "$\{\{B_{n,m}\}_n\}_m$ converges to $\{A_n\}_n$" is equivalent to "$\{\{B_{n,m}\}_n\}_m$ is an a.c.s. for $\{A_n\}_n$". In other words, we have

$$\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n \quad \iff \quad \lim_{m\to\infty} d_{\text{a.c.s.}}(\{B_{n,m}\}_n, \{A_n\}_n) = 0.$$

We remark that the a.c.s. distance $d_{\text{a.c.s.}}$ is a pseudometric but not a metric because $d_{\text{a.c.s.}}(\{A_n\}_n, \{B_n\}_n)$ can be zero even if $\{A_n\}_n$ is not equal to $\{B_n\}_n$. In fact, we have

$$d_{\text{a.c.s.}}(\{A_n\}_n, \{B_n\}_n) = 0 \quad \iff \quad \{A_n - B_n\}_n \text{ is zero-distributed;} \tag{2.5}$$

see [19, Section 5.2.1]. Whenever one of the equivalent conditions in (2.5) is satisfied, we say that $\{A_n\}_n$ is a.c.s.-equivalent to $\{B_n\}_n$ and we write $\{A_n\}_n \equiv_{\text{a.c.s.}} \{B_n\}_n$. We refer the reader to [19, Chapter 5] and [1, 5] for deeper insights into the a.c.s. convergence.

2.4. GLT sequences.

Definition 2.3 (GLT sequence). *Let $\{A_n\}_n$ be a matrix sequence and let $\kappa : [0,1]\times[-\pi,\pi] \to \mathbb{C}$ be measurable. We say that $\{A_n\}_n$ is a GLT sequence with symbol κ, and we write $\{A_n\}_n \sim_{\text{GLT}} \kappa$, if there exist functions $a_{i,m}$, $f_{i,m}$, $i = 1,\ldots,N_m$, with $a_{i,m} : [0,1] \to \mathbb{C}$ continuous a.e. and $f_{i,m} : [-\pi,\pi] \to \mathbb{C}$ in $L^1([-\pi,\pi])$, such that*

- $\kappa_m(x,\theta) = \sum_{i=1}^{N_m} a_{i,m}(x) f_{i,m}(\theta) \to \kappa(x,\theta)$ *in measure over* $[0,1]\times[-\pi,\pi]$,
- $\{A_{n,m}\}_n = \left\{\sum_{i=1}^{N_m} D_n(a_{i,m}) T_n(f_{i,m})\right\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$.

We report below the properties of GLT sequences that we need in this paper. The corresponding proofs can be found in [19, Chapter 8].

GLT 1. If $\{A_n\}_n \sim_{\text{GLT}} \kappa$ then $\{A_n\}_n \sim_\sigma \kappa$. If $\{A_n\}_n \sim_{\text{GLT}} \kappa$ and the matrices A_n are Hermitian, then $\{A_n\}_n \sim_\lambda \kappa$.

GLT 2. We have

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- $\{T_n(f)\}_n \sim_{\rm GLT} \kappa(x,\theta) = f(\theta)$ if $f \in L^1([-\pi,\pi])$,
- $\{D_n(a)\}_n \sim_{\rm GLT} \kappa(x,\theta) = a(x)$ if $a : [0,1] \to \mathbb{C}$ is continuous a.e.,
- $\{Z_n\}_n \sim_{\rm GLT} \kappa(x,\theta) = 0$ if and only if $\{Z_n\}_n$ is zero-distributed.

GLT 3. If $\{A_n\}_n \sim_{\rm GLT} \kappa$ and $\{B_n\}_n \sim_{\rm GLT} \xi$, then
- $\{A_n^*\}_n \sim_{\rm GLT} \overline{\kappa}$,
- $\{\alpha A_n + \beta B_n\}_n \sim_{\rm GLT} \alpha\kappa + \beta\xi$ for all $\alpha, \beta \in \mathbb{C}$,
- $\{A_n B_n\}_n \sim_{\rm GLT} \kappa\xi$.

GLT 4. If $\{A_n\}_n \sim_{\rm GLT} \kappa$ and the matrices A_n are Hermitian, then $\{f(A_n)\}_n \sim_{\rm GLT} f(\kappa)$ for every continuous function $f : \mathbb{C} \to \mathbb{C}$.

GLT 5. $\{A_n\}_n \sim_{\rm GLT} \kappa$ if and only if there exist GLT sequences $\{B_{n,m}\}_n \sim_{\rm GLT} \kappa_m$ such that $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ and $\kappa_m \to \kappa$ in measure.

GLT 6. Suppose $\{A_n\}_n \sim_{\rm GLT} \kappa$ and $\{B_{n,m}\}_n \sim_{\rm GLT} \kappa_m$. Then, $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ if and only if $\kappa_m \to \kappa$ in measure.

3. Main results. We have now collected all the necessary ingredients to state our main results. Throughout this paper, we denote by O_n and I_n the $n \times n$ zero matrix and the $n \times n$ identity matrix, respectively. Moreover, we denote by

$$F_n = \frac{1}{\sqrt{n}}\left[\mathrm{e}^{-2\pi \mathrm{i} jk/n}\right]_{j,k=0}^{n-1} = \frac{1}{\sqrt{n}}\left[\mathrm{e}^{-2\pi \mathrm{i}(j-1)(k-1)/n}\right]_{j,k=1}^{n} \tag{3.6}$$

the unitary discrete Fourier transform of size n. For every $m, n \in \mathbb{N}$ with $m \le n$, we define the normal matrix

$$Q_{n,m} = \begin{bmatrix} \boxed{F_{\lfloor n/m \rfloor}} & & & & \\ & \boxed{F_{\lfloor n/m \rfloor}} & & & \\ & & \ddots & & \\ & & & \boxed{F_{\lfloor n/m \rfloor}} & \\ & & & & \boxed{I_{n \bmod m}} \end{bmatrix}, \tag{3.7}$$

where it is understood that the last identity block $I_{n \bmod m}$ is not present if n is a multiple of m. Note that the matrix $Q_{n,m}$ (with the last block $I_{n \bmod m}$ replaced by $F_{n \bmod m}$) was already considered in [23, p. 310] for the construction of an efficient preconditioner for matrices belonging to a GLT sequence. A matrix sequence formed by diagonal (resp., unitary, normal) matrices is referred to as a diagonal (resp., unitary, normal) sequence. The first main result of this paper (Theorem 3.1) delivers a normal form for every GLT sequence.

THEOREM 3.1 (normal form for GLT sequences). *There exists a unitary sequence $\{Q_n\}_n$ with the following property.*

> *For every GLT sequence $\{A_n\}_n \sim_{\rm GLT} \kappa$ there exists a diagonal sequence $\{D_n\}_n$ such that $\{A_n\}_n \equiv_{\rm a.c.s.} \{Q_n D_n Q_n^*\}_n$, $\{D_n\}_n \sim_\lambda \kappa$, and D_n is real for every n if κ is real.*

Moreover, if $\{m(n)\}_n$ is any sequence such that $m(n) \le n$, $m(n) \to \infty$ and $m(n) = o(n)$ as $n \to \infty$, then Q_n can be chosen as $Q_n = Q_{n,m(n)}$.

As we shall see, the matrices D_n are built in order to contain an ordered approximation of the samples of the symbol κ over $[0,1] \times [-\pi,\pi]$. Note that a unitary transformation does not change the eigenvalues, so $\{Q_n D_n Q_n^*\}_n \sim_\lambda \kappa$. We refer to the sequence $\{Q_n D_n Q_n^*\}_n$ as a normal form for the GLT sequence $\{A_n\}_n$.

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The second main result of this paper (Theorem 3.2) provides the spectral distribution of matrix sequences of the form $\{X_n + Y_n\}_n$, where $\{X_n\}_n$ is a normal sequence such that $\{X_n\}_n \sim_\lambda f$ and Y_n is a "small" perturbation of X_n. Throughout this paper, we use the natural convention $1/\infty = 0$.

Theorem 3.2 (spectral distribution of perturbed normal sequences). *Let $\{X_n\}_n$ be a normal sequence such that $\{X_n\}_n \sim_\lambda f$ and let $\{Y_n\}_n$ be a matrix sequence. Suppose that one of the following conditions is met.*

1. $\|Y_n\|_p = o(1)$ *for some* $1 \le p \le 2$.
2. $\|Y_n\|_p = o(n^{2/p-1})$ *for some* $2 \le p \le \infty$.
3. $\{Y_n\}_n$ *is zero-distributed and the matrices* $X_n + Y_n$ *are normal.*

Then $\{X_n + Y_n\}_n \sim_\lambda f$.

The last two main results of this paper (Theorems 3.3 and 3.4) generalize properties **GLT 1** and **GLT 4** to the case where the matrices A_n are normal.

Theorem 3.3 (spectral distribution of normal GLT sequences). *If $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ and the matrices A_n are normal, then $\{A_n\}_n \sim_\lambda \kappa$.*

Theorem 3.4 (functions of normal GLT sequences). *If $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ and the matrices A_n are normal, then $\{f(A_n)\}_n \sim_{\mathrm{GLT}} f(\kappa)$ for every continuous function $f : \mathbb{C} \to \mathbb{C}$.*

Note that $f(A_n)$ is normal whenever A_n is normal. Thus, under the hypotheses of Theorem 3.4, we have $\{f(A_n)\}_n \sim_\lambda f(\kappa)$ for every continuous function $f : \mathbb{C} \to \mathbb{C}$ by Theorem 3.3.

4. Proofs of the main results.

4.1. Proof of Theorem 3.1.

The proof of Theorem 3.1 requires several preliminary results. In particular, it requires Theorem 4.1, which is analogous to the main a.c.s. approximation results [19, Corollaries 5.1 and 5.2], and it is therefore of interest also in itself.

4.1.1. Diagonal sequences.

If $A = U\Sigma V$ is an SVD of $A \in \mathbb{C}^{n\times n}$ (recall the the singular values $\sigma_1(A) \ge \ldots \ge \sigma_n(A)$ are arranged in decreasing order in Σ by definition of SVD), then we set, for every $i = 0, \ldots, n$,

$$\hat{A}^{(i)} = U\hat{\Sigma}^{(i)}V = U\,\mathrm{diag}(\sigma_1(A), \ldots, \sigma_i(A), 0, \ldots, 0)\,V,$$
$$\tilde{A}^{(i)} = U\tilde{\Sigma}^{(i)}V = U\,\mathrm{diag}(0, \ldots, 0, \sigma_{i+1}(A), \ldots, \sigma_n(A))\,V.$$

For every $A \in \mathbb{C}^{n\times n}$ and every SVD $A = U\Sigma V$, the scalar quantity $p(A)$ in (2.4) satisfies

$$p(A) = \min_{i=0,\ldots,n}\Big(\frac{i}{n} + \sigma_{i+1}(A)\Big) = \min_{i=0,\ldots,n}\Big(\frac{\mathrm{rank}(\hat{A}^{(i)})}{n} + \|\tilde{A}^{(i)}\|\Big),$$

because the minimum in both sides of the last equality is reached for $i \in \{0, \ldots, \mathrm{rank}(A)\}$, since for $i \ge \mathrm{rank}(A)$ we have $\|\tilde{A}^{(i)}\| = \sigma_{i+1}(A) = 0$.

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LEMMA 4.1. *Let $\{A_n\}_n$ and $\{B_{n,m}\}_n$ be matrix sequences, let $A_n - B_{n,m} = U_{n,m}\Sigma_{n,m}V_{n,m}$ be an SVD of $A_n - B_{n,m}$, define*

$$R_{n,m}^{(i)} = \widehat{(A_n - B_{n,m})}^{(i)}, \quad i = 0, \dots, n,$$
$$N_{n,m}^{(i)} = \widetilde{(A_n - B_{n,m})}^{(i)}, \quad i = 0, \dots, n,$$

and let $i_{n,m} \in \{0, \dots, n\}$ be the index such that

$$p(A_n - B_{n,m}) = \min_{i=0,\dots,n}\Big(\frac{\operatorname{rank}(R_{n,m}^{(i)})}{n} + \|N_{n,m}^{(i)}\|\Big) = \frac{\operatorname{rank}(R_{n,m}^{(i_{n,m})})}{n} + \|N_{n,m}^{(i_{n,m})}\|.$$

Then, $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ if and only if for every m there exists n_m such that, for $n \ge n_m$,

$$\operatorname{rank}(R_{n,m}^{(i_{n,m})}) \le c(m)n, \qquad \|N_{n,m}^{(i_{n,m})}\| \le \omega(m),$$

where $n_m, c(m), \omega(m)$ depend only on m and

$$\lim_{m\to\infty} c(m) = \lim_{m\to\infty} \omega(m) = 0.$$

Proof. We have

$$\begin{aligned}
\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n \quad &\iff \quad \lim_{m\to\infty} d_{\text{a.c.s.}}(\{A_n\}_n, \{B_{n,m}\}_n) = 0\\
&\iff \quad \lim_{m\to\infty} \limsup_{n\to\infty} p(A_n - B_{n,m}) = 0\\
&\iff \quad \lim_{m\to\infty} \limsup_{n\to\infty}\Big(\frac{\operatorname{rank}(R_{n,m}^{(i_{n,m})})}{n} + \|N_{n,m}^{(i_{n,m})}\|\Big) = 0\\
&\iff \quad \lim_{m\to\infty} \limsup_{n\to\infty} \frac{\operatorname{rank}(R_{n,m}^{(i_{n,m})})}{n} = 0 \quad \wedge \quad \lim_{m\to\infty} \limsup_{n\to\infty} \|N_{n,m}^{(i_{n,m})}\| = 0.
\end{aligned}$$

The last two conditions are satisfied if and only if for every m there exists n_m such that, for $n \ge n_m$,

$$\operatorname{rank}(R_{n,m}^{(i_{n,m})}) \le c(m)n, \qquad \|N_{n,m}^{(i_{n,m})}\| \le \omega(m),$$

where $n_m, c(m), \omega(m)$ depend only on m and $\lim_{m\to\infty} c(m) = \lim_{m\to\infty} \omega(m) = 0$. ∎

COROLLARY 4.1. *Let $\{A_n\}_n$ and $\{B_{n,m}\}_n$ be diagonal sequences, and suppose that $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$. Then, for every m there exists n_m such that, for $n \ge n_m$,*

$$A_n = B_{n,m} + R_{n,m} + N_{n,m}, \qquad \operatorname{rank}(R_{n,m}) \le c(m)n, \qquad \|N_{n,m}\| \le \omega(m),$$

where $R_{n,m}, N_{n,m}$ are diagonal matrices, $n_m, c(m), \omega(m)$ depend only on m, and

$$\lim_{m\to\infty} c(m) = \lim_{m\to\infty} \omega(m) = 0.$$

Proof. Since $A_n - B_{n,m}$ is diagonal, there exists an SVD $A_n - B_{n,m} = U_{n,m}\Sigma_{n,m}V_{n,m}$ with $U_{n,m}$ and $V_{n,m}$ diagonal. By applying Lemma 4.1 with this SVD, we conclude that for every m there exists n_m such that, for $n \ge n_m$,

$$A_n = B_{n,m} + R_{n,m}^{(i_{n,m})} + N_{n,m}^{(i_{n,m})}, \qquad \operatorname{rank}(R_{n,m}^{(i_{n,m})}) \le c(m)n, \qquad \|N_{n,m}^{(i_{n,m})}\| \le \omega(m),$$

where $n_m, c(m), \omega(m)$ depend only on m and $\lim_{m\to\infty} c(m) = \lim_{m\to\infty} \omega(m) = 0$. Since $R_{n,m}^{(i_{n,m})}$ and $N_{n,m}^{(i_{n,m})}$ are diagonal by definition (and by the fact that $U_{n,m}$ and $V_{n,m}$ are diagonal), the result is proved. ∎

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In what follows, the cardinality of a set E is denoted by $\#E$.

Theorem 4.1. *Let $\{A_n\}_n$, $\{B_{n,m}\}_n$ be matrix sequences formed by diagonal matrices and let $f, f_m : D \subset \mathbb{R}^k \to \mathbb{C}$ be measurable functions defined on a set D with $0 < \mu_k(D) < \infty$. Suppose that*

1. $\{B_{n,m}\}_n \sim_\lambda f_m$ *for every m,*
2. $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$,
3. $f_m \to f$ *in measure.*

Then $\{A_n\}_n \sim_\lambda f$.

Proof. Let $F \in C_c(\mathbb{C})$. For all n, m we have

$$\begin{aligned}\left|\frac{1}{n}\sum_{j=1}^{n} F(\lambda_j(A_n)) - \frac{1}{\mu_k(D)}\int_D F(f(\mathbf{x}))\mathrm{d}\mathbf{x}\right| &\le \left|\frac{1}{n}\sum_{j=1}^{n} F(\lambda_j(A_n)) - \frac{1}{n}\sum_{j=1}^{n} F(\lambda_j(B_{n,m}))\right| \\ &+ \left|\frac{1}{n}\sum_{j=1}^{n} F(\lambda_j(B_{n,m})) - \frac{1}{\mu_k(D)}\int_D F(f_m(\mathbf{x}))\mathrm{d}\mathbf{x}\right| \\ &+ \left|\frac{1}{\mu_k(D)}\int_D F(f_m(\mathbf{x}))\mathrm{d}\mathbf{x} - \frac{1}{\mu_k(D)}\int_D F(f(\mathbf{x}))\mathrm{d}\mathbf{x}\right|. \end{aligned} \tag{4.8}$$

The second term in the right-hand side of (4.8) tends to 0 as $n \to \infty$ because $\{B_{n,m}\}_n \sim_\lambda f_m$ for every m by hypothesis. The third term in the right-hand side of (4.8) tends to 0 as $m \to \infty$ because $f_m \to f$ in measure by hypothesis, which implies that $\int_D F(f_m(\mathbf{x}))\mathrm{d}\mathbf{x} \to \int_D F(f(\mathbf{x}))\mathrm{d}\mathbf{x}$; see, e.g., [19, Lemma 2.5]. We show that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{j=1}^{n} F(\lambda_j(A_n)) - \frac{1}{n}\sum_{j=1}^{n} F(\lambda_j(B_{n,m}))\right| = 0, \tag{4.9}$$

after which the thesis follows by passing first to the $\limsup_{n\to\infty}$ and then to the $\lim_{m\to\infty}$ in (4.8). To prove (4.9), recall that $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ and $A_n, B_{n,m}$ are diagonal. Hence, by Corollary 4.1, for every m there exists n_m such that, for $n \ge n_m$,

$$A_n = B_{n,m} + R_{n,m} + N_{n,m}, \qquad \mathrm{rank}(R_{n,m}) \le c(m)n, \qquad \|N_{n,m}\| \le \omega(m),$$

where $R_{n,m}, N_{n,m}$ are diagonal and $\lim_{m\to\infty} c(m) = \lim_{m\to\infty}\omega(m) = 0$. It follows that

$$\#\Big\{j \in \{1,\dots,n\} : |\lambda_j(A_n) - \lambda_j(B_{n,m})| > \|N_{n,m}\|\Big\} \le \mathrm{rank}(R_{n,m}).$$

Indeed, since

$$\lambda_j(A_n) - \lambda_j(B_{n,m}) = \lambda_j(A_n - B_{n,m}) = \lambda_j(R_{n,m} + N_{n,m}) = \lambda_j(R_{n,m}) + \lambda_j(N_{n,m}),$$

if we have $|\lambda_j(A_n) - \lambda_j(B_{n,m})| > \|N_{n,m}\|$, then we also have $\lambda_j(R_{n,m}) \ne 0$, because $|\lambda_j(N_{n,m})| \le \|N_{n,m}\|$. Thus, for every m and every $n \ge n_m$, if we denote by ω_F the modulus of continuity of F, we have

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$$\begin{aligned}\left|\frac{1}{n}\sum_{j=1}^{n}F(\lambda_j(A_n))-\frac{1}{n}\sum_{j=1}^{n}F(\lambda_j(B_{n,m}))\right| &\le \frac{1}{n}\sum_{\substack{j\in\{1,\dots,n\}:\\ |\lambda_j(A_n)-\lambda_j(B_{n,m})|\le\|N_{n,m}\|}}|F(\lambda_j(A_n))-F(\lambda_j(B_{n,m}))|\\ &\quad+\frac{1}{n}\sum_{\substack{j\in\{1,\dots,n\}:\\ |\lambda_j(A_n)-\lambda_j(B_{n,m})|>\|N_{n,m}\|}}|F(\lambda_j(A_n))-F(\lambda_j(B_{n,m}))|\\ &\le \omega_F(\|N_{n,m}\|)+2\|F\|_\infty\frac{\operatorname{rank}(R_{n,m})}{n}\\ &\le \omega_F(\omega(m))+2\|F\|_\infty c(m),\end{aligned}$$

which tends to 0 as $m\to\infty$. □

4.1.2. Circulant sequences. If $n\in\mathbb{N}$ and $f:[-\pi,\pi]\to\mathbb{C}$ is a function in $L^1([-\pi,\pi])$, the nth circulant matrix generated by f is the $n\times n$ matrix

$$C_n(f)=\sum_{k=-(n-1)}^{n-1}f_kC_n^k,$$

where the numbers f_k are the Fourier coefficients of f defined in (2.3) and

$$C_n=\begin{bmatrix}0 & & & 1\\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0\end{bmatrix}.$$

C_n is called the generator of circulant matrices of size n, and $\{C_n(f)\}_n$ is called the circulant sequence generated by f. For the proof of the next theorem, see [19, Section 6.4]. For more details on circulant matrices, see [17]. In what follows, if $n\in\mathbb{N}$ and $g:[0,2\pi]\to\mathbb{C}$, we denote by $\Delta_n(g)$ the diagonal sampling matrix

$$\Delta_n(g)=\operatorname*{diag}_{j=0,\dots,n-1}g\Big(\frac{2\pi j}{n}\Big).$$

Moreover, if $t\in\mathbb{N}$ and $f\in L^1([-\pi,\pi])$, we denote by $f^{(t)}$ the truncated Fourier series

$$f^{(t)}(\theta)=\sum_{k=-t}^{t}f_k\mathrm{e}^{\mathrm{i}k\theta}.$$

THEOREM 4.2. *Let $n\in\mathbb{N}$ and $f\in L^1([-\pi,\pi])$. Then,*

$$C_n(f)=F_n\Delta_n(f^{(n-1)})F_n^*,$$

where F_n is defined in (3.6). In particular, $C_n(f)$ is a normal matrix whose spectrum is given by

$$\Big\{f^{(n-1)}\Big(\frac{2\pi j}{n}\Big):\,j=0,\dots,n-1\Big\}.$$

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4.1.3. LT operator. If X, Y are matrices of any size, say $X \in \mathbb{C}^{m_1\times m_2}$ and $Y \in \mathbb{C}^{\ell_1\times \ell_2}$, the tensor (Kronecker) product of X and Y is the $m_1\ell_1 \times m_2\ell_2$ matrix defined by

$$X \otimes Y = \left[x_{ij}Y\right]_{\substack{i=1,\ldots,m_1\\ j=1,\ldots,m_2}} = \begin{bmatrix} x_{11}Y & \cdots & x_{1m_2}Y \\ \vdots & & \vdots \\ x_{m_11}Y & \cdots & x_{m_1m_2}Y \end{bmatrix},$$

and the direct sum of X and Y is the $(m_1+\ell_1)\times(m_2+\ell_2)$ matrix defined by

$$X \oplus Y = \mathrm{diag}(X, Y) = \begin{bmatrix} X & O \\ O & Y \end{bmatrix}.$$

Definition 4.1 (locally Toeplitz operator). *Let $m, n \in \mathbb{N}$ with $m \le n$, let $a : [0,1] \to \mathbb{C}$ and let $f \in L^1([-\pi,\pi])$. The locally Toeplitz (LT) operator is defined as the following $n \times n$ matrix:*

$$LT_n^m(a,f) = D_m(a) \otimes T_{\lfloor n/m\rfloor}(f) \;\oplus\; O_{n \bmod m} = \mathop{\mathrm{diag}}_{i=1,\ldots,m}\left[a\Big(\frac{i}{m}\Big)T_{\lfloor n/m\rfloor}(f)\right] \;\oplus\; O_{n \bmod m}$$

$$= \begin{bmatrix} \boxed{a(\frac{1}{m})T_{\lfloor n/m\rfloor}(f)} & & & & \\ & \boxed{a(\frac{2}{m})T_{\lfloor n/m\rfloor}(f)} & & & \\ & & \ddots & & \\ & & & \boxed{a(1)T_{\lfloor n/m\rfloor}(f)} & \\ & & & & \boxed{O_{n \bmod m}} \end{bmatrix}.$$

It is understood that the zero block $O_{n \bmod m}$ is not present when n is a multiple of m. Moreover, here and in what follows, the tensor product operation $\otimes$ is always applied before the direct sum $\oplus$, exactly as in the case of numbers, where multiplication is always applied before addition.

For our purposes, we are interested only in $LT_n^m(a,f)$ with $m = m(n)$ such that $m(n) \to \infty$ and $m(n) = o(n)$ as $n \to \infty$. In what follows, if $1 \le p \le \infty$, we denote by $|\cdot|_p$ both the p-norm of vectors and the associated induced norm for matrices. The 2-norm $|\cdot|_2$ is the classical spectral (Euclidean) norm and is preferably denoted by $\|\cdot\|$. An important inequality involving the p-norms with $p = 1, 2, \infty$ is the following:

$$\|X\| \le \sqrt{|X|_1|X|_\infty}, \qquad X \in \mathbb{C}^{m\times m}; \tag{4.10}$$

see, e.g., [21, Corollary 2.3.2].

Lemma 4.2. *Let $a : [0,1] \to \mathbb{C}$ be continuous and let f be a trigonometric polynomial. Then,*

$$\{LT_n^{m(n)}(a,f)\}_n \equiv_{\text{a.c.s.}} \{D_n(a)T_n(f)\}_n$$

for every $m(n) \to \infty$ such that $m(n) = o(n)$.

Proof. Throughout this proof, $m = m(n)$ and $m(n)$ has the properties specified in the statement. We have to show that $\{Z_n\}_n$ is zero-distributed, where

$$Z_n = LT_n^m(a,f) - D_n(a)T_n(f).$$

The matrix Z_n is banded with bandwidth $2r+1$, where r is the degree of f. Let R_n be the matrix whose nonzero entries are the nonzero entries of Z_n lying outside the diagonal blocks of $LT_n^m(a,f)$ or in its last

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zero block of size $n \bmod m$. The matrix R_n has at most $2rm + n \bmod m = o(n)$ nonzero rows. Hence, $\mathrm{rank}(R_n) = o(n)$. Now, let $N_n = Z_n - R_n$. The only nonzero entries of N_n lie inside the diagonal blocks of $LT_n^m(a, f)$ (excluding the last zero block). Each nonzero entry of N_n is of the form $(a(x) - a(y))\tau$, where $x, y \in [0, 1]$ and τ is a coefficient of $T_n(f)$. Moreover, the following properties hold.

- The distance between x and y is small. Indeed, if we are in the ith block with $1 \le i \le m$, then

$$x = \frac{i}{m}, \qquad ny \in \Big[(i-1)\Big\lfloor\frac{n}{m}\Big\rfloor + 1,\, i\Big\lfloor\frac{n}{m}\Big\rfloor\Big].$$

  Since

$$\frac{i}{m} \ge \frac{i}{n}\Big\lfloor\frac{n}{m}\Big\rfloor,$$

  we have

$$|x - y| \le \frac{i}{m} - \frac{i-1}{n}\Big\lfloor\frac{n}{m}\Big\rfloor \le \frac{i}{m} - \frac{i-1}{n}\Big(\frac{n}{m} - 1\Big) = \frac{1}{m} + \frac{i-1}{n} \le \frac{1}{m} + \frac{m}{n}.$$

  It follows that $|a(x) - a(y)| \le \omega_a(|x - y|) \le \omega_a(1/m + m/n)$.
- $|\tau| \le M$, where $M = \max_{k=-r,\dots,r} |f_k|$.

Since N_n is banded with bandwidth $2r + 1$, we infer from (4.10) that

$$\|N_n\| \le \sqrt{|N_n|_1 |N_n|_\infty} \le (2r + 1)\,\omega_a\Big(\frac{1}{m} + \frac{m}{n}\Big)\, M = o(1).$$

In conclusion, $Z_n = R_n + N_n$ with $\mathrm{rank}(R_n) = o(n)$ and $\|N_n\| = o(1)$, which means that $\{Z_n\}_n$ is zero-distributed. ∎

4.1.4. LC operator. The matrix $LT_n^m(a, f)$ is not normal in general, but if we replace in its definition the Toeplitz matrix $T_{\lfloor n/m\rfloor}(f)$ with the circulant matrix $C_{\lfloor n/m\rfloor}(f)$, then we obtain a normal matrix.

Definition 4.2 (locally circulant operator). *Let $m, n \in \mathbb{N}$ with $m \le n$, let $a : [0, 1] \to \mathbb{C}$ and let $f \in L^1([-\pi, \pi])$. The locally circulant (LC) operator is defined as the following $n \times n$ normal matrix:*

$$LC_n^m(a, f) = D_m(a) \otimes C_{\lfloor n/m\rfloor}(f) \,\oplus\, O_{n \bmod m} = \mathop{\mathrm{diag}}_{i=1,\dots,m}\Big[a\Big(\frac{i}{m}\Big)C_{\lfloor n/m\rfloor}(f)\Big] \,\oplus\, O_{n \bmod m}$$

$$= \begin{bmatrix} a(\frac{1}{m})C_{\lfloor n/m\rfloor}(f) & & & & \\ & a(\frac{2}{m})C_{\lfloor n/m\rfloor}(f) & & & \\ & & \ddots & & \\ & & & a(1)C_{\lfloor n/m\rfloor}(f) & \\ & & & & O_{n \bmod m} \end{bmatrix}.$$

Note that, by Theorem 4.2,

$$LC_n^m(a, f) = Q_{n,m}\Delta_{n,m}(a, f^{(\lfloor n/m\rfloor - 1)})Q_{n,m}^*, \tag{4.11}$$

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where $Q_{n,m}$ is defined in (3.7) and $\Delta_{n,m}(a,f)$ is the diagonal matrix given by

$$\Delta_{n,m}(a,f) = \begin{bmatrix} a(\frac{1}{m})\Delta_{\lfloor n/m\rfloor}(f^{(\lfloor n/m\rfloor-1)}) & & & & \\ & a(\frac{2}{m})\Delta_{\lfloor n/m\rfloor}(f^{(\lfloor n/m\rfloor-1)}) & & & \\ & & \ddots & & \\ & & & a(1)\Delta_{\lfloor n/m\rfloor}(f^{(\lfloor n/m\rfloor-1)}) & \\ & & & & O_{n \bmod m} \end{bmatrix}.$$

For our purposes, we are interested only in $LC_n^m(a,f)$ with $m = m(n)$ such that $m(n) \to \infty$ and $m(n) = o(n)$ as $n \to \infty$.

LEMMA 4.3. *Let $a : [0,1] \to \mathbb{C}$ and let f be a trigonometric polynomial. Then,*

$$\{LC_n^{m(n)}(a,f)\}_n \equiv_{\text{a.c.s.}} \{LT_n^{m(n)}(a,f)\}_n$$

for every $m(n) \to \infty$ such that $m(n) = o(n)$.

Proof. Throughout this proof, $m = m(n)$ and $m(n)$ has the properties specified in the statement. Let r be the degree of f. By writing explicitly $T_k(f)$ and $C_k(f)$, we see that the difference $T_k(f) - C_k(f)$ has at most $2r$ nonzero rows. This implies that $\text{rank}(T_k(f) - C_k(f)) \le 2r$ for all $k \in \mathbb{N}$. Hence, for every n,

$$\text{rank}(LC_n^m(a,f) - LT_n^m(a,f)) \le m\,\text{rank}(C_{\lfloor n/m\rfloor}(f) - T_{\lfloor n/m\rfloor}(f)) \le 2rm = o(n).$$

It follows that $\{LC_n^m(a,f) - LT_n^m(a,f)\}_n$ is zero-distributed, and the result is proved. ◻

Given $m, n \in \mathbb{N}$ with $m \le n$ and $\zeta : [0,1] \times [0,2\pi] \to \mathbb{C}$, we denote by $\Delta_{n,m}(\zeta)$ the diagonal matrix

$$\Delta_{n,m}(\zeta) = \begin{bmatrix} \Delta_{\lfloor n/m\rfloor}(\zeta(\frac{1}{m},\cdot)) & & & & \\ & \Delta_{\lfloor n/m\rfloor}(\zeta(\frac{2}{m},\cdot)) & & & \\ & & \ddots & & \\ & & & \Delta_{\lfloor n/m\rfloor}(\zeta(1,\cdot)) & \\ & & & & O_{n \bmod m} \end{bmatrix}. \tag{4.12}$$

LEMMA 4.4. *Let $\zeta : [0,1] \times [0,2\pi] \to \mathbb{C}$ be continuous and let $m(n) \to \infty$ such that $m(n) = o(n)$ as $n \to \infty$. Then, $\{\Delta_{n,m(n)}(\zeta)\}_n \sim_\lambda \zeta$.*

Proof. Throughout this proof, $m = m(n)$ and $m(n)$ has the properties specified in the statement. The lemma is proved by direct computation. The eigenvalues of the diagonal matrix $\Delta_{n,m}(\zeta)$ are just the diagonal entries and so they are given by

$$\zeta\Big(\frac{i}{m}, \frac{2\pi j}{\lfloor n/m\rfloor}\Big), \qquad i = 1,\dots,m, \qquad j = 1,\dots,\lfloor n/m\rfloor,$$

plus further $n \bmod m$ zero eigenvalues. Thus, for every $F \in C_c(\mathbb{C})$,

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^n F(\lambda_k(\Delta_{n,m}(\zeta))) &= \frac{(n \bmod m)F(0)}{n} + \frac{1}{n}\sum_{i=1}^m \sum_{j=1}^{\lfloor n/m\rfloor} F\Big(\zeta\Big(\frac{i}{m}, \frac{2\pi j}{\lfloor n/m\rfloor}\Big)\Big) \\ &= \frac{(n \bmod m)F(0)}{n} + \frac{m\lfloor n/m\rfloor}{2\pi n}\cdot\frac{2\pi}{m\lfloor n/m\rfloor}\sum_{i=1}^m \sum_{j=1}^{\lfloor n/m\rfloor} F\Big(\zeta\Big(\frac{i}{m}, \frac{2\pi j}{\lfloor n/m\rfloor}\Big)\Big) \\ &\xrightarrow{n\to\infty} \frac{1}{2\pi}\int_{[0,1]\times[0,2\pi]} F(\zeta(x,\theta))\mathrm{d}x\mathrm{d}\theta,\end{aligned}$$

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where the last limit follows from the assumptions on m and the fact that $\frac{2\pi}{m\lfloor n/m\rfloor}\sum_{i=1}^{m}\sum_{j=1}^{\lfloor n/m\rfloor}F(\zeta(\frac{i}{m},\frac{2\pi j}{\lfloor n/m\rfloor}))$ is a Riemann sum for the continuous (and hence Riemann-integrable) function $F(\zeta(x,\theta))$. ∎

4.1.5. Finalization of the proof. The last results we need to prove Theorem 3.1 are the following two technical lemmas.

Lemma 4.5. *Let $\kappa:[0,1]\times[-\pi,\pi]\to\mathbb{C}$ be measurable. Then, there exists functions $a_{i,\ell}$, $f_{i,\ell}$, $i=1,\dots,N_\ell$, with $a_{i,\ell}\in C^\infty([0,1])$ and $f_{i,\ell}$ trigonometric polynomial, such that*

$$\sum_{i=1}^{N_\ell}a_{i,\ell}(x)f_{i,\ell}(\theta)\to\kappa(x,\theta)\ \text{ a.e. in } [0,1]\times[-\pi,\pi].$$

Moreover, if κ is real, then $a_{i,\ell}$, $f_{i,\ell}$ can be chosen to be real for all $i=1,\dots,N_\ell$ and all ℓ.

Proof. See [19, Lemma 2.8]. ∎

Lemma 4.6. *Let $\{A_n\}_n$ be a matrix sequence and let $\{B_{n,m}\}_n\xrightarrow{\text{a.c.s.}}\{A_n\}_n$. Then, there exists a sequence $\{m(n)\}_n$ such that $m(n)\to\infty$ and $\{A_n\}_n\equiv_{\text{a.c.s.}}\{B_{n,m(n)}\}_n$.*

Proof. Since $\{B_{n,m}\}_n\xrightarrow{\text{a.c.s.}}\{A_n\}_n$, for every m there exists n_m such that, for $n\ge n_m$,

$$A_n=B_{n,m}+R_{n,m}+N_{n,m},\qquad \operatorname{rank}(R_{n,m})\le c(m)n,\qquad \|N_{n,m}\|\le\omega(m),$$

where $\lim_{m\to\infty}c(m)=\lim_{m\to\infty}\omega(m)=0$. Without loss of generality, we can assume that the integers n_m have been chosen so that the sequence $\{n_m\}_m$ is strictly increasing. Define

$$m(n)=\begin{cases}1, & \text{if } n<n_2,\\ 2, & \text{if } n_2\le n<n_3,\\ 3, & \text{if } n_3\le n<n_4,\\ \dots\end{cases}$$

It is clear that $m(n)\to\infty$. Moreover, for every $n\ge n_2$, we have $n\ge n_{m(n)}$ and so

$$A_n=B_{n,m(n)}+R_{n,m(n)}+N_{n,m(n)},\qquad \operatorname{rank}(R_{n,m(n)})\le c(m(n))n,\qquad \|N_{n,m(n)}\|\le\omega(m(n)).$$

Since $c(m(n))\to0$ and $\omega(m(n))\to0$ (because $m(n)\to\infty$), the matrix sequence $\{R_{n,m(n)}+N_{n,m(n)}\}_n$ is zero-distributed and we conclude that $\{A_n\}_n\equiv_{\text{a.c.s.}}\{B_{n,m(n)}\}_n$. ∎

Proof of Theorem 3.1. Let $\{A_n\}_n\sim_{\text{GLT}}\kappa$, let $\{m(n)\}_n$ be a sequence such that $m(n)\le n$, $m(n)\to\infty$ and $m(n)=o(n)$ as $n\to\infty$, and let $Q_n=Q_{n,m(n)}$, where $Q_{n,m}$ is defined in (3.7). By Lemma 4.5, there exist functions $a_{i,\ell}$, $f_{i,\ell}$, $i=1,\dots,N_\ell$, with $a_{i,\ell}\in C^\infty([0,1])$ and $f_{i,\ell}$ trigonometric polynomial, such that $a_{i,\ell}$, $f_{i,\ell}$ are real for all i,ℓ if κ is real and

$$\sum_{i=1}^{N_\ell}a_{i,\ell}(x)f_{i,\ell}(\theta)\to\kappa(x,\theta)\ \text{ a.e.}$$

By **GLT 2 – GLT 3**, we have

$$\left\{\sum_{i=1}^{N_\ell}D_n(a_{i,\ell})T_n(f_{i,\ell})\right\}_n\sim_{\text{GLT}}\sum_{i=1}^{N_\ell}a_{i,\ell}(x)f_{i,\ell}(\theta).$$

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By **GLT 6**, we obtain

$$\left\{\sum_{i=1}^{N_\ell} D_n(a_{i,\ell})T_n(f_{i,\ell})\right\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n.$$

By Lemmas 4.2 and 4.3, for every i,ℓ, we have

$$\{D_n(a_{i,\ell})T_n(f_{i,\ell})\}_n \equiv_{\text{a.c.s.}} \{LT_n^{m(n)}(a_{i,\ell},f_{i,\ell})\}_n \equiv_{\text{a.c.s.}} \{LC_n^{m(n)}(a_{i,\ell},f_{i,\ell})\}_n.$$

Thus,

$$\left\{\sum_{i=1}^{N_\ell} D_n(a_{i,\ell})T_n(f_{i,\ell})\right\}_n \equiv_{\text{a.c.s.}} \left\{\sum_{i=1}^{N_\ell} LC_n^{m(n)}(a_{i,\ell},f_{i,\ell})\right\}_n,$$

and, consequently,

$$\left\{\sum_{i=1}^{N_\ell} LC_n^{m(n)}(a_{i,\ell},f_{i,\ell})\right\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n.$$

By Lemma 4.6, there exists a sequence $\{\ell(n)\}_n$ such that $\ell(n)\to\infty$ and

$$\{A_n\}_n \equiv_{\text{a.c.s.}} \left\{\sum_{i=1}^{N_{\ell(n)}} LC_n^{m(n)}(a_{i,\ell(n)},f_{i,\ell(n)})\right\}_n.$$

Hence,

$$\left\{\sum_{i=1}^{N_\ell} LC_n^{m(n)}(a_{i,\ell},f_{i,\ell})\right\}_n \xrightarrow{\text{a.c.s.}} \left\{\sum_{i=1}^{N_{\ell(n)}} LC_n^{m(n)}(a_{i,\ell(n)},f_{i,\ell(n)})\right\}_n \equiv_{\text{a.c.s.}} \{A_n\}_n. \tag{4.13}$$

By (4.11),

$$\sum_{i=1}^{N_\ell} LC_n^{m(n)}(a_{i,\ell},f_{i,\ell}) = \sum_{i=1}^{N_\ell} Q_n\Delta_{n,m(n)}(a_{i,\ell},f_{i,\ell})Q_n^* = Q_n\left[\sum_{i=1}^{N_\ell}\Delta_{n,m(n)}(a_{i,\ell},f_{i,\ell})\right]Q_n^*. \tag{4.14}$$

The matrix inside the square brackets is diagonal and we denote it by $D_{n,\ell}$. We also set $D_n = D_{n,\ell(n)}$. By (4.13)–(4.14),

$$\{Q_nD_nQ_n^*\}_n \equiv_{\text{a.c.s.}} \{A_n\}_n.$$

Note that $\{Q_n\}_n$ is a unitary sequence that does not depend on $\{A_n\}_n$, so it only remains to prove that $\{D_n\}_n \sim_\lambda \kappa$. By definition, the a.c.s. convergence is not affected by a unitary base change. Hence, from (4.13)–(4.14), we obtain

$$\left\{\sum_{i=1}^{N_\ell}\Delta_{n,m(n)}(a_{i,\ell},f_{i,\ell})\right\}_n \xrightarrow{\text{a.c.s.}} \{D_n\}_n. \tag{4.15}$$

Note that $\sum_{i=1}^{N_\ell}\Delta_{n,m(n)}(a_{i,\ell},f_{i,\ell}) = \Delta_{n,m(n)}(\zeta)$, where $\zeta(x,\theta)=\sum_{i=1}^{N_\ell}a_{i,\ell}(x)f_{i,\ell}(\theta)$ is continuous on $[0,1]\times\mathbb{R}$. Thus, Lemma 4.4 yields

$$\left\{\sum_{i=1}^{N_\ell}\Delta_{n,m(n)}(a_{i,\ell},f_{i,\ell})\right\}_n \sim_\lambda \sum_{i=1}^{N_\ell}a_{i,\ell}(x)f_{i,\ell}(\theta). \tag{4.16}$$

Note that, according to Lemma 4.4, (4.16) holds if we consider for the function $\sum_{i=1}^{N_\ell}a_{i,\ell}(x)f_{i,\ell}(\theta)$ the domain $[0,1]\times[0,2\pi]$, but actually (4.16) holds even if we replace $[0,1]\times[0,2\pi]$ with $[0,1]\times[-\pi,\pi]$ because $\sum_{i=1}^{N_\ell}a_{i,\ell}(x)f_{i,\ell}(\theta)$ is 2π-periodic in the variable θ. The thesis $\{D_n\}_n\sim_\lambda\kappa$ now follows from Theorem 4.1, whose hypotheses are satisfied in view of (4.15)–(4.16) and the fact that $\sum_{i=1}^{N_\ell}a_{i,\ell}(x)f_{i,\ell}(\theta)\to\kappa(x,\theta)$ a.e. ∎

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4.2. Proof of Theorem 3.2. The proof of Theorem 3.2 requires two preliminary results. The first one is Theorem 4.3, which was proved in [2]. The second one is Theorem 4.4, which is proved below and is of interest also in itself.

4.2.1. GLT sequences and diagonal sequences enjoying a spectral distribution. The next theorem is the main result of [2]. It plays a central role in the proof of Theorem 3.2.

THEOREM 4.3. *Let $g : [0,1] \to \mathbb{C}$ be measurable and let $\{D_n\}_n$ be a diagonal sequence such that $\{D_n\}_n \sim_\lambda g$. Then, there exists a sequence of permutation matrices $\{P_n\}_n$ such that $\{P_n D_n P_n^T\}_n \sim_{\rm GLT} \kappa(x,\theta) = g(x)$.*

Proof. See [2, Theorem 2]. ∎

4.2.2. Normal sequences: from singular value distribution to spectral distribution. In what follows, a radial function $G : \mathbb{C} \to \mathbb{C}$ is a function of the form $G(z) = H(|z - z_0|)$ with $H : \mathbb{R} \to \mathbb{C}$ and $z_0 \in \mathbb{C}$. It was proved in [16] that any $F \in C_c(\mathbb{C})$ can be arbitrarily approximated in ∞-norm by linear combinations of Gaussian functions, which are special cases of radial functions. It follows that the vector space generated by the set of radial functions

$$\mathscr{R} = \{H(|z - z_0|) : \, H \in C_c(\mathbb{R}), \; z_0 \in \mathbb{C}\}$$

is dense in $C_c(\mathbb{C})$ with respect to the ∞-norm. In other words, if $F \in C_c(\mathbb{C})$ and $\varepsilon > 0$, then there exists a linear combination L of radial functions belonging to the set $\mathscr{R}$ such that $\|F - L\|_\infty < \varepsilon$.

THEOREM 4.4. *Let $\{A_n\}_n$ be a normal sequence such that $\{A_n - cI_n\}_n \sim_\sigma f - c$ for all $c \in \mathbb{C}$ and for some measurable $f : D \subset \mathbb{R}^k \to \mathbb{C}$ with $0 < \mu_k(D) < \infty$. Then $\{A_n\}_n \sim_\lambda f$.*

Proof. It is enough to prove that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} G(\lambda_i(A_n)) = \frac{1}{\mu_k(D)}\int_D G(f(\mathbf{x}))\mathrm{d}\mathbf{x}$$

for all radial functions G belonging to the set $\mathscr{R}$. Let $G(z) = H(|z - z_0|)$ be a radial function belonging to the set $\mathscr{R}$. The matrices $A_n - z_0 I_n$ are normal by assumption, and so $\sigma_i(A_n - z_0 I_n) = |\lambda_i(A_n - z_0 I_n)|$ for all $i = 1, \ldots, n$. Using the hypothesis $\{A_n - z_0 I_n\}_n \sim_\sigma f - z_0$, we obtain

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} G(\lambda_i(A_n)) &= \frac{1}{n}\sum_{i=1}^{n} H(|\lambda_i(A_n) - z_0|) = \frac{1}{n}\sum_{i=1}^{n} H(|\lambda_i(A_n - z_0 I_n)|) \\ &\xrightarrow{n\to\infty} \frac{1}{\mu_k(D)}\int_D H(|f(\mathbf{x}) - z_0|)\mathrm{d}\mathbf{x} = \frac{1}{\mu_k(D)}\int_D G(f(\mathbf{x}))\mathrm{d}\mathbf{x},\end{aligned}$$

and the theorem is proved. ∎

4.2.3. Finalization of the proof. We have now collected all the necessary ingredients to prove Theorem 3.2.

Proof of Theorem 3.2. In the case where Y_n satisfies the first or the second condition in the statement of the theorem, the thesis $\{X_n + Y_n\}_n \sim_\lambda f$ is a direct consequence of [9, Problem VI.8.2]. We prove the thesis in the case where Y_n satisfies the third condition. Let $A_n = X_n + Y_n$ and let g be a rearranged version

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of f on $[0,1]$, i.e., a measurable function $g:[0,1]\to\mathbb{C}$ such that

$$\int_0^1 F(g(x))\mathrm{d}x = \int_0^1 F(f(\mathbf{x}))\mathrm{d}\mathbf{x}, \qquad \forall\, F\in C_c(\mathbb{C}).$$

We remark that such a function g exists by [2, Lemma 6]. We also note that $\{X_n\}_n \sim_\lambda f$ is equivalent to $\{X_n\}_n \sim_\lambda g$. Since $\{X_n\}_n$ is a normal sequence with $\{X_n\}_n \sim_\lambda g$, there exists a unitary sequence $\{U_n\}_n$ such that $X_n = U_n D_n U_n^*$, where $\{D_n\}_n$ is a diagonal sequence with $\{D_n\}_n \sim_\lambda g$. In view of Theorem 4.3, by replacing D_n with $P_n D_n P_n^T$ for a suitable permutation matrix P_n (if necessary), we can assume that $\{D_n\}_n \sim_{\rm GLT} g(x)$. Thus,

$$U_n^* A_n U_n = U_n^* X_n U_n + U_n^* Y_n U_n = D_n + Z_n,$$

where $\{Z_n\}_n = \{U_n^* Y_n U_n\}_n$ is zero-distributed like $\{Y_n\}_n$, and so $\{U_n^* A_n U_n\}_n \sim_{\rm GLT} g(x)$ by **GLT 2 – GLT 3**. It follows from **GLT 2 – GLT 3** and the relation $\{I_n\}_n = \{T_n(1)\}_n \sim_{\rm GLT} 1$ that $\{U_n^* A_n U_n - cI_n\}_n \sim_{\rm GLT} g(x) - c$ for all $c\in\mathbb{C}$. We conclude that $\{U_n^* A_n U_n - cI_n\}_n \sim_\sigma g - c$ for all $c\in\mathbb{C}$ (by **GLT 1**) and $\{U_n^* A_n U_n\}_n \sim_\lambda g$ (by Theorem 4.4). The latter is equivalent to $\{A_n\}_n \sim_\lambda g$ by definition of spectral distribution. This means that $\{A_n\}_n \sim_\lambda f$ because g is just a rearranged version of f. ∎

4.3. Proof of Theorem 3.3. Theorem 3.3 is a direct consequence of Theorems 3.1 and 3.2.

Proof of Theorem 3.3. Let $\{A_n\}_n \sim_{\rm GLT} \kappa$ and assume that the matrices A_n are normal. By Theorem 3.1, there exist a unitary sequence $\{Q_n\}_n$ and a zero-distributed sequence $\{Z_n\}_n$ such that $A_n = Q_n D_n Q_n^* + Z_n$, where $\{D_n\}_n$ is a diagonal sequence and $\{D_n\}_n \sim_\lambda \kappa$. The thesis $\{A_n\}_n \sim_\lambda \kappa$ now follows from Theorem 3.2 applied with $X_n = Q_n D_n Q_n^*$ and $Y_n = Z_n$. ∎

4.4. Proof of Theorem 3.4. The proof of Theorem 3.4 requires some preliminary results on matrix functions and matrix polynomials as well as on the so-called sparsely unbounded matrix sequences.

4.4.1. Matrix functions and matrix polynomials. We first recall the notion of matrix functions in the case of a diagonalizable matrix. For more details on matrix functions, see [22].

Definition 4.3 (function of a diagonalizable matrix). *Let $A\in\mathbb{C}^{n\times n}$ be a diagonalizable matrix and let $f:\Lambda(A)\to\mathbb{C}$ be a function defined on the spectrum of A, denoted by $\Lambda(A)$. We define $f(A)$ as the unique matrix in $\mathbb{C}^{n\times n}$ such that $f(A)\mathbf{u} = f(\lambda)\mathbf{u}$ whenever $A\mathbf{u} = \lambda\mathbf{u}$.*

By Definition 4.3, if

$$A = XDX^{-1}, \qquad D = \mathrm{diag}(\lambda_1,\ldots,\lambda_n),$$

is a spectral decomposition of the diagonalizable matrix A, then a spectral decomposition of $f(A)$ is given by

$$f(A) = Xf(D)X^{-1}, \qquad f(D) = \mathrm{diag}(f(\lambda_1),\ldots,f(\lambda_n)).$$

Definition 4.4 (complex bivariate polynomial). *A complex bivariate polynomial is a function $p(x,y):\mathbb{C}\to\mathbb{C}$ of the form*

$$p(x,y) = a(x,y) + \mathrm{i}\, b(x,y),$$

where $a(x,y), b(x,y):\mathbb{R}^2\to\mathbb{R}$ are real bivariate polynomials. Note that we write $p(x,y)$ instead of $p(z)$ for convenience (x and y are the real part and the imaginary part of z, respectively). Note also that a complex bivariate polynomial $p(x,y)$ should not be confused with a polynomial $q(z)$ of the variable z, i.e., a (analytic) function of the form $q(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_m z^m$.

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DEFINITION 4.5 (complex bivariate polynomial of commuting matrices). *If $p(x,y):\mathbb{C}\to\mathbb{C}$ is a complex bivariate polynomial and A,B are commuting matrices in $\mathbb{C}^{n\times n}$, then we define $p(A,B)$ as the matrix in $\mathbb{C}^{n\times n}$ obtained by replacing x with A and y with B in the expression $p(x,y)$ (with the usual convention that a constant α independent of x and y in the expression of $p(x,y)$ must be interpreted as αx^0y^0 and replaced with αI_n). Note that this definition is well-posed because A,B commute and so the resulting matrix $p(A,B)$ does not depend on the way in which we write the expression $p(x,y)$.*

For example, if $p(x,y)=2+yx+xy^2+\mathrm{i}(2xy+yx^3)$ and A,B are commuting matrices in $\mathbb{C}^{n\times n}$, then $p(A,B)=2I_n+BA+AB^2+\mathrm{i}(2AB+BA^3)$. If we write $p(x,y)=2+xy+xy^2+\mathrm{i}(2xy+x^3y)$, the resulting matrix $p(A,B)=2I_n+AB+AB^2+\mathrm{i}(2AB+A^3B)$ is the same because A and B commute. In what follows, whenever $f(x,y):\mathbb{C}\to\mathbb{C}$ is a complex function written as $f(x,y)$ instead of $f(z)$, it is understood that a writing such as $f(z_0)$, with a unique argument z_0, must be interpreted as the complex number obtained by replacing x with $\mathrm{Re}\,z_0$ and y with $\mathrm{Im}\,z_0$ in the expression $f(x,y)$.

LEMMA 4.7. *Let $A\in\mathbb{C}^{n\times n}$ be a normal matrix and let $p(x,y):\mathbb{C}\to\mathbb{C}$ be a complex bivariate polynomial. Then, $p(A)$ as given by Definition 4.3 coincides with $p(\mathrm{Re}\,A,\mathrm{Im}\,A)$, where*

$$\mathrm{Re}\,A=\frac{A+A^*}{2},\qquad \mathrm{Im}\,A=\frac{A-A^*}{2\mathrm{i}}$$

are commuting matrices (because A is normal).

Proof. Since $p(x,y)$ can be written as a linear combination of monic monomials in the variables x,y, it is enough to prove the result for such monomials. Let $q(x,y)=x^ry^s$ be a monic monomial in the variables x,y. Consider a spectral decomposition of A,

$$A=QDQ^*,\qquad D=\mathrm{diag}(\lambda_1,\ldots,\lambda_n),\qquad Q\text{ unitary}.$$

Note that

$$\begin{aligned}\mathrm{Re}\,A&=Q(\mathrm{Re}\,D)Q^*, &\quad \mathrm{Re}\,D&=\mathrm{diag}(\mathrm{Re}\,\lambda_1,\ldots,\mathrm{Re}\,\lambda_n),\\ \mathrm{Im}\,A&=Q(\mathrm{Im}\,D)Q^*, &\quad \mathrm{Im}\,D&=\mathrm{diag}(\mathrm{Im}\,\lambda_1,\ldots,\mathrm{Im}\,\lambda_n).\end{aligned}$$

By Definition 4.3, we have

$$\begin{aligned}q(A)&=Qq(D)Q^*\\ &=Q\,\mathrm{diag}(q(\lambda_1),\ldots,q(\lambda_n))Q^*\\ &=Q\,\mathrm{diag}((\mathrm{Re}\,\lambda_1)^r(\mathrm{Im}\,\lambda_1)^s,\ldots,(\mathrm{Re}\,\lambda_n)^r(\mathrm{Im}\,\lambda_n)^s)Q^*\\ &=(Q\,\mathrm{diag}(\mathrm{Re}\,\lambda_1,\ldots,\mathrm{Re}\,\lambda_n)Q^*)^r(Q\,\mathrm{diag}(\mathrm{Im}\,\lambda_1,\ldots,\mathrm{Im}\,\lambda_n)Q^*)^s\\ &=(\mathrm{Re}\,A)^r(\mathrm{Im}\,A)^s\\ &=q(\mathrm{Re}\,A,\mathrm{Im}\,A).\end{aligned}$$

□

4.4.2. Sparsely unbounded matrix sequences. We report in this section the notion of sparsely unbounded matrix sequences and some related results that are necessary for the proof of Theorem 3.4.

DEFINITION 4.6 (sparsely unbounded matrix sequence). *A matrix sequence $\{A_n\}_n$ is said to be sparsely unbounded (s.u.) if for every $M>0$ there exists n_M such that, for $n\ge n_M$,*

$$\frac{\#\{i\in\{1,\ldots,n\}:\,\sigma_i(A_n)>M\}}{n}\le r(M),$$

where $\lim_{M\to\infty}r(M)=0$.

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Any matrix sequence enjoying a singular value distribution as per Definition 2.1 is s.u.

LEMMA 4.8. *If* $\{A_n\}_n$ *is a matrix sequence such that* $\{A_n\}_n \sim_\sigma f$ *for some function* f*, then* $\{A_n\}_n$ *is s.u.*

Proof. See [19, Proposition 5.4]. ∎

LEMMA 4.9. *Let* $\{A_n\}_n$ *be an s.u. normal sequence. Then, the following property holds: for every* $M > 0$ *there exists* n_M *such that, for* $n \geq n_M$,

$$A_n = \hat{A}_{n,M} + \tilde{A}_{n,M}, \qquad \operatorname{rank}(\hat{A}_{n,M}) \leq r(M)n, \qquad \|\tilde{A}_{n,M}\| \leq M, \tag{4.17}$$

where $\lim_{M\to\infty} r(M) = 0$*, the matrices* $\hat{A}_{n,M}$ *and* $\tilde{A}_{n,M}$ *are normal, and for all functions* $g : \mathbb{C} \to \mathbb{C}$ *satisfying* $g(0) = 0$ *we have*

$$g(\hat{A}_{n,M} + \tilde{A}_{n,M}) = g(\hat{A}_{n,M}) + g(\tilde{A}_{n,M}).$$

Proof. Since the matrices A_n are normal, the singular values $\sigma_i(A_n)$, $i = 1, \ldots, n$, coincide with the moduli of the eigenvalues $|\lambda_i(A_n)|$, $i = 1, \ldots, n$. Since $\{A_n\}_n$ is s.u., for every $M > 0$ there exists n_M such that, for $n \geq n_M$,

$$\frac{\#\{i \in \{1, \ldots, n\} : |\lambda_i(A_n)| > M\}}{n} \leq r(M),$$

where $\lim_{M\to\infty} r(M) = 0$. Let $A_n = U_n\Lambda_nU_n^*$ be a spectral decomposition of A_n. Let $\hat{\Lambda}_{n,M}$ be the matrix obtained from Λ_n by setting to 0 all the eigenvalues of A_n whose modulus is less than or equal to M, and let $\tilde{\Lambda}_{n,M} = \Lambda_n - \hat{\Lambda}_{n,M}$ be the matrix obtained from Λ_n by setting to 0 all the eigenvalues of A_n whose modulus is greater than M. Then, for $M > 0$ and $n \geq n_M$,

$$A_n = U_n\Lambda_nU_n^* = U_n\hat{\Lambda}_{n,M}U_n^* + U_n\tilde{\Lambda}_{n,M}U_n^* = \hat{A}_{n,M} + \tilde{A}_{n,M},$$

where $\hat{A}_{n,M} = U_n\hat{\Lambda}_{n,M}U_n^*$ and $\tilde{A}_{n,M} = U_n\tilde{\Lambda}_{n,M}U_n^*$. The matrices $\hat{A}_{n,M}$, $\tilde{A}_{n,M}$ constructed in this way are normal, satisfy the properties (4.17) and, moreover, for all functions $g : \mathbb{C} \to \mathbb{C}$ satisfying $g(0) = 0$ we have

$$g(\hat{A}_{n,M} + \tilde{A}_{n,M}) = g(\hat{A}_{n,M}) + g(\tilde{A}_{n,M}).$$ ∎

4.4.3. Finalization of the proof. We have now collected all the necessary ingredients to prove Theorem 3.4.

Proof of Theorem 3.4. Let $\{A_n\}_n \sim_{\mathrm{GLT}} \kappa$ with A_n normal for every n, and let $f(x,y) : \mathbb{C} \to \mathbb{C}$ be continuous. For each $M > 0$, let $p_{m,M}(x,y) : \mathbb{C} \to \mathbb{C}$ be a sequence of complex bivariate polynomials that converges uniformly to $f(x,y)$ over $[-M,M]^2$:

$$\lim_{m\to\infty} \|f - p_{m,M}\|_{\infty,[-M,M]^2} = 0.$$

Note that such a sequence exists by the Stone–Weierstrass theorem; see, e.g., [24, Theorem 7.33]. By replacing $p_{m,M}$ with $p_{m,M} + f(0) - p_{m,M}(0)$ if necessary, we can assume, without loss of generality, that $p_{m,M}(0) = f(0)$. Since any GLT sequence is s.u. (by **GLT 1** and Lemma 4.8), the sequence $\{A_n\}_n$ is s.u. Hence, by Lemma 4.9, for all $M > 0$ there exists n_M such that, for $n \geq n_M$,

$$A_n = \hat{A}_{n,M} + \tilde{A}_{n,M}, \qquad \operatorname{rank}(\hat{A}_{n,M}) \leq r(M)n, \qquad \|\tilde{A}_{n,M}\| \leq M, \tag{4.18}$$

where $r(M) \to 0$ as $M \to \infty$, the matrices $\hat{A}_{n,M}$ and $\tilde{A}_{n,M}$ are normal, and for all functions $g : \mathbb{C} \to \mathbb{C}$ satisfying $g(0) = 0$ we have

$$g(\hat{A}_{n,M} + \tilde{A}_{n,M}) = g(\hat{A}_{n,M}) + g(\tilde{A}_{n,M}).$$

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Taking into account that $(f-p_{m,M})(0)=0$, for every $M>0$, every m and every $n\ge n_M$, we can write

$$\begin{aligned} f(A_n) &= p_{m,M}(A_n)+f(A_n)-p_{m,M}(A_n)\\ &= p_{m,M}(A_n)+(f-p_{m,M})(\hat{A}_{n,M})+(f-p_{m,M})(\tilde{A}_{n,M})\\ &= p_{m,M}(A_n)+R_{n,m,M}+N_{n,m,M}, \end{aligned} \tag{4.19}$$

where, in view of (4.18), the matrices $R_{n,m,M}=(f-p_{m,M})(\hat{A}_{n,M})$ and $N_{n,m,M}=(f-p_{m,M})(\tilde{A}_{n,M})$ satisfy

$$\begin{aligned} \operatorname{rank}(R_{n,m,M}) &\le \operatorname{rank}(\hat{A}_{n,M})\le r(M)n,\\ \|N_{n,m,M}\| &\le \|f-p_{m,M}\|_{\infty,[-M,M]^2}. \end{aligned}$$

Choose a sequence $\{M_m\}_m$ such that

$$M_m\to\infty,\qquad \|f-p_{m,M_m}\|_{\infty,[-M_m,M_m]^2}\to 0. \tag{4.20}$$

Then, for every m and every $n\ge n_{M_m}$,

$$\begin{aligned} f(A_n) &= p_{m,M_m}(A_n)+R_{n,m,M_m}+N_{n,m,M_m},\\ \operatorname{rank}(R_{n,m,M_m}) &\le r(M_m)n,\\ \|N_{n,m,M_m}\| &\le \|f-p_{m,M_m}\|_{\infty,[-M_m,M_m]^2}, \end{aligned}$$

which implies that

$$\{p_{m,M_m}(A_n)\}_n \xrightarrow{\text{a.c.s.}} \{f(A_n)\}_n.$$

Moreover, by Lemma 4.7, we have $p_{m,M_m}(A_n)=p_{m,M_m}(\operatorname{Re}A_n,\operatorname{Im}A_n)$, where $\{\operatorname{Re}A_n\}_n\sim_{\rm GLT}\operatorname{Re}\kappa$ and $\{\operatorname{Im}A_n\}_n\sim_{\rm GLT}\operatorname{Im}\kappa$ by **GLT 3**. Hence, again by **GLT 3**,

$$\{p_{m,M_m}(A_n)\}_n=\{p_{m,M_m}(\operatorname{Re}A_n,\operatorname{Im}A_n)\}_n\sim_{\rm GLT} p_{m,M_m}(\operatorname{Re}\kappa,\operatorname{Im}\kappa)=p_{m,M_m}(\kappa).$$

Finally, by (4.20),

$$p_{m,M_m}(\kappa)\to f(\kappa)\ \text{ a.e.}$$

We conclude that $\{f(A_n)\}_n\sim_{\rm GLT} f(\kappa)$ by **GLT 5**. □

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REFERENCES

[1] G. Barbarino. Equivalence between GLT sequences and measurable functions. *Linear Algebra Appl.*, 529:397–412, 2017.
[2] G. Barbarino. Spectral measures. *Springer INdAM Ser.*, 30:1–24, 2019.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 1-20, January 2025.

[3] G. Barbarino. A systematic approach to reduced GLT. *BIT Numer. Math.*, 62:681–743, 2022.

[4] G. Barbarino, S.-E. Ekström, C. Garoni, D. Meadon, S. Serra-Capizzano, and P. Vassalos. From asymptotic distribution and vague convergence to uniform convergence, with numerical applications. *J. Numer. Math.*, to appear.

[5] G. Barbarino and C. Garoni. From convergence in measure to convergence of matrix-sequences through concave functions and singular values. *Electron. J. Linear Algebra*, 32:500–513, 2017.

[6] G. Barbarino, C. Garoni, M. Mazza, and S. Serra-Capizzano. Rectangular GLT sequences. *Electron. Trans. Numer. Anal.*, 55:585–617, 2022.

[7] G. Barbarino, C. Garoni, and S. Serra-Capizzano. Block generalized locally Toeplitz sequences: theory and applications in the unidimensional case. *Electron. Trans. Numer. Anal.*, 53:28–112, 2020.

[8] G. Barbarino, C. Garoni, and S. Serra-Capizzano. Block generalized locally Toeplitz sequences: theory and applications in the multidimensional case. *Electron. Trans. Numer. Anal.*, 53:113–216, 2020.

[9] R. Bhatia. *Matrix Analysis.* Springer, New York, 1997.

[10] D. Bianchi. Analysis of the spectral symbol associated to discretization schemes of linear self-adjoint differential operators. *Calcolo*, 58:38, 2021.

[11] D. Bianchi and C. Garoni. On the asymptotic spectral distribution of increasing size matrices: test functions, spectral clustering, and asymptotic estimates of outliers. *Linear Algebra Appl.*, 697:615–638, 2024.

[12] J.M. Bogoya, A. Böttcher, S.M. Grudsky, and E.A. Maximenko. Maximum norm versions of the Szegő and Avram–Parter theorems for Toeplitz matrices. *J. Approx. Theory*, 196:79–100, 2015.

[13] J.M. Bogoya, A. Böttcher, and E.A. Maximenko. From convergence in distribution to uniform convergence. *Bol. Soc. Mat. Mex.*, 22:695–710, 2016.

[14] A. Böttcher, C. Garoni, and S. Serra-Capizzano. Exploration of Toeplitz-like matrices with unbounded symbols is not a purely academic journey. *Sb. Math.*, 208:1602–1627, 2017.

[15] A. Böttcher and B. Silbermann. *Introduction to Large Truncated Toeplitz Matrices.* Springer, New York, 1999.

[16] F.L. Cao and T.F. Xie. The rate of approximation of Gaussian radial basis neural networks in continuous function space. *Acta Math. Sin. Engl. Ser.*, 29:295–302, 2013.

[17] P.J. Davis. *Circulant Matrices*, second edition. AMS Chelsea Publishing, Rhode Island, 1994.

[18] C. Garoni. Introduction to the theory of generalized locally Toeplitz sequences and its applications. Submitted.

[19] C. Garoni and S. Serra-Capizzano. *Generalized Locally Toeplitz Sequences: Theory and Applications*, Vol. I. Springer, Cham, 2017.

[20] C. Garoni and S. Serra-Capizzano. *Generalized Locally Toeplitz Sequences: Theory and Applications*, Vol. II. Springer, Cham, 2018.

[21] G.H. Golub and C.F. Van Loan. *Matrix Computations*, fourth edition. The Johns Hopkins University Press, Baltimore, 2013.

[22] N.J. Higham. *Functions of Matrices: Theory and Computation.* SIAM, Philadelphia, 2008.

[23] V.B. Kiran Kumar, R. Rajan, and N.S. Sarathkumar. Banach spaces of GLT sequences and function spaces. *Electron. J. Linear Algebra*, 38:295–316, 2022.

[24] W. Rudin. *Principles of Mathematical Analysis*, third edition. McGraw-Hill, New York, 1976.