



NEW SUFFICIENT CONDITIONS FOR SUBDIRECT SUMS OF NEKRASOV MATRICES*

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Abstract. Some new sufficient conditions ensuring that the k -subdirect sum of strictly diagonally dominant matrices and Nekrasov matrices is in the class of Nekrasov matrices are given. These sufficient conditions are different from those in [*Electron. J. Linear Algebra*, 38:339–346, 2022] and [*Linear Multilinear Algebra*, 64:208–218, 2016; 72:1044–1055, 2023]. In addition, some examples are given to illustrate the conditions presented.

Key words. Subdirect sums, Nekrasov matrices, Strictly diagonally dominant matrices.

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1. Introduction. As a generalization of the usual sum of matrices [3], the concept of k -subdirect sums of square matrices was introduced by Fallat and Johnson [8]. Its definition is listed as below, see Definition 1.1. Since the subdirect sum could be applied in a variety of practical applications, such as in matrix completion problems, overlapping subdomains in domain decomposition methods, and global stiffness matrices in finite elements (for details, see [2, 8, 9, 22]), many interesting and important results on its closure property appeared now and then. For example, Fallat and Johnson in [8] showed that the k -subdirect sum of positive definite matrices, or symmetric M -matrices, is a positive definite matrix or a symmetric M -matrix, respectively. Bru et al. in [2] gave sufficient conditions ensuring that the k -subdirect sum of two nonsingular M -matrices is a nonsingular M -matrix. We refer to [3, 4, 5, 10, 12, 15, 16, 18, 24, 25, 26] for the closure property on the subdirect sum of other classes of matrices like doubly diagonally dominant matrices [25], S -strictly diagonally dominant matrices [3], Σ -strictly diagonally dominant matrices [4], Nekrasov matrices [15, 23], QN-matrices [10], α_1 and α_2 -matrices [5], weakly chained diagonally dominant matrices [16], $\{P_1, P_2\}$ -Nekrasov Matrices [12], $SDD(p)$ matrices [18], and H -matrices [26].

DEFINITION 1.1. [8, 15] Let A and B be two square matrices of order n_1 and n_2 , respectively, and let k be an integer such that $1 \leq k \leq \min\{n_1, n_2\}$. Let A and B be partitioned into 2×2 blocks as follows,

$$(1.1) \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where A_{22} and B_{11} are square matrices of order k . We call the square matrix of order $n_1 + n_2 - k$ given by

$$(1.2) \quad C = \left(\begin{array}{c|c|c} A_{11} & A_{12} & 0 \\ \hline A_{21} & A_{22} + B_{11} & B_{12} \\ \hline 0 & B_{21} & B_{22} \end{array} \right),$$

the k -subdirect sum of A and B , denoted by $C = A \oplus_k B$. Furthermore, if we let $C = A \oplus_k B = (c_{ij})$, $A = (a_{ij})$ and $B = (b_{ij})$, then

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$$c_{ij} = \begin{cases} a_{ij}, & i \in S_1, \quad j \in S_1 \cup S_2, \\ 0, & i \in S_1, \quad j \in S_3, \\ a_{ij}, & i \in S_2, \quad j \in S_1, \\ a_{ij} + b_{i-t, j-t}, & i \in S_2, \quad j \in S_2, \\ b_{i-t, j-t}, & i \in S_2, \quad j \in S_3, \\ 0, & i \in S_3, \quad j \in S_1, \\ b_{i-t, j-t}, & i \in S_3, \quad j \in S_2 \cup S_3, \end{cases}$$

where

$$(1.3) \quad S_1 = \{1, 2, \dots, n_1 - k\}, \quad S_2 = \{n_1 - k + 1, \dots, n_1\}, \quad S_3 = \{n_1 + 1, \dots, n\}$$

and $t = n_1 - k$, $S_1 \cup S_2 \cup S_3 = N := \{1, 2, \dots, n\}$.

Nekrasov matrices including strictly diagonally dominant (SDD) matrices, as one important subclass of H -matrices, appear now and then in various fields, for details, see [6, 11]. In order to solve problems arising in different fields, it needs various properties for Nekrasov matrices, which leads to various problems related to Nekrasov matrices, such as Schur complement problem for Nekrasov matrices [19], estimating the infinity norm for the inverse of Nekrasov matrices [6, 21], error bound for linear complementarity problems [11, 17], etc.

DEFINITION 1.2. [1, 7, 13] A matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ is called a strictly diagonally dominant (SDD) matrix if

$$|a_{ii}| > r_i(A), \quad i \in N,$$

where $r_i(A) = \sum_{j \in N, j \neq i} |a_{ij}|$.

DEFINITION 1.3. [6, 14] A matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ is called a Nekrasov matrix if

$$|a_{11}| > r_1(A) = \sum_{j=2}^n |a_{1j}|, \quad |a_{ii}| > h_i(A), \quad i = 2, 3, \dots, n,$$

where $h_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^n |a_{ij}|$, $i = 2, 3, \dots, n$.

As for the subdirect sum of Nekrasov matrices, Li et al. in [15] first presented some sufficient conditions such that

$$C = A \oplus_k B$$

is a Nekrasov matrix, where A is a Nekrasov matrix and B is an SDD matrix. Subsequently, some new sufficient conditions for the closure property for Nekrasov matrices case were given in [15, 20, 23]. However, these sufficient conditions are established for the case that A is a Nekrasov matrix, and B is an SDD matrix, for example, see Theorem 1.4 (Theorem 2.8 in [15]) and Theorem 1.5 (Theorem 2.1 in [23]). As stated in [15], for a Nekrasov matrix A and an SDD matrix B , $B \oplus_k A$ may not be a Nekrasov matrix although $A \oplus_k B$ is a Nekrasov matrix, for details see Example 2.9 and Remark 2 in [15]. Hence, this motivates us to find some conditions such that the closure property of the k -subdirect sum $A \oplus_k B$ holds for the case that A is an SDD matrix, and B is a Nekrasov matrix.

THEOREM 1.4. [15, Theorem 2.8] Let $A = (a_{ij})$ and $B = (b_{ij})$ be square matrices of order $n_1 \geq 2$ and $n_2 \geq 2$ partitioned as in (1.1), respectively, and k be an integer such that $1 \leq k \leq \min\{n_1, n_2\}$ which defines the sets S_1, S_2, S_3 as in (2). We assume that A is a Nekrasov matrix and B is an SDD matrix. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), and for any $i \in S_2$,

$$a_{ij} = 0, \quad i > j, j \in S_2,$$

then the k -subdirect sum $C = A \oplus_k B$ for $k \geq 2$ is a Nekrasov matrix.

THEOREM 1.5. [23, Theorem 2.1] Let A be a Nekrasov matrix of order $n_1 \geq 2$, B be an SDD matrix of order $n_2 \geq 2$, k, S_1, S_2, S_3 , and t be the same as Definition 1.1, where $n = n_1 + n_2 - k$. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), and

$$|a_{t+m,t+m}| + |b_{mm}| - h_{t+m}(A) - r_m(B) > \sum_{j=1}^{m-1} \eta_{m,j}, \quad m = 2, \dots, k, \forall i \in S_2 \setminus \{t+1\},$$

then the k -subdirect sum $C = A \oplus_k B$ is a Nekrasov matrix, where

$$\eta_{m,j} = |a_{t+m,t+j} + b_{m,j}| \frac{H_{t+j}(A, B)}{|a_{t+j,t+j}| + |b_{jj}|} - |a_{t+m,t+j}| \frac{h_{t+j}(A)}{|a_{t+j,t+j}|} - |b_{m,j}|,$$

and

$$H_{t+m}(A, B) := h_{t+m}(A) + r_m(B) + \sum_{j=1}^{m-1} \eta_{m,j}.$$

The outline of the paper is as follows. In Section 2, we provide some sufficient conditions such that the k -subdirect sum $A \oplus_k B$ is also a Nekrasov matrix if A is an SDD matrix and B is a Nekrasov matrix. In particular, it is shown that $A \oplus_1 B$ is a Nekrasov matrix under some conditions. In Section 3, we end the paper by some concluding remarks.

2. The k -subdirect sum of SDD matrices and Nekrasov matrices. We in this section consider the k -subdirect sum of an SDD matrix A and a Nekrasov matrix B . First, we give a sufficient condition for the closure property of the 1-subdirect sum $A \oplus_1 B$.

THEOREM 2.1. Let $A = (a_{ij})$ and $B = (b_{ij})$ be square matrices of order n_1 and n_2 partitioned as in (1.1), respectively. And let $k = 1$, $S_1 = \{1, 2, \dots, n_1 - 1\}$, $S_2 = \{n_1\}$, $S_3 = \{n_1 + 1, \dots, n\}$ and t be the same as Definition 1.1, where $n = n_1 + n_2 - 1$. Assume that A is SDD, and B is a Nekrasov matrix. If all diagonal entries of A_{22} and B_{11} are positive (or all negative), and

$$(2.4) \quad \frac{h_{n_1}(A)}{|a_{n_1 n_1}|} \leq \frac{h_1(B)}{|b_{11}|},$$

then the 1-subdirect sum $C = A \oplus_1 B$ is a Nekrasov matrix.

Proof. Since A is SDD and B is a Nekrasov matrix, we have that for each $i \in \{1, 2, \dots, n_1\}$, $|a_{ii}| > r_i(A)$, and

$$|b_{11}| > r_1(B) = h_1(B), \quad |b_{ii}| > h_i(B), \quad i = 2, 3, \dots, n_2.$$

Case 1 For any $i \in S_1$, we have

$$|c_{ii}| = |a_{ii}| > r_i(A) \geq \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n_1} |a_{ij}| = \sum_{j=1}^{i-1} |c_{ij}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=i+1}^n |c_{ij}| = h_i(C).$$

Case 2 For $i = n_1 \in S_2$, we have

$$\begin{aligned} |c_{n_1, n_1}| &= |a_{n_1, n_1}| + |b_{11}| > r_{n_1}(A) + h_1(B) \\ &= \sum_{j=1}^{n_1-1} |a_{n_1, j}| + \sum_{j=2}^{n_2} |b_{1, j}| \\ &\geq \sum_{j=1}^{n_1-1} |a_{n_1, j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=2}^{n_2} |b_{1, j}| \\ &= \sum_{j=1}^{n_1-1} |c_{n_1, j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1+1}^n |c_{n_1, j}| \\ &= h_{n_1}(C) = h_{n_1}(A) + h_1(B). \end{aligned}$$

Furthermore, the above inequality yields

$$1 > \frac{h_{n_1}(A) + h_1(B)}{|a_{n_1 n_1}| + |b_{11}|} = \frac{h_{n_1}(C)}{|a_{n_1 n_1}| + |b_{11}|}.$$

In addition, by (2.4) we have

$$\begin{aligned} \frac{h_{n_1}(A)}{|a_{n_1 n_1}|} &\leq \frac{h_1(B)}{|b_{11}|} \\ &\Leftrightarrow |b_{11}| h_{n_1}(A) \leq |a_{n_1 n_1}| h_1(B) \\ &\Leftrightarrow |b_{11}| (h_{n_1}(A) + h_1(B)) \leq (|a_{n_1 n_1}| + |b_{11}|) h_1(B) \quad (a_{n_1 n_1}, b_{11} \text{ are positive (or all negative)}) \\ &\Leftrightarrow \frac{h_{n_1}(A) + h_1(B)}{|a_{n_1 n_1}| + |b_{11}|} \leq \frac{h_1(B)}{|b_{11}|}, \end{aligned}$$

and

$$\begin{aligned} \frac{h_{n_1}(A)}{|a_{n_1 n_1}|} &\leq \frac{h_1(B)}{|b_{11}|} \\ &\Leftrightarrow |b_{11}| h_{n_1}(A) \leq |a_{n_1 n_1}| h_1(B) \\ &\Leftrightarrow h_{n_1}(A) (|b_{11}| + |a_{n_1 n_1}|) \leq |a_{n_1 n_1}| (h_1(B) + h_{n_1}(A)) \quad (a_{n_1 n_1}, b_{11} \text{ are positive (or all negative)}) \\ &\Leftrightarrow \frac{h_{n_1}(A)}{|a_{n_1 n_1}|} \leq \frac{h_{n_1}(A) + h_1(B)}{|a_{n_1 n_1}| + |b_{11}|}. \end{aligned}$$

Therefore,

$$(2.5) \quad \frac{h_{n_1}(A)}{|a_{n_1 n_1}|} \leq \frac{h_{n_1}(C)}{|a_{n_1 n_1}| + |b_{11}|} = \frac{h_{n_1}(A) + h_1(B)}{|a_{n_1 n_1}| + |b_{11}|} \leq \frac{h_1(B)}{|b_{11}|}.$$

Case 3 For $i = n_1 + 1 \in S_3$, we have

$$\begin{aligned} |c_{n_1+1, n_1+1}| &= |b_{22}| > h_2(B) = |b_{21}| \frac{h_1(B)}{|b_{11}|} + \sum_{j=3}^{n_2} |b_{2j}| \\ &\geq |b_{21}| \frac{h_{n_1}(A) + h_1(B)}{|a_{n_1 n_1} + b_{11}|} + \sum_{j=3}^{n_2} |b_{2j}| \quad (\text{by (2.5)}) \\ &= \sum_{j=1}^{n_1-1} |c_{n_1+1, j}| \frac{h_j(A)}{|a_{jj}|} + |b_{21}| \frac{h_{n_1}(A) + h_1(B)}{|a_{n_1 n_1} + b_{11}|} + \sum_{j=3}^{n_2} |b_{2j}| \\ &= \sum_{j=1}^{n_1-1} |c_{n_1+1, j}| \frac{h_j(C)}{|c_{jj}|} + |c_{n_1+1, n_1}| \frac{h_{n_1}(C)}{|c_{n_1, n_1}|} + \sum_{j=n_1+2}^n |c_{n_1+1, j}| \\ &= h_{n_1+1}(C), \end{aligned}$$

and hence

$$1 > \frac{h_2(B)}{|b_{22}|} \geq \frac{h_{n_1+1}(C)}{|b_{22}|} = \frac{h_{n_1+1}(C)}{|c_{n_1+1, n_1+1}|}.$$

Next we prove that

$$(2.6) \quad 1 > \frac{h_{i-t}(B)}{|b_{i-t, i-t}|} \geq \frac{h_i(C)}{|b_{i-t, i-t}|} = \frac{h_i(C)}{|c_{ii}|}$$

holds for $i = n_1 + 2, n_1 + 3, \dots, n_1 + u$ by mathematical induction, where $t = n_1 - 1$, u is a positive integer and $u \leq n_2 - 1$. Suppose that (2.6) holds for $i = n_1 + 2, n_1 + 3, \dots, n_1 + u - 1$, note that (2.6) is equivalent to

$$(2.7) \quad |c_{ii}| = |b_{i-t, i-t}| > h_{i-t}(B) \geq h_i(C),$$

we have

$$\begin{aligned} |c_{n_1+u, n_1+u}| &= |b_{1+u, 1+u}| > h_{1+u}(B) \\ &= |b_{1+u, 1}| \frac{h_1(B)}{|b_{11}|} + \sum_{j=2}^u |b_{1+u, j}| \frac{h_j(B)}{|b_{jj}|} + \sum_{j=u+2}^{n_2} |b_{1+u, j}| \\ &\geq \sum_{j=1}^t |c_{n_1+u, j}| \frac{h_j(A)}{|a_{jj}|} + |b_{1+u, 1}| \frac{h_{n_1}(A) + h_1(B)}{|a_{n_1, n_1} + b_{11}|} \\ &\quad + \sum_{j=2}^u |b_{1+u, j}| \frac{h_{t+j}(C)}{|b_{jj}|} + \sum_{j=u+2}^{n_2} |b_{1+u, j}| \quad (\text{by (2.5)}) \\ &= \sum_{j=1}^t |c_{n_1+u, j}| \frac{h_j(C)}{|c_{jj}|} + |c_{n_1+u, n_1}| \frac{h_{n_1}(C)}{|c_{n_1, n_1}|} + \sum_{j=n_1+1}^{n_1+u-1} |c_{n_1+u, j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1+u+1}^n |c_{n_1+u, j}| \\ &= h_{n_1+u}(C), \end{aligned}$$

and hence (2.6) holds for $i = n_1 + u$. Therefore, $|c_{ii}| > h_i(C)$ for all $i \in S_3$.

From Cases 1, 2, and 3, $|c_{ii}| > h_i(C)$ holds for any $i \in S_1 \cup S_2 \cup S_3$. This implies that $C = A \oplus_1 B$ is a Nekrasov matrix. The conclusion follows. ■

Next, we give an example to show that the $C = A \oplus_1 B$ is a Nekrasov matrix by Theorem 2.1.

EXAMPLE 2.2. Consider the following matrices:

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 2 & 2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & 1 \\ 6 & -7 & 2 \\ 7 & 3 & -10 \end{pmatrix},$$

where A is an SDD matrix and B is a Nekrasov matrix. By computation, we have

$$\frac{h_3(A)}{|a_{33}|} = \frac{22}{45} < \frac{2}{3} = \frac{h_1(B)}{|b_{11}|}.$$

Hence, $C = A \oplus_1 B$ is a Nekrasov matrix by Theorem 2.1. In fact, the 1-subdirect sum of A and B is

$$C = A \oplus_1 B = \begin{pmatrix} 3 & 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 2 & 2 & 8 & 1 & 1 \\ 0 & 0 & 6 & -7 & 2 \\ 0 & 0 & 7 & 3 & -10 \end{pmatrix},$$

and

$$\begin{cases} h_1(C) = 2 < 3 = |c_{11}|, \\ h_2(C) = \frac{5}{3} < 3 = |c_{22}|, \\ h_3(C) = \frac{40}{9} < 8 = |c_{33}|, \\ h_4(C) = \frac{16}{3} < 7 = |c_{44}|, \\ h_5(C) = \frac{389}{63} < 10 = |c_{55}|. \end{cases}$$

Hence, by Definition 1.3 C is indeed a Nekrasov matrix. On the other hand, the 1-subdirect sum of B and A is

$$\widehat{C} = B \oplus_1 A = \begin{pmatrix} 3 & 1 & 1 & 0 & 0 \\ 6 & -7 & 2 & 0 & 0 \\ 7 & 3 & -7 & 1 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 2 & 4 \end{pmatrix},$$

and hence \widehat{C} is not a Nekrasov matrix because $h_3(\widehat{C}) = \frac{194}{21} > 7 = |\widehat{c}_{33}|$. Therefore, the 1-subdirect sum $B \oplus_1 A$ of a Nekrasov matrix B and an SDD matrix A is not a Nekrasov matrix in general although $A \oplus_1 B$ is a Nekrasov matrix. Furthermore, the 2-subdirect sum of A and B is

$$\widetilde{C} = A \oplus_2 B = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 6 & 2 & 1 \\ 2 & 8 & -2 & 2 \\ 0 & 7 & 3 & -10 \end{pmatrix}.$$

However, \widetilde{C} is not a Nekrasov matrix because $h_3(\widetilde{C}) = \frac{74}{9} > 2 = |\widetilde{c}_{33}|$. Therefore, the 2-subdirect sum of A and B is not a Nekrasov matrix in general.

Example 2.2 motivates us to find some conditions such that $C = A \oplus_k B$ for $k \geq 2$ is a Nekrasov matrix when A is an SDD matrix and B is a Nekrasov matrix.

THEOREM 2.3. Let $A = (a_{ij})$ and $B = (b_{ij})$ be square matrices of order n_1 and n_2 partitioned as in (1.1), respectively. And let k be an integer such that $1 \leq k \leq \min\{n_1, n_2\}$, $S_1 = \{1, 2, \dots, n_1 - k\}$, $S_2 = \{n_1 - k + 1, \dots, n_1\}$, $S_3 = \{n_1 + 1, \dots, n\}$ and $t = n_1 - k$ be the same as Definition 1.1, where $n = n_1 + n_2 - k$. Assume that A is an SDD matrix, and B is a Nekrasov matrix. If the diagonal element $a_{t+j, t+j}$ of matrix A_{22} and the diagonal element b_{jj} of matrix B_{11} is either positive or negative at the same time, i.e., $\text{sign}(a_{t+j, t+j}) = \text{sign}(b_{jj})$ for $j = 1, 2, \dots, k$, $b_{ij} = 0$ for $i \in \{2, 3, \dots, k\}$, $j < i$, $j \in \{1, 2, \dots, k - 1\}$, and

$$(2.8) \quad \frac{r_{t+j}(A)}{|a_{t+j, t+j}|} \leq \frac{h_j(B)}{|b_{jj}|}, \quad j = 1, 2, \dots, k,$$

then the k -subdirect sum $C = A \oplus_k B$ is a Nekrasov matrix.

Proof. Since A is SDD and B is a Nekrasov matrix with $b_{ij} = 0$ for $j < i$, $i = 1, 2, \dots, k$, we have that for each $i \in \{1, 2, \dots, n_1\}$, $|a_{ii}| > r_i(A)$, and

$$|b_{ii}| > h_i(B) = r_i(B), \quad i = 1, 2, \dots, k, \quad |b_{ii}| > h_i(B), \quad i = k + 1, k + 2, \dots, n_2.$$

Case 1. For any $i \in S_1$, we have

$$|c_{ii}| = |a_{ii}| > r_i(A) \geq \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n_1} |a_{ij}| = \sum_{j=1}^{i-1} |c_{ij}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=i+1}^n |c_{ij}| = h_i(C).$$

Case 2 For any $i = t + m \in S_2$, $1 \leq m \leq k$, we have

$$\begin{aligned} |c_{t+m, t+m}| &= |a_{t+m, t+m}| + |b_{mm}| > r_{t+m}(A) + h_m(B) \\ &= \sum_{j=1}^t |a_{t+m, j}| + \sum_{j=t+1}^{t+m-1} |a_{t+m, j}| + \sum_{j=t+m+1}^{n_1} |a_{t+m, j}| + \sum_{j=m+1}^k |b_{m, j}| + \sum_{j=k+1}^{n_2} |b_{m, j}| \\ &\geq \sum_{j=1}^t |a_{t+m, j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=t+1}^{t+m-1} |a_{t+m, j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=m+1}^k |a_{t+m, t+j} + b_{m, j}| + \sum_{j=k+1}^{n_2} |b_{m, j}| \\ &= \sum_{j=1}^t |c_{t+m, j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+1}^{t+m-1} |c_{t+m, j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+m+1}^{n_1} |c_{t+m, j}| + \sum_{j=n_1+1}^n |c_{t+m, j}| \\ &= \sum_{j=1}^{t+m-1} |c_{t+m, j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+m+1}^n |c_{t+m, j}| \\ &= h_{t+m}(C). \end{aligned}$$

Furthermore, the above inequality yields

$$1 > \frac{r_{t+m}(A) + h_m(B)}{|a_{t+m, t+m}| + |b_{mm}|} \geq \frac{h_{t+m}(C)}{|a_{t+m, t+m}| + |b_{mm}|}.$$

In addition, by (2.8) we have

$$\frac{r_{t+m}(A)}{|a_{t+m, t+m}|} \leq \frac{h_m(B)}{|b_{mm}|}$$

$$\begin{aligned} &\Leftrightarrow |b_{mm}|r_{t+m}(A) \leq |a_{t+m,t+m}|h_m(B) \\ &\Leftrightarrow |b_{mm}|(r_{t+m}(A) + h_m(B)) \leq (|a_{t+m,t+m}| + |b_{mm}|)h_m(B) \\ &\Leftrightarrow \frac{r_{t+m}(A) + h_m(B)}{|a_{t+m,t+m}| + |b_{mm}|} \leq \frac{h_m(B)}{|b_{mm}|}. \end{aligned}$$

Therefore,

$$(2.9) \quad 1 > \frac{h_m(B)}{|b_{mm}|} \geq \frac{r_{t+m}(A) + h_m(B)}{|a_{t+m,t+m}| + |b_{mm}|} \geq \frac{h_{t+m}(C)}{|a_{t+m,t+m}| + |b_{mm}|}.$$

Case 3 For $i = n_1 + 1 \in S_3$, we have

$$\begin{aligned} |c_{n_1+1,n_1+1}| &= |b_{k+1,k+1}| > h_{k+1}(B) \\ &= \sum_{j=1}^k |b_{k+1,j}| \frac{h_j(B)}{|b_{jj}|} + \sum_{j=k+2}^{n_2} |b_{k+1,j}| \\ &\geq \sum_{j=1}^t |c_{n_1+1,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=1}^k |b_{k+1,j}| \frac{h_{t+j}(C)}{|a_{t+j,t+j} + b_{jj}|} + \sum_{j=k+2}^{n_2} |b_{k+1,j}| \quad (\text{by (2.9)}) \\ &= \sum_{j=1}^t |c_{n_1+1,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+1}^{n_1} |c_{n_1+1,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1+2}^n |c_{n_1+1,j}| \\ &= h_{n_1+1}(C), \end{aligned}$$

and hence

$$1 > \frac{h_{k+1}(B)}{|b_{k+1,k+1}|} \geq \frac{h_{n_1+1}(C)}{|b_{k+1,k+1}|} = \frac{h_{n_1+1}(C)}{|c_{n_1+1,n_1+1}|}.$$

Next we prove that (2.6) holds for $i = n_1 + 2, n_1 + 3, \dots, n_1 + u$ by mathematical induction, where $t = n_1 - k$, u is a positive integer and $u \leq n_2 - k$. Suppose that (2.6) holds for $i = n_1 + 2, n_1 + 3, \dots, n_1 + u - 1$, note that (2.6) is equivalent to (2.7), we have

$$\begin{aligned} |c_{n_1+u,n_1+u}| &= |b_{k+u,k+u}| > h_{k+u}(B) \\ &= \sum_{j=1}^k |b_{k+u,j}| \frac{h_j(B)}{|b_{jj}|} + \sum_{j=k+1}^{k+u-1} |b_{k+u,j}| \frac{h_j(B)}{|b_{jj}|} + \sum_{j=k+u+1}^{n_2} |b_{k+u,j}| \\ &\geq \sum_{j=1}^t |c_{n_1+u,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=1}^k |b_{k+u,j}| \frac{h_{t+j}(C)}{|a_{t+j,t+j} + b_{jj}|} \\ &\quad + \sum_{j=k+1}^{k+u-1} |b_{k+u,j}| \frac{h_{t+j}(C)}{|b_{jj}|} + \sum_{j=k+u+1}^{n_2} |b_{k+u,j}| \quad (\text{by (2.9)}) \\ &= \sum_{j=1}^t |c_{n_1+u,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+1}^{n_1} |c_{n_1+u,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1+1}^{n_1+u-1} |c_{n_1+u,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1+u+1}^n |c_{n_1+u,j}| \\ &= h_{n_1+u}(C), \end{aligned}$$

and hence Inequality (2.6) holds for $i = n_1 + u$. Therefore, $|c_{ii}| > h_i(C)$ for all $i \in S_3$.

From Cases 1, 2, and 3, $|c_{ii}| > h_i(C)$ holds for any $i \in S_1 \cup S_2 \cup S_3$. This implies that $C = A \oplus_k B$ is a Nekrasov matrix. The conclusion follows. ■

EXAMPLE 2.4. Consider the following matrices:

$$A = \left(\begin{array}{c|ccc} 4 & 1 & 1 & 0 \\ \hline 1 & 6 & 1 & 1 \\ 1 & 2 & -8 & 1 \\ 2 & 2 & 1 & 10 \end{array} \right), \quad B = \left(\begin{array}{ccc|cc} 4 & 1 & 1 & 1 & 0 \\ 0 & -6 & 2 & 1 & 1 \\ 0 & 0 & 8 & 4 & 2 \\ \hline 4 & 6 & 4 & 12 & -1 \\ 4 & 6 & 4 & 12 & -22 \end{array} \right),$$

where A is SDD and B is a Nekrasov matrix. The 3-subdirect sum of A and B is

$$C = A \oplus_3 B = \left(\begin{array}{cccc|cc} 4 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 10 & 1 & 1 & 1 & 0 \\ 1 & 2 & -14 & 3 & 1 & 1 \\ 2 & 2 & 1 & 18 & 4 & 2 \\ \hline 0 & 4 & 6 & 4 & 12 & -1 \\ 0 & 4 & 6 & 4 & 12 & -22 \end{array} \right).$$

By computation,

$$\frac{r_2(A)}{|a_{22}|} = \frac{r_3(A)}{|a_{33}|} = \frac{r_4(A)}{|a_{44}|} = \frac{1}{2} < \frac{2}{3} = \frac{h_2(B)}{|b_{22}|} < \frac{3}{4} = \frac{h_1(B)}{|b_{11}|} = \frac{h_3(B)}{|b_{33}|}.$$

By Theorem 2.3, $C = A \oplus_3 B$ is a Nekrasov matrix. In fact,

$$\begin{cases} h_1(C) = 2 < 4 = |c_{11}|, \\ h_2(C) = 3.5 < 10 = |c_{22}|, \\ h_3(C) = 6.2 < 14 = |c_{33}|, \end{cases} \quad \begin{cases} h_4(C) = 8.1429 < 18 = |c_{44}|, \\ h_5(C) = 6.867 < 12 = |c_{55}|, \\ h_6(C) = 12.7346 < 22 = |c_{66}|, \end{cases}$$

and hence by Definition 1.3 C is indeed a Nekrasov matrix.

THEOREM 2.5. Let $A = (a_{ij})$ and $B = (b_{ij})$ be square matrices of order n_1 and n_2 partitioned as in (1.1), respectively. And let k be an integer such that $1 \leq k \leq \min\{n_1, n_2\}$, $S_1 = \{1, 2, \dots, n_1 - k\}$, $S_2 = \{n_1 - k + 1, \dots, n_1\}$, $S_3 = \{n_1 + 1, \dots, n\}$ and $t = n_1 - k$ be the same as Definition 1.1, where $n = n_1 + n_2 - k$. Assume that A is an SDD matrix, and B is a Nekrasov matrix. If the diagonal element $a_{t+j,t+j}$ of matrix A_{22} and the diagonal element b_{jj} of matrix B_{11} is either positive or negative at the same time, i.e., $\text{sign}(a_{t+j,t+j}) = \text{sign}(b_{jj})$ for $j = 1, 2, \dots, k$, and

$$(2.10) \quad \frac{h_{t+j}(A, B)}{|a_{t+j,t+j} + b_{jj}|} \leq \frac{h_j(B)}{|b_{jj}|}, j = 1, 2, \dots, k,$$

then the k -subdirect sum $C = A \oplus_k B$ is a Nekrasov matrix, where $h_{t+j}(A, B) = h_{t+j}(A \oplus_k B)$.

Proof. Since A is SDD and B is a Nekrasov matrix, we have that for each $i \in \{1, 2, \dots, n_1\}$, $|a_{ii}| > r_i(A)$, and

$$|b_{11}| > r_1(B) = h_1(B), \quad |b_{ii}| > h_i(B), \quad i = 2, 3, \dots, n_2.$$

Case 1. For any $i \in S_1$, we have

$$|c_{ii}| = |a_{ii}| > r_i(A) \geq \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n_1} |a_{ij}| = \sum_{j=1}^{i-1} |c_{ij}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=i+1}^n |c_{ij}| = h_i(C).$$

Case 2. For $i = n_1 - k + 1 \in S_2$, we have

$$\begin{aligned} |c_{n_1-k+1, n_1-k+1}| &= |a_{n_1-k+1, n_1-k+1}| + |b_{11}| > r_{n_1-k+1}(A) + h_1(B) \\ &= \sum_{j=1}^{n_1-k} |a_{n_1-k+1, j}| + \sum_{j=2}^{n_2} |b_{1, j}| + \sum_{j=n_1-k+2}^{n_1} |a_{n_1-k+1, j}| \\ &\geq \sum_{j=1}^{n_1-k} |a_{n_1-k+1, j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=2}^k (|b_{1, j}| + |a_{n_1-k+1, n_1-k+j}|) + \sum_{j=k+1}^{n_2} |b_{1, j}| \\ &\geq \sum_{j=1}^{n_1-k} |a_{n_1-k+1, j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=2}^k |b_{1, j} + a_{n_1-k+1, n_1-k+j}| + \sum_{j=k+1}^{n_2} |b_{1, j}| \\ &= \sum_{j=1}^{n_1-k} |c_{n_1-k+1, j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1-k+2}^{n_1} |c_{n_1-k+1, j}| + \sum_{j=n_1+1}^n |c_{n_1-k+1, j}| \\ &= \sum_{j=1}^{n_1-k} |c_{n_1-k+1, j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1-k+2}^n |c_{n_1-k+1, j}| \\ &= h_{n_1-k+1}(C). \end{aligned}$$

Furthermore, for any $i = t + m \in S_2$, $2 \leq m \leq k$, we have

$$\begin{aligned} |c_{t+m, t+m}| &= |a_{t+m, t+m}| + |b_{mm}| > r_{t+m}(A) + h_m(B) \\ &= \sum_{j=1}^t |a_{t+m, j}| + \sum_{j=t+1}^{t+m-1} |a_{t+m, j}| + \sum_{j=t+m+1}^{n_1} |a_{t+m, j}| \\ &\quad + \sum_{j=1}^{m-1} |b_{m, j}| \frac{h_j(B)}{|b_{jj}|} + \sum_{j=m+1}^k |b_{m, j}| + \sum_{j=k+1}^{n_2} |b_{m, j}| \\ &\geq \sum_{j=1}^t |a_{t+m, j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=1}^{m-1} (|a_{t+m, t+j}| + |b_{m, j}|) \frac{h_j(B)}{|b_{jj}|} \\ &\quad + \sum_{j=m+1}^k (|a_{t+m, t+j}| + |b_{m, j}|) + \sum_{j=k+1}^{n_2} |b_{m, j}| \\ &\geq \sum_{j=1}^t |a_{t+m, j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=1}^{m-1} |a_{t+m, t+j} + b_{m, j}| \frac{h_{t+j}(A, B)}{|a_{t+j, t+j}| + |b_{jj}|} \\ &\quad + \sum_{j=m+1}^k |a_{t+m, t+j} + b_{m, j}| + \sum_{j=k+1}^{n_2} |b_{m, j}| \text{ (by (2.10))} \\ &= \sum_{j=1}^t |c_{t+m, j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+1}^{t+m-1} |c_{t+m, j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+m+1}^{n_1} |c_{t+m, j}| + \sum_{j=n_1+1}^n |c_{t+m, j}| \end{aligned}$$

$$\begin{aligned} &= \sum_{j=1}^{t+m-1} |c_{t+m,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+m+1}^n |c_{t+m,j}| \\ &= h_{t+m}(C). \end{aligned}$$

Case 3. For $i = n_1 + 1 \in S_3$, we have

$$\begin{aligned} |c_{n_1+1,n_1+1}| &= |b_{k+1,k+1}| > h_{k+1}(B) \\ &= \sum_{j=1}^k |b_{k+1,j}| \frac{h_j(B)}{|b_{jj}|} + \sum_{j=k+2}^{n_2} |b_{k+1,j}| \\ &\geq \sum_{j=k+2}^{n_2} |b_{k+1,j}| + \sum_{j=1}^k |b_{k+1,j}| \frac{h_{t+j}(A, B)}{|a_{t+j,t+j} + b_{jj}|} \text{ (by (2.10))} \\ &= \sum_{j=1}^t |c_{n_1+1,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=1}^k |b_{k+1,j}| \frac{h_{t+j}(A, B)}{|a_{t+j,t+j} + b_{jj}|} + \sum_{j=k+2}^{n_2} |b_{k+1,j}| \\ &= \sum_{j=1}^t |c_{n_1+1,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=t+1}^{n_1} |c_{n_1+1,j}| \frac{h_j(C)}{|c_{jj}|} + \sum_{j=n_1+2}^n |c_{n_1+1,j}| \\ &= h_{n_1+1}(C). \end{aligned}$$

Similarly to Case 3 in the proof of Theorem 2.3, it is easy to obtain

$$|c_{ii}| > h_i(C), i = n_1 + 2, n_1 + 3, \dots, n_1 + u$$

by mathematical induction. Therefore, $|c_{ii}| > h_i(C)$ for all $i \in S_3$.

From Cases 1, 2, and 3, $|c_{ii}| > h_i(C)$ holds for any $i \in S_1 \cup S_2 \cup S_3$. This implies that $C = A \oplus_k B$ is a Nekrasov matrix. The conclusion follows. ■

Remark here that $h_{t+j}(A, B) = h_{t+j}(A \oplus_k B)$ in (2.10) could be only determined by the entries of A and B . In fact, note that

$$h_{t+1}(A, B) = \sum_{j=1}^{n_1-k} |a_{n_1-k+1,j}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=2}^k |b_{1,j} + a_{n_1-k+1,n_1-k+j}| + \sum_{j=k+1}^{n_2} |b_{1,j}|$$

is completely determined by the entries of A and B . Furthermore,

$$h_{t+2}(A, B) = \sum_{j=1}^t |a_{t+2,j}| \frac{h_j(A)}{|a_{jj}|} + |a_{t+2,t+1} + b_{21}| \frac{h_{t+1}(A, B)}{|a_{t+1,t+1}| + |b_{11}|} + \sum_{j=3}^k |a_{t+2,t+j} + b_{2,j}| + \sum_{j=k+1}^{n_2} |b_{2,j}|,$$

and hence $h_{t+2}(A, B)$ involves A, B and $h_{t+1}(A, B)$. This implies $h_{t+2}(A, B)$ can be also completely determined by the entries of A and B . Similarly, $h_{t+j}(A, B)$, $j = 3, \dots, k$ are only completely determined by the entries of A and B because $h_{t+j}(A, B)$ involves $h_{t+j-1}(A, B)$. Therefore, inequalities (2.10) is not difficult to be determined in general.

EXAMPLE 2.6. Consider the following matrices:

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & -3 & 1 \\ 2 & 2 & 5 \end{pmatrix}, B = \begin{pmatrix} -3 & 1 & 1 \\ 6 & 7 & 2 \\ 7 & 3 & 10 \end{pmatrix},$$

where A is SDD and B is a Nekrasov matrix. Obviously, the 2-subdirect sum of A and B is

$$C = A \oplus_2 B = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & -6 & 2 & 1 \\ 2 & 8 & 12 & 2 \\ 0 & 7 & 3 & 10 \end{pmatrix}.$$

Therefore, the 2-subdirect sum of A and B is a Nekrasov matrix by Theorem 2.5. In fact,

$$\frac{h_2(A, B)}{|a_{22} + b_{11}|} = \frac{|a_{21}| \frac{h_1(A)}{|a_{11}|} + |b_{12} + a_{23}| + |b_{13}|}{|a_{22} + b_{11}|} = \frac{\frac{11}{3}}{|-3-3|} = \frac{11}{18} < \frac{2}{3} = \frac{h_1(B)}{|b_{11}|},$$

$$\frac{h_3(A, B)}{|a_{33} + b_{22}|} = \frac{|a_{31}| \frac{h_1(A)}{|a_{11}|} + |b_{21} + a_{32}| \frac{h_2(A, B)}{|a_{22} + b_{11}|} + |b_{23}|}{|a_{33} + b_{22}|} = \frac{\frac{2}{3} \times 2 + |2+6| \times \frac{11}{18} + 2}{|5+7|} = \frac{\frac{74}{9}}{12} = \frac{37}{54} < \frac{6}{7} = \frac{h_2(B)}{|b_{22}|}.$$

In addition,

$$\begin{cases} h_1(C) = 2 < 3 = |c_{11}|, \\ h_2(C) = \frac{11}{3} < 6 = |c_{22}|, \\ h_3(C) = \frac{74}{9} < 12 = |c_{33}|, \\ h_4(C) = \frac{19}{3} < 10 = |c_{44}|, \end{cases}$$

and hence by Definition 1.3 C is indeed a Nekrasov matrix. From this example it is also shown that Inequalities (2.10) in Theorem 2.5 depends only on the elements of matrices A and B .

3. Conclusions. In this paper, we give some sufficient conditions such that the k -subdirect sum $A \oplus_k B$ of an SDD matrix A and a Nekrasov matrix B is in the class of Nekrasov matrices. These conditions are different from those in [10, 15, 20, 23] in which the sufficient conditions for the closure property are given for the k -subdirect sum $B \oplus_k A$. In the future, we will search for conditions such that $A \oplus_k B$ is a Nekrasov matrix for the case that A and B are all Nekrasov matrices.

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