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# DIGRAPHS WITH LARGE EXPONENT*

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**Abstract.** Primitive digraphs on $n$ vertices with exponents at least $\lfloor \omega_n/2 \rfloor + 2$, where $\omega_n = (n-1)^2 + 1$, are considered. For $n \geq 3$, all such digraphs containing a Hamilton cycle are characterized; and for $n \geq 6$, all such digraphs containing a cycle of length $n-1$ are characterized. Each eigenvalue of any stochastic matrix having a digraph in one of these two classes is proved to be geometrically simple.

**Key words.** primitive directed graph, exponent.

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## 1. Introduction.

A directed graph (digraph) $D$ is *primitive* if for some positive integer $m$ there is a (directed) walk of length $m$ between any two vertices $u$ and $v$ (including $u = v$). The minimum such $m$ is the *exponent* of $D$, denoted by $exp(D)$. It is well known that $D$ is primitive iff it is strongly connected and the *gcd* of its cycle lengths is 1. A nonnegative matrix $A$ is primitive if $A^m$ is entrywise positive for some positive integer $m$. If $D = D(A)$, the digraph of a primitive matrix $A$, then $exp(D) = exp(A)$, which is the minimum $m$ such that $A^m$ is entrywise positive.

Denoting $(n-1)^2 + 1$ by $\omega_n$, the best upper bound for $exp(D)$ when a primitive digraph $D$ has $n \geq 2$ vertices is given by $exp(D) \leq \omega_n$, with equality holding iff $D = D(W_n)$ where $W_n$ is a Wielandt matrix; see, e.g., [2, Theorem 3.5.6]. When $n = 2$, then $D(W_2)$, consisting of a 1 cycle and a 2 cycle, has exponent equal to 2. Henceforth we assume that $n \geq 3$. The digraph $D(W_n)$ consists of a Hamilton cycle (i.e., a cycle of length $n$) and one more arc, between a pair of vertices that are distance two apart on the Hamilton cycle, giving a cycle of length $n-1$.

The following result of Lewin and Vitek [6, Theorem 3.1], see also [2, Theorem 3.5.8], is the basis for our discussion of digraphs with large exponent.

THEOREM 1.1. *If $D$ has $n \geq 3$ vertices and is primitive with sufficiently large exponent, namely*

$$exp(D) \geq \lfloor \omega_n/2 \rfloor + 2, \textit{with } \omega_n = (n-1)^2 + 1, \tag{1}$$

*then $D$ has cycles of exactly two different lengths $j, k$ with $n \geq k > j$.*

We say that a primitive digraph $D$ on $n$ vertices satisfying (1) has a *large exponent.* Note that in Theorem 1.1, $gcd(j,k) = 1$ since $D$ is primitive. If $gcd(j,k) = 1$, then

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every integer greater than or equal to $(j-1)(k-1)$ can be written as $c_1 j + c_2 k$, where $c_i$ are nonnegative integers. The value $(j-1)(k-1)$ is the smallest such integer, and is called the Frobenius-Schur index for the two relatively prime integers $j$ and $k$; see, e.g., [2, Lemma 3.5.5].

The Frobenius-Schur index is used to prove the following result that gives a necessary and sufficient condition for the existence of a primitive digraph with large exponent and cycles of two specified lengths.

THEOREM 1.2. *Let $k$ and $j$ be such that $gcd(j,k) = 1$ and $n \geq k > j$. There exists a primitive digraph $D$ on $n$ vertices having only cycle lengths $k$ and $j$ and $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$ iff $j(k-2) \geq \lfloor \omega_n/2 \rfloor + 2 - n$.*

*Proof.* Suppose that $D$ is a digraph with large exponent and cycle lengths $k$ and $j < k \leq n$. We claim that for any pair of vertices $u$ and $v$, there is a walk from $u$ to $v$ of length at most $k + n - j - 1 \geq n$ that goes through a vertex on a $k$ cycle and a vertex on a $j$ cycle. To prove this claim, note that from the proof of Theorem 1 in [4], there are no pairs of vertex disjoint cycles in $D$; that is, any pair of cycles share at least one common vertex. If there is a walk from $u$ to $v$ of length less than or equal to $n$ that passes through at least one vertex on a $k$ cycle and at least one vertex on a $j$ cycle, then the claim is proved.

So suppose that this is not the case. In particular, assume that $u$ and $v$ are only on $k$ (resp. $j$) cycles, and any path from $u$ to $v$ passes only through vertices not on any $j$ (resp. $k$) cycle. Consider the first case. Let $l$ be the number of vertices *not* on a $j$ cycle, and note that $2 \leq l \leq n - j$. Since a shortest path from $u$ to $v$ goes only through vertices not on a $j$ cycle, the length $p$ of such a path satisfies $p \leq l - 1$. Consider the walk from $u$ to $v$ formed by first traversing a $k$ cycle at $u$ (necessarily going through a vertex on a $j$ cycle), then taking the path of length $p$ from $u$ to $v$. This generates a walk from $u$ to $v$ that goes through a vertex on a $k$ cycle and one on a $j$ cycle, and its length is $k + p \leq k + l - 1 \leq k + n - j - 1$. The second case follows by interchanging $k$ and $j$ and noting that $j + n - k - 1 < k + n - j - 1$. Thus the claim is proved. By the Frobenius-Schur index, there is a walk from $u$ to $v$ of length $k + n - j - 1 + (k-1)(j-1) = n + j(k-2)$ for any pair $u, v$. Thus $n + j(k-2) \geq exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$, giving the condition on $k$ and $j$.

For the converse, assume the condition on $k$ and $j$, and consider the digraph $D$ consisting of the $k$ cycle $1 \to k \to k-1 \to \cdots \to k+j-n+1 \to k+j-n \to k+j-n-1 \to \cdots \to 2 \to 1$, and arcs $1 \to k+1 \to k+2 \to \cdots \to n-1 \to n \to k+j-n$. Thus $D$ has exactly one $k$ cycle and one $j$ cycle. Consider the length of a walk from $k$ to $k+j-n+1$. Such a walk has length $n-j-1$ or $k+n-j-1+c_1 k + c_2 j$ for some nonnegative integers $c_i$, and (from the Frobenius-Schur index) there is no walk of length $k+n-j-1+(k-1)(j-1)-1$. Thus

$$exp(D) \geq k + n - j - 1 + (k-1)(j-1) = n + j(k-2) \geq \lfloor \omega_n/2 \rfloor + 2. \quad \square$$

Note that for $D$ primitive with only cycles of lengths $k$ and $j$ with $j < k \leq n$, the bound on $exp(D)$ found in the above proof, namely $exp(D) \leq n + j(k-2)$, improves the bound in [4, Lemma 1] and includes the converse. Furthermore, Theorem 1.2 does not include additional assumptions as in [6, Theorem 4.1].

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We assume that $D$ has a large exponent and focus on the graph theoretic aspects of this condition. In Section 2, we characterize the case when $D$ has a Hamilton cycle ($k = n \geq 3$); and in Section 3, we characterize the case $k = n-1$. Our characterizations give some information on the case for general $k \leq n$ when $n \geq 4$, since a result of Beasley and Kirkland [1, Theorem 1] implies that any induced subdigraph on $k$ vertices that is primitive also has large exponent (relative to $\lfloor \omega_k/2 \rfloor + 2$), so the structure of some such induced subdigraphs is known from our results. It is known from results in [6] exactly which numbers $\geq \lfloor \omega_n/2 \rfloor + 2$ are attainable as exponents of primitive digraphs. (Note that there are some gaps in this exponent set.) Our work in Sections 2 and 3 focuses on describing the corresponding digraphs when $k \geq n-1$.

Some algebraic consequences of the large exponent condition (1) for a stochastic matrix $A$ with $D(A) = D$ have been investigated in [4] and [5]. The characteristic polynomial of $A$ has a simple form (see [4, Theorem 1]), and, if $n$ is sufficiently large, then about half of the eigenvalues of $A$ have modulus close to 1. Kirkland and Neumann [5] considered the magnitudes of the entries in the group generalized inverse of $I - A$ (which measures stability of the left Perron vector of $A$ under perturbations). In Section 4 we use results of Sections 2 and 3 to investigate the multiplicities of eigenvalues of stochastic matrices with large exponents.

**2. The Hamiltonian Case.** Assuming that $D$ has large exponent and a Hamilton cycle, we begin by finding possible lengths for other cycles in $D$.

LEMMA 2.1. *Suppose that $D$ is a primitive digraph on $n \geq 3$ vertices with $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$ and that $D$ has a Hamilton cycle. Then $D$ has precisely one Hamilton cycle, and all other cycles have length $j$, where $n > j \geq \lceil (n-1)/2 \rceil$.*

*Proof.* By Theorem 1.1, $D$ contains cycles of exactly two lengths, $n = k > j$. W.l.o.g. take the given Hamilton cycle as $1 \to n \to n-1 \to \cdots \to 2 \to 1$, and assume that the arc $1 \to j$ lies on a second Hamilton cycle. Note that the only possible arcs from any vertex $i$ are $i \to i-1 \pmod n$ and $i \to i+j-1 \pmod n$. Since the arc $j+1 \to j$ is not on the second Hamilton cycle, this cycle must include the arc $j+1 \to (j+1)+j-1 = 2j \pmod n$. Similarly, there is an arc on the second Hamilton cycle from $(m-1)j+1$ to $mj \pmod n$, for $m = 1, \ldots, n$. As $gcd(j,n) = 1$, $D$ contains the digraph of a primitive circulant. By [3, Theorem 2.1], $exp(D) \leq (n-1)$ or $exp(D) \leq \lfloor n/2 \rfloor$, thus $exp(D) < \lfloor \omega_n/2 \rfloor + 2$ . Hence, there is no second Hamilton cycle in $D$. For the lower bound on $j$, take $k = n$ in Theorem 1.2; see also [4, Theorem 1]. $\square$

If $D$ has large exponent and $k = n = 3$, then Lemma 2.1 implies that $j \in \{1, 2\}$. For $j = 1$, $D$ consisting of a 3 cycle and a 1 cycle has exponent equal to $4 = \lfloor \omega_3/2 \rfloor + 2$. For $j = 2 = n-1$, either $D = D(W_3)$ with exponent equal to $5 = \omega_3$, or $D$ consists of a 3 cycle with two 2 cycles and has exponent equal to 4. This last case is an example of the result that a digraph $D$ on $n$ vertices has $exp(D) = (n-1)^2$ iff $D$ is isomorphic to an $n$ cycle with two additional arcs from consecutive vertices forming two $n-1$ cycles; see, e.g., [2, pp. 82–83].

These observations motivate our next two theorems, which describe the Hamiltonian digraphs with large exponent. Most cases are covered in Theorem 2.2, but, if $n$

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is odd, then the case $j = (n-1)/2$ is slightly different and is given in Theorem 2.3.

THEOREM 2.2. *Suppose that $j \geq n/2$. Then $D$ is a primitive digraph on $n \geq 3$ vertices with $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$ and cycle lengths $n$ and $j$ iff $D$ is isomorphic to a (primitive) subdigraph of the digraph formed by taking the cycle $1 \to n \to n-1 \to \cdots \to 2 \to 1$, and adding in the arcs $i \to i+j-1$ for $1 \leq i \leq n-j+1$.*

*Proof.* Assume that $D$ is primitive with large exponent and has a Hamilton cycle. Then by Lemma 2.1, $D$ has only one Hamilton cycle and other cycles of length $j$, which by assumption is at least $n/2$. W.l.o.g. assume that the Hamilton cycle is $1 \to n \to n-1 \to \cdots \to 2 \to 1$, and that $D$ contains the arc $1 \to j$. Since $D$ has cycles of just two different lengths, each vertex $i$ of $D$ has outdegree $\leq 2$, and if the outdegree is 2, then the outarcs from vertex $i$ are $i \to i-1$ and $i \to i+j-1$. Here and throughout the proof, all indices are mod $n$. As $1 \to j$, the outdegree of vertex $i$ is 1 for each $i \in \{n-j+2, \ldots, j\}$, since otherwise $1 \to j \to j-1 \to \cdots \to i \to i+j-1-n \to i+j-2-n \to \cdots \to 2 \to 1$ is a cycle of length less than $j$. Consequently if the outdegree of vertex $i \in \{2, \ldots, j\}$ is 2, then in fact $i \in \{2, \ldots, n-j+1\}$. If there is no such $i$, then $D$ has the desired structure, since $D$ has at most $n-j+1$ consecutive vertices on the Hamilton cycle (namely 1 and $j+1, \ldots, n$) of outdegree 2. Henceforth suppose that there exists $i \in \{2, \ldots, n-j+1\}$ with outdegree 2, and let $i_1$ be the maximum such $i$; thus $i_1 \to i_1 + j - 1 \in \{j+1, \ldots, n\}$. As before, the outdegree is 1 for each vertex $\in \{n-j+i_1+1, \ldots, j+i_1-1\}$. In particular, if $n-j+i_1+1 \leq j+1$, then the only vertices that can have outdegree 2 are $1, \ldots, i_1$ and $j+i_1, \ldots, n$, that is $n-j+1$ consecutive vertices, as desired. So suppose henceforth that $n-j+i_1 > j$, that is $i_1 > 2j-n \geq 0$. Suppose also that there exists $i_2$ such that $n-j+i_1 \geq i_2 \geq j+1$ with $i_2$ having outdegree 2. Then $i_2 \to i_2+j-1$. Now $n+i_1-1 \geq i_2+j-1 \geq 2j$, so that $i_2+j-1 \pmod{n} = i_2+j-1-n \in \{2j-n, \ldots, i_1-1\}$. But then there is a cycle $i_1 \to i_1+j-1 \to i_1+j-2 \to \cdots \to i_2 \to i_2+j-1-n \to i_2+j-2-n \to \cdots \to 2 \to 1 \to j \to j-1 \to \cdots \to i_1+1 \to i_1$, which has length $3j-n$. As there is only one Hamilton cycle (Lemma 2.1), this implies that $3j-n = j$, giving a contradiction, since $gcd(n,j) = 1$. Thus again each of vertices $i_1+1, \ldots, j+i_1-1$ has outdegree 1, and so at most $n-j+1$ consecutive vertices have outdegree 2, as desired.

For the converse, consider the maximal such digraph $D$ with the above Hamilton cycle and the $n-j+1$ additional arcs. Note that each of the vertices $n-j+2, \ldots, n$ has outdegree 1, and each of the vertices $1, 2, \ldots, j-1$ has indegree 1, so the only path from $n$ to 1 is $n \to n-1 \to \cdots \to 1$ with length $n-1$. By Frobenius-Schur, it follows that there is no walk from $n$ to 1 of length $n-1+(n-1)(j-1)-1$; hence $exp(D) \geq j(n-1)$. Since $gcd(n,j) = 1$, it follows that $j = n/2$ is inadmissible. Thus $j \geq n/2$ implies that $j \geq (n+1)/2$, and so $j(n-1) \geq (n^2-1)/2 \geq \lfloor \omega_n/2 \rfloor + 2$. Since $D$ is maximal, any primitive subdigraph has exponent at least as large as $exp(D)$. □

THEOREM 2.3. *Suppose that $n \geq 3$ is odd and $j = (n-1)/2$. Then $D$ is a primitive digraph on $n$ vertices with $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$ and cycle lengths $n$ and $j$ iff $D$ is isomorphic to a (primitive) subdigraph of the digraph formed by taking the cycle $1 \to n \to n-1 \to \cdots \to 2 \to 1$, and adding in the arcs $i \to i+j-1$ for $1 \leq i \leq (n-1)/2 = j$.*

*Proof.* First assume that $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2 = (n-1)^2/2 + 2$. Observe that if vertex $i$ is on a $j$ cycle, then (by Frobenius-Schur) there is a walk of length

$\leq (n-1) + (n-1)(j-1) = j(n-1) = (n-1)^2/2$ from $i$ to each vertex of $D$. It follows that there must be a vertex with distance 2 to the nearest $j$ cycle. W.l.o.g. that vertex is $n$, with vertex $n-2$ on a $j$ cycle. In fact that $j$ cycle is $n-2 \to n-3 \to \cdots \to (n-1)/2 = j \to n-2$, otherwise $n-1$ or $n$ is on a $j$ cycle. None of the vertices $j+1, j+2, \cdots, n$ can have outdegree 2 (otherwise one of $n-1$ or $n$ is on a $j$ cycle). However, the $j-1$ additional arcs $i \to i+j-1$ for $i = 1, 2, \ldots, j-1$ may be included in $D$. Thus it follows that $D$ is a subdigraph of the digraph that has the $n-1$ cycle and the additional $j$ arcs as in the theorem statement.

For the converse, note that if $D$ is isomorphic to a subdigraph of the specified digraph, then a walk from $n$ to $n-1$ of length greater than 1 must traverse the entire Hamilton cycle, so walks from $n$ to $n-1$ have length 1 or $n+1+c_1 n + c_2 j$ where $c_1$ and $c_2$ are nonnegative integers. Thus (by Frobenius-Schur) there is no walk from $n$ to $n-1$ of length $n+1+(n-1)(n-3)/2-1 = (n-1)^2/2+1$, so that $exp(D) \geq (n-1)^2/2+2$, as desired. ☐

Using the structures of Hamiltonian digraphs $D$ with large exponents given in Theorems 2.2 and 2.3, we determine the exact value of $exp(D)$ in terms of a parameter $a$ that depends on which $j$ cycles occur in $D$.

COROLLARY 2.4. *Suppose that $D$ is a primitive digraph on $n \geq 3$ vertices with $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$, a Hamilton cycle and all other cycles of length $j$, where $n > j \geq \lceil (n-1)/2 \rceil$. Suppose that the Hamilton cycle is $1 \to n \to n-1 \to \cdots \to 2 \to 1$. Let $1 \leq a \leq n-j+1$ if $j \geq n/2$, and $1 \leq a \leq j$ if $j = (n-1)/2$. Suppose that $D$ also contains the arc(s) $1 \to j$ and $a \to a+j-1$, and that if $i$ is a vertex of outdegree 2, then $1 \leq i \leq a$. Then $exp(D) = n-a+1+(n-2)j$.*

*Proof.* The shortest walk from $n$ to $a+j$ that passes through a vertex on a $j$ cycle has length $n-a-j+n$, so it follows (by Frobenius Schur) that there is no walk from $n$ to $a+j$ of length $n-a-j+n+(n-1)(j-1)-1$. Thus $exp(D) \geq n-a+1+(n-2)j$. Further, since there is a walk between any two vertices of length at most $n-a-j+n$ that goes through a vertex on a $j$ cycle, it follows that $exp(D) \leq n-a+1+(n-2)j$, and thus $exp(D) = n-a+1+(n-2)j$. ☐

If $j \geq n/2$, note that $exp(D) = n-a+1+(n-2)j \geq j(n-1)$ for $1 \leq a \leq n-j+1$, giving the result of [6, Corollary 3.1] when $k = n$ without the additional assumption. Also note that if $j = n-1$ and $a = 1$, then $exp(D)$ achieves its maximum value of $\omega_n$, and $D = D(W_n)$, as described in Section 1. It is interesting to note that in the above corollary, it is only the value of $a$ that influences the value of the exponent; if $2 \leq i \leq a-1$, the presence or absence of the arc $i \to i+j-1$ does not affect the exponent. For fixed $n$ and $j$, this result gives a range of values of $exp(D)$ in which there are no gaps; see [6].

**3. The Case $k = n-1$.** If $D$ on $n$ vertices has large exponent with cycle lengths $n-1$ and $j < n-1$, then Theorem 1.2 shows that $j \geq \lceil n/2 \rceil$ provided that $n \geq 5$. (There are no such digraphs for $n \leq 4$.) Our next two theorems characterize these digraphs for $n \geq 6$. As in the Hamiltonian case, most digraphs are covered by the first result (Theorem 3.3), but the case $j = n/2$ (when $n$ is even) is different, and is given by the second result (Theorem 3.4). Before proving our main results, we give a definition and a preliminary Lemma. Note that since there is a cycle of length $n-1$, indices are taken mod $(n-1)$. Vertex $n$ *replicates vertex* $v \in \{1,\ldots,n-1\}$ in a digraph $D$ on $n$ vertices if for all $a,b \in \{1,\ldots,n-1\}, a \to n$ iff $a \to v$ and $n \to b$ iff $v \to b$. Thus in the adjacency matrix $A$ with $D = D(A)$, the rows (and columns) corresponding to vertices $n$ and $v$ are the same.

LEMMA 3.1. *Let $D$ be a strongly connected digraph on $n \geq 5$ vertices, with cycle lengths $n-1$ and $j$, where $n-1 > j \geq 3$. Suppose that $1 \to n-1 \to \cdots \to 2 \to 1$ is an $n-1$ cycle, and that $c \to n$. Then $n$ has outdegree at most 2, with either $n \to c-2$ or $n \to c+j-2$ or both. Furthermore, if the outdegree of $n$ is 2, then the indegree of $n$ is 1.*

*Proof.* First suppose that there is an arc $n \to a$. Then there is a cycle $n \to a \to a-1 \to \cdots \to c \to n$ of length $a-c+2$ if $a > c$, or length $n+1+a-c$ if $c > a$. In the former case, $a-c+2 = j$ or $n-1$, from which it follows that $a = c+j-2$ or $c-2$; in the latter case similarly $a = c+j-2$ or $c-2$. This establishes the possible outarcs from $n$. Finally, assume that $n \to c-2$ and $n \to c+j-2$. Suppose that $d \to n$ for some $d \neq c$. As above the two outarcs from $n$ can be written as $d-2$ and $d+j-2$. As $d \neq c$, it follows that $d-2 = c+j-2$ and $c-2 = d+j-2$. Hence $d-c = j$ and $c-d = j$, giving a contradiction. Thus the indegree of $n$ is 1. ☐

COROLLARY 3.2. *Let $D$ be as in Lemma 3.1. If $n \to c$, then either $c+2 \to n$ or $c+2-j \to n$ or both. Furthermore, if the indegree of $n$ is 2, then the outdegree of $n$ is 1.*

*Proof.* Form $D'$ by reversing the orientation of each arc in $D$. Then Lemma 3.1 applies to $D'$, and the result follows. ☐

THEOREM 3.3. *Suppose that $n \geq 6$ and $n-1 > j > n/2$. Then $D$ is a primitive digraph on $n$ vertices with $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$ and cycle lengths $n-1$ and $j$ iff (up to relabeling of vertices and reversal of each arc) $D$ is a (primitive) subdigraph of a digraph formed by taking an $n-1$ cycle $1 \to n-1 \to n-2 \to \cdots \to 2 \to 1$, adding in the arcs $a \to a+j-1$ for $1 \leq a \leq n-j$, and one of the following:*

*(a) arcs so that $n$ replicates $i_0$ for a fixed $i_0 \in \{1,\ldots,n-1\}$,*

*(b) arcs $1 \to n, n \to n-2$ and $n \to j-1$.*

*Proof.* First suppose that $D$ is primitive with $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$ and cycle lengths $n-1$ and $j$. By relabeling the vertices and/or reversing each arc in $D$ if necessary, we may assume that the $n-1$ cycle is as above, and that vertex $n$ has indegree 1 (Lemma 3.1 and Corollary 3.2). If the subdigraph induced by $\{1,\ldots,n-1\}$ is not primitive, then this subdigraph is just the $n-1$ cycle, and without loss of generality $1 \to n$, so by Lemma 3.1 the outarcs of $n$ are a subset of those given in (b). So suppose that the subdigraph induced by $\{1,\ldots,n-1\}$ is primitive. It follows from a result of Beasley and Kirkland [1, Theorem 1], that the exponent of this induced subdigraph is at least $\lfloor \omega_n/2 \rfloor$, which in turn is at least $\lfloor \omega_{n-1}/2 \rfloor + 2$. Hence without loss of generality, take the subdigraph to contain the arc $1 \to j$, and (by Theorem 2.2 ) to have the property that if $a \to a+j-1$, then $1 \leq a \leq n-j$. Let $a_0$ be the maximum such $a$. Suppose that $i \to n$ and note from Lemma 3.1 that the only possible outarcs from $n$ are $n \to i-2$ and $n \to i+j-2$. Consider the two cases: (i) $n \not\to i+j-2$, (ii) $n \to i+j-2$.

Case (i) $n \not\to i+j-2$: Vertex $n$ has outdegree 1 with $n \to i-2$ (and indegree 1 with $i \to n$). From the structure of the subgraph induced by $\{1,\ldots,n-1\}$ (described above), $D$ is a subdigraph of one constructed as in (a) (with $i_0 = i - 1$).

Case (ii): $n \to i+j-2$: If $1 \leq i-1 \leq n-j$ or $n-1 \geq i-1 \geq a_0+j-1$, then $D$ is a subdigraph of one of the ones constructed in (a) (if $i \neq 1$, with $i_0 = i+j-1$) or in (b) (if $i = 1$). Suppose now that $n-j+1 \leq i-1 \leq a_0+j-2$. Then $n \leq i+j-2 \leq a_0+2j-3$, so that $1 \leq i+j-2-(n-1) \leq a_0+2j-3-(n-1) < a_0-2$. Note that $D$ contains the closed walk $a_0 \to a_0+j-1 \to a_0+j-2 \to ... \to i \to n \to i+j-2-(n-1) \to i+j-3-(n-1) \to ... \to 1 \to j \to j-1 \to ... \to a_0$, which has length $3j-(n-1)$. Any closed walk can be decomposed into cycles, thus $3j-(n-1) = c_1 j + c_2(n-1)$ for some nonnegative integers $c_1, c_2$. Since $j < 3j-(n-1) < 2(n-1)$, the only possible cases are that $3j-(n-1)$ is one of $n-1$ (with $c_1 = 0, c_2 = 1$), $2j$ (with $c_1 = 2, c_2 = 0$) and $j+n-1$ (with $c_1 = 1, c_2 = 1$). The last two of these imply that $j = n-1$ (a contradiction). The first of these three can only occur if $3j = 2(n-1)$, and since $j$ and $n-1$ are relatively prime, this is also impossible. Consequently, it must be the case that $1 \leq i-1 \leq n-j$ or $n-1 \geq i-1 \geq a_0+j-1$, so that $D$ is a subgraph of one of the ones constructed in (a) or (b).

For the converse, consider a maximal digraph $H$ constructed as in (a). Since $n$ replicates $i_0$, $exp(H) = exp(H')$ where $H'$ is formed from $H$ by deleting $n$ and its incident arcs. Now $H'$ is Hamiltonian on $n-1$ vertices and has the digraph structure of Theorem 2.2, thus $exp(H') \geq \lfloor \omega_{n-1}/2 \rfloor + 2$. Applying Corollary 2.4 to $H'$ with $n$ replaced by $n-1$ and $a = n-j$, $exp(H') = j(n-2) \geq \lfloor \omega_n/2 \rfloor + 2$, since $j > n/2$ and $n \geq 6$. For case (b), observe that there is no walk from $n-1$ to 1 of length $(n-2)j-1$ (by the usual Frobenius- Schur argument), so that the exponent is at least $(n-2)j$, giving the required result as in (a). ☐

Note that the result of Theorem 3.3 does not hold for small values of $n$. For example, if $n = 5$ a digraph as in (a) of Theorem 3.3 with exponent equal to $9 < 10 = \lfloor \omega_5/2 \rfloor + 2$ can be constructed by taking a Hamiltonian digraph on 4 vertices with two additional arcs from consecutive vertices forming two $3-$cycles (see, e.g., [2, pp. 82-83]) and vertex 5 replicating vertex 1.

THEOREM 3.4. *Suppose that $n \geq 6$ is even and $j = n/2$. Then $D$ is a primitive digraph on $n$ vertices with $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$ and cycle lengths $n-1$ and $j$ iff (up to relabeling of vertices and reversal of each arc) $D$ is a (primitive) subdigraph of a digraph formed by taking an $n-1$ cycle $1 \to n-1 \to n-2 \to \cdots \to 2 \to 1$, adding in the arcs $i \to i+j-1$ for $1 \leq i \leq n/2-3$, and one of the constructions (a) or (b) in Theorem 3.3.*

*Proof.* First suppose that $D$ is primitive with $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2$ and cycle lengths $n-1$ and $j$. As in the proof of Theorem 3.3, assume that the $n-1$ cycle is as above, that the subdigraph induced by $\{1, \ldots, n-1\}$ is primitive, with $1 \to j$, and with the property that if $a \to a+j-1$, then $1 \leq a \leq n-j$. Finally, also suppose that $i \to n$. By Lemma 3.1 and Corollary 3.2 there are two cases to consider: (i) $D$ contains exactly one of the arcs $n \to i+j-2$ and $i-j \to n$, (ii) $D$ contains neither the arc $n \to i+j-2$ nor the arc $i-j \to n$.

Case (i): We claim that we may assume that $n \to i+j-2$. To see the claim, observe that if instead we have the arc $i-j \to n$ (and thus, by Lemma 3.1, $n \to i-2$), we can reverse every arc in $D$ and relabel vertices $1, \ldots, n-1$ by sending $t$ to $n-t$ for each such $t$. With this relabeling, it follows from Lemma 3.1 that $n-i+2 \to n$ and

$n \to n-i+j$. With $n-i+2$ replaced by $i$, this digraph contains the arc $n \to i+j-2$. So without loss of generality, we assume that the arc $n \to i+j-2$ is in $D$. Since vertex $n$ is on a $j$-cycle and since $D$ has diameter at most $n-1$, it follows that there is a walk from $n$ to any vertex of length $n-1+(n-2)(n/2-1)=(n^2-2n+2)/2$, and similarly that from any vertex in $D$ there is a walk to $n$ of length $(n^2-2n+2)/2$. Since $exp(D) \geq \lfloor \omega_n/2 \rfloor + 2 = (n^2-2n+6)/2$, it must be the case that there are vertices $u$ and $v \in \{1,\ldots,n-1\}$ such that there is no walk from $u$ to $v$ of length $(n^2-2n+4)/2$. Observe that for any vertex $w \in \{1,\ldots,n-1\}$ that is on a $j$-cycle, there is a walk from $w$ to every vertex in $\{1,\ldots,n-1\}$ of length $n-2+(n-2)(n/2-1)=(n^2-2n)/2$. As a result, the shortest walk from $u$ to a vertex in $\{1,\ldots,n-1\}$ that is on a $j$-cycle must have length at least 3. It follows from this that in fact vertex $n-1$ must be at least 3 steps from the nearest $j$-cycle, so that in particular, none of $n-1$, $n-2$ and $n-3$ can be on a $j$-cycle. Thus in $D$, $n-j \not\to n-1$, $n-j-1 \not\to n-2$ and $n-j-2 \not\to n-3$, and so if $a \to a+j-1$, then $a \leq n-j-3 = n/2-3$. Further, it must be the case that $1 \leq i \leq n-j-2$, otherwise one of vertices $n-1$, $n-2$ and $n-3$ is on a $j$-cycle (involving vertices $i$ and $n$). Consequently, $D$ can be relabeled to yield a subdigraph of one of those constructed in (a) with $i_0 = i-1$ (if $2 \leq i \leq n-j-2$), or (b) (if $i=1$).

Case (ii): If $D$ contains neither the arc $n \to i+j-2$ nor the arc $i-j \to n$, then $n$ has both indegree and outdegree 1, with $i \to n \to i-2$. Now if $D$ contains either of the arcs $i-1 \to i+j-2$ or $i-j \to i-1$, then the labels of vertices $i-1$ and $n$ can be exchanged and case (i) above applies. On the other hand if $D$ contains neither of those two arcs, then $i-1$ has indegree and outdegree 1, with $i \to i-1 \to i-2$, so that vertex $n$ replicates vertex $i-1$. Thus $exp(D) = exp(D')$ where $D'$ is formed from $D$ by deleting vertex $n$ and the arcs incident with it. From Corollary 2.4 with $n$ replaced by $n-1$, $exp(D') = n-1-a+1+(n-3)j$ where $a = max\{b$ is a vertex in $D'$ : the arc $b \to b+j-1$ is in $D'\}$. Thus $exp(D') = exp(D) = n-a+(n-3)n/2 \geq (n^2-2n+6)/2$, which implies that $a \leq n/2-3$. Consequently $D$ is a subdigraph of one of those constructed in (a) with $i_0 = i-1$.

For the converse, consider a digraph $H$ constructed as in (a). Since $n$ replicates $i_0$, $exp(H) = exp(H')$, where $H'$ is formed from $H$ by deleting $n$ and its incident arcs. Appealing to Corollary 2.4 with $n$ replaced by $n-1$, $a = n/2-3$, and $j = n/2$, $exp(H') = (n^2-2n+6)/2 = \lfloor \omega_n/2 \rfloor + 2$ if $n$ is even. Finally, consider the digraph $H$ constructed in (b). Evidently the walks from vertex $n-1$ to $n-3$ can only have lengths equal to 2, or to $2+n-1+c_1(n-1)+c_2 j$ for nonnegative integers $c_1$ and $c_2$. It follows that there is no walk from $n-1$ to $n-3$ of length $(n^2-2n+4)/2$, so that $exp(H) \geq (n^2-2n+6)/2$. $\square$

**4. Eigenvalue Results.** In this section we explore results on the multiplicities of eigenvalues of primitive stochastic matrices having large exponent. These complement eigenvalue results in [4]. Our first theorem gives conditions for a stochastic matrix with large exponent to have a multiple nonzero eigenvalue. This result, which is not restricted to $k=n$ or $k=n-1$, shows that a multiple nonzero eigenvalue must be negative with algebraic multiplicity 2.

THEOREM 4.1. *Let $A$ be a primitive, row stochastic $n$-by-$n$ matrix with $n \geq 3$ and $exp(A) \geq \lfloor \omega_n/2 \rfloor + 2$. Let $k$ and $j$ be the two cycle lengths in $D(A)$ with $n \geq k > j$. Then $A$ has a multiple nonzero eigenvalue $\lambda$ iff $\lambda = -r$, where $r$ is the unique positive root of $kx^j + jx^k = k - j$. When this is the case, $k$ is odd and $j$ is even.*

*Proof.* By Theorem 1 in [4], the characteristic equation of $A$ is $z^n - \alpha z^{n-j} - (1-\alpha)z^{n-k} = 0$, for some $\alpha \in (0,1)$. Thus a nonzero eigenvalue satisfies

$$z^k - \alpha z^{k-j} - (1-\alpha) = 0. \tag{2}$$

Note that 1 is always an eigenvalue, and (by Descartes' rule of signs) there is no other positive eigenvalue. Let $\lambda = \rho e^{i\theta}$ be an eigenvalue with $\rho > 0$ and $0 < \theta < 2\pi$. By differentiating, if $\lambda$ is a multiple eigenvalue, then it also satisfies $\lambda^j = \alpha(k-j)/k$, giving $\rho^j = \alpha(k-j)/k$ and $\theta = 2\pi l/j$ for some positive integer $l < j$. Further differentiation shows that the algebraic multiplicity of $\lambda$ is 2. By taking imaginary parts of the characteristic equation, $\rho^k sin(k\theta) = \alpha\rho^{k-j} sin((k-j)\theta)$. On substituting for $\rho^j$, this gives $(k-j)sin(k\theta) = ksin((k-j)\theta) = ksin((k-j)2\pi l/j) = ksin(k\theta)$. Thus $sin(k\theta) = 0$, so that $\theta = \pi m/k$ for some positive integer $m$. Hence $2lk = mj$, and since $gcd(k,j) = 1$ and $j$ divides $2l$, it must be that $j = 2l$. As a result $\theta = \pi, \lambda = -\rho$, $j$ is even, $k$ is odd and $\alpha = k\rho^j/(k-j)$. Substituting into (2) gives $k\rho^j + j\rho^k = k - j$. The converse is straightforward. □

From the characteristic equation, a matrix satisfying the conditions of Theorem 4.1 has zero as an eigenvalue iff $k < n$, and its algebraic multiplicity is $n - k$.

The digraph characterizations in Sections 2 and 3 lead to results about the geometric multiplicities of eigenvalues of primitive, stochastic matrices with large exponent.

THEOREM 4.2. *Let $A$ be a primitive, row stochastic $n$-by-$n$ matrix with $n \geq 3$ and $exp(A) \geq \lfloor \omega_n/2 \rfloor + 2$. If $D(A)$ is Hamiltonian, then each eigenvalue of $A$ is geometrically simple.*

*Proof.* Let the length of the shorter cycle(s) in $D(A)$ be $j \geq \lceil (n-1)/2 \rceil$ by Lemma 2.1. For $j \geq n/2$ take $p = n - j + 1$, and for $j = (n-1)/2$ take $p = (n-1)/2$. Then by Theorems 2.2 and 2.3, without loss of generality by permutation similarity $A = [a_{ij}]$ has the following form: $a_{1,n} = 1 - \alpha_1$; $a_{i,i-1} = 1 - \alpha_i$ for $2 \leq i \leq p$; $a_{i,i-1} = 1$ for $p + 1 \leq i \leq n$; $a_{i,i+j-1} = \alpha_i$ for $1 \leq i \leq p$; and all other $a_{ij} = 0$. Here $\alpha_i$ satisfy $0 < \alpha_1 < 1$ and $0 \leq \alpha_i < 1$ for $2 \leq i \leq p$. Thus for all $j \geq \lceil (n-1)/2 \rceil$, $A$ is an unreduced Hessenberg matrix. By deleting row 1 and column $n$, it can be seen that rank $A \geq (n-1)$ [7, Exercise 22, p. 274]. Similarly, rank $(A - \lambda I) = n - 1$ for each eigenvalue $\lambda$ of $A$. This implies that each eigenvalue has geometric multiplicity one. □

As an example of the above eigenvalue results, consider the 3-by-3 row stochastic matrix $A$ having $k = 3$ and $j = 2$ as in the proof of Theorem 4.2 with $\alpha_1 = \alpha_2 = 1/2$. Note that $exp(A) = 4$. The characteristic equation of $A$ is $z^3 - \alpha z - (1-\alpha) = 0$, with $\alpha = 3/4$; thus $A$ has eigenvalues $1, -1/2, -1/2$. Here $-1/2$ is an eigenvalue of algebraic multiplicity 2 (as predicted by Theorem 4.1), but geometric multiplicity 1 (as predicted by Theorem 4.2).

THEOREM 4.3. *Let $A$ be a primitive, row stochastic $n$-by-$n$ matrix with $n \geq 6$ and $exp(A) \geq \lfloor \omega_n/2 \rfloor + 2$. If the maximal cycle length in $D(A)$ is $n - 1$, then each* *eigenvalue of $A$ is geometrically simple.*

*Proof.* Since $k = n-1$, $\lambda = 0$ is a simple eigenvalue of $A$. Let the length of the shorter cycle(s) in $D(A)$ be $j \geq \lceil n/2 \rceil$ by Theorem 1.2. For simplicity, only the proof for the case $j > n/2$ is given, the case $j = n/2$ is essentially the same. For $j > n/2$, by Theorem 3.3, without loss of generality by permutation similarity $A = [a_{ij}]$, or its transpose, must have one of two forms corresponding to (a) or (b).

In case (a), without loss of generality $n$ can be taken to replicate a vertex with outdegree 1. (This is because, by Lemma 3.1, $n$ has either indegree or outdegree 1, so, if necessary, take $A^T$.) Let vertex $n$ replicate vertex $i$ where $n-1 \geq i > n-j$. Consider the matrix $A - \lambda I$, where $\lambda \neq 0$ and the digraph of $A$ is as in Theorem 3.3(a). Form $B$ from $A - \lambda I$ by deleting the first row and the last column. Then $B$ is block upper triangular with a $(1,1)$ block of order $i-2$ and a $(2,2)$ block of order $n-i+1$. Since the $(1,1)$ block is upper triangular with positive diagonal entries, it is nonsingular. The $(2,2)$ block has the first $n-i$ diagonal entries positive, $-\lambda$ in each superdiagonal entry, and a 1 in the last row first column. Every other entry in the $(2,2)$ block is zero. By expanding about the first row, the determinant of the $(2,2)$ block has magnitude $\lambda^{n-i}$. As a result, $B$ is nonsingular, so that $A - \lambda I$ has a submatrix of rank $n-1$.

In case (b), $a_{i,i+j-1} = \alpha_i$ for $1 \leq i \leq n-j$; $a_{1,n-1} = \beta_1$; $a_{1,n} = 1 - \alpha_1 - \beta_1$; $a_{i,i-1} = 1 - \alpha_i$ for $2 \leq i \leq n-j$; $a_{i,i-1} = 1$ for $n-j+1 \leq i \leq n-1$; $a_{n,j-1} = \gamma_n$; $a_{n,n-2} = 1 - \gamma_n$; and all other $a_{ij} = 0$. Here the parameters satisfy: $0 \leq \alpha_1 < 1$; $0 < \beta_1 < 1$; $1 - \alpha_1 - \beta_1 > 0$; $0 \leq \alpha_i < 1$ for $2 \leq i \leq n-j$; and $0 < \gamma_n \leq 1$, such that $A$ is primitive. Deleting row $n$ and column $n-1$, the remaining submatrix of $A - \lambda I$ is upper Hessenberg, and has rank $n-1$ for all values of $\lambda$, because it has a unique nonzero transversal of length $n-1$ (from the subdiagonal and $(1,n)$ entries of $A - \lambda I$).

Thus rank $(A - \lambda I) = n-1$ for every eigenvalue $\lambda$ of $A$, and the geometric multiplicity of each eigenvalue is one. ∎

We close the paper with a class of examples to show that for $k \leq n-2$, a row stochastic matrix with large exponent can have an eigenvalue of large geometric multiplicity.

EXAMPLE 4.4. For a fixed $n$, take $k \leq n-2$ so that $\omega_k \geq \lfloor \omega_n/2 \rfloor + 2$. Select $\alpha$ such that $0 < \alpha < 1$, and form the primitive row stochastic $n$-by-$n$ matrix $A$ with nonzero entries as follows: $a_{1,k-1} = \alpha$, $a_{1k} = 1-\alpha$, $a_{i,i-1} = 1$ for $i \in \{2,3\} \cup \{5,\ldots,k\}$, $a_{4i} = 1/(n-k+1)$ for $i \in \{3\} \cup \{k+1,\ldots,n\}$, and $a_{i2} = 1$ for $i \in \{k+1,\ldots n\}$. The digraph of $A$ can be formed by starting from $D(W_k)$ and taking each of the vertices $k+1,\ldots,n$ replicating vertex 3. Since vertex 3 is replicated $n-k$ times, there is a walk involving any of the vertices $k+1,\ldots,n$ in $D(A)$ iff there is a corresponding walk involving vertex 3 in $D(W_k)$. Thus $exp(A) = exp(D(W_k)) = \omega_k \geq \lfloor \omega_n/2 \rfloor + 2$. Observe that since each of rows $k+1$ through $n$ is a copy of row 3, $A$ has nullity at least $n-k$. Further, from the statement after Theorem 4.1, the algebraic multiplicity of 0 as an eigenvalue of $A$ is equal to $n-k$. Thus the algebraic and geometric multiplicities of 0 coincide, with common value $n-k \geq 2$ . The smallest example in this class has $n = 9, k = 7$ with other cycles of length 6. In this case, 0 is an eigenvalue of (algebraic and geometric) multiplicity 2.

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