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SPECTRAL EXTREMA ON GRAPHS WITH GIVEN SIZE FORBIDDING SMALL FANS*

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Abstract. The k-fan H_k is the graph obtained by taking the join of a vertex with a path order k. Brualdi–Hoffman–Turán problem asks what is the maximal spectral radius of an H-free graph G on m edges? Clearly, H_2 is a triangle, and H_3 is a book. Brualdi–Hoffman problems on triangles and books were solved by Nosal and Nikiforov. Recently, Yu, Li, and Peng showed that if G is an H_4-free graph of size $m \geq 8$, then $\rho(G) \leq \rho(S_{\frac{m+3}{2},2})$, with equality if and only if m is odd and $G \cong S_{\frac{m+3}{2},2}$. In this note, by pure spectral techniques, we show that if $m \geq 14$ and $\rho(G) \geq \frac{1}{2}(1+\sqrt{4m-5})$, then G contains a copy of H_4 unless $G \in \{S_{\frac{m+3}{2},2}, S^{-}_{\frac{m+4}{2},2}\}$. Our result not only determines the spectral extremal value of H_4 for both even and odd m but also implies a result on 5-cycle-free graphs by Min, Lou, and Huang.

Key words. Fan, Friendship graph, Spectral radius, Extremal graph.

AMS subject classifications. 05C50, 05C35.

1. Introduction. We shall assume that all graphs considered here have no isolated vertices. The *join* of G and H, denoted by $G \vee H$, is the graph obtained from G and H by joining each vertex of G to each vertex of H. Let $A(G)$ be the adjacency matrix of G. Since $A(G)$ is real and symmetric, its eigenvalues are all real numbers, the largest of which is called the spectral radius of G and denoted by $\rho(G)$. The study on the extremal value of $\rho(G)$ is a very important topic in spectral graph theory. A classic problem, proposed by Brualdi and Hoffman [1], asks what is the maximal spectral radius of a graph on m edges? In 1988, Rowlinson [16] solved this problem. A graph is said to be *H-free*, if it does not contain a subgraph isomorphic to H. A variation of Brualdi–Hoffman problem asks what is the maximal spectral radius of an H-free graph on m edges? In 1970, Nosal [15] proved that $\rho(G) \leq \sqrt{m}$ for every triangle-free graph G on m edges. Nikiforov [11, 12] extended Nosal's result by showing that $\rho(G) \leq \sqrt{2m(1-1/r)}$ for K_{r+1}-free graphs, where the equality was also characterized.

In this paper, we focus on a new extension of Nosal's result on triangles. Let F_k be the friendship graph of order $2k+1$, and let H_k denote the *k-fan*, which is isomorphic to $K_1 \vee P_k$. Brualdi–Hoffman problem on triangles and books was solved by Nosal and Nikiforov. Recently, Yu, Li and Peng [19], and independently, Zhang and Wang [23], obtained the following result.

THEOREM 1.1. *Let G be an H_4-free graph of size $m \geq 8$ and $S_{n,k} = K_k \vee (n-k)K_1$. Then $\rho(G) \leq \frac{1}{2}(1+\sqrt{4m-3})$, with equality if and only if $G \cong S_{\frac{m+3}{2},2}$.*

Theorem 1.1 implies some previous results, such as 5-cycle-free [20], F_2-free [6], and chorded 5-cycle-free graphs [17]. For additional results on the Brualdi–Hoffman–Turán problem, we refer to [3, 4, 5, 7, 8, 9, 13, 14, 18, 21, 22]. Since the extremal graph in Theorem 1.1 has odd size, one may be interested in the case

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that m is even. Let $S_{n,k}^{-r}$ be the graph obtained from $S_{n,k}$ by deleting an edge-subset E_0, where E_0 consists of edges from a dominating vertex to r vertices of degree k. Clearly, $S_{n,k}^{-r}$ has m edges for $n = \frac{m+r}{k} + \frac{k+1}{2}$ and every nonnegative integer r. Particularly, we denote $S_{n,k}^{-r}$ by $S_{n,k}^{-}$ when $r = 1$. Min, Lou, and Huang obtained the following result.

Theorem 1.2. *(Gao-Lou-Huang [10]) Let G be a C_5-free graph of even size $m \geq 14$. Then $\rho(G) \leq \rho(S^{-}_{\frac{m+4}{2},2})$, with equality if and only if $G \cong S^{-}_{\frac{m+4}{2},2}$.*

In this paper, we investigate graphs of size m with spectral radius $\rho \geq \frac{1}{2}(1+\sqrt{4m-5})$. The main result is as follows.

Theorem 1.3. *Let G be a graph of size $m \geq 14$. If $\rho(G) \geq \frac{1}{2}(1+\sqrt{4m-5})$, then G contains a copy of H_4 unless $G \in \{S_{\frac{m+3}{2},2}, S^{-}_{\frac{m+4}{2},2}\}$.*

As a corollary, we get that $\rho(G) \leq \rho(S_{\frac{m+3}{2},2})$ for every H_4-free graph G of odd size and $\rho(G) \leq \rho(S^{-}_{\frac{m+4}{2},2})$ for every H_4-free graph G of even size. Since C_5 is a subgraph of H_4, our result also extends Theorem 1.2.

2. Preliminaries.

In this section, we first introduce the definitions of equitable partition and quotient matrix. A vertex partition Π: $V_1 \cup V_2 \cup \ldots \cup V_k$ of a graph G is said to be *equitable* if, for each $u \in V_i$, $|V_j \cap N(u)| = b_{ij}$ is a constant depending only on i, j $(1 \leq i, j \leq k)$. The matrix $B_\Pi = (b_{ij})$ is called the *quotient matrix* of G with respect to Π.

Lemma 2.1. *(Cvetković-Rowlinson-Simić [2]) Let Π: $V_1 \cup \ldots \cup V_k$ be an equitable partition of G with quotient matrix B_Π. Then $det(xI - B_\Pi) \mid det\big(xI - A(G)\big)$. Furthermore, the largest eigenvalue of B_Π is just the spectral radius of G.*

Lemma 2.2. *$\rho(S^{-r}_{\frac{m+r+3}{2},2})$ is the maximum root of the following polynomial:*

$$x^4 - mx^2 - (m-r-1)x + \frac{1}{2}r(m-r-1).$$

Proof. Note that $S^{-r}_{\frac{m+r+3}{2},2}$ contains an edge v_1v_2, where v_1, v_2 are of degrees $\frac{1}{2}(m+r+1)$ and $\frac{1}{2}(m-r+1)$, respectively. Let V_3 be the set of vertices of degree one and V_4 be the set of other vertices. Then $|V_3| = r$. Moreover, $|V_4| = \frac{1}{2}(m-r-1)$ and every vertex in V_4 is of degree two. It is easy to see that $S_{\frac{m+r+3}{2},2}$ has an equitable partition $\Pi : \{v_1\} \cup \{v_2\} \cup V_3 \cup V_4$. The quotient matrix is given by

$$B_\Pi = \begin{bmatrix} 0 & 1 & r & \frac{m-r-1}{2} \\ 1 & 0 & 0 & \frac{m-r-1}{2} \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Straightforward calculation gives that

$$det(xI - B_\Pi) = x^4 - mx^2 - (m-r-1)x + \frac{1}{2}r(m-r-1).$$

By Lemma 2.1, $\rho(S^{-r}_{\frac{m+r+3}{2},2})$ is the maximum root of $det(xI - B_\Pi)$. □

Assume now that $m \geq 14$. Let G^* be an H_4-free graph on m edges with $\rho(G^*) \geq \frac{1}{2}(1+\sqrt{4m-5})$. By Perron–Frobenius theorem, there exists a nonnegative unit eigenvector X with respect to $\rho(G^*)$. Let

$X = (x_1, x_2, \ldots, x_{|G^*|})^T$ and choose a vertex $u^* \in V(G^*)$ such that $x_{u^*} = \max_{u\in V(G^*)} x_u$. Denote $A = N_{G^*}(u^*)$ and $B = V(G^*) \setminus (A \cup \{u^*\})$. Furthermore, we partition A into two subsets A_0 and A_1, where $A_0 = \{v \in A : d_A(v) = 0\}$ and $A_1 = A \setminus A_0$. Denote $\rho = \rho(G^*)$ for simplicity. Then we have

$$\rho x_{u^*} = \sum_{v\in A_0} x_v + \sum_{v\in A_1} x_v, \tag{2.1}$$

$$\rho^2 x_{u^*} = |A|x_{u^*} + \sum_{v\in A_1} d_A(v)x_v + \sum_{w\in B} d_A(w)x_w. \tag{2.2}$$

In addition, we shall notice that

$$m = |A| + e(A_1) + e(A, B) + e(B), \tag{2.3}$$

Notice that $\frac{1}{2}(1+\sqrt{4m-5})$ is the maximum root of the polynomial $x^2-x-(m-\frac{3}{2})$. Since $\rho \geq \frac{1}{2}(1+\sqrt{4m-5})$, we can see that

$$\rho^2 - \rho \geq m - \frac{3}{2}. \tag{2.4}$$

In view of (2.1), (2.2) and (2.4), we obtain

$$\big(m - \frac{3}{2}\big)x_{u^*} \leq (\rho^2 - \rho)x_{u^*} = |A|x_{u^*} + \sum_{v\in A_1} (d_A(v) - 1)x_v + \sum_{w\in B} d_A(w)x_w - \sum_{v\in A_0} x_v. \tag{2.5}$$

Since G^* is H_4-free, it is clear that $G^*[A]$ is P_4-free. Hence, every connected component in $G^*[A_1]$ is isomorphic to a triangle or a star. Assume that $G^*[A_1]$ contains s stars and t triangles. Then $e(A_1) = |A_1| - s$ and thus

$$\sum_{v\in A_1} (d_A(v) - 1) = 2e(A_1) - |A_1| = e(A_1) - s. \tag{2.6}$$

Claim 1. $e(B) + s \leq \frac{3}{2} - \sum_{v\in A_0} x_v$.

Proof. Obviously, $\sum_{w\in B} d_A(w)x_w \leq e(A, B)x_{u^*}$. Combining (2.5), we can see that

$$\big(m - \frac{3}{2}\big)x_{u^*} \leq \big(|A| + e(A, B)\big)x_{u^*} + \sum_{v\in A_1} \big(d_A(v) - 1\big)x_v - \sum_{v\in A_0} x_v. \tag{2.7}$$

In view of (2.3) and (2.7), we obtain that

$$\big(e(A_1) + e(B) - \frac{3}{2}\big)x_{u^*} \leq \sum_{v\in A_1} \big(d_A(v) - 1\big)x_v - \sum_{v\in A_0} x_v. \tag{2.8}$$

Combining (2.6) and (2.8), we further have that $e(B) + s \leq \frac{3}{2} - \sum_{v\in A_0} x_v$. □

Note that X is a nonnegative eigenvector. By Claim 1, we can see that $e(B) + s \leq \frac{3}{2}$, which implies that $e(B) + s \leq 1$. If $e(B) + s = 1$, we will get some more properties.

Claim 2. *If $e(B)+s=1$, then $\sum_{v\in A_0} x_v \le \frac{1}{2}x_{u^*}$ and $\sum_{w\in B} d_A(w)(x_{u^*}-x_w)\le \frac{1}{2}x_{u^*}$.*

Proof. If $e(B)+s=1$, then by Claim 1, we obtain $\sum_{v\in A_0} x_v \le \frac{1}{2}x_{u^*}$ immediately. In the following, we show $\sum_{w\in B} d_A(w)(x_{u^*}-x_w)\le \frac{1}{2}x_{u^*}$. Combining (2.5) and (2.6), we get that

$$(m-\frac{3}{2})x_{u^*} \le |A|x_{u^*} + \big(e(A_1)-s\big)x_{u^*} + \sum_{w\in B} d_A(w)x_w.$$

Note that $e(A,B)x_{u^*} = \sum_{w\in B} d_A(w)x_{u^*}$. In view of (2.3), we can further see that

$$\big(e(B)+e(A,B)+s-\frac{3}{2}\big)x_{u^*} \le \sum_{w\in B} d_A(w)x_w,$$

which yields that $\sum_{w\in B} d_A(w)(x_{u^*}-x_w)\le \frac{1}{2}x_{u^*}$. ☐

Claim 3. *If $e(B)+s=1$, then $|A_1|\ge \rho-\frac{1}{2}$.*

Proof. By Claim 2, we know that $\sum_{v\in A_0} x_v \le \frac{1}{2}x_{u^*}$. Thus, $\rho x_{u^*} = \sum_{v\in A_1} x_v + \sum_{v\in A_0} x_v \le (|A_1| + \frac{1}{2})x_{u^*}$, which implies that $|A_1|\ge \rho-\frac{1}{2}$. ☐

Now denote by A_1' the set of vertices in triangle components of $G^*[A_1]$.

Claim 4. *If $e(B)+s=1$, then $\sum_{v\in A_1'} x_v \ge (|A_1'|-\frac{1}{2})x_{u^*}$.*

Proof. Observe that $d_A(v)=2$ for each $v\in A_1'$. Then $\sum_{v\in A_1'}(d_A(v)-1)x_v = \sum_{v\in A_1'} x_v$. If $\sum_{v\in A_1'} x_v < (|A_1'|-\frac{1}{2})x_{u^*}$, then $\sum_{v\in A_1'}(d_A(v)-1)x_v < \sum_{v\in A_1'}\big(d_A(v)-1\big)x_{u^*} - \frac{1}{2}x_{u^*}$ and

$$\sum_{v\in A_1}\big(d_A(v)-1\big)x_v < \sum_{v\in A_1}\big(d_A(v)-1\big)x_{u^*} - \frac{1}{2}x_{u^*}.$$

Combining (2.6), we have that $\sum_{v\in A_1}\big(d_A(v)-1\big)x_v < (e(A_1)-s-\frac{1}{2})x_{u^*}$. In view of (2.8), we further obtain that $(e(A_1)+e(B)-\frac{3}{2})x_{u^*} < (e(A_1)-s-\frac{1}{2})x_{u^*}$, which contradicts the assumption that $e(B)+s=1$. ☐

3. Proof of Theorem 1.3. In this section, we give the proof of Theorem 1.3. Recall that $G^*[A_1]$ contains s stars and t triangles. By Claim 1, $e(B)+s\le 1$. Now we prove three key lemmas.

Lemma 3.1. *If $e(B)=1$, then the unique edge within B is an isolated edge of G^*.*

Proof. Let $E(G^*[B])=\{w_1w_2\}$. Then $s=0$, and hence $G^*[A_1]$ consists of t triangles. Now suppose to the contrary that w_1w_2 is not an isolated edge. Then $d_A(w_1)+d_A(w_2)\ge 1$. Assume without loss of generality that $x_{w_1}\ge x_{w_2}$. Since G^* is H_4-free, w_1 has at most one neighbor in every connected component of $G^*[A_1]$. Hence, $d_{A_1}(w_1)\le t$.

Now we shall show that $d_{A_1}(w_1)\ge 2$. By Claim 2, $\sum_{v\in A_0} x_v \le \frac{1}{2}x_{u^*}$, and hence,

$$\rho x_{w_1} \le x_{w_2} + \sum_{v\in N_{A_1}(w_1)} x_v + \sum_{v\in A_0} x_v \le x_{w_1} + d_{A_1}(w_1)x_{u^*} + \frac{1}{2}x_{u^*},$$

which yields that $x_{w_1}\le \frac{d_{A_1}(w_1)+\frac{1}{2}}{\rho-1}x_{u^*}$. Recall that $m\ge 14$ and $\rho\ge \frac{1+\sqrt{4m-5}}{2}$. It is easy to see that $\rho>4$. If $d_{A_1}(w_1)=1$, then $x_{w_1}\le \frac{3}{2(\rho-1)}x_{u^*} < \frac{1}{2}x_{u^*}$ and $d_A(w_1)(x_{u^*}-x_{w_1}) \ge d_{A_1}(w_1)(x_{u^*}-x_{w_1}) > \frac{1}{2}x_{u^*}$, which contradicts Claim 2. If $d_{A_1}(w_1)=0$, then $x_{w_1}\le \frac{1}{2(\rho-1)}x_{u^*} < \frac{1}{6}x_{u^*}$ and thus $x_{w_2}\le x_{w_1} < \frac{1}{2}x_{u^*}$. Recall that $d_A(w_1)+d_A(w_2)\ge 1$. Then we have $\sum_{i\in\{1,2\}} d_A(w_i)(x_{u^*}-x_{w_i}) > \frac{1}{2}x_{u^*}$, which also contradicts Claim 2.

Now denote $A_1'' = A_1 \setminus N_{A_1}(w_1)$. Then $A_1'' \subseteq N_{G^*}(u^*) \setminus N_{G^*}(w_1)$ and $w_2 \in N_{G^*}(w_1) \setminus N_{G^*}(u^*)$. Hence, $\rho(x_{u^*} - x_{w_1}) \ge \sum_{v \in A_1''} x_v - x_{w_2} \ge \sum_{v \in A_1''} x_v - x_{w_1}$. Recall that A_1' denotes the set of vertices in triangle-components of $G^*[A_1]$. Since $s = 0$, we have $A_1 = A_1'$ and thus $\sum_{v \in A_1} x_v \ge (|A_1| - \frac{1}{2})x_{u^*}$ by Claim 4. This implies that $\sum_{v \in A_1''} x_v \ge (|A_1''| - \frac{1}{2})x_{u^*}$. It follows that $\rho(x_{u^*} - x_{w_1}) \ge \big(|A_1''| - \frac{1}{2}\big)x_{u^*} - x_{w_1}$, which yields that

$$x_{w_1} \le \frac{\rho - |A_1''| + \frac{1}{2}}{\rho - 1}x_{u^*} = x_{u^*} - \frac{|A_1''| - \frac{3}{2}}{\rho - 1}x_{u^*}. \tag{3.9}$$

Recall that $|A_1| = 3t$ and $|A_1''| = |A_1| - d_{A_1}(w_1) = 3t - d_{A_1}(w_1)$. By Claim 3, we have $\rho \le |A_1| + \frac{1}{2}$ and so $\rho - 1 \le 3t - \frac{1}{2}$. According to (3.9), we get that

$$d_A(w_1)(x_{u^*} - x_{w_1}) \ge d_{A_1}(w_1)\frac{|A_1''| - \frac{3}{2}}{\rho - 1}x_{u^*} \ge \frac{d_{A_1}(w_1)(3t - d_{A_1}(w_1) - \frac{3}{2})}{3t - \frac{1}{2}}x_{u^*}.$$

Since $2 \le d_{A_1}(w_1) \le t$, we have $d_{A_1}(w_1)(3t - d_{A_1}(w_1) - \frac{3}{2}) \ge 2(3t - \frac{7}{2})$. Thus, $d_A(w_1)(x_{u^*} - x_{w_1}) \ge \frac{2(3t - \frac{7}{2})}{3t - \frac{1}{2}}x_{u^*} > \frac{1}{2}x_{u^*}$, contradicting Claim 2. Hence, w_1w_2 is an isolated edge. ☐

Recall that $G^*[A_1]$ contains s stars and t triangles. Moreover, by Lemma 3.1, we have that either $e(B) = 0$ or $G^*[B]$ contains exactly one edge, which is an isolated edge of G^*. In the following, we show that $G^*[A_1]$ contains no triangle components.

Lemma 3.2. $t = 0$.

Proof. Suppose to the contrary that $t \ge 1$. Then we can find three vertices $v_1, v_2, v_3 \in A_1$ such that $\{v_1, v_2, v_3\}$ induces a triangle. Let $B_i = N_B(v_i)$ for $i \in \{1, 2, 3\}$. Since G^* is H_4-free, B_1, B_2, and B_3 are pairwise disjoint. It is easy to check that $\sum_{i=1}^{3} \rho x_{v_i} = 3x_{u^*} + \sum_{i=1}^{3} 2x_{v_i} + \sum_{w \in \cup_{i=1}^{3} B_i} x_w$. Recall that $\rho > 4$. Then

$$\sum_{i=1}^{3} x_{v_i} = \frac{3x_{u^*} + \sum_{w \in \cup_{i=1}^{3} B_i} x_w}{\rho - 2} < \frac{3}{2}x_{u^*} + \frac{\sum_{i=1}^{3} |B_i|}{\rho - 2}x_{u^*}. \tag{3.10}$$

Moreover, $\sum_{i=1}^{3}(d_A(v_i) - 1)(x_{v_i} - x_{u^*}) = \sum_{i=1}^{3}(x_{v_i} - x_{u^*})$. Combining (3.10), we have

$$\begin{aligned} \sum_{v \in A_1}(d_A(v) - 1)x_v &< \sum_{v \in A_1}(d_A(v) - 1)x_{u^*} + \big(\tfrac{3}{2}x_{u^*} + \tfrac{\sum_{i=1}^{3} |B_i|}{\rho - 2}x_{u^*}\big) - 3x_{u^*} \\ &= \big(e(A_1) - s - \tfrac{3}{2} + \tfrac{\sum_{i=1}^{3} |B_i|}{\rho - 2}\big)x_{u^*}, \end{aligned} \tag{3.11}$$

where the last equality follows from (2.6).

In the following, we have to prove three claims. **First, we claim that $\cup_{i=1}^{3} B_i \ne \emptyset$.** Suppose that $\cup_{i=1}^{3} B_i = \emptyset$. Then combining (2.8) and (3.11) gives that $(e(A_1) + e(B) - \frac{3}{2})x_{u^*} < (e(A_1) - s - \frac{3}{2})x_{u^*}$. If follows that $e(B) + s < 0$, a contradiction. Hence, $\cup_{i=1}^{3} B_i \ne \emptyset$.

Now assume without loss of generality that $x_{v_1} \ge x_{v_2} \ge x_{v_3}$. Since $\cup_{i=1}^{3} B_i \ne \emptyset$, we will see that $B_1 \ne \emptyset$ (otherwise, we may assume that $B_2 \ne \emptyset$, then $\rho x_{v_2} = x_{u^*} + x_{v_1} + x_{v_3} + \sum_{w \in B_2} x_w$ and $\rho x_{v_1} = x_{u^*} + x_{v_2} + x_{v_3}$, thus we have $(\rho + 1)(x_{v_2} - x_{v_1}) = \sum_{w \in B_2} x_w > 0$, which contradicts the assumption that $x_{v_1} \ge x_{v_2}$). Let $x_{v_2} + x_{v_3} = \alpha x_{u^*}$. **Second, we claim that $\alpha \ge 1$.** Indeed,

$$\sum_{v \in A_1}(d_A(v) - 1)x_v \le \sum_{v \in A_1}(d_A(v) - 1)x_{u^*} - (2 - \alpha)x_{u^*} = \big(e(A_1) - s - 2 + \alpha\big)x_{u^*}. \tag{3.12}$$

Choose a vertex $w_1 \in B_1$. By Lemma 3.1, $d_B(w_1) = 0$ and so $\rho x_{w_1} \le d_A(w_1)x_{u^*}$. If $d_A(w_1) \le \frac{\rho}{2}$, then $x_{w_1} \le \frac{1}{2}x_{u^*}$. It follows that

$$\sum_{w\in B} d_A(w)x_w \le e(A,B)x_{u^*} - \frac{1}{2}x_{u^*}. \tag{3.13}$$

Combining (3.12) and (3.13) with (2.5), we obtain that

$$\big(m-\frac{3}{2}\big)x_{u^*} \le \big(|A| + e(A_1) + e(A,B) - s - \frac{5}{2} + \alpha\big)x_{u^*}.$$

Note that $m = |A| + e(A_1) + e(A,B) + e(B)$. Hence, $\alpha \ge e(B) + s + 1 \ge 1$.

Now we consider the case $d_A(w_1) > \frac{\rho}{2}$. Since $v_2, v_3 \in N_{G^*}(u^*) \setminus N_{G^*}(w_1)$, we have $\rho(x_{u^*} - x_{w_1}) \ge x_{v_2} + x_{v_3} = \alpha x_{u^*}$. In fact, we can similarly obtain that $\rho(x_{u^*} - x_w) \ge \alpha x_{u^*}$ for every $w \in \cup_{i=1}^{3} B_i$ as $x_{v_1} \ge x_{v_2} \ge x_{v_3}$. From $d_A(w_1) > \frac{\rho}{2}$, we can see that $d_A(w_1)(x_{u^*} - x_{w_1}) \ge \frac{1}{2}\rho(x_{u^*} - x_{w_1}) \ge \frac{\alpha}{2}x_{u^*}$. Thus,

$$\sum_{w\in B} d_A(w)x_w \le e(A,B)x_{u^*} - \frac{\alpha}{2}x_{u^*}. \tag{3.14}$$

Combining (3.12) and (3.14) with (2.5), we obtain that

$$\big(m-\frac{3}{2}\big)x_{u^*} \le \big(|A| + e(A_1) + e(A,B) - s - 2 + \frac{\alpha}{2}\big)x_{u^*},$$

which also gives $\alpha \ge 2e(B) + 2s + 1 \ge 1$.

Third, we claim that $d_A(w)(x_{u^*} - x_w) \ge \frac{x_{u^*}}{\rho-2}$ **for every vertex** $w \in \cup_{i=1}^{3} B_i$**.** Indeed, if $d_A(w) \le \frac{\rho(\rho-3)}{\rho-2}$, then $\rho x_w \le d_A(w)x_{u^*}$ and so $x_w \le \frac{\rho-3}{\rho-2}x_{u^*}$. It follows that $d_A(w)(x_{u^*} - x_w) \ge x_{u^*} - x_w \ge \frac{x_{u^*}}{\rho-2}$ for each $w \in \cup_{i=1}^{3} B_i$. On the other hand, recall that $\rho(x_{u^*} - x_w) \ge \alpha x_{u^*} \ge x_{u^*}$ for every $w \in \cup_{i=1}^{3} B_i$. If $d_A(w) > \frac{\rho(\rho-3)}{\rho-2}$, then $d_A(w)(x_{u^*} - x_w) \ge \frac{\rho-3}{\rho-2}x_{u^*}$. Since $\rho > 4$, we also have $d_A(w)(x_{u^*} - x_w) \ge \frac{x_{u^*}}{\rho-2}$.

Now by the third claim, we can see that

$$\sum_{w\in B} d_A(w)x_w \le e(A,B)x_{u^*} - \frac{\sum_{i=1}^{3}|B_i|}{\rho-2}x_{u^*}. \tag{3.15}$$

Combining (3.11) and (3.15) with (2.5), we obtain that

$$\big(m-\frac{3}{2}\big)x_{u^*} < \big(|A| + e(A_1) + e(A,B) - s - \frac{3}{2}\big)x_{u^*}.$$

Recall that $m = |A| + e(A_1) + e(A,B) + e(B)$. Hence, $e(B) + s < 0$, a contradiction. Therefore, $t = 0$. This completes the proof. ◻

By Lemma 3.2, we have $t = 0$. Combining Lemma 3.1, we can see that if $s = 0$, then G^* is bipartite and so $\rho(G^*) \le \sqrt{m}$, which contradicts the fact that $\rho(G^*) \ge \frac{1+\sqrt{4m-5}}{2}$. Hence, $s \ge 1$. Recall that $e(B) + s \le 1$ by Claim 1. Thus, we immediately conclude that $s = 1$ and $e(B) = 0$. Now assume that H is the unique component in $G^*[A_1]$. Then H is a star of order $|A_1|$. By Claim 3, $|A_1| \ge \rho - \frac{1}{2}$. Recall that $\rho > 4$. Thus $|A_1| \ge 4$.

Lemma 3.3. $|B| = 0$.

Proof. Suppose to the contrary that $|B| \geq 1$. Choose a vertex $w^* \in B$ and denote by v^* the central vertex of H. Since $e(B) = 0$, we have $d_A(w^*) \geq 1$.

First, we claim that $w^*v^* \notin E(G^*)$. If $w^*v^* \in E(G^*)$, then w^* has exactly one neighbor v^* in A_1 as $d_A(v^*) = |A_1| - 1 \geq 3$ and G^* is H_4-free. Thus, $\rho x_{w^*} \leq x_{v^*} + \sum_{v \in A_0} x_v$. Moreover, $\sum_{v\in A_0} x_v \leq \frac{1}{2}x_{u^*}$ by Claim 2. It follows that $\rho x_{w^*} \leq \frac{3}{2}x_{u^*}$, which gives that $x_{w^*} \leq \frac{3}{2\rho}x_{u^*} < \frac{1}{2}x_{u^*}$ as $\rho > 4$. Thus, $d_A(w^*)(x_{u^*} - x_{w^*}) > \frac{1}{2}x_{u^*}$, contradicting Claim 2. Therefore, $w^*v^* \notin E(G^*)$.

Second, we claim that $d_A(w^*) > \frac{\rho}{2}$. If $d_A(w^*) = 1$, then $x_{w^*} \leq \frac{1}{\rho}x_{u^*} < \frac{1}{2}x_{u^*}$ as $\rho > 4$. Hence, $d_A(w^*)(x_{u^*} - x_{w^*}) > \frac{1}{2}x_{u^*}$. If $2 \leq d_A(w^*) \leq \frac{\rho}{2}$, then $\rho x_{w^*} \leq d_A(w^*)x_{u^*} \leq \frac{\rho}{2}x_{u^*}$, and so $x_{w^*} \leq \frac{1}{2}x_{u^*}$. Thus, $d_A(w^*)(x_{u^*} - x_{w^*}) \geq x_{u^*} > \frac{1}{2}x_{u^*}$, In either case, we get a contradiction with Claim 2. Therefore, $d_A(w^*) > \frac{\rho}{2}$.

Now let $x_{v^*} = \beta x_{u^*}$. Then $\beta \leq 1$. Since $w^*v^* \notin E(G^*)$, we have $\rho(x_{u^*} - x_{w^*}) \geq x_{v^*} = \beta x_{u^*}$. Since $d_A(w^*) > \frac{\rho}{2}$, we further get that $d_A(w^*)(x_{u^*} - x_{w^*}) > \frac{\beta}{2}x_{u^*}$. Thus,

$$\sum_{w\in B} d_A(w)x_w < e(A,B)x_{u^*} - \frac{\beta}{2}x_{u^*}. \tag{3.16}$$

On the other hand, since $d_A(v^*) = e(A_1)$ and $d_A(v) = 1$ for $v \in A_1 \setminus \{v^*\}$, we have

$$\sum_{v\in A_1} \big(d_A(v) - 1\big)x_v = \big(d_A(v^*) - 1\big)x_{v^*} = \big(e(A_1) - 1\big)\beta x_{u^*}. \tag{3.17}$$

Combining (3.16) and (3.17) with (2.5), we obtain that

$$\big(m - \frac{3}{2}\big)x_{u^*} < \big(|A| + e(A,B) - \frac{\beta}{2} + (e(A_1) - 1)\beta\big)x_{u^*}.$$

Recall that $m = |A| + e(A_1) + e(A,B)$ and $|A_1| \geq 4$. Hence, $(e(A_1) - \frac{3}{2})x_{u^*} < (e(A_1) - \frac{3}{2})\beta x_{u^*}$. It follows that $\beta > 1$, a contradiction. Therefore, $|B| = 0$. ☐

By Lemmas 3.1, 3.2, and 3.3, we can see that $V(G^*) = A \cup \{u^*\}$ and $G^*[A]$ is the union of a star and $|A_0|$ isolated vertices. Therefore, $G^* \cong S^{-|A_0|}_{|A|+1,2}$. Now we are ready to complete the proof of Theorem 1.3.

Proof. Set $|A_0| = r$. Since $m = |A| + e(A_1)$ and $e(A_1) = |A_1| - 1 = |A| - r - 1$, we have $|A| = \frac{1}{2}(m+r+1)$ and $G^* \cong S^{-r}_{\frac{m+r+3}{2},2}$. By Lemma 2.2, $\rho(S^{-r}_{\frac{m+r+3}{2},2})$ is the maximum root of the following polynomial:

$$h(x) := x^4 - mx^2 - (m - r - 1)x + \frac{1}{2}r(m - r - 1).$$

Set $g(x) := (x^2 + x - \frac{1}{2})(x^2 - x - m + \frac{3}{2})$. One can check that

$$h(x) = g(x) + \big(r - 1\big)\big(x + \frac{m - r - 3}{2}\big) + \frac{1}{2}(r - \frac{3}{2}).$$

Note that $\frac{1}{2}(1 + \sqrt{4m - 5})$ is the maximum root of $x^2 - x - (m - \frac{3}{2})$. Clearly, $g(x) \geq 0$ provide that $x \geq \frac{1}{2}(1 + \sqrt{4m - 5})$. Recall that $|A_1| \geq 4$. Then $m = |A_0| + 2|A_1| - 1 > |A_0| + 3 = r + 3$.

On the one hand, if $r \geq 2$ and $x \geq \frac{1}{2}(1 + \sqrt{4m - 5})$, then $h(x) > g(x) \geq 0$. Therefore, $\rho(S^{-r}_{\frac{m+r+3}{2},2}) < \frac{1}{2}(1 + \sqrt{4m - 5})$ for each $r \geq 2$.

On the other hand, if $r \in \{0,1\}$ and $x = \frac{1}{2}(1+\sqrt{4m-5})$, then $h(x) < g(x) = 0$, and hence $\rho(S^{-r}_{\frac{m+r+3}{2},2}) > \frac{1}{2}(1+\sqrt{4m-5})$. Therefore, $G^* \cong S_{\frac{m+3}{2},2}$ for $r = 0$ and $G^* \cong S^{-}_{\frac{m+4}{2},2}$ for $r = 1$. This completes the proof of Theorem 1.3. □

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