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BARYCENTER OF THE ARITHMETIC-HARMONIC QUANTUM DIVERGENCE*

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Abstract. A notion of divergence is a very important and useful tool to measure the difference between probability distributions or between data (information). We consider a quantum divergence constructed by the difference of two-variable weighted arithmetic and harmonic means on the open convex cone of positive definite Hermitian matrices, called the arithmetic-harmonic quantum divergence. We see its invariance properties and study the barycenter minimizing the weighted sum of arithmetic-harmonic quantum divergences to given variables. We provide the lower bound for the barycenter of the arithmetic-harmonic quantum divergence in terms of Loewner order and its upper bound in terms of operator norm.

Key words. Quantum divergence, Arithmetic mean, Harmonic mean, Geometric mean, Barycenter.

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1. Introduction. A divergence function is originated from differences between probability distributions in statistics and the degree of difference between data (information) in the fields of information theory and data science. In mathematics, one can see that the divergence is a generalization of a squared distance. More precisely, for a set S of data, the function $D : S \times S \to \mathbb{R}$ is called a *divergence* if $D(x,y) \geq 0$ and $D(x,y) = 0 \Longleftrightarrow x = y$ for any two data (points) $x, y \in S$. In general, it is not necessary to satisfy symmetry and triangle inequality. For the ease of calculation, a divergence is used in various applications such as signal processing [11], medical image analysis [10, 13], econometrics [14], and machine learning [9, 17]. In addition, the k-mean, at which the weighted sum of divergence values to each data point is minimized, is the most important concept in the data clustering algorithm [2, 6, 7].

A divergence on the setting $\mathbb{P}_m$ of $m \times m$ positive definite Hermitian matrices is called a *quantum divergence*. See Definition 2.1 for more details. The canonical examples are the Kullback–Leibler divergence Φ_{KL} and the Bregman divergence Φ_B:

$$\Phi_{KL}(A,B) = \operatorname{tr}(B^{-1}A - I) - \log\det(B^{-1}A),$$
$$\Phi_B(A,B) = \phi(A) - \phi(B) - D\phi(B)(A-B),$$

for any $A, B \in \mathbb{P}_m$, where $\phi : \mathbb{P}_m \to \mathbb{R}$ is a differentiable and strictly convex function. Recently, the family of generalized quantum Hellinger divergences of the form

$$\Psi(A,B) = \operatorname{tr}\left((1-\alpha)A + \alpha B - A\sigma B\right),$$

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where σ is a Kubo-Ando's operator mean and $\alpha \in (0,1)$ is the weight of σ, has been introduced in [12].

Generally, since a quantum divergence Φ may not satisfy symmetry, one can consider two different minimization problems as follows:

$$\underset{X\in\mathbb{P}_m}{\arg\min}\sum_{i=1}^{n}\Phi(X,A_i), \qquad \underset{X\in\mathbb{P}_m}{\arg\min}\sum_{i=1}^{n}\Phi(A_i,X). \tag{1.1}$$

We call, respectively, the *left mean* and *right mean*, when the minimizers uniquely exist. Such means are generalizations of the least squares mean on a given metric space, and the existence and uniqueness of the solution are very important results from a pure mathematical point of view. In most cases, using the convexity of the objective function and fixed point theorem, the existence and uniqueness of the solution can be proved.

In this paper, we consider a quantum divergence for $\alpha \in (0,1)$

$$\Phi_\alpha(A,B) = \operatorname{tr}\left[(1-\alpha)A + \alpha B - ((1-\alpha)A^{-1} + \alpha B^{-1})^{-1}\right],$$

called the *arithmetic-harmonic quantum divergence*, as a special case of quantum divergences in [12]. We show its invariance properties under unitary congruence transformation and tensor product with a density matrix, which imply the data processing inequality. Next, we prove the existence and the uniqueness of the minimization problem (1.1), which provides us the barycenter (right mean and left mean) of the arithmetic-harmonic quantum divergence. Moreover, we verify fundamental properties for the barycenter of the arithmetic-harmonic quantum divergence and the relationship with other matrix means.

2. The arithmetic-harmonic quantum divergence. Let $\mathbb{H}_m$ be the real vector space of all $m \times m$ Hermitian matrices with complex entries. For any $A, B \in \mathbb{H}_m$ we denote as $A < B$ if $B - A$ is positive definite, and $A \leq B$ if $B - A$ is positive semi-definite. Note that $\leq$ is known as the Loewner order. Let $\mathbb{P}_m \subset \mathbb{H}_m$ be the open convex cone of all $m \times m$ positive definite matrices.

Definition 2.1. [1, 4] A quantum divergence on the open convex cone $\mathbb{P}_m$ is defined as a smooth function $\Phi : \mathbb{P}_m \times \mathbb{P}_m \to \mathbb{R}$ that satisfies the following properties:

(i) $\Phi(A,B) \geq 0$, and the equality holds if and only if $A = B$;

(ii) the derivative $D\Phi$ with respect to the second variable vanishes on the diagonal, that is,

$$D\Phi(A,B)|_{B=A} = 0;$$

(iii) the second derivative is nonnegative on the diagonal, that is,

$$D^2\Phi(A,B)|_{B=A}(Y,Y) \geq 0,$$

for any Hermitian matrix Y.

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For given $A, B \in \mathbb{P}_m$ and $\alpha \in [0,1]$ the weighted arithmetic, geometric, and harmonic means are defined, respectively, by

$$A\nabla_\alpha B = (1-\alpha)A + \alpha B,$$
$$A\#_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2},$$
$$A!_\alpha B = ((1-\alpha)A^{-1} + \alpha B^{-1})^{-1}.$$

Note that the weighted arithmetic and geometric means can be considered as the geodesics on $\mathbb{P}_m$ for Euclidean distance $d(A,B) = \|A-B\|_2$ and Riemannian trace distance $\delta(A,B) = \|\log A^{-1/2}BA^{-1/2}\|_2$, respectively. Furthermore, the weighted arithmetic-geometric-harmonic mean inequalities hold:

$$A!_\alpha B \leq A\#_\alpha B \leq A\nabla_\alpha B. \tag{2.2}$$

For convenience, we simply denote as $A\nabla B$, $A\#B$, and $A!B$ the arithmetic, geometric, and harmonic means for $\alpha = 1/2$.

One can see from [12] that the map $\Psi : \mathbb{P}_m \times \mathbb{P}_m \to \mathbb{R}$ defined by

$$\Psi(A,B) = \operatorname{tr}(A\nabla_\alpha B - A\sigma_\alpha B),$$

is a quantum divergence, where $A\sigma_\alpha B$ is the Kubo-Ando's operator mean with weight $\alpha \in [0,1]$. On the other hand, we give more details of the proof when $A\sigma_\alpha B = A!_\alpha B$, which we mainly concern in this paper.

Theorem 2.2. *The map $\Phi_\alpha : \mathbb{P}_m \times \mathbb{P}_m \to \mathbb{R}$ defined by*

$$\Phi_\alpha(A,B) = \operatorname{tr}(A\nabla_\alpha B - A!_\alpha B), \tag{2.3}$$

is a quantum divergence.

Proof. For $\alpha = 0$ or $\alpha = 1$, the map Φ_α is obviously a quantum divergence, since $\Phi_\alpha(A,B) = 0$ for any $A, B \in \mathbb{P}_m$.

We prove that the map Φ_α for $\alpha \in (0,1)$ satisfies the items (i) through (iii) in Definition 2.1.

(i) Since $A\nabla_\alpha B \geq A!_\alpha B$ by (2.2), obviously $\Phi_\alpha(A,B) \geq 0$. If $\Phi_\alpha(A,B) = 0$ then $A\nabla_\alpha B = A!_\alpha B$. Taking the congruence transformation by $A^{-1/2}$, we obtain

$$(1-\alpha)I + \alpha Z = ((1-\alpha)I + \alpha Z^{-1})^{-1},$$

where $Z = A^{-1/2}BA^{-1/2} \in \mathbb{P}_m$. It is equivalent to $(1-\alpha) + \alpha\lambda = ((1-\alpha) + \alpha\lambda^{-1})^{-1}$ for any eigenvalue λ of Z, and thus, $\lambda = 1$. It means $Z = I$, that is, $A = B$.

(ii) Let $F(A,B) = \operatorname{tr} A\nabla_\alpha B$ and $G(A,B) = \operatorname{tr} A!_\alpha B$. Then $\Phi_\alpha(A,B) = F(A,B) - G(A,B)$. We can easily check that

$$\frac{\partial F}{\partial B}(A,B)(Y) = \operatorname{tr} \alpha Y,$$

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from the fact that (see [8, Theorem 3.23])

$$\frac{d}{dt}\operatorname{tr} f(A+tB)|_{t=0} = \operatorname{tr}(f'(A)B). \tag{2.4}$$

Set $g(B) = (1-\alpha)A^{-1} + \alpha B^{-1}$. Since $\{f(B)^{-1}\}' = -f(B)^{-1}f'(B)f(B)^{-1}$, using the chain rule and (2.4) we get that

$$\begin{aligned}\frac{\partial G}{\partial B}(A,B)(Y) &= \operatorname{tr}(-g(B)^{-1}Dg(B)(Y)g(B)^{-1}) \\ &= -\operatorname{tr}(\alpha g(B)^{-1}D(B^{-1})(Y)g(B)^{-1}).\end{aligned}$$

Thus,

$$\frac{\partial G}{\partial B}(A,B=A)(Y) = \operatorname{tr}(\alpha A(A^{-1}YA^{-1})A) = \operatorname{tr}\alpha Y.$$

Hence, we have

$$\frac{\partial \Phi_\alpha}{\partial B}(A,B=A)(Y) = 0.$$

(iii) Moreover,

$$\begin{aligned}\frac{\partial^2 G}{\partial B^2}(A,B)(Y,Y) =& \operatorname{tr}(2\alpha^2 g(B)^{-1}D(B^{-1})(Y)g(B)^{-1}D(B^{-1})(Y)g(B)^{-1}) \\ &- \operatorname{tr}(g(B)^{-1}D^2g(B)(Y,Y)g(B)^{-1}).\end{aligned}$$

So, we get that

$$\frac{\partial^2 G}{\partial B^2}(A,B=A)(Y,Y) = \operatorname{tr}(2\alpha^2 YA^{-1}Y) - \operatorname{tr}(AD^2(A)(Y,Y)A).$$

Since $D(A^{-1})(Y) = -A^{-1}YA^{-1}$, $\operatorname{tr}(AD(A^{-1})(Y)A) = -\operatorname{tr} Y$. Taking the derivative on both sides, we see that

$$\operatorname{tr}(D(A)(Y)D(A^{-1})(Y)A + AD^2(A^{-1})(Y,Y)A + AD(A^{-1})(Y)D(A)(Y)) = 0.$$

Therefore, $\operatorname{tr}(AD^2(A^{-1})(Y,Y)A) = -\operatorname{tr}(YD(A^{-1})(Y)A + AD(A^{-1})(Y)Y)$, and so,

$$\begin{aligned}\frac{\partial^2 G}{\partial B^2}(A,B=A)(Y,Y) &= \operatorname{tr}(2\alpha^2 YA^{-1}Y - 2YA^{-1}Y) \\ &= \operatorname{tr}((2\alpha^2-2)YA^{-1}Y) \le 0.\end{aligned}$$

Hence, we get that $\frac{\partial^2 \Phi_\alpha}{\partial B^2}(A,B=A)(Y,Y) \ge 0$. □

We call Φ_α the *arithmetic-harmonic quantum divergence*, and one can easily see that $\Phi_\alpha(A,B) = \Phi_{1-\alpha}(B,A)$ for any $A,B \in \mathbb{P}_m$.

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Remark 2.3. Several quantum divergences as a matrix extension of the Hellinger distance of discrete probability vectors have been introduced in [4]:

$$\Psi(A,B) = \operatorname{tr}\left(\frac{A+B}{2}\right) - \operatorname{tr}\, G(A,B),$$

where $G(A,B)$ is different kinds of the geometric mean of A,B such as

(1) the fidelity $F(A,B) = (A^{1/2}BA^{1/2})^{1/2}$,
(2) the metric geometric mean $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$,
(3) the log-Euclidean mean $L(A,B) = \exp\left(\frac{\log A + \log B}{2}\right)$,
(4) the Hölder (generalized) mean $H_p(A,B) = \left(\frac{A^p + B^p}{2}\right)^{\frac{1}{p}}$, $p \in (0,1)$.

We denote as $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ the quantum divergences constructed from (1) through (4), respectively. Note from [5] and (2.2) that $\operatorname{tr}(A!B) \le \operatorname{tr}(A\#B) \le \operatorname{tr} L(A,B) \le \operatorname{tr} F(A,B)$, and hence,

$$\Phi_{1/2}(A,B) \ge \Psi_2(A,B) \ge \Psi_3(A,B) \ge \Psi_1(A,B).$$

Moreover, since $|||H_p(A,B)|||$ is an increasing function of $p \in \mathbb{R}$ for any unitarily invariant norm from [5] and $H_p(A,B)$ converges to the log-Euclidean mean $L(A,B)$ as $p \to 0$, we have $\operatorname{tr}(A!B) \le \operatorname{tr} L(A,B) \le \operatorname{tr} H_p(A,B)$, and thus,

$$\Phi_{1/2}(A,B) \ge \Psi_3(A,B) \ge \Psi_4(A,B).$$

We note some properties of the arithmetic-harmonic quantum divergence.

Theorem 2.4. *Let $A, B \in \mathbb{P}_m$ and $\alpha \in [0,1]$. Then*

(i) $\Phi_\alpha(UAU^*, UBU^*) = \Phi_\alpha(A,B)$ *for any unitary matrix U;*
(ii) $\Phi_\alpha(A\otimes\rho, B\otimes\rho) = \Phi_\alpha(A,B)$ *for any invertible density matrix ρ.*

Proof. Since the weighted arithmetic and harmonic means are invariant under congruence transformation, one can easily see (i).

Let ρ be an invertible density matrix, that is, a positive definite Hermitian matrix with $\operatorname{tr}\rho = 1$.

$$\begin{aligned}
&\Phi_\alpha(A\otimes\rho, B\otimes\rho)\\
&= \operatorname{tr}((A\otimes\rho)\nabla_\alpha(B\otimes\rho) - (A\otimes\rho)!_\alpha(B\otimes\rho))\\
&= \operatorname{tr}\left((1-\alpha)A\otimes\rho + \alpha B\otimes\rho - ((1-\alpha)A^{-1}\otimes\rho^{-1} + \alpha B^{-1}\otimes\rho^{-1})^{-1}\right)\\
&= \operatorname{tr}\left((A\nabla_\alpha B)\otimes\rho - (A!_\alpha B)\otimes\rho\right),
\end{aligned}$$

where the equalities follow from the facts that the tensor product is linear and $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$ for any invertible matrices A and B. Since $\operatorname{tr}(A\otimes B) = \operatorname{tr} A \operatorname{tr} B$, we see

$$\Phi_\alpha(A\otimes\rho, B\otimes\rho) = \operatorname{tr}\left((A\nabla_\alpha B - A!_\alpha B)\otimes\rho\right) = \Phi_\alpha(A,B). \qquad \square$$

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3. Barycenters of the arithmetic-harmonic quantum divergence. For an n-tuple $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}_m^n$ and $\omega = (w_1, \ldots, w_n) \in \Delta_n$, we consider the minimization problem

$$\underset{X\in\mathbb{P}_m}{\arg\min} \sum_{i=1}^{n} w_i \Phi_\alpha(A_i, X), \tag{3.5}$$

where the quantum divergence Φ_α is given by (2.3). To show that the above minimization has a unique solution in $\mathbb{P}_m$, we use the following lemma.

Lemma 3.1. *Given $A \in \mathbb{P}_m$ and $0 < \alpha < 1$, the map $f : \mathbb{P}_m \to \mathbb{R}$ defined by $f(X) = \operatorname{tr} A!_\alpha X$ is strictly concave. That is,*

$$\operatorname{tr} A!_\alpha \left(\frac{X+Y}{2}\right) \geq \frac{\operatorname{tr} A!_\alpha X + \operatorname{tr} A!_\alpha Y}{2}, \tag{3.6}$$

for any $X, Y \in \mathbb{P}_m$ and equality holds if and only if $X = Y$.

Proof. For given $A \in \mathbb{P}_m$ and $\alpha \in (0, 1)$, the map $\mathbb{P}_m \ni X \mapsto A!_\alpha X$ is concave. That is,

$$A!_\alpha \left(\frac{X+Y}{2}\right) \geq \frac{A!_\alpha X + A!_\alpha Y}{2},$$

for $X, Y \in \mathbb{P}_m$. Since the trace map is linear and $\operatorname{tr} H \leq \operatorname{tr} K$ whenever $H \leq K$ for $H, K \in \mathbb{H}_m$, the map $f : \mathbb{P}_m \to \mathbb{R}$, $f(X) = \operatorname{tr} A!_\alpha X$ is concave.

Suppose that the equality in (3.6) holds. Then

$$A!_\alpha \left(\frac{X+Y}{2}\right) = \frac{A!_\alpha X + A!_\alpha Y}{2},$$

because $\operatorname{tr} A = 0$ for a positive semi-definite Hermitian matrix A implies $A = 0$. Since $(A^{-1} + B^{-1})^{-1} = A - A(A + B)^{-1}A$ in [16, Theorem 2.5], we obtain by a simple computation

$$\left(\alpha A + (1-\alpha)\frac{X+Y}{2}\right)^{-1} = \frac{1}{2}((\alpha A + (1-\alpha)X)^{-1} + (\alpha A + (1-\alpha)Y)^{-1}).$$

Since the inversion map $A \mapsto A^{-1}$ is strictly convex on $\mathbb{P}_m$, so is the map $\mathbb{P}_m \ni X \mapsto (\alpha A+(1-\alpha)X)^{-1}$. Thus, $\alpha A + (1-\alpha)X = \alpha A + (1-\alpha)Y$, so we obtain $X = Y$. ∎

Remark 3.2. The data processing inequality states that for a quantum divergence Φ, the following inequality

$$\Phi(f(A), f(B)) \leq \Phi(A, B), \quad A, B \in \mathbb{P}_m$$

holds for any completely positive trace-preserving map f. This means that the data information cannot increase under any local physical operation. According to [15, Theorem 5.16], if a map Φ is jointly convex and invariant under unitary congruence transformation and tensor product with density matrices, then it satisfies the data processing inequality. Lemma 3.1 yields the joint convexity of quantum divergence Φ_α, and thus, by Theorem 2.4 Φ_α fulfills the data processing inequality.

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From Lemma 3.1, the map $\mathbb{P}_m \ni X \mapsto \Phi_\alpha(A, X)$ is strictly convex. So the minimization (3.5) has a unique solution in $\mathbb{P}_m$.

Definition 3.3. Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}_m^n$, $\omega = (w_1, \ldots, w_n) \in \Delta_n$. We denote such a unique minimizer of (3.5) in $\mathbb{P}_n$ as $\mathcal{Q}_\alpha(\omega; \mathbb{A})$ and call the *barycenter of the arithmetic-harmonic quantum divergence*.

By vanishing the gradient of the objective function, we obtain that $\mathcal{Q}_\alpha(\omega; \mathbb{A})$ coincides with the unique solution $X \in \mathbb{P}_m$ of the nonlinear matrix equation in the following theorem.

Theorem 3.4. *Let $0 < \alpha < 1$. The barycenter of the arithmetic-harmonic quantum divergence $\mathcal{Q}_\alpha(\omega; \mathbb{A})$ is the unique positive definite solution X of the following equation*

$$X = \left[\sum_{i=1}^n w_i (A_i !_\alpha X)^2\right]^{1/2}, \tag{3.7}$$

equivalently $X^2 = \sum_{i=1}^n w_i (A_i !_\alpha X)^2$.

Proof. Let $\Psi(X) = \sum_{i=1}^n w_i \Phi_\alpha(A_i, X)$. Recall that

$$\frac{\partial \Phi_\alpha}{\partial B}(A, B)(Y) = \mathrm{tr}[\alpha Y + \alpha (A !_\alpha B) D(B^{-1})(Y)(A !_\alpha B)].$$

Since $D(B^{-1})(Y) = -B^{-1} Y B^{-1}$, we have

$$\begin{aligned} D\Psi(X)(Y) &= \sum_{i=1}^n w_i \,\mathrm{tr}\left[\alpha Y - \alpha (A_i !_\alpha X) X^{-1} Y X^{-1} (A_i !_\alpha X)\right] \\ &= \alpha \,\mathrm{tr}\left[\left(I - \sum_{i=1}^n w_i X^{-1} (A_i !_\alpha X)^2 X^{-1}\right) Y\right]. \end{aligned}$$

Then the critical point of $\Psi(X)$ is the solution of the following equation

$$I = \sum_{i=1}^n w_i X^{-1} (A_i !_\alpha X)^2 X^{-1}.$$

Applying the congruence transformation by X and taking the square root map on both sides, we obtain that the critical point of $\Psi(X)$ satisfies the equation (3.7).

We now show that the equation (3.7) has a unique solution. Note from Lemma 3.1 that $\Psi(X)$ is strictly convex. Thus, if the equation (3.7) has a solution, then it is unique.

Assume that $\alpha I \leq A_i \leq \beta I$ for some $0 < \alpha \leq \beta$. Let

$$F(X) = \left[\sum_{i=1}^n w_i ((1-\alpha) A_i^{-1} + \alpha X^{-1})^{-2}\right]^{1/2}.$$

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Then F is a self-map on the closed interval $[\alpha,\beta] := \{A \in \mathbb{H}_m : \alpha I \le A \le \beta I\}$. Indeed, for $X \in [\alpha,\beta]$

$$\beta^{-1} I \le (1-\alpha)A_i^{-1} + \alpha X^{-1} \le \alpha^{-1} I,$$

so

$$\alpha^2 I \le ((1-\alpha)A_i^{-1} + \alpha X^{-1})^{-2} \le \beta^2 I.$$

By the order preserving of the weighted sum and square root map $\alpha I \le F(X) \le \beta I$. Since the closed interval $[\alpha,\beta]$ is a compact subset in $\mathbb{H}_m$, by Brouwer's fixed point theorem the self-map F on $[\alpha,\beta]$ has a fixed point. ∎

As an analog of such barycenter, one can consider the minimization problem

$$\underset{X \in \mathbb{P}_m}{\arg\min} \sum_{i=1}^{n} w_i \Phi_\alpha(X, A_i). \tag{3.8}$$

Corollary 3.5. *There exists a unique minimizer of the problem (3.8) that coincides with the unique positive definite solution of the equation*

$$X^2 = \sum_{i=1}^{n} w_i (X !_\alpha A_i)^2,$$

equivalently $X = \left[\sum_{i=1}^{n} w_i (X !_\alpha A_i)^2\right]^{1/2}$.

Remark 3.6. Theorem 3.4 and Corollary 3.5 give us the right mean and left mean, respectively, for the arithmetic-harmonic quantum divergence. So, they are the same when $\alpha = 1/2$.

4. Properties of the barycenter $\mathcal{Q}_\alpha$.

Now, we provide several interesting properties of the barycenter of the arithmetic-harmonic quantum divergence.

Let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}_m^n$, $\omega = (w_1, \ldots, w_n) \in \Delta_n$, $\sigma \in S_n$ a permutation on an n-letters, and let M be the $m \times m$ invertible matrix. For convenience, we denote as

$$\begin{aligned} \omega_\sigma &:= (w_{\sigma(1)}, \ldots, w_{\sigma(n)}) \in \Delta_n, \\ \mathbb{A}_\sigma &:= (A_{\sigma(1)}, \ldots, A_{\sigma(n)}) \in \mathbb{P}_m^n, \\ M\mathbb{A}M^* &:= (MA_1M^*, \ldots, MA_nM^*) \in \mathbb{P}_m^n, \end{aligned}$$

and

$$\begin{aligned} \omega^{(k)} &:= \frac{1}{k}(\underbrace{w_1, \ldots, w_n}, \ldots, \underbrace{w_1, \ldots, w_n}) \in \Delta_{nk}, \\ \mathbb{A}^{(k)} &:= (\underbrace{A_1, \ldots, A_n}, \ldots, \underbrace{A_1, \ldots, A_n}) \in \mathbb{P}_m^{nk} \end{aligned}$$

of which number of tuples is a natural number k.

Using Theorem 3.4, we can obtain the following.

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Lemma 4.1. *The barycenter $\mathcal{Q}_\alpha(\omega;\mathbb{A})$ of the arithmetic-harmonic quantum divergence satisfies the following properties:*

(1) $\mathcal{Q}_\alpha(\omega;c\mathbb{A}) = c\mathcal{Q}_\alpha(\omega;\mathbb{A})$ *for any* $c>0$*;*
(2) $\mathcal{Q}_\alpha(\omega_\sigma;\mathbb{A}_\sigma) = \mathcal{Q}_\alpha(\omega;\mathbb{A})$ *for any permutation* $\sigma\in S_n$*;*
(3) $\mathcal{Q}_\alpha(\omega;U\mathbb{A}U^*) = U\mathcal{Q}_\alpha(\omega;\mathbb{A})U^*$ *for any unitary matrix* U*;*
(4) $\mathcal{Q}_\alpha(\omega^{(k)};\mathbb{A}^{(k)}) = \mathcal{Q}_\alpha(\omega;\mathbb{A})$ *for any natural number* k*;*
(5) $X = \mathcal{Q}_\alpha(\omega;A_1,\ldots,A_{n-1},X)$ *implies that* $X = \mathcal{Q}_\alpha(\hat{\omega};A_1,\ldots,A_{n-1})$*, where* $\hat{\omega} = \frac{1}{1-w_n}(w_1,\ldots,w_{n-1})\in\Delta_{n-1}$ *;*
(6) $\mathcal{Q}_\alpha(\omega;\mathbb{A}) = \mathcal{Q}_\alpha\left(\sum_{i=1}^k w_i, w_{k+1},\ldots,w_n;A_1,A_{k+1},\ldots,A_n\right)$ *if* $A_1=\cdots=A_k$ *for* $1\le k<n$*.*

Theorem 4.2. *The barycenter $\mathcal{Q}_\alpha(\omega;\mathbb{A})$ of the arithmetic-harmonic quantum divergence satisfies*

$$\mathcal{Q}_\alpha(\omega;\mathbb{A}) \ge \left(\sum_{i=1}^n w_i A_i^{-1}\right)^{-1}.$$

Proof. Set $X=\mathcal{Q}_\alpha(\omega;\mathbb{A})$. Then we have

$$X = \left[\sum_{i=1}^n w_i (A_i !_\alpha X)^2\right]^{1/2}.$$

Since the map $\mathbb{P}_m \ni A \mapsto A^{1/2}$ is concave,

$$X \ge \sum_{i=1}^n w_i A_i !_\alpha X.$$

Equivalently,

$$X^{-1} \le \left[\sum_{i=1}^n w_i A_i !_\alpha X\right]^{-1}.$$

From the convexity of the inversion map,

$$X^{-1} \le \sum_{i=1}^n w_i (1-\alpha) A_i^{-1} + \alpha X^{-1}.$$

Thus, we obtain the desired inequality. ∎

Corollary 4.3. *Let Φ be a strictly positive linear unital map. Assume that $pI\le A_i\le qI$ for all i, where $p,q>0$ are constants. Then*

$$\Phi\left(\mathcal{Q}_\alpha(\omega;\mathbb{A})\right) \ge \frac{4pq}{(p+q)^2}\left[\sum_{i=1}^n w_i \Phi(A_i)^{-1}\right]^{-1}.$$

Proof. Taking the strictly positive linear unital map Φ on the $\mathcal{Q}_\alpha$-harmonic mean inequality in Theorem 4.2 and applying Choi's inequality $\Phi(A)^{-1}\le\Phi(A^{-1})$ for any $A\in\mathbb{P}_m$ in [3, Theorem 2.3.6], we

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have

$$\Phi(\mathcal{Q}_\alpha(\omega;\mathbb{A})) \geq \Phi\left(\left(\sum_{i=1}^n w_i A_i^{-1}\right)^{-1}\right) \geq \left[\sum_{i=1}^n w_i \Phi(A_i^{-1})\right]^{-1}.$$

From Proposition 2.7.8 in [3], we obtain that

$$\Phi(A_i^{-1}) \leq \frac{(p+q)^2}{4pq}\Phi(A_i)^{-1}.$$

Thus, we get that

$$\Phi(\mathcal{Q}_\alpha(\omega;\mathbb{A})) \geq \left[\frac{(p+q)^2}{4pq}\sum_{i=1}^n w_i \Phi(A_i)^{-1}\right]^{-1} = \frac{4pq}{(p+q)^2}\left[\sum_{i=1}^n w_i \Phi(A_i)^{-1}\right]^{-1}.$$ □

We give an upper bound for the barycenter of the arithmetic-harmonic quantum divergence in terms of operator norm.

Theorem 4.4. *For* $0 \leq \alpha < 1$

$$\|\mathcal{Q}_\alpha(\omega;\mathbb{A})\| \leq \left[\sum_{i=1}^n w_i \|A_i\|^{2(1-\alpha)}\right]^{\frac{1}{2(1-\alpha)}},$$

where $\|\cdot\|$ *denotes the operator norm.*

Proof. Note that $A \leq B$ for $A, B \in \mathbb{P}_m$ implies $\|A\| \leq \|B\|$. Indeed, let $\lambda(A)$ be the largest eigenvalue of A with unit eigenvector $x \in \mathbb{C}^m$. Then $A \leq B$ implies $x^*(B-A)x \geq 0$, so

$$\|A\| = \lambda(A) = x^*Ax \leq x^*Bx \leq \max_{\|z\|=1} z^*Bz = \|B\|.$$

Set $X = \mathcal{Q}_\alpha(\omega;\mathbb{A})$. Then we have

$$X^2 = \sum_{i=1}^n w_i (A_i !_\alpha X)^2,$$

and

$$\begin{aligned}\|X\|^2 = \|X^2\| &\leq \sum_{i=1}^n w_i \|(A_i !_\alpha X)^2\| = \sum_{i=1}^n w_i \|(A_i !_\alpha X)\|^2 \\ &\leq \sum_{i=1}^n w_i \|A_i \#_\alpha X\|^2 \leq \sum_{i=1}^n w_i \|A_i\|^{2(1-\alpha)} \|X\|^{2\alpha}.\end{aligned}$$

The first inequality follows from the triangle inequality, and the second identity holds from the fact that $\|A^p\| = \|A\|^p$ for any $A \in \mathbb{P}_m$ and $p \geq 0$. The second inequality follows from the weighted geometric-harmonic mean inequality with the preceding argument, and the last inequality holds from [5, Theorem 3] and the sub-multiplicity of operator norm that

$$\|A \#_t B\| \leq \|A^{1-t}B^t\| \leq \|A\|^{1-t}\|B\|^t,$$

for any $t \in [0,1]$. Therefore, we obtain the desired result by solving the above inequality for $\|X\|$. □

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Remark 4.5. One of the important properties for the barycenter G as a multivariable geometric mean on the open convex cone $\mathbb{P}_m$ is the boundedness in terms of Loewner order, such as the weighted arithmetic-geometric-harmonic mean inequalities:

$$\left(\sum_{i=1}^{n} w_i A_i^{-1}\right)^{-1} \leq G(\omega; A_1, \ldots, A_n) \leq \sum_{i=1}^{n} w_i A_i.$$

Theorem 4.2 provides the lower bound for the barycenter $\mathcal{Q}_\alpha$ of the arithmetic-harmonic quantum divergence in terms of Loewner order. On the other hand, it is an open question to find its upper bound. Theorem 4.4 for the upper bound of the barycenter $\mathcal{Q}_\alpha$ in terms of operator norm gives us an affirmative answer for this problem.

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