

EVEN AND ODD TOURNAMENT MATRICES WITH MINIMUM RANK OVER FINITE FIELDS*

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Abstract. The $(0, 1)$ -matrix A of order n is a tournament matrix provided

$$A + A^T + I = J,$$

where I is the identity matrix, and $J = J_n$ is the all 1's matrix of order n . It was shown by de Caen and Michael that the rank of a tournament matrix A of order n over a field of characteristic p satisfies $\text{rank}_p(A) \geq (n-1)/2$ with equality if and only if n is odd and $AA^T = O$. This article shows that the rank of a tournament matrix A of even order n over a field of characteristic p satisfies $\text{rank}_p(A) \geq n/2$ with equality if and only if after simultaneous row and column permutations

$$AA^T = \begin{bmatrix} \pm J_m & O \\ O & O \end{bmatrix},$$

for a suitable integer m . The results and constructions for even order tournament matrices are related to and shed light on tournament matrices of odd order with minimum rank.

Key words. Tournament matrix, Rank.

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1. Introduction. A *tournament matrix* of order n is a $(0, 1)$ -matrix A that satisfies

$$(1.1) \quad A + A^T + I = J,$$

where I denotes the identity matrix of order n , and $J = J_n$ denotes the all 1's matrix of order n . The tournament matrix A records the results of a round-robin tournament among n players; the (i, j) -entry is 1 provided player i defeats player j and is 0 otherwise. The number of players defeated by player i is the *score* of player i ; it equals the sum of the entries in row i of A . The *joint score* of players i and j is the number of players defeated by both i and j ; this equals the (i, j) -entry of the product AA^T .

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Because each entry of A is 0 or 1, we may view A as a matrix over any field. It is known [3, 5] that a tournament matrix A of order n satisfies

$$\text{rank}(A) \geq n - 1$$

over any field of characteristic 0. No satisfactory characterization of the tournament matrices with the minimal rank $n - 1$ is known.

Our emphasis in this article is on ranks of tournament matrices over fields of prime characteristic p . Without loss of generality the underlying field is the field \mathbb{Z}_p of integers modulo p . The p -rank of the matrix A is the rank of A over a field of characteristic p and is denoted by

$$\text{rank}_p(A).$$

The p -rank of a tournament matrix is an important parameter in the study of skew Hadamard block designs [7, 8].

Let O denote a matrix (of appropriate size) of 0's, and let $J_{a,b}$ denote the a by b matrix of 1's. Let

$$\chi_A(x) \quad \text{and} \quad \mu_A(x)$$

denote the characteristic and minimal polynomials, respectively, of the matrix A over a given field.

1.1. Tournament matrices with minimum rank. The following theorem gives a general lower bound for p -ranks of tournament matrices and summarizes the known results about the case of equality.

THEOREM 1.1. (de Caen, Michael) *Let A be a tournament matrix of order n ($n \geq 2$), and let p be a prime.*

(a) *The p -rank of A satisfies*

$$(1.2) \quad \text{rank}_p(A) \geq \frac{n-1}{2}.$$

(b) *The following assertions are equivalent:*

- (i) $\text{rank}_p(A) = (n-1)/2$,
 - (ii) n is odd and $AA^T = O$,
 - (iii) $\chi_A(x) = x^{(n+1)/2}(x+1)^{(n-1)/2}$ and $\mu_A(x) = x(x+1)$.
- (c) *The equality $\text{rank}_p(A) = (n-1)/2$ implies that $(n-1)/2 \equiv 0 \pmod{p}$.*
- (d) *The equality $\text{rank}_p(A) = (n-1)/2$ implies that $(n-1)/2 \equiv 0 \pmod{2p}$ if $p \equiv 3 \pmod{4}$.*

The fundamental inequality (1.2) was established by de Caen [1] in 1991. In 1995, the equivalence of (i) and (ii) in (b) was demonstrated in [6], as was the congruence condition in (c) for equality. Condition (ii) in (b) gives a combinatorial characterization of equality: All scores and joint scores of the tournament must be divisible by p . Examples of tournament matrices satisfying equality in (1.2) were also constructed in [6]. Condition (iii) can be extracted from the work in [6]; we provide an explicit proof in Section 4.2. The congruence condition in (d) was discovered by de Caen [2].

For future reference we note that if the tournament matrix A of order n satisfies $\text{rank}_p(A) = (n-1)/2$, then the conditions in Theorem 1.1 imply that

$$(1.3) \quad O = AA^T = A(J - A - I) = AJ - A(A + I) = AJ.$$

1.2. Even tournament matrices with minimum rank. Now suppose that A is a tournament matrix of *even* order n . Then (1.2) and the integrality of $\text{rank}_p(A)$ imply that

$$(1.4) \quad \text{rank}_p(A) \geq \frac{n}{2}.$$

Our main theorem characterizes tournament matrices for which equality holds in (1.4); our characterizations of equality are analogous to those in Theorem 1.1.

THEOREM 1.2. *Let A be a tournament matrix of order n ($n \geq 4$), and let p be a prime.*

(a) *If n is even, then*

$$(1.5) \quad \text{rank}_p(A) \geq \frac{n}{2}.$$

(b) *The following assertions are equivalent:*

(i) $\text{rank}_p(A) = n/2$,

(ii) n is even, and after simultaneous row and column permutations

$$AA^T = \begin{bmatrix} +J_m & O \\ O & O \end{bmatrix} \quad \text{for some } m \in \{1, 2, \dots, n-1\}, \text{ or}$$

$$AA^T = \begin{bmatrix} -J_m & O \\ O & O \end{bmatrix} \quad \text{for some } m \in \{0, 1, \dots, n\},$$

(iii) *The corresponding characteristic and minimal polynomials of A are*

$$\chi_A(x) = x^{(n/2)+1}(x+1)^{(n/2)-1} \quad \text{and} \quad \mu_A(x) = x^2(x+1), \text{ or}$$

$$\chi_A(x) = x^{n/2}(x+1)^{n/2} \quad \text{and} \quad \mu_A(x) = x(x+1)^2 \text{ or } x(x+1).$$

The extreme cases $m = 0$ and $m = n$ occur in (ii) if and only if $\mu_A(x) = x(x+1)$.

- (c) The equality $\text{rank}_p(A) = n/2$ implies that $m \equiv n/2 \equiv 1$ or $0 \pmod{p}$, according as A satisfies the upper or lower conditions, respectively, in (ii) and (iii).
- (d) The equality $\text{rank}_p(A) = n/2$ implies that $n/2 \equiv 0 \pmod{2p}$ if A satisfies the lower conditions in (ii) and (iii), and $p \equiv 3 \pmod{4}$.

Condition (ii) has a combinatorial interpretation: All scores and joint scores are divisible by p , except for scores of players in the set $\{1, \dots, m\}$ and joint scores involving two players in this same set; the exceptional scores and joint scores are all 1 more or all 1 less than a multiple of p .

In Section 2, we provide examples of tournament matrices for which equality holds in Theorem 1.2, and in Section 3, we discuss relationships between the tournament matrices with minimum rank of even and odd orders. The proof of Theorem 1.2 occurs in Section 5 and relies on the lemmas we establish in Section 4.

2. Equality: $(\pm J)$ -type tournament matrices. In this section, we construct some even order tournament matrices of minimum rank. These constructions shed light on Theorem 1.2.

A tournament matrix A_n of even order n is a $(+J)$ -type or a $(-J)$ -type matrix of sub-order m provided that after simultaneous row and column permutations

$$(2.1) \quad A_n A_n^T = \begin{bmatrix} +J_m & O \\ O & O \end{bmatrix} \quad \text{or} \quad A_n A_n^T = \begin{bmatrix} -J_m & O \\ O & O \end{bmatrix},$$

respectively, over \mathbb{Z}_p . When $p = 2$, we adopt the convention that A_n is a $(+J)$ -type (respectively, $(-J)$ -type) tournament matrix of sub-order m provided $A_n A_n^T$ is of the form (2.1), and m is odd (respectively, even).

Note that Theorem 1.2 tells us that if A_n is a tournament matrix of even order n satisfying $A_n A_n^T = O$, then A_n is a $(-J)$ -type matrix of sub-order $m = 0$.

According to Theorem 1.2, the $(\pm J)$ -type tournament matrices are precisely the even order tournament matrices of minimum rank. We shall see that $(\pm J)$ -type tournament matrices are intimately related to the *odd* order tournament matrices of minimum rank, which are characterized in Theorem 1.1. Throughout our constructions we rely on the characterizations in Theorem 1.1 and Theorem 1.2.

2.1. Transposes. We begin with an observation based directly on matrix manipulations.

PROPOSITION 2.1. *The matrix A_n is a $(+J)$ -type tournament matrix of order n and sub-order m if and only if A_n^T is a $(+J)$ -type tournament matrix of order n and sub-order $n - m$. The same assertion holds for $(-J)$ -type tournament matrices of order n and sub-order m .*

Proof. Let A_n be a $(\pm J)$ -type tournament matrix of order n and sub-order m . Then

$$(2.2) \quad A_n A_n^T = \begin{bmatrix} \pm J_m & O \\ O & O \end{bmatrix} \quad \text{and} \quad A_n J_n = \begin{bmatrix} \pm J_{m,n} \\ O \end{bmatrix}.$$

The first equation is the definition of a $(\pm J)$ -type matrix. Because the i th diagonal element of AA^T is the score of player i modulo p , (that is, the number of 1's in row i of A), the second equation follows from the first. Now from (1.1) and (2.2) it follows that

$$\begin{aligned} A_n^T A_n &= (J_n - I - A_n)(J_n - I - A_n^T) \\ &= J_n^2 - J_n - J_n A^T - J_n + I + A_n^T - A_n J_n + A_n + A_n A_n^T \\ &= (n-1)J_n - (A_n J_n + (A_n J_n)^T) + A_n A_n^T = \begin{bmatrix} O & O \\ O & \pm J_{n-m} \end{bmatrix}, \end{aligned}$$

where the last step uses the congruence $m \equiv n/2 \equiv 1$ or $0 \pmod{p}$ from Theorem 1.2. Thus, A_n^T is a $(\pm J)$ -type tournament matrix of order n and sub-order $n-m$. Replace m by $n-m$ in the above computation to establish the reverse implication. \square

2.2. Construction using two odd tournament matrices. Our first construction produces a $(+J)$ -type tournament matrix from two minimum rank tournament matrices of odd orders.

PROPOSITION 2.2. *Suppose that A_m and A_{n-m} are tournament matrices of odd orders m and $n-m$ with*

$$\text{rank}_p(A_m) = \frac{m-1}{2} \quad \text{and} \quad \text{rank}_p(A_{n-m}) = \frac{n-m-1}{2}.$$

Then the tournament matrix

$$A = \begin{bmatrix} A_m & J_{m,n-m} \\ O & A_{n-m} \end{bmatrix}$$

of even order n satisfies

$$\text{rank}_p(A) = \frac{n}{2}.$$

Moreover, A is a $(+J)$ -type tournament matrix of sub-order m .

Proof. The result is clear if $n = 2$. Suppose that $n \geq 4$. Then $A_m A_m^T = O$ and $A_{n-m} A_{n-m}^T = O$. Also, $(n-m-1)/2 \equiv 0 \pmod{p}$ and so $n-m \equiv 1 \pmod{p}$. By

block multiplication and (1.3)

$$\begin{aligned} AA^T &= \begin{bmatrix} A_m & J_{m,n-m} \\ O & A_{n-m} \end{bmatrix} \begin{bmatrix} A_m^T & O \\ J_{n-m,m} & A_{n-m}^T \end{bmatrix} \\ &= \begin{bmatrix} A_m A_m^T + (n-m)J_m & J_{m,n-m} A_{n-m}^T \\ A_{n-m} J_{n-m,m} & A_{n-m} A_{n-m}^T \end{bmatrix} = \begin{bmatrix} J_m & O \\ O & O \end{bmatrix}, \end{aligned}$$

which is of the required form in (2.1). Therefore, $\text{rank}(A) = n/2$, and A is a $(+J)$ -type tournament matrix of sub-order m . \square

2.3. Construction using two even tournament matrices. Our second construction produces a $(-J)$ -type tournament matrix from two minimum rank tournament matrices of even orders.

PROPOSITION 2.3. *Suppose that A_m and A_{n-m} are $(-J)$ -type tournament matrices of even orders m and $n-m$ with*

$$\text{rank}_p(A_m) = \frac{m}{2} \quad \text{and} \quad \text{rank}_p(A_{n-m}) = \frac{n-m}{2}$$

and of extreme sub-orders m and 0 , respectively. Then the tournament matrix

$$A = \begin{bmatrix} A_m & J_{m,n-m} \\ O & A_{n-m} \end{bmatrix}$$

of even order n satisfies

$$\text{rank}_p(A) = \frac{n}{2}.$$

Moreover, A is a $(-J)$ -type tournament matrix of sub-order m .

Proof. The proof is similar to the preceding one. We have $A_m A_m^T = -J_m$ and $A_{n-m} A_{n-m}^T = O$. The second equation tells us that the number of 1's in each row of A_{n-m} is 0 modulo p . Thus, $A_{n-m} J_{n-m,m} = O$. Also, $m \equiv n-m \equiv 0 \pmod{p}$, and thus $J_{m,n-m} J_{n-m,m} = O$. By block multiplication

$$\begin{aligned} AA^T &= \begin{bmatrix} A_m & J_{m,n-m} \\ O & A_{n-m} \end{bmatrix} \begin{bmatrix} A_m^T & O \\ J_{n-m,m} & A_{n-m}^T \end{bmatrix} \\ &= \begin{bmatrix} A_m A_m^T + J_{m,n-m} J_{n-m,m} & J_{m,n-m} A_{n-m}^T \\ A_{n-m} J_{n-m,m} & A_{n-m} A_{n-m}^T \end{bmatrix} = \begin{bmatrix} -J_m & O \\ O & O \end{bmatrix}, \end{aligned}$$

which is of the required form in (2.1). Therefore, $\text{rank}(A) = n/2$, and A is a $(-J)$ -type tournament matrix of sub-order m . \square

2.4. Doubly regular tournament matrices. A *doubly regular tournament matrix* D_{4t-1} is a $(0, 1)$ -matrix of order $4t - 1$ that satisfies

$$D_{4t-1} + D_{4t-1}^T + I = J \quad \text{and} \quad D_{4t-1} D_{4t-1}^T = tI + (t-1)J$$

(over the field of rational numbers) for some positive integer t .

Doubly regular tournament matrices arise in the construction of Hadamard matrices and are conjectured to exist for all positive integers t ; they are known to exist whenever $4t - 1$ is a prime power and for many other orders (e.g., see [4]). Suppose that the prime p divides t . Then D_{4t-1} is a tournament matrix that satisfies

$$D_{4t-1} D_{4t-1}^T = -J$$

over the field \mathbb{Z}_p . Note that this matrix equation does not imply that D_{4t-1} is a $(-J)$ -type tournament matrix over \mathbb{Z}_p ; after all, D_{4t-1} has odd order. However, we may readily transform D_{4t-1} to an extreme $(-J)$ -type matrix by a bordering technique.

PROPOSITION 2.4. *Let D_{4t-1} be a doubly regular tournament matrix of order $4t - 1$, and let p be a prime divisor of t . Then over \mathbb{Z}_p the matrix*

$$A = \left[\begin{array}{ccc|c} & & & 1 \\ & D_{4t-1} & & \vdots \\ & & & 1 \\ \hline 0 & \cdots & 0 & 0 \end{array} \right]$$

is a $(-J)$ -type tournament matrix of order $4t$ and sub-order 0. Also, A^T is a $(-J)$ -type tournament matrix of order $4t$ and sub-order $4t$.

Proof. Clearly, A is a tournament matrix of even order $4t$. Block multiplication shows that $AA^T = O$. The last assertion follows from Proposition 2.1. \square

3. Even and odd tournament matrices. Let $\mathcal{M}_p(n)$ denote the set of tournament matrices of order n with minimum rank over \mathbb{Z}_p , that is, the tournament matrices of order n with p -rank equal to $\lfloor n/2 \rfloor$. For even n let $\mathcal{M}_p^+(n)$ and $\mathcal{M}_p^-(n)$ denote the subsets of $\mathcal{M}_p(n)$ consisting of the respective $(\pm J)$ -type matrices.

In this section, we establish relationships among the sets $\mathcal{M}_p(n-1)$, $\mathcal{M}_p^+(n)$, $\mathcal{M}_p^-(n)$, and $\mathcal{M}_p(n+1)$, as indicated in Figure 3.1.

For each $(0, 1)$ -matrix Y of size r by s , we define

$$\tilde{Y} = (J_{r,s} - Y)^T.$$

Thus, the s by r matrix \tilde{Y} is the transpose of the $(0, 1)$ -complement of Y .

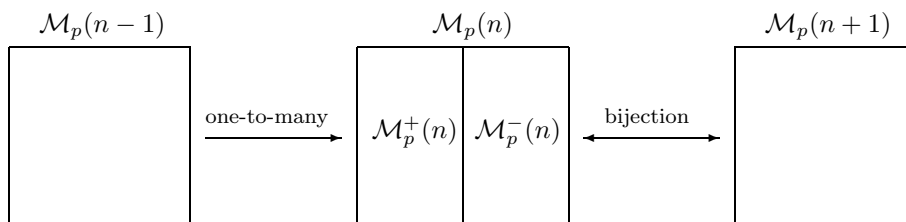


FIG. 3.1. Relationships among sets of tournament matrices with minimum rank (n even).

PROPOSITION 3.1. Suppose that n is even and p is a prime. Then there is a bijection from the set $\mathcal{M}_p(n+1)$ to the set $\mathcal{M}_p^-(n)$: simply delete row $n+1$ and column $n+1$ from each matrix in $\mathcal{M}_p(n+1)$.

Proof. Let A_{n+1} be in $\mathcal{M}_p(n+1)$. Simultaneously permute rows and columns to bring all the 1's in column $n+1$ of A_{n+1} to the leading positions. Thus,

$$A_{n+1} = \begin{bmatrix} A_n & Y \\ \tilde{Y} & 0 \end{bmatrix},$$

where the leading principal tournament submatrix A_n has even order n , and

$$Y = [1, \dots, 1, 0, \dots, 0]^T$$

is an n by 1 matrix. Suppose that there are m leading 1's in Y . Theorem 1.1 tells us that $A_{n+1}A_{n+1}^T = O$. It follows from block multiplication that A_n satisfies the second equation in (2.1) and thus is a $(-J)$ -type tournament matrix of order n and sub-order m . The process is reversible, and we may pass from the matrix in $\mathcal{M}_p^-(n)$ to a matrix in $\mathcal{M}_p(n+1)$ by appending a suitable row and column. \square

PROPOSITION 3.2. Suppose that n is even. Then there is a one-to-many function from $\mathcal{M}_p(n-1)$ to $\mathcal{M}_p^+(n)$.

Proof. Let A_{n-1} be in $\mathcal{M}_p(n-1)$ so that

$$\text{rank}_p(A_{n-1}) = \frac{n}{2} - 1.$$

Let Y be any $(0,1)$ -vector (i.e., $n-1$ by 1 matrix) in the column space of A_{n-1} . For instance, Y could be identical to any column of A_{n-1} or to a vector of 0's. Define the tournament matrix

$$A_n = \begin{bmatrix} A_{n-1} & Y \\ \tilde{Y} & 0 \end{bmatrix}$$

of order n . The rank of the leading $n-1$ by n submatrix of A_n is also $(n/2)-1$, and appending the last row of A_n to this submatrix can increase the rank by at most 1. Thus,

$$\frac{n}{2} \leq \text{rank}_p(A_n) \leq 1 + \text{rank}_p(A_{n-1}) = \frac{n}{2}.$$

Therefore, A_n is an even order tournament matrix with minimum rank. We know that $A_{n-1}A_{n-1}^T = O$, and block multiplication shows that A_n is a $(+J)$ -type tournament matrix of sub-order m , where m is the number of 1's in Y . \square

4. Three lemmas. In this section, we establish some preliminary results for our proof of Theorem 1.2.

4.1. The rank of A and the rank of $A+I$. The first lemma is of some interest in its own right.

LEMMA 4.1. *Let A be a tournament matrix of order n . Then over any field the geometric multiplicities of 0 and -1 as eigenvalues of A differ by at most 1. That is,*

$$\text{rank}(A) - \text{rank}(A+I) \in \{-1, 0, 1\}.$$

Proof. The rank of a sum of matrices does not exceed the sum of their ranks, and thus (1.1) implies that over any field

$$\text{rank}(A+I) = \text{rank}(J - A^T) \leq \text{rank}(J) + \text{rank}(-A^T) = 1 + \text{rank}(A).$$

Similarly,

$$\text{rank}(A) = \text{rank}(J + (-A^T - I)) \leq \text{rank}(J) + \text{rank}(-A^T - I) = 1 + \text{rank}(A+I),$$

and the result follows. \square

4.2. The proof of condition (iii) in Theorem 1.1(b). We use Lemma 4.1 to justify the inclusion of condition (iii) in Theorem 1.1(b). We show that conditions (iii) and (i) are equivalent.

(iii) \Rightarrow (i): Suppose that

$$\chi_A(x) = x^{(n+1)/2}(x+1)^{(n-1)/2} \quad \text{and} \quad \mu_A(x) = x(x+1).$$

The linearity of the factors in $\mu_A(x)$ tells us that the rank of A equals the sum of the multiplicities of its nonzero eigenvalues. Thus, $\text{rank}_p(A) = (n-1)/2$, and (i) holds.

(i) \Rightarrow (iii): Suppose that $\text{rank}_p(A) = (n-1)/2$. Then the eigenvalue 0 of A has geometric multiplicity $(n+1)/2$. By Lemma 4.1 the eigenvalue -1 of A has

geometric multiplicity at least $(n-1)/2$. The sum of the geometric multiplicities of the eigenvalues is at most n . It follows that -1 has geometric multiplicity exactly $(n-1)/2$. Therefore, $\chi_A(x) = x^{(n+1)/2}(x+1)^{(n-1)/2}$ and $\mu_A(x) = x(x+1)$. \square

4.3. Row and column regularity. Our second lemma is helpful in establishing the congruence conditions that arise when a tournament matrix satisfies a regularity condition.

The tournament matrix A' of order n' is *row α -regular* (*column α -regular*) modulo p provided the sum of the elements in each row (respectively, column) of A' is α modulo p . In matrix terms, row and column α -regularity of A' modulo p are equivalent, respectively, to the matrix equations

$$A'J_{n'} = \alpha J_{n'} \quad \text{and} \quad J_{n'}A' = \alpha J_{n'}.$$

over \mathbb{Z}_p .

LEMMA 4.2. *The tournament matrix A' of order n' is row α -regular modulo p if and only if A' is column $(n' - 1 - \alpha)$ -regular modulo p . Moreover, if A' is row α -regular, then*

$$\frac{n'(n'-1)}{2} \equiv n'\alpha \equiv n'(n'-1-\alpha) \pmod{p}.$$

Proof. The total number of 1's in row i and column i of A' is $n'-1$ for $i = 1, \dots, n'$, and the first assertion follows. Also, the total number of 1's in A' is $n'(n'-1)/2$, which must be congruent to $n'\alpha$ and to $n'(n'-1-\alpha)$ modulo p if A' is row α -regular and column $(n'-1-\alpha)$ -regular modulo p . \square

4.4. Main lemma. Our third lemma contains the key elements of the proof of Theorem 1.2.

LEMMA 4.3. *Let A be a tournament matrix of order n with $\text{rank}_p[A(A+I)] = 1$. Then after a simultaneous permutation of rows and columns A satisfies*

$$(4.1) \quad AJ = \begin{bmatrix} \sigma J_{m,n} \\ \tau J_{n-m,n} \end{bmatrix}, \quad A(A+I) = \begin{bmatrix} O & (\sigma - \tau)J_{m,n-m} \\ O & O \end{bmatrix},$$

where σ and τ are distinct elements in \mathbb{Z}_p , and $m \in \{1, \dots, n-1\}$. Moreover,

$$(4.2) \quad \frac{n(n-1)}{2} \equiv m\sigma + (n-m)\tau \pmod{p}.$$

Proof. Because $\text{rank}_p[A(A+I)] = 1$, we may write

$$A^2 + A = A(A+I) = VW^T,$$

where $V = [v_1, \dots, v_n]^T$ and $W^T = [w_1, \dots, w_n]$ are non-zero matrices. Because A is a tournament matrix, each diagonal element of both A and A^2 equals 0. Hence, $v_i w_i = 0$ for $i = 1, \dots, n$. Thus, we may simultaneously permute the rows and columns of A so that $v_i \neq 0$ for $i = 1, \dots, m$ and $v_{m+1} = \dots = v_n = 0$ for some index m . We must have $w_1 = \dots = w_m = 0$. It follows that

$$(4.3) \quad A(A+I) = \begin{bmatrix} O & Y \\ O & O \end{bmatrix}$$

for some m by $n-m$ matrix Y . We also know that

$$A(A+I) = A(J - A^T) = AJ - AA^T.$$

Let s_i denote the sum of the elements in row i of A , and let s_{ij} denote the (i, j) -element of AA^T over \mathbb{Z}_p . Each element of row i of AJ equals s_i , and hence the (i, j) -entry of $A(A+I)$ is $s_i - s_{ij}$. Because AA^T is symmetric, equation (4.3) now implies that $s_1 = \dots = s_m = \sigma$, say, and $s_{m+1} = \dots = s_n = \tau$, say. Hence, AJ has the form specified in (4.1). Moreover, $Y = (\sigma - \tau)J_{m, n-m}$. Note that $\sigma \neq \tau$, and $1 \leq m \leq n-1$, for otherwise $A(A+I) = O$, contrary to the hypothesis that $\text{rank}_p[A(A+I)] = 1$.

Now (4.1) implies that $\text{trace}(AJ) = m\sigma + (n-m)\tau$. On the other hand, $\text{trace}(AJ)$ is the sum of the entries in the tournament matrix A , i.e., $\text{trace}(AJ) = n(n-1)/2$. Therefore, congruence (4.2) holds. \square

For future reference we note that (4.1) implies that

$$(4.4) \quad AA^T = A(J - I - A) = AJ - A(A+I) = \begin{bmatrix} \sigma J_m & \tau J_{m, n-m} \\ \tau J_{n-m, m} & \tau J_{n-m} \end{bmatrix}.$$

Part of our strategy to prove Theorem 1.2(b) involves showing that $\sigma = \pm 1$ and $\tau = 0$ so that AA^T has the form stated in (ii).

5. The proof of Theorem 1.2.

5.1. Proof of the rank inequality in (a). Inequality (1.5) follows immediately from the de Caen inequality (1.2) when n is even, but we include a short proof for the sake of completeness. The nullspaces of A and $A+I$ intersect trivially, and thus

$$(n - \text{rank}_p(A)) + (n - \text{rank}_p(A+I)) \leq n.$$

By Lemma 4.1 we have $\text{rank}_p(A + I) \leq \text{rank}_p(A) + 1$, and it follows that

$$n \leq \text{rank}_p(A) + \text{rank}_p(A + I) \leq 2 \cdot \text{rank}_p(A) + 1,$$

which implies that $\text{rank}_p(A) \geq (n-1)/2$. Because n is even, we must have $\text{rank}_p(A) \geq n/2$, and inequality (1.5) is established. \square

5.2. Proof of the characterizations of equality in (b) and the congruence in (c). We shall show that (ii) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (ii). The congruence condition in Theorem 1.2(c) is established within the implication (iii) \Rightarrow (ii).

5.2.1. (ii) \Rightarrow (i). Suppose that n is even and that AA^T is of one of the forms in (ii). Then $\text{rank}_p(AA^T) = 0$ or 1 . Sylvester's law for the rank of a matrix product [9, p. 162] tells us that

$$1 \geq \text{rank}_p(AA^T) \geq \text{rank}_p(A) + \text{rank}_p(A^T) - n = 2 \cdot \text{rank}_p(A) - n.$$

Hence, $\text{rank}_p(A) \leq (n+1)/2$. Because n is even, (1.5) implies that $\text{rank}_p(A) = n/2$.

5.2.2. (i) \Rightarrow (iii). Suppose that $\text{rank}_p(A) = n/2$. By Lemma 4.1 two eigenvalues of A are 0 and -1 with respective geometric multiplicities $n/2$ and at least $(n/2) - 1$. Thus, the characteristic polynomial of A is of the form

$$\chi_A(x) = x^{n/2}(x+1)^{(n/2)-1}(x-\lambda),$$

where λ is the eigenvalue not yet accounted for. Because A is a tournament matrix, we must have $\text{trace}(A) = \text{trace}(A^2) = 0$. The trace of a matrix is the sum of its eigenvalues, and thus

$$0 = \text{trace}(A) = \left(\frac{n}{2} - 1\right)(-1) + \lambda,$$

$$0 = \text{trace}(A^2) = \left(\frac{n}{2} - 1\right)(-1)^2 + \lambda^2.$$

Addition of these two equations yields $0 = \lambda + \lambda^2$. Thus, $\lambda = 0$ or $\lambda = -1$. An inspection of the algebraic and geometric multiplicities of 0 and -1 as eigenvalues of A reveals that the only feasible characteristic and minimal polynomials of A are those listed in (iii).

5.2.3. (iii) \Rightarrow (ii). This is the most difficult implication. Suppose that (iii) holds. Then clearly n is even. There are two cases.

Case 1: Suppose that $\chi_A(x) = x^{(n/2)+1}(x+1)^{(n/2)-1}$ and $\mu_A(x) = x^2(x+1)$. The equality $0 = \text{trace}(A) = ((n/2) - 1)(-1)$ implies that

$$(5.1) \quad \frac{n}{2} \equiv 1 \pmod{p}.$$

We must have $\text{rank}_p[A(A+I)] = 1$ for otherwise the geometric multiplicities of the eigenvalues 0 and -1 differ by more than 1, contrary to Lemma 4.1. By Lemma 4.3 we may write

$$(5.2) \quad A = \begin{bmatrix} B_m & X \\ \tilde{X} & C_{n-m} \end{bmatrix},$$

where B_m and C_{n-m} are tournament matrices of orders m and $n-m$, respectively, and the matrices AJ and $A(A+I)$ have the forms in (4.1) with $\sigma \neq \tau$.

Use the block matrix expression for $A(A+I)$ given in Lemma 4.3 and the minimal polynomial $\mu_A(x) = x^2(x+1)$ to see that

$$O = A[A(A+I)] = \begin{bmatrix} B_m & X \\ \tilde{X} & C_{n-m} \end{bmatrix} \begin{bmatrix} O & (\sigma - \tau)J_{m,n-m} \\ O & O \end{bmatrix}.$$

It follows that $B_m J_{m,n-m} = O$ (that is, B is row 0-regular) and $\tilde{X} J_{m,n-m} = O$. By Lemma 4.2

$$(5.3) \quad \frac{m(m-1)}{2} \equiv 0 \pmod{p}.$$

Similarly,

$$O = [A(A+I)]A = \begin{bmatrix} O & (\sigma - \tau)J_{m,n-m} \\ O & O \end{bmatrix} \begin{bmatrix} B_m & X \\ \tilde{X} & C_{n-m} \end{bmatrix}.$$

Hence, $J_{m,n-m}C_{n-m} = O$ (that is, C is column 0-regular and row $(n-m-1)$ -regular), and by Lemma 4.2

$$(5.4) \quad \frac{(n-m)(n-m-1)}{2} \equiv 0 \pmod{p},$$

Now (5.1), (5.3), and (5.4) imply that $m \equiv 1 \pmod{p}$.

Congruence (4.2) gives

$$(5.5) \quad 1 \equiv \frac{n(n-1)}{2} \equiv m\sigma + (n-m)\tau \equiv \sigma + \tau \pmod{p}.$$

Also, from Lemma 4.3 and our earlier work

$$\begin{bmatrix} \sigma J_{m,n} \\ \tau J_{n-m,n} \end{bmatrix} = AJ = \begin{bmatrix} B_m & X \\ \tilde{X} & C_{n-m} \end{bmatrix} \begin{bmatrix} J_{m,n} \\ J_{n-m,n} \end{bmatrix} = \begin{bmatrix} XJ_{n-m,n} \\ CJ_{n-m,n} \end{bmatrix},$$

which tells us that the tournament matrix C of order $n-m$ is row τ -regular. We already saw that C is column 0-regular, and so Lemma 4.2 gives

$$\tau \equiv n-m-1 \equiv 2-1-1 \equiv 0 \pmod{p}.$$

Now (5.5) shows that $\sigma \equiv 1 \pmod{p}$. Therefore, AA^T has the form stated in (ii) in this case by (4.4). Note that our proof is valid when $p = 2$ by our convention for $(\pm J)$ -type tournament matrices.

Case 2: Suppose that $\chi_A(x) = x^{n/2}(x+1)^{n/2}$. Then the equation $0 = \text{trace}(A) = (n/2)(-1)$ implies that

$$(5.6) \quad \frac{n}{2} \equiv 0 \pmod{p}.$$

There are two possible minimal polynomials of A , which we treat in turn.

Subcase 2.1: Suppose that $\mu_A(x) = x(x+1)$. Then $O = A(A+I) = A(J-A^T)$, and so $AA^T = AJ$. Because AA^T is symmetric and the columns of AJ are identical, we must have $AJ = AA^T = \sigma J$ for some scalar σ . Moreover,

$$A^T J = (J - I - A)J = nJ - J - \sigma J = -(\sigma + 1)J.$$

Hence,

$$O = \sigma nJ = \sigma J^2 = (\sigma J)J = (AA^T)J = A(A^T J) = -(\sigma + 1)AJ = -\sigma(\sigma + 1)J.$$

Thus, $\sigma = 0$ or $\sigma = -1$. Therefore, $AA^T = O$ or $AA^T = -J$. These are the extreme $(-J)$ -type tournament matrices of sub-orders $m = 0$ and $m = n$ in (ii). Note that $m \equiv 0 \pmod{p}$.

Subcase 2.2: Suppose that $\mu_A(x) = x(x+1)^2$. The argument will be similar to Case 1. Again we have $\text{rank}_p[A(A+I)] = 1$. Lemma 4.3 implies that AJ and $A(A+I)$ have the forms in (4.1), and

$$0 \equiv \frac{n}{2}(n-1) \equiv m\sigma + (n-m)\tau \equiv m(\sigma - \tau) \pmod{p}.$$

Thus, $m \equiv 0 \pmod{p}$.

We again write A in the block form (5.2). Because $\mu_A(x) = x(x+1)^2$, we have $-A(A+I) = [A(A+I)]A$, and by the block matrix expression for $A(A+I)$ given in Lemma 4.3

$$-\begin{bmatrix} O & (\sigma - \tau)J_{m,n-m} \\ O & O \end{bmatrix} = \begin{bmatrix} O & (\sigma - \tau)J_{m,n-m} \\ O & O \end{bmatrix} \begin{bmatrix} B_m & X \\ \tilde{X} & C_{n-m} \end{bmatrix}.$$

Therefore, $J_{m,n-m}C = -J_{m,n-m}$ and $J_{m,n-m}\tilde{X} = O$. The latter equation implies that $XJ_{n-m,m} = O$. Similarly, the equation $-A(A+I) = A[A(A+I)]$ gives

$$B_m J_{m,n-m} = -J_{m,n-m} \quad \text{and} \quad \tilde{X} J_{m,n-m} = O.$$

From Lemma 4.3 it now follows that

$$\begin{bmatrix} \sigma J_{m,n} \\ \tau J_{n-m,n} \end{bmatrix} = AJ = \begin{bmatrix} B_m & X \\ \tilde{X} & C_{n-m} \end{bmatrix} \begin{bmatrix} J_{m,n} \\ J_{n-m,n} \end{bmatrix} = \begin{bmatrix} -J_{m,n} \\ CJ_{n-m} \end{bmatrix}.$$

Thus, $\sigma \equiv -1 \pmod{p}$. Also, the tournament matrix C of order $n - m$ is row τ -regular and column (-1) -regular modulo p . By Lemma 4.2 we have $\tau \equiv 0 \pmod{p}$. Therefore, AA^T has the form stated in (ii). \square

Note that our proof is valid for $p = 2$ by our convention for $(\pm J)$ -type tournament matrices.

5.3. Proof of the congruence in (d). Suppose that $p \equiv 3 \pmod{4}$. Append a suitable row and column to A , as in the proof of Proposition 3.1, to obtain a tournament matrix A_{n+1} of odd order $n + 1$ with $\text{rank}_p(A_{n+1}) = ((n + 1) - 1)/2$. Apply Theorem 1.1(d) to the matrix A_{n+1} to conclude that $((n + 1) - 1)/2 \equiv 1 \pmod{2p}$. \square

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