Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 348-362, March 2011

ELA

THE ALGEBRAIC RICCATI EQUATION WITH TOEPLITZ MATRICES AS COEFFICIENTS*

ALBRECHT BÖTTCHER†

Abstract. It is shown that, under appropriate assumptions, the continuous algebraic Riccati equation with Toeplitz matrices as coefficients has Toeplitz-like solutions. Both infinite and sequences of finite Toeplitz matrices are considered, and also studied is the finite section method, which consists in approximating infinite systems by large finite truncations. The results are proved by translating the problem into C^*-algebraic language and by using theorems on the Riccati equation in general C^*-algebras. The paper may serve as another illustration of the usefulness of C^*-algebra techniques in matrix theory.

Key words. Algebraic Riccati equation, Toeplitz matrix, C^*-algebra.

AMS subject classifications. 47N70, 15A24, 15B05, 46L89, 47B35, 93C15.

1. Introduction. We consider the (continuous) algebraic Riccati equation in the form

$$XDX - XA - A^*X - C = 0. \tag{1.1}$$

Suppose the coefficients are infinite Toeplitz matrices generated by continuous functions,

$$XT(d)X - XT(a) - T(\overline{a})X - T(c) = 0. \tag{1.2}$$

Does equation (1.2) have a solution X which is Toeplitz-like, say $X = T(\varphi) + K$ with a continuous function φ and a compact operator K? A perhaps more important situation is the one when the coefficients are $n \times n$ Toeplitz matrices,

$$X_nT_n(d)X_n - X_nT_n(a) - T_n(\overline{a})X_n - T_n(c) = 0. \tag{1.3}$$

Does this equation possess a Toeplitz-like solution X_n? Defining Toeplitz-likeness for an individual $n \times n$ matrix is a delicate matter, but the concept makes perfect sense for the sequence $\{X_n\}_{n=1}^{\infty}$. Namely, we say that $\{X_n\}_{n=1}^{\infty}$ is a Toeplitz-like sequence if there exist a continuous function φ, compact operators K and L, and a sequence

*Received by the editors on December 4, 2010. Accepted for publication on March 11, 2011. Handling Editor: Christian Mehl.

†Fakultät für Mathematik, TU Chemnitz, 09107 Chemnitz, Germany (aboettch@mathematik.tu-chemnitz.de).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 348-362, March 2011

of $n \times n$ matrices U_n whose spectral norms $\|U_n\|$ go to zero such that

$$X_n = T_n(\varphi) + P_n K P_n + W_n L W_n + U_n, \tag{1.4}$$

where P_n is the projection onto the first n coordinates and W_n stands for P_n followed by reversal of the coordinates. As the main mass of the matrix of a compact operator is contained in its upper-left corner, the main mass of the $n \times n$ perturbation matrices $P_n K P_n + W_n L W_n$ in (1.4) is concentrated in the upper-left and lower-right corners.

Searching for Toeplitz-like solutions of equations (1.2) and (1.3) is a special case of the problem of considering equation (1.1) in a C^*-algebra $\mathcal{A}$. Thus, given $D, A, C \in \mathcal{A}$, is there a solution $X \in \mathcal{A}$? This general question has already been explored, and the purpose of this paper is to show how these general C^*-algebraic results yield answers in the case of Toeplitz coefficients in a very quick and elegant way.

The question whether (1.3) has Toeplitz-like solutions was studied in [17], and that paper was in fact the motivation for the present paper. Paper [17] is based on a theorem from [23], [24], which states that if $\mathcal{A}$ is an operator algebra and $D, A, A^*, C \in \mathcal{A}$, then, under certain assumptions, equation (1.1) has a solution in $\mathcal{A}$. Accordingly, a special operator algebra $\mathcal{T}_{\delta,\varrho,\alpha}$ of so-called almost Toeplitz matrices is constructed in [17] and the theorem is then applied to $\mathcal{A} = \mathcal{T}_{\delta,\varrho,\alpha}$. However, it remains a critical issue what exactly an operator algebra in this context is, and a counterexample to the theorem of [23], [24] is given in [7]. Moreover, the algebra $\mathcal{T}_{\delta,\varrho,\alpha}$ is quite complicated. In contrast to this, working with (1.4) as the definition of Toeplitz-likeness has proven extremely useful since Silbermann's paper [26]. In [2], it was observed (and significantly exploited) that the set of all sequences $\{X_n\}_{n=1}^{\infty}$ of the form (1.4) is a C^*-algebra, and this favorable circumstance has been taken advantage of since then in many instances; see, for example, [1], [3], [20]. In this light, the idea to invoke results on the Riccati equation in C^*-algebras, which are the nicest operator algebras at all, in order to treat equations (1.2) and (1.3) emerges very naturally.

We remark that the questions considered here are of even greater interest in the case where the coefficients of the Riccati equation are block Toeplitz matrices, the case of 2×2 blocks being the perhaps most important. Our results can be carried over to this more general setting, although the hypotheses of the theorems are then no longer as simple as, for example, (i) to (iii) of Theorem 3.2. However, we see the main purpose of this paper in illustrating how C^*-algebra arguments can be used to tackle certain questions for the Riccati equation, and as this intention could be fogged by the technical details of the block case, we confine ourselves to the scalar case.

The paper is organized as follows. In Section 2, we cite two results on the Riccati equation in general C^*-algebras. These results are then applied to C^*-algebras of Toeplitz matrices in Sections 3 and 4. Theorems 3.2 and 4.2 as well as Corollary 4.3 are our main results. In Section 5, we give an alternative proof to a known result

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 348-362, March 2011

which relates the Riccati equation with large Toeplitz matrices to the Riccati equation with Laurent matrices as coefficients. Finally, Section 6 is devoted to additional issues concerning Riccati equations in C^*-algebras and W^*-algebras.

2. The Riccati equation in C^*-algebras. Let $\mathcal{A}$ be a unital C^*-algebra and denote the unit by I. The reader is referred to [14] or [16] for an introduction to C^*-algebras. We let $\sigma(H)$ stand for the spectrum of $H \in \mathcal{A}$. An element $H \in \mathcal{A}$ is called Hermitian if $H = H^*$, it is said to be positive, $H \geq 0$, if $H = H^*$ and $\sigma(H) \subset [0, \infty)$, and it is referred to as a positive definite element, $H > 0$, if $H = H^*$ and $\sigma(H) \subset (0, \infty)$. We write $G \leq H$ or $H \geq G$ if G and H are Hermitian and $H - G$ is positive.

In the case where $\mathcal{A} = \mathbf{C}^{N \times N}$, the Riccati equation arises from optimal control as follows; see, e.g., [18], [22]. Let $A, B, C, R \in \mathbf{C}^{N \times N}$ and suppose $C \geq 0$ and $R > 0$. The control $u(t)$ of the system $\dot{x} = Ax + Bu$ which minimizes the cost functional

$$\int_0^\infty (x^* C x + u^* R u)\, \mathrm{d}t \tag{2.1}$$

is given by $u(t) = -R^{-1} B^* X_+ x(t)$ where X_+ is the maximal Hermitian solution of the equation

$$XBR^{-1}B^*X - XA - A^*X - C = 0. \tag{2.2}$$

A Hermitian solution X_+ of (2.2) is said to be maximal if $X_+ \geq X$ for every Hermitian solution X of (2.2). Clearly, $D := BR^{-1}B^*$ is positive. Notice also that the feedback input $u = -Fx + v$ transforms the system $\dot{x} = Ax + Bu$ into the new system $\dot{x} = (A - BF)x + Bv$.

Now let $\mathcal{A}$ be a general unital C^*-algebra, let $D, A, C \in \mathcal{A}$, and suppose $D \geq 0$, $C \geq 0$. The pair (A, D) is called stabilizable in $\mathcal{A}$ if there exists an $F \in \mathcal{A}$ such that $\sigma(A - DF)$ is contained in the open left half-plane $\mathbf{C}_-$. Given a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded linear operators on $\mathcal{H}$. It is well known that every C^*-algebra $\mathcal{A}$ may be identified with a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some $\mathcal{H}$. Bunce [4] (see also [15, Corollary 3.3]) showed that if the pair (A, D) is stabilizable in $\mathcal{B}(\mathcal{H})$, then it is automatically stabilizable in $\mathcal{A}$. A solution $X_+ \in \mathcal{A}$ of equation (1.1) is called maximal in $\mathcal{A}$ if X_+ is Hermitian and $X_+ \geq X$ for every Hermitian solution $X \in \mathcal{A}$ of (1.1). Dobovišek [15, pp. 74–75] observed that if $X_+ \in \mathcal{A}$ is a maximal solution in $\mathcal{A}$, then in fact $X_+ \geq X$ for every Hermitian solution $X \in \mathcal{B}(\mathcal{H})$. Thus, stabilizability and maximality do not depend on the algebra $\mathcal{A}$ and we therefore omit the "in $\mathcal{A}$". Curtain and Rodman [11] proved that if the pair (A, D) is stabilizable, then equation (1.1) has a maximal solution $X_+ \in \mathcal{B}(\mathcal{H})$ and this solution is positive. The big problem is whether this solution X_+ is in $\mathcal{A}$ or not. The following two theorems provide us with partial answers.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 348-362, March 2011

ELA

THEOREM 2.1. (Bunce [4]) *If $D \geq 0$, $C = I$, and (A, D) is stabilizable, then equation (1.1) has a maximal solution $X_+ \in \mathcal{A}$ and this solution is positive.*

Note that the restriction to $C = I$ means that all state variables in (2.1) are considered to have equal rights.

THEOREM 2.2. (Dobovišek [15, Theorem 3.7]) *If $D \geq 0$, $C \geq 0$, (A, D) is stabilizable, and $\sigma(A)$ does not intersect the imaginary axis, then equation (1.1) has a maximal solution $X_+ \in \mathcal{A}$ and this solution is positive.*

In [4] and [15], it is shown that under the hypotheses of Theorems 2.1 or 2.2 the spectrum of $A - DX_+$ is a subset of $\mathbf{C}_-$, that is, $X_+ \in \mathcal{A}$ stabilizes (A, D).[1]

3. Infinite Toeplitz matrices. Let $\mathbf{T}$ be the complex unit circle and let $C(\mathbf{T})$ be the C^*-algebra of all continuous complex-valued functions on $\mathbf{T}$. We abbreviate $C(\mathbf{T})$ to $\mathcal{C}$. Given $f \in \mathcal{C}$ with Fourier coefficients

$$f_k := \frac{1}{2\pi} \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}x}) \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}x, \quad k \in \mathbf{Z},$$

we consider the infinite Toeplitz matrix $T(f) := (f_{j-k})_{j,k=1}^{\infty}$. The function f is called the generating function or the symbol of $T(f)$. It is well known that $T(f)$ induces a bounded linear operator on $\ell^2 := \ell^2(\mathbf{N})$. Moreover, $\|T(f)\| = \|f\|_\infty$, where $\| \cdot \|_\infty$ is the L^∞ norm. A classical result by Gohberg says that the spectrum of $T(f)$ is the union of the range $f(\mathbf{T})$ and the points in $\mathbf{C} \setminus f(\mathbf{T})$ whose winding number with respect to $f(\mathbf{T})$ is nonzero; see, e.g., [3, Theorem 1.17]. In particular, if $f(\mathbf{T}) \subset \mathbf{C}_-$, then $T(f)$ is invertible.

We denote by $\mathcal{T}$ the smallest norm closed subalgebra of $\mathcal{B} := \mathcal{B}(\ell^2)$ which contains the set $T(\mathcal{C}) := \{T(f) : f \in \mathcal{C}\}$. The algebra $\mathcal{T}$ is a C^*-algebra and it turns out that the set $\mathcal{K} := \mathcal{K}(\ell^2)$ of all compact operators is a subset and thus a closed two-sided ideal of $\mathcal{T}$. Coburn [5], [6] showed that $\mathcal{T}/\mathcal{K}$ is isometrically *-isomorphic to $\mathcal{C}$, the map $f \mapsto T(f) + \mathcal{K}$ being an isometric *-isomorphism of $\mathcal{C}$ onto $\mathcal{T}/\mathcal{K}$. Thus, an operator X belongs to $\mathcal{T}$ if and only if $X = T(f) + K$ with $f \in \mathcal{C}$ and $K \in \mathcal{K}$. This decomposition is unique, that is, $T(\mathcal{C}) \cap \mathcal{K} = \{0\}$.

Theorems 2.1 and 2.2 are applicable to $\mathcal{A} = \mathcal{T}$ and show that, under their hypotheses, equation (1.1) with $D, A, C \in \mathcal{T}$ has a maximal solution $X_+ \in \mathcal{T}$. To be more specific, we consider the equation

$$X\Big(T(d) + T(b)T^{-1}(r)T(\overline{b})\Big)X - XT(a) - T(\overline{a})X - T(c) = 0. \tag{3.1}$$

[1] I am grateful to one of the referees for pointing out that exponentially stabilizing solutions of equation (1.1) are unique and so the maximal solutions in Theorems 2.1 and 2.2 are the unique exponentially stabilizing ones.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 348-362, March 2011

Note that the adjoint of $T(f)$ is just $T(\overline{f})$ where $\overline{f}(t) = \overline{f(t)}$ and the bar on the right stands for complex conjugation. We also write $T^{-1}(r) := [T(r)]^{-1}$. If $b = 0$, then (3.1) becomes (1.2), while if $d = 0$, equation (3.1) takes the form

$$XT(b)T^{-1}(r)T(\overline{b})X - XT(a) - T(\overline{a})X - T(c) = 0,$$

which is (2.2) in Toeplitz matrices. In what follows, if $f \in \mathcal{C}$, then $f > 0$ means that f is a positive definite element of $\mathcal{C}$, that is, f is real-valued and $f(t) > 0$ for all $t \in \mathbf{T}$. Analogously, we write $f \geq 0$ if f is real-valued and $f(t) \geq 0$ for all $t \in \mathbf{T}$.

Lemma 3.1. *Let $b, r \in \mathcal{C}$ and suppose $r > 0$. Then*

$$T(b)T^{-1}(r)T(\overline{b}) = T(|b|^2 r^{-1}) + L$$

with some compact operator $L \geq 0$.

Proof. Suppose first that r is smooth. Then r has a Wiener-Hopf factorization $r = r_+ \overline{r}_+$ with a function $r_+ \in \mathcal{C}$ such that $r_+^{-1} \in \mathcal{C}$ and all Fourier coefficients with negative indices of r_+ and r_+^{-1} vanish; see, e.g, [3, Theorem 1.14]. Standard computations with Toeplitz matrices, [3, Section 1.5], give

$$\begin{aligned} T(b)T^{-1}(r)T(\overline{b}) &= T(b)T(r_+^{-1})T(\overline{r}_+^{-1})T(\overline{b}) = T(br_+^{-1})T(\overline{r}_+^{-1}\overline{b}) \\ &= T(|b|^2 r^{-1}) - H(g)H(\widetilde{\overline{g}}) =: T(|b|^2 r^{-1}) + L, \end{aligned}$$

where $g := br_+^{-1}$, $\widetilde{f}$ is defined by $\widetilde{f}(t) := f(1/t)$ for $t \in \mathbf{T}$, and $H(f)$ is the Hankel operator induced by the matrix $(f_{j+k-1})_{j,k=1}^{\infty}$. Since $H(\widetilde{\overline{g}})$ is the adjoint operator of $H(g)$, the operator L is positive, and since g is continuous, the operator L is compact.

An arbitrary continuous function $r > 0$ may be approximated uniformly by smooth functions $r_n > 0$. Then $T^{-1}(r_n)$ and $T(|b|^2 r_n^{-1})$ converge in the norm to $T^{-1}(r)$ and $T(|b|^2 r^{-1})$, respectively. Since $T(b)T^{-1}(r_n)T(\overline{b}) - T(|b|^2 r_n^{-1})$ was shown to be positive and compact, so also is $T(b)T^{-1}(r)T(\overline{b}) - T(|b|^2 r^{-1})$. $\square$

We denote the real part of a function f and an operator H by $\operatorname{Re} f$ and $\operatorname{Re} H$.

Theorem 3.2. *Let $a, b, c, d, r \in \mathcal{C}$ with $d \geq 0$, $r > 0$ and suppose one of the following conditions is satisfied:*

(i) $d + |b|^2 r^{-1} > 0$ *and* $c = 1$;

(ii) $d + |b|^2 r^{-1} > 0$, $c \geq 0$, *and* $\operatorname{Re} a > 0$;

(iii) $c \geq 0$ *and* $\operatorname{Re} a < 0$.

Then equation (3.1) has a maximal solution $X_+ \in \mathcal{T}$, this solution is positive, and $X_+ = T(\varphi) + K$ where $\varphi \in \mathcal{C}$, $\varphi \geq 0$, and $K = K^*$ is compact. Furthermore, φ satisfies the equation

$$(d + |b|^2 r^{-1})\varphi^2 - (a + \overline{a})\varphi - c = 0. \tag{3.2}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 348-362, March 2011

ELA

Proof. First of all, $C := T(c) \geq 0$ and $D := T(d) + T(b)T^{-1}(r)T(\overline{b}) \geq 0$. Put $A := T(a)$. If (iii) holds, then $\operatorname{Re} A < 0$ and hence (A, D) is trivially stabilizable. So assume (i) or (ii) is in force. Then $d + |b|^2 r^{-1} > 0$. From Lemma 3.1 we infer that if $\gamma \in (0, \infty)$ is any constant, then

$$\operatorname{Re}(A - \gamma D) = T(\operatorname{Re} a - \gamma(d + |b|^2 r^{-1})) - \gamma L,$$

and choosing γ large enough we can achieve that $\operatorname{Re}(A - \gamma D)$ is negative definite as the sum of a negative definite operator and a negative operator. Thus, (A, D) is stabilizable. The spectrum of $T(a)$ is in the left and right open half-planes if $\operatorname{Re} a < 0$ and $\operatorname{Re} a > 0$, respectively. Theorem 2.1 now yields a maximal and positive solution $X_+ \in \mathcal{T}$ under the hypothesis (i), while Theorem 2.2 does this if (ii) or (iii) is satisfied. Writing $X_+ = T(\varphi) + K$ and taking into account that passage from $\mathcal{T}$ to $\mathcal{T}/\mathcal{K} \cong \mathcal{C}$ preserves positivity, we conclude that $\varphi = \overline{\varphi} \geq 0$ and hence $K = K^*$. Inserting $X_+ = T(\varphi) + K$ in (3.1), and considering the resulting equation modulo compact operators we get

$$T(\varphi)\Big(T(d) + T(b)T^{-1}(r)T(\overline{b})\Big)T(\varphi) - T(\varphi)T(a) - T(\overline{a})T(\varphi) - T(c) \in \mathcal{K}.$$

Taking into account that $T^{-1}(r) - T(r^{-1})$ is compact (Lemma 3.1 with $b = 1$) and that $T(f)T(g) - T(fg)$ is compact for arbitrary $f, g \in \mathcal{C}$ (Coburn), we obtain that

$$T((d + |b|^2 r^{-1})\varphi^2 - (a + \overline{a})\varphi - c) \in \mathcal{K}.$$

Using that the only compact Toeplitz operator is the zero operator, we finally arrive at (3.2). ☐

4. Finite Toeplitz matrices. For $f \in \mathcal{C}$, we denote by $T_n(f)$ the $n \times n$ Toeplitz matrix $(f_{j-k})_{j,k=1}^n$. It is well known that if $\operatorname{Re} f < 0$ or $\operatorname{Re} f > 0$, then $T_n(f)$ is invertible for all $n \geq 1$,

$$\sup_{n \geq 1} \|T_n^{-1}(f)\| < \infty, \tag{4.1}$$

and $T_n^{-1}(f)P_n \to T^{-1}(f)$ strongly as $n \to \infty$, where P_n is projection onto the first n co-ordinates; see, e.g., [3, Proposition 2.17]. Here and throughout what follows, $\|\cdot\|$ is always the operator norm on ℓ^2 (= spectral norm in the case of matrices).

Let $\mathbf{B}$ denote the set of all sequences $F = \{F_n\}_{n=1}^\infty$ such that $F_n \in \mathbf{C}^{n \times n}$ and

$$\|F\| := \sup_{n \geq 1} \|F_n\| < \infty. \tag{4.2}$$

We henceforth abbreviate $\{F_n\}_{n=1}^\infty$ to $\{F_n\}$. With termwise operations, $\{F_n\} + \{G_n\} := \{F_n + G_n\}$, $\alpha\{F_n\} := \{\alpha F_n\}$, $\{F_n\}\{G_n\} := \{F_n G_n\}$, $\{F_n\}^* := \{F_n^*\}$,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 348-362, March 2011

and with the norm (4.2), $\mathbf{B}$ is a C^*-algebra. We denote by $\mathbf{B}_0$ the elements $\{F_n\} \in \mathbf{B}$ for which $\|F_n\| \to 0$ as $n \to \infty$. Clearly, $\mathbf{B}_0$ is a closed two-sided ideal of $\mathbf{B}$. Let P_n and W_n be the operators

$$P_n : \ell^2 \to \ell^2, \ \{x_1, x_2, \ldots\} \mapsto \{x_1, \ldots, x_n, 0, 0, \ldots\},$$
$$W_n : \ell^2 \to \ell^2, \ \{x_1, x_2, \ldots\} \mapsto \{x_n, \ldots, x_1, 0, 0, \ldots\}.$$

Note that $W_n = W_n^* \to 0$ weakly, so that $W_n K W_n \to 0$ strongly whenever K is compact. For $H \in \mathcal{B}(\ell^2)$, we think of $P_n H P_n$ and $W_n H W_n$ as $n \times n$ matrices. Let $\mathbf{S}$ be the smallest closed subalgebra of $\mathbf{B}$ which contains the set $\{\{T_n(f)\} : f \in \mathcal{C}\}$. One can show that $\mathbf{S}$ is a C^*-algebra and that actually $\mathbf{S}$ coincides with the set of all sequences $\{F_n\}$ of the form

$$F_n = T_n(f) + P_n K P_n + W_n L W_n + U_n \tag{4.3}$$

with $f \in \mathcal{C}$, $K \in \mathcal{K}$, $L \in \mathcal{K}$, $\{U_n\} \in \mathbf{B}_0$; see [2] or [3, Proposition 2.33]. Moreover, the set $\mathbf{J}$ of all sequences $\{F_n\}$ of the form $F_n = P_n K P_n + W_n L W_n + U_n$ with $K \in \mathcal{K}$, $L \in \mathcal{K}$, $\{U_n\} \in \mathbf{B}_0$ (that is those of the form (4.3) with $f = 0$) is closed two-sided ideal of $\mathbf{S}$ and the map $f \mapsto \{T_n(f)\} + \mathbf{J}$ is an isometric *-isomorphism of $\mathcal{C}$ onto $\mathbf{B}/\mathbf{J}$.

Using Theorems 2.1 and 2.2 with $\mathcal{A} = \mathbf{S}$, we get results for the Riccati equation (1.1) with $D = \{D_n\}$, $A = \{A_n\}$, $C = \{C_n\}$, $X = \{X_n\}$ is $\mathbf{S}$. We illustrate things for the equation

$$X_n \Big(T_n(d) + T_n(b) T_n^{-1}(r) T_n(\bar{b}) \Big) X_n - X_n T_n(a) - T_n(\overline{a}) X_n - T_n(c) = 0. \tag{4.4}$$

Lemma 4.1. *If $b, r \in \mathcal{C}$ and $r > 0$, then*

$$T_n(b) T_n^{-1}(r) T_n(\bar{b}) = T_n(|b|^2 r^{-1}) + P_n K' P_n + W_n L' W_n + U_n'$$

with compact operators $K' \geq 0$ and $L' \geq 0$ and Hermitian matrices U_n' such that $\|U_n'\| \to 0$ as $n \to \infty$.

Proof. By virtue of (4.1), $\{T_n(r)\}$ is invertible in $\mathbf{B}$ and thus also in $\mathbf{S}$. We therefore have

$$T_n(b) T_n^{-1}(r) T_n(\bar{b}) = T_n(s) + P_n K' P_n + W_n L' W_n + U_n' \tag{4.5}$$

with $s \in \mathcal{C}$, $K' \in \mathcal{K}$, $L' \in \mathcal{K}$, $\{U_n'\} \in \mathbf{B}_0$. Passing to the strong limit $n \to \infty$ in (4.5) we obtain $T(b) T^{-1}(r) T(b) = T(s) + K'$, and Lemma 3.1 now implies that $s = |b|^2 r^{-1}$ and that K' is positive and compact. Multiplying (4.5) from the left and the right by W_n, taking into account that $W_n T_n(f) W_n = T_n(\widetilde{f})$, and then passing to the strong limit $n \to \infty$ we arrive at the equality $T(\widetilde{b}) T^{-1}(\widetilde{r}) T(\widetilde{\bar{b}}) = T(\widetilde{s}) + L'$. Lemma 3.1

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 348-362, March 2011

shows again that L' is positive and compact. It is then immediate from (4.5) that the matrices U_n' are Hermitian. ☐

THEOREM 4.2. *Let $a, b, c, d, r \in \mathcal{C}$ with $d \geq 0$, $r > 0$ and let at least one of the following conditions be satisfied:*

(i) $d + |b|^2 r^{-1} > 0$ *and* $c = 1$*;*

(ii) $d + |b|^2 r^{-1} > 0$*,* $c \geq 0$*, and* $\operatorname{Re} a > 0$*;*

(iii) $c \geq 0$ *and* $\operatorname{Re} a < 0$.

Then there is an n_0 such that for every $n \geq n_0$ equation (4.4) has a maximal solution $X_n^+ \in \mathbf{C}^{n\times n}$. This solution is positive and there exist $\varphi \in \mathcal{C}$, $\varphi \geq 0$, $K = K^* \in \mathcal{K}$, $L = L^* \in \mathcal{K}$, $U_n = U_n^* \in \mathbf{C}^{n\times n}$, $\|U_n\| \to 0$ such that

$$X_n^+ = T_n(\varphi) + P_n K P_n + W_n L W_n + U_n \tag{4.6}$$

for $n \geq n_0$. Moreover, φ satisfies the equation

$$(d + |b|^2 r^{-1})\varphi^2 - (a + \overline{a})\varphi - c = 0. \tag{4.7}$$

Proof. Put $D := \{T_n(d) + T_n(b)T_n^{-1}(r)T_n(\overline{b})\}$, $A := \{T_n(a)\}$, $C := \{T_n(c)\}$. Obviously, $D \geq 0$ and $C \geq 0$. It is clear that $\operatorname{Re} A < 0$ if (iii) holds. So consider the cases (ii) and (iii), where $d + |b|^2 r^{-1} > 0$. By Lemma 4.1,

$$\operatorname{Re}(A - \gamma D) = \{T_n(\operatorname{Re} a - \gamma(d + |b|^2 r^{-1})) - \gamma P_n K' P_n - \gamma W_n L' W_n - \gamma U_n'\}$$

for each constant $\gamma \in (0, \infty)$, and hence $\operatorname{Re}(A - \gamma D) \leq -2I - \gamma\{U_n'\}$ for all $n \geq 1$ if γ is sufficiently large. Since the matrices U_n' are Hermitian and $\|U_n'\| \to 0$, there is an n_0 such that $-2I - \gamma U_n' \leq -I$ for all $n \geq n_0$. Thus, considering only sequences of the form $\{F_n\}_{n=n_0}^{\infty}$ and denoting the corresponding C^*-algebras by $\mathbf{B}(n_0)$, $\mathbf{B}_0(n_0)$, $\mathbf{S}(n_0)$, we see that (A, D) is stabilizable in $\mathbf{S}(n_0)$. Clearly, $\sigma(\{T_n(a)\})$ does not intersect the imaginary axis if $\operatorname{Re} a < 0$ or $\operatorname{Re} a > 0$. From Theorems 2.1 and 2.2 we now deduce that equation (1.1) has a positive maximal solution $X_+ = \{X_n^+\}_{n=n_0}^{\infty}$ in $\mathbf{S}(n_0)$.

The reasoning of the previous paragraph is applicable to each individual number $n \geq n_0$, that is, $D_n \geq 0$, $C_n \geq 0$, (A_n, D_n) is stabilizable, and $\sigma(T_n(a))$ has no points on the imaginary axis in the cases (ii) and (iii). Consequently, equation (4.4) has a positive maximal solution X_n for each $n \geq n_0$. We want to show that $X_n = X_n^+$. The matrix X_n can be obtained by an iterative procedure such as in [18]. This procedure yields Hermitian matrices $X_n^{(1)}, X_n^{(2)}, \ldots$ satisfying $X_n^{(1)} \geq X_n^{(2)} \geq \ldots \geq X_n$ and $X_n^{(k)} \to X_n$ as $k \to \infty$. In particular, $X_n \leq X_n^{(1)}$. We have

$$X_n^{(1)} = \int_0^\infty \mathrm{e}^{t(A_n - \gamma D_n)^*}(\gamma^2 D_n + C_n)\mathrm{e}^{t(A_n - \gamma D_n)}\,\mathrm{d}t.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 348-362, March 2011

There is a constant $M < \infty$ such that $\|D_n\| \le M$ (due to (4.1)) and $\|C_n\| \le M$ (obvious). As $\mathrm{Re}\,(A_n - \gamma D_n) \le -I$, it follows that

$$\|X_n^{(1)}\| \le (\gamma^2+1)M \int_0^\infty \mathrm{e}^{-2t}\,\mathrm{d}t = (\gamma^2+1)M/2 =: M_0.$$

Consequently, $\|X_n\| \le M_0$ for $n \ge n_0$, which implies that $\{X_n\}_{n=n_0}^\infty$ is a Hermitian solution of equation (1.1) in $\mathbf{B}(n_0)$. As noted in Section 2, we then necessarily have $\{X_n\}_{n=n_0}^\infty \le \{X_n^+\}_{n=n_0}^\infty$, that is $X_n \le X_n^+$ for all $n \ge n_0$. As X_n is the maximal solution of the nth equation and X_n^+ is a Hermitian solution of that equation, we also have $X_n^+ \le X_n$. Thus, $X_n = X_n^+$ for $n \ge n_0$.

As components of an element in $\mathbf{S}(n_0)$, the matrices X_n^+ are of the form (4.6) with $\varphi \in \mathcal{C}$, $K \in \mathcal{K}$, $L \in \mathcal{K}$, $\{U_n\} \in \mathbf{B}_0(n_0)$. We also know that $X_n^+ \ge 0$. As $n \to \infty$, the strong limit of (4.6) is $T(\varphi)+K$. This is positive operator if and only if $\varphi = \overline{\varphi} \ge 0$ and $K = K^*$ (recall the proof of Theorem 3.2). Multiplying (4.6) from the left and the right by W_n and passing to the strong limit $n \to \infty$, we obtain analogously that $L = L^*$. As $T_n(\varphi)$, K, L are Hermitian, so also is U_n.

Finally, passing in (4.4) to the strong limit as $n \to \infty$, we arrive at (3.1). Theorem 3.2 therefore implies that φ must satisfy (4.7). $\square$

The following result on the finite section method for equation (3.1) is immediate from Theorems 3.2 and 4.2. We silently already used it in the last paragraph of the previous proof to establish (4.7).

Corollary 4.3. *Under the hypotheses of Theorems 3.2 and 4.2, equations (4.4) have* (*unique*) *positive maximal solutions X_n^+ for all sufficiently large n, and $X_n^+ P_n$ converges strongly to the* (*unique*) *positive maximal solution X_+ of equation (3.1).*

5. The Riccati equation with Laurent matrices as coefficients. The Laurent matrix $L(f)$ generated by a function $f \in \mathcal{C}$ is the doubly-infinite Toeplitz matrix $(f_{j-k})_{j,k=-\infty}^\infty$. This matrix induces a bounded operator on $\ell^2(\mathbf{Z})$ and is unitarily equivalent to the operator of multiplication by f on $L^2(\mathbf{T})$. The Riccati equation (1.1) in Laurent matrices reads

$$XL(d)X - XL(a) - L(\overline{a})X - L(c) = 0. \tag{5.1}$$

The smallest norm closed subalgebra of $\mathcal{B}(\ell^2(\mathbf{Z}))$ which contains the set $L(\mathcal{C}) := \{L(f) : f \in \mathcal{C}\}$ is in fact $L(\mathcal{C})$ itself, it is a (commutative) unital C^*-algebra, and the map $f \mapsto L(f)$ is an isometric *-isomorphism of $\mathcal{C}$ onto $L(\mathcal{C})$. Using Theorems 2.1 and 2.2, we see that if $d, a, c \in \mathcal{C}$ and at least one of the conditions (i) $d > 0$ and $c = 1$, (ii) $d > 0$, $c \ge 0$, and $\mathrm{Re}\,a > 0$, (iii) $c \ge 0$ and $\mathrm{Re}\,a < 0$ is satisfied, then equation (5.1) has a maximal solution $X_+ \in L(\mathcal{C})$, and this solution is actually $X_+ = L(\varphi)$ where

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 348-362, March 2011

$\varphi \in \mathcal{C}$, $\varphi \geq 0$, and $d\varphi^2 - (a+\overline{a})\varphi - c = 0$. The finite section method for equation (5.1) consists in passing to the equations

$$Y_n L_n(d) Y_n - Y_n L_n(a) - L_n(\overline{a}) Y_n - L_n(c) = 0. \tag{5.2}$$

where $L_n(f) := (f_{j-k})_{j,k=-n}^{n}$ and $Y_n = (y_{jk}^{(n)})_{j,k=-n}^{n}$. Define the projection $\widetilde{P}_n$ on $\ell^2(\mathbf{Z})$ by $(\widetilde{P}_n x)_j = x_j$ if $|j| \leq n$ and $(\widetilde{P}_n x)_j = 0$ if $|j| > n$.

THEOREM 5.1. *If $d, a, c \in \mathcal{C}$ and one of the hypotheses $(i), (ii), (iii)$ of the preceding paragraph is satisfied, then equations (5.2) have (unique) positive maximal solutions Y_n^+ for all sufficiently large n, and $Y_n^+ \widetilde{P}_n$ converges strongly to the (unique) positive maximal solution $X_+ = L(\varphi)$ of equation (5.1).*

Proof. Clearly, equation (5.2) is nothing but equation (1.3) with n replaced by $2n+1$, that is,

$$X_{2n+1} T_{2n+1}(d) X_{2n+1} - X_{2n+1} T_{2n+1}(a) - T_{2n+1}(\overline{a}) X_{2n+1} - T_{2n+1}(c) = 0. \tag{5.3}$$

Theorem 4.2 shows that, under the hypotheses of the preceding paragraph, the maximal solutions X_{2n+1}^+ of (5.3) are of the form

$$X_{2n+1}^+ = T_{2n+1}(\varphi) + P_{2n+1} K P_{2n+1} + W_{2n+1} L W_{2n+1} + C_{2n+1} \tag{5.4}$$

with compact operators K and L on $\ell^2(\mathbf{N})$ and $\|C_{2n+1}\| \to 0$. Let $U : \ell^2(\mathbf{Z}) \to \ell^2(\mathbf{Z})$ be the forward shift, $(Ux)_j = x_{j-1}$. If $M = (m_{jk})_{j,k=0}^{2n+1}$ is a $(2n+1) \times (2n+1)$ matrix which is thought of as an operator on $\ell^2(\mathbf{N})$, then $U^{-n} P_{2n+1} M P_{2n+1} U^n$ may be identified with the matrix $(m_{j+n,k+n})_{j,k=-n}^{n}$ regarded as an operator on $\ell^2(\mathbf{Z})$. This implies that the maximal solution of (5.2) is

$$Y_n^+ = U^{-n} P_{2n+1} X_{2n+1}^+ P_{2n+1} U^n. \tag{5.5}$$

Since $P_{2n+1} U^n$ and $W_{2n+1} U^n$ converge weakly to zero and K and L are compact, it follows that $U^{-n} P_{2n+1} K P_{2n+1} U^n \to 0$ and $U^{-n} W_{2n+1} L W_{2n+1} U^n \to 0$ strongly. We also have $U^{-n} P_{2n+1} T_{2n+1}(\varphi) P_{2n+1} U^n = L_n(\varphi)$. Thus, combining (5.4) and (5.5) we arrive at the conclusion that $Y_n^+ \widetilde{P}_n$ converges strongly to the maximal solution $X_+ = L(\varphi)$ of equation (5.1). ∎

Results like Theorem 5.1 are known and were established in [10], [21], [25] employing different methods. Our proof is new. It makes use of the precise knowledge of the structure of X_{2n+1}^+, after which the rest is soft functional analysis.

We should also emphasize the different intentions behind Corollary 4.3 and Theorem 5.1. The corollary allows us to solve a Riccati equation with infinite Toeplitz matrices as coefficients by approximating it by a Riccati equation with large but finite Toeplitz matrices. The latter might nevertheless be a numerical challenge, but it is

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 348-362, March 2011

good progress when compared with working with infinite Toeplitz matrices. On the other hand, Theorem 5.1 interprets finite Toeplitz matrices as central sections of an infinite Laurent matrix and enables us to pass from large but finite Toeplitz matrices to Laurent matrices and *thus* to the equation $d\varphi^2 - (a+\overline{a})\varphi - c = 0$ for functions on the unit circle. And in certain respects, an equation in functions is simpler than an equation in matrices.

Interesting problems for equation (5.1) arise when d, a, c are taken from a subset $\mathcal{W}$ of $\mathcal{C}$ and one is interested in the question whether special (e.g., maximal) solutions of the equation are of the form $X = L(\varphi)$ with $\varphi \in \mathcal{W}$. Usually $\mathcal{W}$ is characterized by decay properties of the Fourier coefficients, and an important case is where $\mathcal{W}$ is a weighted Wiener algebra (which is not a C^*-algebra). More generally, one takes d, a, c from $\mathcal{W}^{N\times N}$, the $N \times N$ matrix functions with entries from $\mathcal{W}$, and looks for solutions of equation (5.1) of the form $X = L(\varphi)$ with $\varphi \in \mathcal{W}^{N\times N}$. Note that in this case, even the problem $\varphi d\varphi - \varphi a - a^*\varphi - c = 0$ is non-commutative. For results in this field of research, we refer to [8], [9], [12], [13], [19], for example.

6. The Riccati equation in W^*-algebras. By the Gelfand–Naimark theorem, every C^*-algebra $\mathcal{A}$ is isometrically *-isomorphic to a closed and selfadjoint subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. If that subalgebra is closed in the strong (= pointwise) operator topology for some $\mathcal{H}$, then $\mathcal{A}$ is called a W^*-algebra or a von Neumann algebra. Known constructive procedures for solving Riccati equations [11], [18] are based on Newton-like iterations and yield the solution as the strong limit of iterates belonging to $\mathcal{A}$. This makes W^*-algebras a convenient terrain for Riccati equations. We remark that finite-dimensional C^*-algebras as well as $L^\infty(\mathbf{T})$, $\mathcal{B}(\mathcal{H})$, $\mathbf{B}$ are W^*-algebras, whereas $C(\mathbf{T})$, $\mathcal{T}$, $\mathcal{K}(\mathcal{H})$ with $\dim \mathcal{H} = \infty$, and $\mathbf{S}$ are C^*-algebras but not W^*-algebras.

THEOREM 6.1. (Curtain and Rodman [11]) *Let $\mathcal{A}$ be a unital W^*-algebra and $D, A, C \in \mathcal{A}$. If $D \geq 0$, $C \geq 0$, and (A, D) is stabilizable, then equation (1.1) has a maximal solution $X_+ \in \mathcal{A}$ and this solution is positive.*

Unfortunately, this is of limited use for our purposes because, as said, $\mathcal{T}$ and $\mathbf{S}$ are not W^*-algebras.

If $\mathcal{J}$ is a closed two-sided ideal of some unital C^*-algebra $\mathcal{A}$, then

$$\mathbf{C}I + \mathcal{J} := \{\alpha I + J : \alpha \in \mathbf{C}, J \in \mathcal{J}\}$$

is a unital C^*-subalgebra of $\mathcal{A}$. For example, $\mathbf{C}I + \mathcal{K}(\mathcal{H})$ is a unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. Here is a result on the Riccati equation in such C^*-subalgebras.

THEOREM 6.2. *Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{J} \subset \mathcal{A}$ be a proper closed two-sided ideal. Suppose D, A, C are in $\mathbf{C}I + \mathcal{J}$, $D \geq 0$, $C > 0$ (sic!), and (A, D)*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 348-362, March 2011

ELA

is stabilizable. If $X \in \mathcal{A}$ is a positive solution of equation (1.1), then necessarily $X \in \mathbf{C}I + \mathcal{J}$.

Proof. We abbreviate αI to α. Thus, let $D = d + K_D$, $A = a + K_A$, $C = c + K_C$ with $d, a, c \in \mathbf{C}$ and $K_D, K_A, K_C \in \mathcal{J}$. We claim that $c > 0$. To see this, assume $c \leq 0$. Then $0 \leq -cI < K_C$, and since ideals of C^*-algebras are hereditary (see, e.g., [14, Theorem I.5.3]), it follows that $-cI \in \mathcal{J}$. If $c < 0$, this implies $I \in \mathcal{J}$ and thus $\mathcal{J} = \mathcal{A}$, which contradicts our assumption that $\mathcal{J}$ is proper. Hence $c = 0$, and thus $C = K_C > 0$, which means that K_C is invertible and therefore again leads to the contradiction $\mathcal{J} = \mathcal{A}$. Consequently, $c > 0$. Analogously one can show that $d \geq 0$.

For $Z \in \mathcal{A}$, we denote by Z^π the coset $Z + \mathcal{J} \in \mathcal{A}/\mathcal{J}$. Since (A, D) is stabilizable, there is an $F = f + K_F \in \mathbf{C}I + J$ such that $\sigma(A - DF) \subset \mathbf{C}_-$, which implies that $\sigma(A^\pi - D^\pi F^\pi) \subset \mathbf{C}_-$. As $A^\pi - D^\pi F^\pi = a - df + \mathcal{J}$, it follows that $a - df \in \mathbf{C}_-$. This is impossible if d and $\operatorname{Re} a$ are both zero. Therefore one of d and $\operatorname{Re} a$ is nonzero. We put $\xi := c/(a + \overline{a})$ if $d = 0$, and

$$\xi := \frac{a + \overline{a}}{2d} + \sqrt{\left(\frac{a + \overline{a}}{2d}\right)^2 + \frac{c}{d}} \tag{6.1}$$

if $d > 0$. Then

$$d\xi^2 - (a + \overline{a})\xi - c = 0. \tag{6.2}$$

We may write $X = \xi + V$ with $V \in \mathcal{A}$. Our aim is to show that $V \in \mathcal{J}$. Equivalently, we want to prove that $v := V^\pi$ is the zero in $\mathcal{A}/\mathcal{J}$.

Inserting $X = \xi + V$ in (1.1) and passing to the quotient algebra $\mathcal{A}/\mathcal{J}$, we obtain

$$d(\xi + v)^2 - (a + \overline{a})(\xi + v) - c = 0,$$

and taking into account (6.2) we get

$$dv^2 + (2d\xi - a - \overline{a})v = 0. \tag{6.3}$$

If $d = 0$ and hence $a + \overline{a} \neq 0$, this implies $v = 0$, as desired. So assume $d > 0$. Then (6.3) reads

$$dv\left(v + 2\xi - \frac{a + \overline{a}}{d}\right) = 0.$$

Due to (6.1),

$$2\xi - \frac{a + \overline{a}}{d} = 2\sqrt{\left(\frac{a + \overline{a}}{2d}\right)^2 + \frac{c}{d}} =: 2\varrho > 0$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 348-362, March 2011

ELA

and thus, $v(v+2\varrho)=0$. We can decompose v as $v=v_+-v_-$ with $v_+\ge 0$, $v_-\ge 0$, $v_+v_-=v_-v_+=0$. It follows that $(v_+-v_-)(v_+-v_-+2\varrho)=0$, whence

$$v_+^2+v_-^2+2\varrho(v_+-v_-)=0. \tag{6.4}$$

Since $v_+^2\ge 0$, $v_-^2\ge 0$, $\varrho>0$, we conclude from (6.4) that $v_+-v_-\le 0$. Consequently, $0\le v_+\le v_-$. Multiplying this from the left and the right by v_+, we see that $0\le v_+^3\le v_+v_-v_+=0$, that is, $v_+^3=0$. We may think of v_+ as a positive operator and thus of multiplication by a function $\psi\ge 0$. The equality $v_+^3=0$ shows that $\psi^3=0$, which gives $\psi=0$ and thus $v_+=0$. Now (6.4) becomes $v_-(v_--2\varrho)=0$. We have $0\le X^\pi=\xi+v=\xi-v_-$ and therefore $v_-\le\xi$. Thus,

$$\begin{aligned} v_--2\varrho\le\xi-2\varrho &= \frac{a+\overline{a}}{2d}+\varrho-2\varrho \\ &= \frac{a+\overline{a}}{2d}-\sqrt{\left(\frac{a+\overline{a}}{2d}\right)^2+\frac{c}{d}}<0, \end{aligned}$$

which shows that $v_--2\varrho$ is invertible. As $v_-(v_--2\varrho)=0$, we arrive at the desired equality $v_-=0$. ∎

We remark that Theorem 6.2 is no longer true if instead of $C>0$ we require only that $C\ge 0$. To see this, consider $\mathcal{A}=\ell^\infty$ (bounded sequences) and $\mathcal{J}=c_0$ (sequences converging to zero) with termwise operations. Clearly, $\mathbf{C}I+\mathcal{J}$ is the algebra of all convergent sequences. Take $D=A=\{1,1,\ldots\}$ and $C=\{0,0,\ldots\}$. Then $X=\{0,2,0,2,\ldots\}$ is a positive solution of equation (1.1), but X is obviously not in $\mathbf{C}I+\mathcal{J}$. Note that the set of all solutions is the set of all sequences $\{X_n\}$ with $X_n\in\{0,2\}$. Consequently, all solutions are positive. A solution is in $\mathbf{C}I+\mathcal{J}$ if and only if X_n eventually stabilizes. Thus, there are countably many solutions in $\mathbf{C}I+\mathcal{J}$ but uncountably many solutions in $\mathcal{A}\setminus(\mathbf{C}I+\mathcal{J})$. The maximal solution $X_+=\{2,2,\ldots\}$ is of course in $\mathbf{C}I+\mathcal{J}$, as it should be by virtue of Theorem 2.2.

Corollary 6.3. *Let $\mathcal{A}$ be a unital W^*-algebra and $\mathcal{J}\subset\mathcal{A}$ be a proper closed two-sided ideal. Suppose D,A,C are in $\mathbf{C}I+\mathcal{J}$, $D\ge 0$, $C>0$, and (A,D) is stabilizable. The equation (1.1) has a maximal solution X_+ in $\mathbf{C}I+\mathcal{J}$ and $X_+\ge 0$.*

Proof. The existence of $X_+\in\mathcal{A}$ and the positivity of X_+ are ensured by Theorem 6.1, while Theorem 6.2 shows that actually $X_+\in\mathbf{C}I+\mathcal{J}$. ∎

Corollary 6.3 is in particular applicable to $\mathcal{A}=\mathcal{B}(\mathcal{H})$ and $\mathcal{J}=\mathcal{K}(\mathcal{H})$. The corollary is not applicable to $\mathcal{A}=\mathbf{B}$ (which is a W^*-algebra) and the set $\mathbf{J}$ introduced in Section 4, because $\mathbf{J}$ is not an ideal of $\mathbf{B}$. Let $\mathbf{B}_c\subset\mathbf{B}$ denote the set of all $\{F_n\}\in\mathbf{B}$ for which F_n, F_n^*, $W_nF_nW_n$, $W_nF_n^*W_n$ have strong limits. The set $\mathbf{J}$ is a closed two-sided ideal of $\mathbf{B}_c$; this was discovered by Silbermann [26], and a proof is also in [3, Lemma 2.21]. However, $\mathbf{B}_c$ is not a W^*-algebra, its strong closure being all of $\mathbf{B}$, so

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 348-362, March 2011

that Corollary 6.3 is again not applicable. But using Theorem 6.2 with $\mathcal{A} = \mathbf{B}_c$ and $\mathcal{J} = \mathbf{J}$, we arrive at the following conclusion: if, under the hypothesis of the theorem, $X = \{X_n\} = \{X_n^*\}$ is a positive solution of equation (1.1), then either $X \in \mathbf{C}I + \mathbf{J}$ or $X \notin \mathbf{B}_c$. The latter means that X_n or $W_nX_nW_n$ (or both) do not have a strong limit as $n \to \infty$.

Acknowledgement. I thank Ruth Curtain, Leiba Rodman, and Ilya Spitkovsky for useful discussions and valuable hints. I am also grateful to the referees for their helpful remarks and suggestions.

REFERENCES

[1] A. Böttcher. C^*-algebras in numerical analysis. *Irish Math. Soc. Bulletin*, 45:57–133, 2000.
[2] A. Böttcher and B. Silbermann. The finite section method for Toeplitz operators on the quarter-plane with piecewise continuous symbols. *Math. Nachr.*, 110:279–291, 1983.
[3] A. Böttcher and B. Silbermann. *Introduction to Large Truncated Toeplitz Matrices*. Springer-Verlag, New York, 1999.
[4] J.W. Bunce. Stabilizability of linear systems defined over C^*-algebras. *Math. Systems Theory*, 18:237–250, 1985.
[5] L.A. Coburn. The C^*-algebra generated by an isometry, Part I. *Bull. Amer. Math. Soc.*, 73:722–726, 1967.
[6] L.A. Coburn. The C^*-algebra generated by an isometry, Part II. *Trans. Amer. Math. Soc.*, 137:211–217, 1969.
[7] R.F. Curtain. Comments on "Optimal control of spatially distributed systems". *IEEE Trans. Automat. Control*, 54:1423–1424, 2009.
[8] R.F. Curtain. Algebraic properties of Riccati equations. Manuscript.
[9] R.F. Curtain, O.V. Iftime, and H.J. Zwart. System theoretic properties of a class of spatially distributed systems. *Automatica*, 45:1619–1627, 2009.
[10] R.F. Curtain, O.V. Iftime, and H.J. Zwart. A comparison between LQR control for a long string of SISO systems and LQR control of the infinite spatially invariant version. *Automatica*, 46:1604–1615, 2010.
[11] R.F. Curtain and L. Rodman. Comparison theorems for infinite-dimensional Riccati equations. *Systems Control Lett.*, 15:153–159, 1990.
[12] R.F. Curtain and L. Rodman. Analytic solutions of matrix Riccati equations with analytic coefficients. *SIAM J. Matrix Anal. Appl.*, 31:2075–2092, 2010.
[13] R.F. Curtain and A.J. Sasane. On Riccati equations in Banach algebras. *SIAM J. Control. Optim.*, 49:464-475, 2011.
[14] K.R. Davidson. *C^*-Algebras by Example*. Hindustan Book Agency, New Delhi, 1996.
[15] M. Dobovišek. The Riccati algebraic equation in a locally C^*-algebra. *Integral Equations Operator Theory*, 16:64–81, 1993.
[16] R.G. Douglas. *Banach Algebra Techniques in Operator Theory*, second edition. Springer-Verlag, New York, 1998.
[17] M. Fardad. The operator algebra of almost Toeplitz matrices and the optimal control of large-scale systems. *Proceedings of the 2009 American Control Conference*, 854–859, 2009.
[18] I. Gohberg, P. Lancaster, and L. Rodman. On Hermitian solutions of the symmetric algebraic Riccati equation. *SIAM J. Control Optim.*, 24:1323–1334, 1986.
[19] K. Gröchenig and M. Leinert. Symmetry and inverse-closedness of matrix algebras and functional calculus for infinite matrices. *Trans. Amer. Math. Soc.*, 358:2695–2711, 2006.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 22, pp. 348-362, March 2011

ELA

[20] R. Hagen, S. Roch, and B. Silbermann. *C^*-Algebras and Numerical Analysis*. Marcel Dekker, New York and Basel, 2001.

[21] F. Kappel and D. Salamon. An approximation theorem for the algebraic Riccati equation. *SIAM J. Control Optim.*, 28:1136–1147, 1990.

[22] P. Lancaster and L. Rodman. *Algebraic Riccati Equations*. Clarendon Press, Oxford, 1995.

[23] N. Motee, A. Jadbabaie, and B. Bamieh. On decentralized optimal control and information structures. *Proceedings of the 2008 American Control Conference*, 4985–4990, 2008.

[24] N. Motee and A. Jadbabaie. Optimal control of spatially distributed systems. *IEEE Trans. Automat. Control*, 53:1616–1629, 2008.

[25] J.C. Oostveen, R.F. Curtain, and K. Ito. An approximation theory for strongly stabilizing solutions to the operator LQ Riccati equation. *SIAM J. Control Optim.*, 38:1909–1937, 2000.

[26] B. Silbermann. Lokale Theorie des Reduktionsverfahrens für Toeplitzoperatoren. *Math. Nachr.*, 104:137–146, 1981.