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DEFECTIVE EIGENVALUES OF THE NON-BACKTRACKING MATRIX[*]

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Abstract. We consider graphs for which the non-backtracking matrix has defective eigenvalues or graphs for which the matrix does not have a full set of eigenvectors. The existence of these values results in Jordan blocks of size greater than one, which are called nontrivial. We develop a relationship between the eigenspaces of the non-backtracking matrix and the eigenspaces of a smaller matrix, completely classifying their differences among graphs with at most one cycle. Finally, we provide several constructions of infinite graph families that have nontrivial Jordan blocks for both this smaller matrix and the non-backtracking matrix.

Key words. Non-backtracking matrix, Jordan form, Non-backtracking walks on graphs.

AMS subject classifications. 05C50, 15A20, 05C81.

1. Introduction.

A non-backtracking walk in a graph is any traversal of the vertices of a graph such that no edge is immediately repeated. The non-backtracking matrix encodes if two edges can be traversed in succession in a non-backtracking walk. This matrix was originally introduced by Hashimoto in 1989 [8] and has been used to study percolation [3], community detection [2], and non-recurrent epidemic spread [17].

Of particular interest is the eigen-information of the non-backtracking matrix. The largest eigenvalue is related to the epidemic threshold of the SIR model [17], while the eigenvectors have been used to rank importance of nodes in networks [14]. Non-backtracking walks are also a better tool for community detection by spectral clustering in sparse networks. Developers of these spectral clustering algorithms commend the spectrum of the non-backtracking matrix for its ability to maintain a large gap between bulk eigenvalues and the eigenvalues related to community detection [12].

The non-backtracking matrix is not symmetric, making it one of the only well-studied graph matrices where there may not be a full set of linearly independent eigenvectors. When the algebraic and geometric multiplicity of an eigenvalue are not equal, the eigenvalue is called *defective*, and the matrix is not diagonalizable. Graphs that have a defective eigenvalue we will call *defective graphs*. We look to answer the question, *which graphs are defective for the non-backtracking matrix?* If a graph has a vertex of degree one, then the non-backtracking matrix of the graph will have a defective eigenvalue of $\lambda = 0$. As such, the question can be rephrased: *which graphs of minimum degree two are defective for the non-backtracking matrix?* It was conjectured by Torres in [19] that the answer is never.

We show that this conjecture is false by providing a constructive method to build three infinite graph families with a defective eigenvalue (see Section 4). In some cases, this defective eigenvalue is real. In addition, we provide some numerical data regarding the number of graphs on ten or fewer vertices with

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defective eigenvalues and the eigenvalues for which they are defective. Our constructions use a previously studied matrix, denoted K, which is defined in terms of the adjacency and degree diagonal matrices of a graph and that shares eigenvalues with the non-backtracking matrix.

We solidify the relationship between the Jordan canonical forms of K and the non-backtracking matrix by completely classifying when they are the same and describing what differences can occur (see Section 2). Further, we construct useful results regarding the structure of the generalized eigenvectors of the matrix K for special graph structures in Section 3. These results are also applied to determine conditions on the generalized eigenvectors of graphs containing twin vertices.

1.1. Preliminaries. A *(simple) graph* G consists of a set of vertices $V(G)$ and a set of edges $E(G)$ such that each edge $e \in E(G)$ is a subset of two vertices. Define $n = |V(G)|$ and $m = |E(G)|$. We will only consider simple graphs without multiedges or loops. For ease of notation, ij will be used to denote the edge $\{i, j\}$. We also use $i \sim j$ to denote that i and j share an edge. The set of all vertices j such that $ij \in E(G)$ is called the *neighborhood* of the vertex i. The *degree* of a vertex i, denoted $\deg(i)$, is the size of its neighborhood. For a graph G and a set $S \subseteq V(G)$, let $G \backslash S$ denote the graph G with the vertices in S (and all their associated edges) removed. A *walk* is a traversal of the edges of a graph and a *non-backtracking walk* is a walk in which the edges that are traversed cannot immediately be repeated. All graphs we consider are *connected*, or have a walk between any two vertices. A graph is *bipartite* if there exists a partition $V(G) = \mathcal{A} \cup \mathcal{B}$ such that every edge $ij \in E(G)$ $i \in \mathcal{A}$ and $j \in \mathcal{B}$. A graph is *unicyclic* if the graph contains only one cycle as a subgraph. For more information regarding graph theory, see [20].

For a vector $\mathbf{v}$, we will let $\mathbf{v}_i$ denote the ith entry of the vector. The all ones vector of dimension n will be denoted $\mathbf{1}_n$ (or just $\mathbf{1}$ when the dimension is clear) and the identity matrix of dimension n will be denoted I_n (or just I when the dimension is clear). The ith standard basis vector will be denoted $\mathbf{e}_i$.

For a matrix M, an eigenvalue λ is *defective* if its geometric multiplicity is strictly less than its algebraic multiplicity. A matrix has no defective eigenvalues if and only if it has n linearly independent eigenvectors. Equivalently, a matrix has no defective eigenvalues if and only if it is diagonalizable. For matrices that are not diagonalizable, it is useful to consider their Jordan Canonical Form instead.

The *Jordan Canonical Form* (JCF) of a matrix M has the form

$$\mathrm{JCF}(M) = S^{-1}MS = \begin{bmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & J_{p-1} & 0 \\ 0 & \cdots & 0 & 0 & J_p \end{bmatrix} \text{ where } J_i = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0\cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{bmatrix},$$

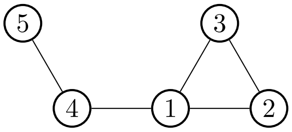
and S is an invertible matrix. Each eigenvalue λ of M corresponds to one or more *Jordan blocks* J_i for $1 \le i \le p$. Furthermore, the geometric multiplicity of λ is the number of Jordan blocks corresponding to λ, and the algebraic multiplicity of λ is the sum of the sizes of all Jordan blocks corresponding to λ. An eigenvalue λ has a *non-trivial* Jordan block, i.e., a block of size at least 2×2, if and only if it is defective. Note that this implies that $\mathrm{JCF}(M)$ is a diagonal matrix if and only if M does not have any defective eigenvalues.

Let $\mathbf{u}^{(1)}$ be an eigenvector for matrix M corresponding to eigenvalue λ. Further, let k be the largest integer such that the system $(M - \lambda I)\mathbf{u}^{(j+1)} = \mathbf{u}^{(j)}$ has a solution for all $j \in \{0, 1, \ldots k-1\}$, where we define $\mathbf{u}^{(0)}$ to be the zero vector. It follows that $k \geq 1$, since $\mathbf{u}^{(1)}$ is an eigenvector. This sequence of vectors $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(k)}$ is a *Jordan chain of length* k of M, and each vector $\mathbf{u}^{(j)}$ for $2 \leq j \leq k$ is a *generalized eigenvector of* M for eigenvalue λ. A full set of Jordan chains for the matrix M, when considered as columns of a matrix, serves as the similarity transformation matrix S for computing JCF(M). The lengths of the Jordan chains for a matrix M correspond to the sizes of its Jordan blocks. That is, M has a Jordan chain of length k for eigenvalue λ if and only if the Jordan canonical form of M contains a Jordan block of size k. Thus, the existence of a Jordan chain of length $k \geq 2$ corresponding to eigenvalue λ for M is sufficient to show that λ is a defective eigenvalue of M. For more information regarding the Jordan Canonical Form, see [9].

1.2. The non-backtracking matrix. To define the non-backtracking matrix of an undirected graph, it is useful to consider each edge ij in a graph as a pair of directed edges (i, j) and (j, i). This is due to the nature of how non-backtracking walks are defined: traveling from i to j along the edge ij will result in a different set of viable next edges for the walk than traveling from j to i along ij. As such, the *non-backtracking matrix* $B(G)$ is indexed by the directed edges of the graph G and defined such that

$$B_{(i,j),(k,\ell)}(G) = \begin{cases} 1 & \text{if } j = k \text{ and } i \neq \ell \\ 0 & \text{otherwise.} \end{cases}$$

In other words, if we can travel along ij and then immediately travel km in a non-backtracking walk, the corresponding matrix entry is one. Otherwise, the matrix entry is zero. See Fig. 1 for an example.



$$\begin{array}{c|cccccccccc|}
 & (1,2) & (1,3) & (1,4) & (2,3) & (4,5) & (2,1) & (3,1) & (4,1) & (3,2) & (5,4) \\
(1,2) & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1,3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
(1,4) & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
(2,3) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
(4,5) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(2,1) & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(3,1) & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(4,1) & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(3,2) & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
(5,4) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}$$

Figure 1. *A graph on five vertices and its non-backtracking matrix.*

The non-backtracking matrix has many eigenvalues and corresponding eigenspaces we understand well; these are $\lambda = \pm 1$ (see [8, 7, 16, 10]). Therefore, the spectral interest of this matrix tends to be on the remaining eigenvalues and corresponding eigenspaces. A useful tool for prioritizing these eigenvalues is Ihara's theorem.

Theorem 1.1 (Ihara's Theorem [11]). *For a graph G with n vertices and m edges, let B be the non-backtracking matrix of G. Let A be the adjacency matrix of G and D the degree diagonal matrix. Then*

$$\det(I - uB) = (1 - u^2)^{m-n} \det(u^2(D - I) - uA + I).$$

Note that this does not directly give us the eigenvalues of B, since the eigenvalues are the solutions to $\det(\lambda I - B)$. Therefore, for the solutions to $\det(I - uB)$, it follows that $u = 1/\lambda$. The matrix polynomial defined by Ihara's Theorem

$$H(t) = I - tA + t^2(D - I),$$

is studied as the deformed graph Laplacian, such as in [7] and [15], or the Bethe-Hessian matrix, such as in [18].

Because Ihara's theorem separates out the eigenvalues of ±1, the deformed graph Laplacian contains the eigenvalues of interest. To calculate these values, we can consider the following matrix K, a version of which is considered in Note 3.5 of [1]. For a graph G with n vertices and m edges, the $2n \times 2n$ matrix $K(G)$ is defined as

$$K = \begin{bmatrix} A & D - I_n \\ -I_n & 0 \end{bmatrix},$$

where A and D are as defined in Theorem 1.1. The matrix K is a linearization of the reversal of $H(t)$ (see [6]), and as such the all the eigenvalues of K (respecting algebraic multiplicity) are eigenvalues of B. This matrix K has seen some interest for those working with the non-backtracking matrix because it is a smaller matrix ($2n \times 2n$ versus $2m \times 2m$) with alike spectral properties.

2. Similarity of eigenspaces of B and K. In this section, we show that except for the cycle graph, the defective eigenspaces K and B are isomorphic to each other. This result will allow us to consider a smaller matrix when looking at defective eigenspaces of B.

Glover and Kempton in [5] showed that B could be decomposed into a block diagonal matrix M with K as one of its blocks. For this decomposition, they consider the following vertex-directed edge incidence matrices. Let S be a $2m \times n$ matrix and U be an $n \times 2m$ matrix where

$$S((u,v),x) = \begin{cases} 1 & v = x \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad U(x,(u,v)) = \begin{cases} 1 & x = u \\ 0 & \text{otherwise} \end{cases}.$$

For this decomposition, we also define R to be a $2m \times 2(m-n)$ matrix, where the columns of R are linearly independent eigenvectors of B for the eigenvalues ±1 (shown to be in the null space of SU by Lubetzky and Peres in [13]).

Theorem 2.1 ([5]). *Let G be a connected graph and B its non-backtracking matrix. Then*

$$BX = X \begin{bmatrix} K & 0 & 0 \\ 0 & I_{m-n} & 0 \\ 0 & 0 & -I_{m-n} \end{bmatrix},$$

and $X = \begin{bmatrix} S & U^T & R \end{bmatrix}$.

Let M be the $2m \times 2m$ matrix

$$M = \begin{bmatrix} K & 0 & 0 \\ 0 & I_{m-n} & 0 \\ 0 & 0 & -I_{m-n} \end{bmatrix}. \tag{2.1}$$

It is clear that M has the same algebraic spectrum as B. Glover and Kempton were able to find new proof techniques for known results about $H(t)$, rephrasing them in terms of K. Grindrod et al. in [7] showed that the geometric multiplicity of $\lambda = 1$ of $H(t)$ is the number of connected components of G and the inverse of the spectral radius of A lower bounds all the absolute values of the eigenvalues of $H(t)$.

Further, Glover and Kempton show in [5] that eigenvectors for K *lift* to eigenvectors of B, i.e., if $\mathbf{v}$ is an eigenvector of K, then $X[\mathbf{v},\mathbf{0},\mathbf{0}]^T$ is an eigenvector for B. We show that this result naturally extends to generalized eigenvectors and Jordan chains.

Proposition 2.2. *Let G be a connected graph with $n\geq 2$ with matrices K and B. Let $\mathbf{u}^{(1)},\dots,\mathbf{u}^{(k)}$ be a Jordan chain of K for the eigenvalue λ and let $\mathbf{v}^{(i)}=\begin{bmatrix}\mathbf{u}^{(i)} & \mathbf{0} & \mathbf{0}\end{bmatrix}^T$. If $X\mathbf{v}^{(1)}\neq\mathbf{0}$, then $X\mathbf{v}^{(1)},\dots,X\mathbf{v}^{(k)}$ is a Jordan chain of B for eigenvalue λ. If $X\mathbf{v}^{(1)}=\cdots=X\mathbf{v}^{(i-1)}=\mathbf{0}$ and $X\mathbf{v}^{(i)}\neq\mathbf{0}$ for some $1<i\leq k$, then $X\mathbf{v}^{(i)},\dots,X\mathbf{v}^{(k)}$ is a Jordan chain of B for eigenvalue λ.*

Proof. Let $\mathbf{u}^{(1)}$ is an eigenvector of K for λ, that is, $K\mathbf{u}^{(1)}=\lambda\mathbf{u}^{(1)}$. Then

$$BX\mathbf{v}^{(1)}=X\begin{bmatrix}K&0&0\\0&I&0\\0&0&-I\end{bmatrix}\begin{bmatrix}\mathbf{u}^{(1)}\\\mathbf{0}\\\mathbf{0}\end{bmatrix}=X\begin{bmatrix}K\mathbf{u}^{(1)}\\\mathbf{0}\\\mathbf{0}\end{bmatrix}=X\lambda\begin{bmatrix}\mathbf{u}^{(1)}\\\mathbf{0}\\\mathbf{0}\end{bmatrix}=\lambda X\mathbf{v}^{(1)}.$$

If $X\mathbf{v}^{(1)}\neq\mathbf{0}$, then $X\mathbf{v}^{(1)}$ is an eigenvector of B for eigenvalue λ.

Let $\mathbf{u}^{(1)},\dots,\mathbf{u}^{(k)}$ be a Jordan chain of K for λ. Therefore, $K\mathbf{u}^{(i)}=\lambda\mathbf{u}^{(i)}+\mathbf{u}^{(i-1)}$ for $1<i\leq k$. Consider

$$BX\mathbf{v}^{(i)}=X\begin{bmatrix}K&0&0\\0&I&0\\0&0&-I\end{bmatrix}\begin{bmatrix}\mathbf{u}^{(i)}\\\mathbf{0}\\\mathbf{0}\end{bmatrix}=X\begin{bmatrix}K\mathbf{u}^{(i)}\\\mathbf{0}\\\mathbf{0}\end{bmatrix}=X\begin{bmatrix}\lambda\mathbf{u}^{(i)}+\mathbf{u}^{(i-1)}\\\mathbf{0}\\\mathbf{0}\end{bmatrix}=\lambda X\mathbf{v}^{(i)}+X\mathbf{v}^{(i-1)}.$$

Further, $BX\mathbf{v}^{(i)}-\lambda X\mathbf{v}^{(i)}=X\mathbf{v}^{(i-1)}$. If $X\mathbf{v}^{(1)}\neq\mathbf{0}$, then $BX\mathbf{v}^{(2)}-\lambda X\mathbf{v}^{(2)}\neq\mathbf{0}$. So $X\mathbf{v}^{(2)}$ is not in the eigenspace of λ nor the zero vector. Therefore, $X\mathbf{v}^{(2)}$ is a generalized eigenvector of B for eigenvalue λ.

For the sake of induction, assume $X\mathbf{v}^{(i-1)}\neq\mathbf{0}$, so $BX\mathbf{v}^{(i)}-\lambda X\mathbf{v}^{(i)}\neq\mathbf{0}$. Therefore, $X\mathbf{v}^{(i)}\neq\mathbf{0}$, so $X\mathbf{v}^{(i)}$ is a generalized eigenvector of B for eigenvalue λ. Therefore, $X\mathbf{v}^{(1)},\dots,X\mathbf{v}^{(k)}$ is a Jordan chain of B for the eigenvalue λ.

If $X\mathbf{v}^{(1)}=\cdots=X\mathbf{v}^{(i-1)}=\mathbf{0}$ and $X\mathbf{v}^{(i)}\neq\mathbf{0}$ for some $1<i\leq k$, then $BX\mathbf{v}^{(i)}=\lambda X\mathbf{v}^{(i)}$. Thus, $X\mathbf{v}^{(i)}$ is an eigenvector of B for λ. By the same argument as in the previous case, $X\mathbf{v}^{(\ell)}\neq\mathbf{0}$ for all $i+1\leq\ell\leq k$. Therefore, $X\mathbf{v}^{(i)},\dots,X\mathbf{v}^{(k)}$ is a Jordan chain of B for the eigenvalue λ. ∎

We now investigate the null space of $\begin{bmatrix}S & U^T\end{bmatrix}$. This will allow us to connect the eigenspace properties of K and B.

Lemma 2.3. *Let G be a connected graph on $n\geq 2$ vertices. If G is not bipartite,* $\operatorname{null}\left(\begin{bmatrix}S & U^T\end{bmatrix}\right)=\operatorname{span}\left\{\begin{bmatrix}\mathbf{1}_n & -\mathbf{1}_n\end{bmatrix}^T\right\}$*. If G is a bipartite graph with partite sets $\mathcal{A}$ and $\mathcal{B}$, then*

$$\operatorname{null}\left(\begin{bmatrix}S & U^T\end{bmatrix}\right)=\operatorname{span}\left\{\begin{bmatrix}\mathbf{1}_{|\mathcal{A}|} & \mathbf{1}_{|\mathcal{B}|} & -\mathbf{1}_{|\mathcal{A}|} & -\mathbf{1}_{|\mathcal{B}|}\end{bmatrix}^T,\begin{bmatrix}\mathbf{1}_{|\mathcal{A}|} & -\mathbf{1}_{|\mathcal{B}|} & \mathbf{1}_{|\mathcal{A}|} & -\mathbf{1}_{|\mathcal{B}|}\end{bmatrix}^T\right\}.$$

Proof. Let G be a simple, connected, undirected graph on $n\geq 2$ vertices. We will use the linear algebra fact that $\begin{bmatrix}S & U^T\end{bmatrix}\mathbf{v}=\mathbf{0}$ if and only if $\begin{bmatrix}S^T\\U\end{bmatrix}\begin{bmatrix}S & U^T\end{bmatrix}\mathbf{v}=\mathbf{0}$.

In [4], van Dooren and Fraikin show that $SU=A$, the adjacency matrix, and by matrix multiplication $S^TS=UU^T=D$, the degree diagonal matrix. For completeness, the results $SU=A$ and $S^TS=UU^T=D$ where also shown in Zager and Verghese [21]. Therefore,

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$$\begin{bmatrix} S^T \\ T \end{bmatrix} \begin{bmatrix} S & U^T \end{bmatrix} = \begin{bmatrix} S^TS & S^TU^T \\ SU & UU^T \end{bmatrix} = \begin{bmatrix} D & A \\ A & D \end{bmatrix}.$$

We will consider the null space of $\begin{bmatrix} D & A \\ A & D \end{bmatrix}$. Let $\mathbf{v} = [\mathbf{x}, \mathbf{y}]^T$ be in the null space of $\begin{bmatrix} D & A \\ A & D \end{bmatrix}$. So $D\mathbf{x} + A\mathbf{y} = \mathbf{0}$ and $A\mathbf{x} + D\mathbf{y} = \mathbf{0}$. Since G is a connected graph on at least two vertices, it follows D is invertible. So

$$\mathbf{x} = -D^{-1}A\mathbf{y} \quad \text{and} \quad \mathbf{y} = -D^{-1}A\mathbf{x}.$$

Therefore,

$$\mathbf{x} = (D^{-1}A)^2\mathbf{x},$$

that is $\mathbf{x}$ is an eigenvector for the probability transition matrix (a stochastic matrix) of G for the eigenvalue $\lambda = \pm 1$. For a simple, connected graph, we know the probability transition matrix always has an eigenvalue $\lambda = 1$ with multiplicity one and an eigenvalue $\lambda = -1$ with multiplicity one if and only if the graph is bipartite.

Further, when $\lambda = 1$, then $\mathbf{x} = \mathbf{1}$. Therefore, $\mathbf{y} = -\mathbf{1}$. When $\lambda = -1$, then $\mathbf{x} = [\mathbf{1}_{|\mathcal{A}|}, -\mathbf{1}_{|\mathcal{B}|}]^T$. Therefore, $\mathbf{y} = [\mathbf{1}_{|\mathcal{A}|}, -\mathbf{1}_{|\mathcal{B}|}]^T$ as desired. ∎

We will now connect the null space of X to the eigenspace of K. This will allow us to specify when the generalized eigenspace of K does not connect to the generalized eigenspace of B.

Lemma 2.4. *Let G be a connected graph with $n \geq 2$. Let $\mathbf{u}$ be an eigenvector of K for eigenvalue λ and let $\mathbf{v} = \begin{bmatrix} \mathbf{u} & \mathbf{0} & \mathbf{0} \end{bmatrix}^T$. If $X\mathbf{v} = \mathbf{0}$, then $\lambda = 1$ if G is not bipartite and $\lambda = \pm 1$ if G is bipartite.*

Proof. For an eigenvector $\mathbf{u}$ of K, let $\mathbf{v} = \begin{bmatrix} \mathbf{u} & \mathbf{0} & \mathbf{0} \end{bmatrix}^T$. If $X\mathbf{v} = \mathbf{0}$, then $\mathbf{u} \in \operatorname{null}\left(\begin{bmatrix} S & U^T \end{bmatrix}\right)$.

If G is not bipartite, then $\operatorname{null}\left(\begin{bmatrix} S & U^T \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} \mathbf{1} & -\mathbf{1} \end{bmatrix}^T\right\}$ by Lemma 2.3. By computation, $\begin{bmatrix} \mathbf{1} & -\mathbf{1} \end{bmatrix}^T$ is an eigenvector for K for the eigenvalue $\lambda = 1$.

If G is bipartite with parts $\mathcal{A}$ and $\mathcal{B}$, then

$$\operatorname{null}\left(\begin{bmatrix} S & U^T \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} \mathbf{1}_{|\mathcal{A}|} & \mathbf{1}_{|\mathcal{B}|} & -\mathbf{1}_{|\mathcal{A}|} & -\mathbf{1}_{|\mathcal{B}|} \end{bmatrix}^T, \begin{bmatrix} \mathbf{1}_{|\mathcal{A}|} & -\mathbf{1}_{|\mathcal{B}|} & \mathbf{1}_{|\mathcal{A}|} & -\mathbf{1}_{|\mathcal{B}|} \end{bmatrix}^T\right\},$$

by Lemma 2.3. By computation, $\begin{bmatrix} \mathbf{1}_{|\mathcal{A}|} & \mathbf{1}_{|\mathcal{B}|} & -\mathbf{1}_{|\mathcal{A}|} & -\mathbf{1}_{|\mathcal{B}|} \end{bmatrix}^T$ is an eigenvector for K for the eigenvalue $\lambda = 1$ and $\begin{bmatrix} \mathbf{1}_{|\mathcal{A}|} & -\mathbf{1}_{|\mathcal{B}|} & \mathbf{1}_{|\mathcal{A}|} & -\mathbf{1}_{|\mathcal{B}|} \end{bmatrix}^T$ is an eigenvector for K for the eigenvalue $\lambda = -1$. Since eigenspaces are disjoint, there is no eigenvector of K that is a nontrivial linear combination of these two vectors. ∎

The next two propositions are results about the algebraic and geometric multiplicity of the eigenvalues ±1 for the non-backtracking matrix.

Proposition 2.5. *[19] Let G have at least two cycles. Then $\lambda = 1$ is an eigenvalue for B with algebraic and geometric multiplicity $m - n + 1$.*

Proposition 2.6. *[19] Let G have at least two cycles. Then $\lambda = -1$ is an eigenvalue for B, with algebraic and geometric multiplicity $m - n + 1$ if G is bipartite and algebraic and geometric multiplicity $m - n$ if G is not bipartite.*

The deformed graph Laplacian, $H(t)$, is known to have a defective eigenvalue of ∞ if a graph has a leaf and no component is the complete graph on two vertices [7]. This corresponds to the eigenvalues $\lambda = 0$ for matrix K. Further, the defectiveness of the eigenvalue $\lambda = 0$ for the non-backtracking matrix has been explicitly found by Torres (see [19]) for graphs with minimum degree one. Such graphs tend to be removed from the discussion about graphs with defective eigenvalues for the non-backtracking matrix because the defectiveness of $\lambda = 0$ is already well understood. We include some of these graphs here for completion, but still require at least one cycle, since M from (2.1) is only well defined when $m \geq n$. We are now able to prove our main result.

Theorem 2.7. *Let G be a connected graph. If G has at least two cycles, then B and M have the same Jordan form.*

Proof. Let G be a graph. Let $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(k)}$ be a Jordan chain for K for the eigenvalue λ and let $\mathbf{v}^{(i)} = \begin{bmatrix}\mathbf{u}^{(i)} & \mathbf{0} & \mathbf{0}\end{bmatrix}^T$. Then $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(k)}$ is a Jordan chain for M for the eigenvalue λ. We note that since the other block of M is a diagonal matrix (and thus $\mathbf{e}_{2n+1}, \ldots, \mathbf{e}_{2m}$, which are linearly independent from $\mathbf{v}^{(i)}$, are are eigenvectors of M), this is the only way to have a nontrivial Jordan chain for M. Also, since M and B have the same spectrum by construction, we only need to show that the Jordan blocks are the same size.

If $X\mathbf{v}^{(1)} \neq \mathbf{0}$, then by Proposition 2.2, $X\mathbf{v}^{(1)}, \ldots, X\mathbf{v}^{(k)}$ is a Jordan chain of B for λ. If $X\mathbf{v}^{(1)} = \mathbf{0}$, then by Lemma 2.4, $\lambda = 1$ if G is not bipartite or $\lambda = \pm 1$ if G is bipartite.

First, let G be bipartite. By Propositions 2.5 and 2.6, we know the algebraic and geometric multiplicity of $\lambda = \pm 1$ for B is $m - n + 1$. We claim that M also has a full set of eigenvectors for the eigenvalues ± 1. If $\lambda = 1$, then eigenvectors of M are $\mathbf{v}^{(1)}, \mathbf{e}_{2n+1}, \ldots, \mathbf{e}_{n+m-1}$ which form a set of $m-n+1$ linearly independent vectors. If $\lambda = -1$, then eigenvectors of M are $\mathbf{v}^{(1)}, \mathbf{e}_{n+m}, \ldots, \mathbf{e}_{2m}$ which form a set of $m - n + 1$ linearly independent vectors.

Now, let G not be bipartite. By Propositions 2.5 and 2.6, we know the algebraic and geometric multiplicity of $\lambda = 1$ for B is $m - n + 1$ and the algebraic and geometric multiplicity of $\lambda = -1$ for B is $m - n$. We claim that M also has a full set of eigenvectors for the eigenvalues ± 1. If $\lambda = 1$, then eigenvectors of M are $\mathbf{v}^{(1)}, \mathbf{e}_{2n+1}, \ldots, \mathbf{e}_{n+m-1}$ which form a set of $m - n + 1$ linearly independent vectors. If $\lambda = -1$, then eigenvectors of M are $\mathbf{e}_{n+m}, \ldots, \mathbf{e}_{2m}$ which form a set of $m - n$ linearly independent vectors.

Therefore, the length of every Jordan chain is the same for M and B for every eigenvalue of λ meaning they have the same dimension of the generalized eigenspaces for every eigenvalue of λ and their Jordan forms are the same. □

The remaining class of graphs for us to connect the eigenspace between M and B for is unicyclic graphs. Grindrod, Higham, and Noferini in [7] completely classified the algebraic and geometric multiplicity for all eigenvalues of the unicyclic graph for $H(t)$. Since K is similar to the companion matrix of the reversal of $H(t)$, the eigenvalues including multiplicity are the same. Moreover, there exists a bijection between the eigenspaces of K and those of the eigenspaces of $H(t)$.

Lemma 2.8. *Let λ be an eigenvalue of K. The vector $[-\lambda\mathbf{x}, \mathbf{x}]$ is an eigenvector for K if and only if $\mathbf{x}$ is an eigenvector of $H(t)$ for eigenvalue $\mu = \frac{1}{\lambda}$ when $\lambda \neq 0$ and $\mu = \infty$ when $\lambda = 0$.*

Proof. This can be verified algebraically. □

This lemma allows us to utilize results about the deformed graph Laplacian.

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Lemma 2.9. *[7] Let $H(t)$ be the deformed graph Laplacian of a connected unicyclic graph with cycle length ν. Then the finite eigenvalues of $H(t)$ are the ν^{th} roots of unity. Moreover, the algebraic multiplicity is always 2 where its geometric multiplicity is 2 if $\lambda \neq \pm 1$ or [the geometric multiplicity] is 1 otherwise.*

For the non-backtracking matrix, Torres gave the following result about cycle graphs.

Proposition 2.10. *[19] Let G be a cycle graph on n vertices. Then eigenvalues [of the non-backtracking matrix of G] are the n^{th} roots of unity, and each has algebraic and geometric multiplicity two.*

Combining this with known results about vertices contained in no cycles, the spectrum and eigenspaces of unicycle graphs are completely understood for the non-backtracking matrix. Using these ideas, along with Theorem 2.7, we are able to classify exactly when the non-backtracking matrix and M have different Jordan forms.

Theorem 2.11. *Let G be a connected graph with at least one cycle. The Jordan forms of B and M differ if and only if G is unicyclic and they only differ for the eigenvalue $\lambda = 1$ when the cycle is odd and for the eigenvalues $\lambda = \pm 1$ when the cycle is even.*

Proof. In the proof of Theorem 2.7, we only used the fact that G has at least two cycles to determine M and B had the same Jordan form for $\lambda = \pm 1$. Therefore, these two eigenvalues in unicyclic graphs are the only places where the Jordan canonical form of B and M have the potential to differ. This result then follows immediately from Lemma 2.9 and Proposition 2.10. ∎

With this result, we are able to look at a smaller matrix when investigating defective eigenvalues for the non-backtracking matrix. Further, if restricting to at least two cycles, K and B are similar matrices making them spectrally interchangeable. For the class of unicyclic graphs, we completely understand the eigenspace differences between these two matrices. The only class of graphs not discussed in our results is trees, which are spectrally categorized by Torres in [19]. We see this result as a major tool in future spectral study of the non-backtracking matrix.

3. Eigenvectors structure for K.

In this section, we establish useful structural results about the eigenvectors and generalized eigenvectors for K. We see these results as being useful tools when showing that a generalized eigenvector does or does not exist.

3.1. Eigenvector and generalized eigenvectors.

The structure of K dictates a structure for its eigenvectors, which we state in the following proposition. The relationship between the top half and the bottom half of the vectors, $\mathbf{v}_{n+i} = \frac{-\mathbf{v}_i}{\lambda}$, has been observed previously, such as in [5]. A similar result has also been observed for the deformed graph Laplacian, such as in [15]. The following result can also be implied from the fact that K is a linearization of the reversal of $H(t)$. However, we include an alternative proof here for completion.

Proposition 3.1. *Let $\mathbf{v}$ be a vector in $\mathbb{C}^{2n}$ and let $\lambda \neq 0$. The vector $\mathbf{v}$ is an eigenvector of K for the eigenvalue λ if and only if*

$$\mathbf{v}_{n+i} = \frac{-\mathbf{v}_i}{\lambda} \quad \text{and} \quad \sum_{j\sim i} \mathbf{v}_j = \mathbf{v}_i \frac{1}{\lambda}\left(\deg(i) - 1 + \lambda^2\right),$$

for all $1 \leq i \leq n$.

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Proof. First assume $\mathbf{v}$ is an eigenvector of K for the eigenvalue λ. Then for $1 \leq i \leq n$,

$$\begin{aligned}(K\mathbf{v})_i &= (A\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}^T)_i + \Big((D-I)\begin{bmatrix}\mathbf{v}_{n+1} & \cdots & \mathbf{v}_{2n}\end{bmatrix}^T\Big)_i \\ \lambda\mathbf{v}_i &= \sum_{j\sim i}\mathbf{v}_j + \mathbf{v}_{n+i}(\deg(i)-1).\end{aligned}$$

Since $\mathbf{v}_{n+i} = \frac{-\mathbf{v}_i}{\lambda}$ (see [5]), we obtain the desired equation

$$\sum_{j\sim i}\mathbf{v}_j = \mathbf{v}_i\frac{1}{\lambda}\left(\deg(i)-1+\lambda^2\right).$$

The other direction follows immediately from reversing the algebra. ∎

We now extend the above result to a generalized eigenvector of K.

Proposition 3.2. *Let $\mathbf{v}$ be an eigenvector of K for the eigenvalue $\lambda \neq 0$ and let $\mathbf{u}$ be a vector in $\mathbb{C}^{2n}$. The vector $\mathbf{u}$ is a generalized eigenvector of K such that $K\mathbf{u} = \lambda\mathbf{u} + \mathbf{v}$ if and only if*

$$\mathbf{u}_{n+i} = \frac{-\mathbf{u}_i}{\lambda} + \frac{\mathbf{v}_i}{\lambda^2} \quad \textit{and} \quad \sum_{j\sim i}\mathbf{u}_j = \mathbf{u}_i\frac{1}{\lambda}(\deg(i)-1+\lambda^2) - \mathbf{v}_i\frac{1}{\lambda^2}(\deg(i)-1-\lambda^2),$$

for all $1 \leq i \leq n$.

Proof. First, assume $\mathbf{u}$ be a generalized eigenvector of K such that $K\mathbf{u} = \lambda\mathbf{u} + \mathbf{v}$. Then for $1 \leq i \leq n$, we have the following equations:

$$(K\mathbf{u})_i = \sum_{j\sim i}\mathbf{u}_j + \mathbf{u}_{n+i}(\deg(i)-1) = \lambda\mathbf{u}_i + \mathbf{v}_i, \tag{3.2}$$

$$(K\mathbf{u})_{i+n} = -\mathbf{u}_i = \lambda\mathbf{u}_{n+i} + \mathbf{v}_{n+i}. \tag{3.3}$$

Recalling that $\mathbf{v}_{n+i} = \frac{-\mathbf{v}_i}{\lambda}$, plugging this into Equation (3.3), and rearranging yields

$$\mathbf{u}_{n+i} = \frac{-\mathbf{u}_i}{\lambda} + \frac{\mathbf{v}_i}{\lambda^2}.$$

Plugging this into Equation (3.2) and rearranging gives

$$\sum_{j\sim i}\mathbf{u}_j = \mathbf{u}_i\frac{1}{\lambda}(\deg(i)-1+\lambda^2) - \mathbf{v}_i\frac{1}{\lambda^2}(\deg(i)-1-\lambda^2),$$

as desired. The other direction follows immediately from reversing the algebra. ∎

These vectors can be the start of a Jordan Chain of K of any length.

Corollary 3.3. *Let $\mathbf{v}$ and $\mathbf{u}$ be vectors in $\mathbb{C}^{2n}$ and let $\lambda \neq 0$. The vectors $\mathbf{v}$ and $\mathbf{u}$ satisfy the claimed equations from Propositions 3.1 and 3.2 if and only if K has a Jordan chain $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \ldots, \mathbf{u}^{(k)}$ for λ with length $k \geq 2$ such that $\mathbf{v} = \mathbf{u}^{(1)}$ and $\mathbf{u} = \mathbf{u}^{(2)}$.*

We have shown the necessary and sufficient conditions two vectors must have in order to be the first two vectors in a Jordan chain for K. In Section 4, we will apply these results to prove that the provided vectors form the start of a Jordan chain for K. But first, we apply them to consider the special case of graphs with twins.

3.2. Twins. Now we will investigate the structure of the eigenvectors and generalized eigenvectors when we have a special graph structure. Two vertices x and y in a graph G are called *twins* if their neighborhoods in $G\backslash\{x,y\}$ are the same, i.e. for all $v \in V(G\backslash\{x,y\})$, $vx \in E(G\backslash\{x,y\})$ if and only if $vy \in E(G\backslash\{x,y\})$. If x and y are adjacent in G, they are called *adjacent* twins; otherwise, they are called *non-adjacent* twins. Twins in graphs relate nicely to matrices because the two columns corresponding to the twins have identical entries (except for the x, y principal submatrix).

We will start by showing that if a graph has twins, we have a known eigenvector entries and a known eigenvalue.

Proposition 3.4. *Let G be a graph on n vertices containing a pair of twin vertices x and y such that $\deg(x) = \deg(y) = d$, and let $\mathbf{v}$ be a vector in $\mathbb{C}^{2n}$. Then the vector $\mathbf{v}$ with entries $\mathbf{v}_x = 1$, $\mathbf{v}_{n+x} = \frac{-1}{\lambda}$, $\mathbf{v}_y = -1$, $\mathbf{v}_{n+y} = \frac{1}{\lambda}$, and $\mathbf{v}_k = \mathbf{v}_{n+k} = 0$ for all $1 \leq k \leq n$ such that $k \notin \{x,y\}$ is an eigenvector of K. The associated eigenvalue is $\lambda = \pm\sqrt{1-d}$ if x, y are nonadjacent and $\lambda = \frac{-1\pm\sqrt{5-4d}}{2}$ if x, y are adjacent. Furthermore, if $\mathbf{v}$ is an eigenvector of K for an eigenvalue $\lambda' \notin \{\lambda, 0\}$, then $\mathbf{v}_x = \mathbf{v}_y$.*

Proof. First, let $\mathbf{v}_x = 1$, $\mathbf{v}_{n+x} = \frac{-1}{\lambda}$, $\mathbf{v}_y = -1$, $\mathbf{v}_{n+y} = \frac{1}{\lambda}$, and $\mathbf{v}_k = \mathbf{v}_{n+k} = 0$ for all $1 \leq k \leq n$ such that $k \notin \{x,y\}$. We will show that such a vector $\mathbf{v}$ is an eigenvector for K for the eigenvalue λ by showing the equations in Proposition 3.1 hold. It is obvious that $\mathbf{v}_{n+i} = \frac{-\mathbf{v}_i}{\lambda}$ for all $1 \leq i \leq n$. Since $\mathbf{v}_x$ and $\mathbf{v}_y$ correspond to twins, for each vertex k in their neighborhood in $G\backslash\{x,y\}$,

$$\sum_{j\sim k} \mathbf{v}_j = 1 - 1 = 0 = \mathbf{v}_k.$$

For vertices not in their neighborhood in $G\backslash\{x,y\}$, there are no nonzero terms in the equations.

For the case where x, y are nonadjacent twins, consider the equations for $\mathbf{v}_x$ and $\mathbf{v}_y$,

$$\sum_{j\sim x} \mathbf{v}_j = 0 = 1\frac{1}{\lambda}\left(d - 1 + \lambda^2\right) \text{ and } \sum_{k\sim y} \mathbf{v}_j = 0 = -1\frac{1}{\lambda}\left(d - 1 + \lambda^2\right).$$

Both of these equations hold when $\lambda = \pm\sqrt{1-d}$, so $\mathbf{v}$ is an eigenvector of K by Proposition 3.1.

For the case x, y are adjacent twins, consider the equations for $\mathbf{v}_x$ and $\mathbf{v}_y$,

$$\sum_{j\sim x} \mathbf{v}_j = -1 = 1\frac{1}{\lambda}\left(d - 1 + \lambda^2\right) \text{ and } \sum_{k\sim y} \mathbf{v}_k = 1 = -1\frac{1}{\lambda}\left(d - 1 + \lambda^2\right).$$

Both of these equations hold when $\lambda = -\frac{1}{2} \pm \frac{\sqrt{5-4d}}{2}$, so $\mathbf{v}$ is an eigenvector of K by Proposition 3.1.

Next, let $\mathbf{v}$ be an eigenvector of K for the eigenvalue $\lambda' \neq \lambda$. Applying Proposition 3.1, the equations corresponding to the vertices x and y are

$$\sum_{j\sim x} \mathbf{v}_j = \mathbf{v}_x \frac{1}{\lambda'}\left(d - 1 + (\lambda')^2\right), \tag{3.4}$$

and

$$\sum_{k\sim y} \mathbf{v}_k = \mathbf{v}_y \frac{1}{\lambda'}\left(d - 1 + (\lambda')^2\right). \tag{3.5}$$

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If the vertices are nonadjacent twins, $\sum_{j\sim x}\mathbf{v}_j=\sum_{k\sim y}\mathbf{v}_k$ and if the vertices are adjacent twins $\sum_{j\sim x}\mathbf{v}_j-\sum_{k\sim y}\mathbf{v}_k=\mathbf{v}_y-\mathbf{v}_x$. Subtracting Equations (3.4) and (3.5), in the nonadjacent twins case, we get

$$\mathbf{v}_x\frac{1}{\lambda'}\left(d-1+(\lambda')^2\right)=\mathbf{v}_y\frac{1}{\lambda'}\left(d-1+(\lambda')^2\right). \tag{3.6}$$

Since $\lambda'\neq\pm\sqrt{1-d}$, we have $d-1+(\lambda')^2\neq 0$. Therefore, $\mathbf{v}_x=\mathbf{v}_y$.

The adjacent twins case follows similarly. ∎

In the case that K has a defective eigenvalue λ with eigenvector $\mathbf{v}$ and generalized eigenvector $\mathbf{u}$, we can further determine the effect of the nonadjacent twin structure on the entries of $\mathbf{v}$ and $\mathbf{u}$.

Proposition 3.5. *Let G be a graph containing a pair of (adjacent) twin vertices x and y such that $\deg(x)=\deg(y)=d$ and let K have a Jordan chain $\mathbf{u}^{(1)},\mathbf{u}^{(2)},\ldots,\mathbf{u}^{(k)}$ for $\lambda\neq 0$ with length $k\geq 2$. Let $\mathbf{v}=\mathbf{u}^{(1)}$ and $\mathbf{u}=\mathbf{u}^{(2)}$. If $\lambda=\pm\sqrt{1-d}$ and x,y are nonadjacent, then $\mathbf{v}_x=\mathbf{v}_y$ and if $\lambda'\notin\{\lambda,0\}$, then $\mathbf{u}_x=\mathbf{u}_y$. If $\lambda=-\frac{1}{2}\pm\frac{\sqrt{5-4d}}{2}$ and x,y are adjacent, then $\mathbf{v}_x=\mathbf{v}_y$ and if $\lambda'\notin\{\lambda,0\}$, then $\mathbf{u}_x=\mathbf{u}_y$.*

Proof. Let $\mathbf{v},\mathbf{u}$ be the first two vectors in a Jordan chain of length at least two for the matrix K for eigenvalue $\lambda\neq 0$. Applying Proposition 3.2, the equations corresponding to the vertices x and y are

$$\sum_{j\sim x}\mathbf{u}_j=\mathbf{u}_x\frac{1}{\lambda}(d-1+\lambda^2)-\mathbf{v}_x\frac{1}{\lambda^2}(d-1-\lambda^2)\text{ and }\sum_{k\sim y}\mathbf{u}_k=\mathbf{u}_y\frac{1}{\lambda}(d-1+\lambda^2)-\mathbf{v}_y\frac{1}{\lambda^2}(d-1-\lambda^2).$$

First, let x,y be nonadjacent twins. Therefore,

$$\sum_{j\sim x}\mathbf{u}_j=\sum_{k\sim y}\mathbf{u}_k,$$

and subtracting the two equations we get

$$(\mathbf{u}_x-\mathbf{u}_y)\frac{(d-1+\lambda^2)}{\lambda}=(\mathbf{v}_x-\mathbf{v}_y)\frac{(d-1-\lambda^2)}{\lambda^2}. \tag{3.7}$$

If $\lambda=\pm\sqrt{1-d}$, then $\frac{(d-1+\lambda^2)}{\lambda}=0$. Note that in this case, $\frac{(d-1-\lambda^2)}{\lambda^2}=\frac{d-1-1+d}{1-d}=-2\neq 0$, so it must be that $\mathbf{v}_x=\mathbf{v}_y$. If $\lambda\neq\pm\sqrt{1-d}$, $\frac{(d-1+\lambda^2)}{\lambda}\neq 0$ and by Proposition 3.4, $\mathbf{v}_x=\mathbf{v}_y$. Therefore, in this case, it must be that $\mathbf{u}_x=\mathbf{u}_y$.

Finally, let x,y be adjacent twins. Therefore,

$$\sum_{j\sim x}\mathbf{u}_j-\sum_{k\sim y}\mathbf{u}_k=\mathbf{u}_y-\mathbf{u}_x,$$

and subtracting the two equations we get

$$(\mathbf{u}_x-\mathbf{u}_y)\frac{(d-1+\lambda^2+\lambda)}{\lambda}=(\mathbf{v}_y-\mathbf{v}_x)\frac{(d-1-\lambda^2)}{\lambda^2}.$$

If $\lambda=-\frac{1}{2}\pm\frac{\sqrt{5-4d}}{2}$, then $\frac{(d-1+\lambda^2+\lambda)}{\lambda}=0$. Note that in this case,

$$\frac{(d-1-\lambda^2)}{\lambda^2}=\frac{1}{\lambda}\left(\frac{(d-1+\lambda^2+\lambda)}{\lambda}\right)-2-\frac{1}{\lambda}=-2-\frac{1}{\lambda}\neq 0,$$

so it must be that $\mathbf{v}_x=\mathbf{v}_y$. If $\lambda\neq-\frac{1}{2}\pm\frac{\sqrt{5-4d}}{2}$, then $\frac{(d-1+\lambda^2+\lambda)}{\lambda}\neq 0$ and by Proposition 3.4, $\mathbf{v}_x=\mathbf{v}_y$. Therefore, in this case, it must be that $\mathbf{u}_x=\mathbf{u}_y$. ∎

These results help us better understand the composition of the eigenvectors and generalized eigenvectors of K when the graph has twins. More generally, they begin a larger exploration of identifying features in the graph and predicting their effect on the eigenvectors and generalized eigenvectors of the graph.

4. Defective graph families. In this section, we provide a constructive technique for producing families of graphs with defective eigenvalues and give some examples of applying the construction. We also provide computational results regarding the number of such graphs on ten or fewer vertices.

To construct infinite families with nontrivial Jordan blocks, we will define the idea of *graph gluing*. Consider the graphs G and H with fixed labelings such that $x \in V(G)$ and $y \in V(H)$. We can construct a new graph $G_x{\circ}_y H$ by identifying x and y together. Specifically, the vertex and edge sets are

$$V(G_x{\circ}_y H) = V(G) \cup V(H) \setminus \{y\}$$

$$E(G_x{\circ}_y H) = E(G) \cup E(H) \cup \{vx : vy \in E(H)\} \setminus \{vy \in E(H)\}.$$

If we would like to glue two graphs together at several vertices, we can define two lists of vertices $X = [x_1, x_2, \dots x_k]$ and $Y = [y_1, y_2, \dots y_k]$ where $x_i \in V(G)$ and $y_i \in V(H)$ for all i. Then $G_X{\circ}_Y H$ is the graph formed by gluing x_i to y_i for all i, ignoring any multiedges that might arise. See an example of this gluing in Fig. 2.

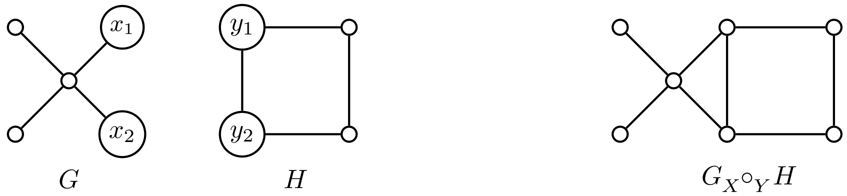

FIGURE 2. *Two graphs G and H with indicated vertex lists $X = [x_1, x_2]$ and $Y = [y_1, y_2]$, respectively. Their glued result is on the right.*

We now show that gluing any graph to a defective graph, under certain conditions, will result in a defective graph.

THEOREM 4.1. *Let G be a graph on n vertices with $x \in V(G)$ and let H be a graph with $y \in V(H)$. If $\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(k)}$ is a Jordan chain of $K(G)$ for the eigenvalue $\lambda \neq 0$ such that $\mathbf{u}_x^{(i)} = 0$ for all $1 \leq i \leq k$, then $G_x{\circ}_y H$ has a Jordan chain of length k for the eigenvalue λ for the matrix $K(G_x{\circ}_y H)$.*

Proof. Let G be a graph with $x \in V(G)$ and $\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(k)}$ be a Jordan chain of $K(G)$ for the eigenvalue $\lambda \neq 0$ with $\mathbf{u}_x^{(i)} = 0$ for all $1 \leq i \leq k$. Since $\mathbf{u}^{(1)}$ is an eigenvector and $\lambda \neq 0$, $\mathbf{u}_{n+x}^{(1)} = 0$ by Proposition 3.1. Furthermore, since $-\mathbf{u}_x^{(i+1)} = \lambda \mathbf{u}_{n+x}^{(i+1)} + \mathbf{u}_{n+x}^{(i)}$ for all $1 \leq i < k$, it is clear by induction that $\mathbf{u}_{n+x}^{(i)} = 0$ for all $1 \leq i \leq k$.

Consider $G_x{\circ}_y H$ for a graph H with $y \in V(H)$ and let $n' = \left|V(G_x{\circ}_y H)\right|$. We construct $\mathbf{v}^{(i)}$ for $G_x{\circ}_y H$ such that $\mathbf{v}_j^{(i)} = \mathbf{u}_j^{(i)}$ and $\mathbf{v}_{n'+j}^{(i)} = \mathbf{u}_{n+j}^{(i)}$ for $j \in V(G)$ and $\mathbf{v}_j^{(i)} = \mathbf{v}_{n'+j}^{(i)} = 0$ otherwise. That is, the new vectors will match the old vectors on G and be zero on H.

For $G_x{\circ}_y H$, consider the jth entry of $K(G_x{\circ}_y H)\mathbf{v}^{(i)}$ which is $(\deg(j) - 1)\mathbf{v}_{n'+j}^{(i)} + \sum_{k \sim j} \mathbf{v}_k^{(i)}$, and the $(n' + j)$th entry is $-\mathbf{v}_j^{(i)}$.

To prove that $\{\mathbf{v}^{(i)}\}$ is a Jordan chain, we consider the vertices in two groups: vertices of $G\backslash\{x\}$ and vertices of $H\cup\{x\}$. First, we will first show that for $j\in V(G)$ and $j\neq x$,

$$(\deg(j)-1)\mathbf{v}^{(i)}_{n'+j}+\sum_{k\sim j}\mathbf{v}^{(i)}_k=\lambda\mathbf{v}^{(i)}_j+\mathbf{v}^{(i-1)}_j, \tag{4.8}$$

and

$$-\mathbf{v}^{(i)}_j=\lambda\mathbf{v}^{(i)}_{n'+j}+\mathbf{v}^{(i-1)}_{n'+j}. \tag{4.9}$$

Then, we will also show for $j\in V(H)$ or $j=x$,

$$(\deg(j)-1)\mathbf{v}^{(i)}_{n'+j}+\sum_{k\sim j}\mathbf{v}^{(i)}_k=\lambda\mathbf{v}^{(i)}_j+\mathbf{v}^{(i-1)}_j=0, \tag{4.10}$$

and

$$-\mathbf{v}^{(i)}_j=\lambda\mathbf{v}^{(i)}_{n+j}+\mathbf{v}^{(i-1)}_{n+j}=0. \tag{4.11}$$

To prove Equations (4.8) and (4.9), let $j\in V(G)$ and $j\neq x$. Then, because the vectors $\mathbf{u}^{(i)}$ form a Jordan chain,

$$\begin{aligned}(\deg(j)-1)\mathbf{v}^{(i)}_{n'+j}+\sum_{k\sim j}\mathbf{v}^{(i)}_k&=(\deg(j)-1)\mathbf{u}^{(i)}_{n+j}+\sum_{k\sim j}\mathbf{u}^{(i)}_k\\&=\lambda\mathbf{u}^{(i)}_j+\mathbf{u}^{(i-1)}_j=\lambda\mathbf{v}^{(i)}_j+\mathbf{v}^{(i-1)}_j,\end{aligned}$$

and

$$-\mathbf{v}^{(i)}_j=-\mathbf{u}^{(i)}_j=\lambda\mathbf{u}^{(i)}_{n+j}+\mathbf{u}^{(i-1)}_{n+j}=\lambda\mathbf{v}^{(i)}_{n'+j}+\mathbf{v}^{(i-1)}_{n'+j}.$$

Now, to prove Equations (4.10) and (4.11), let $j\in V(H)$ or $j=x$. Then

$$\begin{aligned}(\deg(j)-1)\mathbf{v}^{(i)}_{n'+j}+\sum_{k\sim j}\mathbf{v}^{(i)}_k&=(\deg(j)-1)(0)+\sum_{k\sim j,\,k\in V(G)}\mathbf{v}^{(i)}_k+\sum_{k\sim j,\,k\in V(H)}\mathbf{v}^{(i)}_k\\&=\sum_{k\sim j,\,k\in V(G)}\mathbf{u}^{(i)}_k+\sum_{k\sim j,\,k\in V(H)}0\\&=0\\&=\lambda\mathbf{v}^{(i)}_j+\mathbf{v}^{(i-1)}_j,\end{aligned}$$

since $\sum_{k\sim x,k\in V(G)}\mathbf{u}^{(i)}_k=0$ by our assumption on the vectors $\{\mathbf{u}^{(i)}\}$. Further,

$$-\mathbf{v}^{(i)}_j=0=\lambda\mathbf{v}^{(i)}_{n'+j}+\mathbf{v}^{(i-1)}_{n'+j}.$$

Therefore, we have constructed a Jordan chain of length k for λ for the matrix $K(G_x\circ_y H)$. ∎

Corollary 4.2. *Let G be a graph with $X\subseteq V(G)$ and let H be a graph with $Y\subseteq V(H)$ such that $|X|=|Y|$. If $\mathbf{u}^{(1)},\dots\mathbf{u}^{(k)}$ form a Jordan chain of $K(G)$ for the eigenvalue $\lambda\neq 0$ such that for all $x\in X$ and all $1\leq i\leq k$, $\mathbf{u}^{(i)}_x=0$, then $G_X\circ_Y H$ has a Jordan chain of length k for the eigenvalue λ for the matrix $K(G_X\circ_Y H)$.*

This is verified in the same manner as Theorem 4.1.

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4.1. Bipartite base family. The following family is built on regular bipartite graphs, or biparite graphs where all vertices have the same degree. An example of such a graph is the complete bipartite graph $K_{r,r}$, i.e., a bipartite graph with $|\mathcal{A}| = |\mathcal{B}| = r$ where every vertex has degree r.

Let $\mathcal{F}_n$ be the family of graphs built through the following process:

1. Start with G a 4-regular bipartite graph on $2n$ vertices and let $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ and $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ be the partite sets.
2. Let H be a graph on $\ell \geq 1$ vertices. Take the disjoint union $G \cup H$.
3. For all $1 \leq i \leq n$, add an edge from $a_i \in \mathcal{A}$ to any single vertex in H and from $b_i \in \mathcal{B}$ to any single vertex in H, such that each vertex in H is adjacent to the same number of vertices in $\mathcal{A}$ and $\mathcal{B}$ and such that the resulting graph is connected.

An immediate example of this construction is $K_{4,4} + K_1$: start with the complete bipartite graph $K_{4,4}$, add a vertex K_1, and add an edge between this new vertex and all other vertices. Another example is shown in Fig. 3.

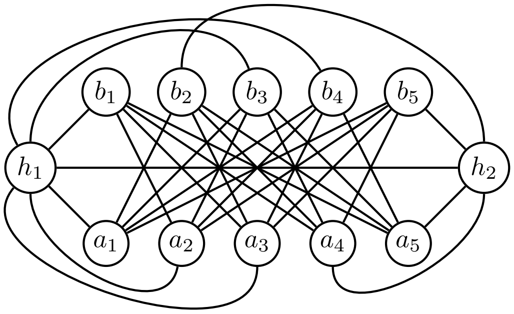

FIGURE 3. *An graph in $\mathcal{F}_5$ constructed by (1) starting with $K_{5,5}$ minus one matching (specifically a_ib_i), (2) allowing $H = K_2$ and (3) adding an edges between h_1 and $\{a_1, a_2, a_3, b_1, b_3, b_4\}$ and h_2 and $\{a_4, a_5, b_2, b_5\}$.*

PROPOSITION 4.3. *For every graph $F \in \mathcal{F}_n$, the eigenvalue $\lambda = -2$ for $K(F)$ is defective.*

Proof. Let F be a graph in $\mathcal{F}_n$ with n' vertices and let $\lambda = -2$. Let the vectors $\mathbf{v}$ and $\mathbf{u}$ be defined as follows:

$$\mathbf{v}_j = \begin{cases} 1 & j \in \mathcal{A} \\ -1 & j \in \mathcal{B} \\ 0 & \text{else} \end{cases} \quad \mathbf{v}_{n'+j} = \begin{cases} -1/2 & j \in \mathcal{A} \\ 1/2 & j \in \mathcal{B} \\ 0 & \text{else} \end{cases} \quad \mathbf{u}_j = \begin{cases} -1/2 & j \in \mathcal{A} \\ 1/2 & j \in \mathcal{B} \\ 0 & \text{else} \end{cases} \quad \mathbf{u}_{n'+j} = \begin{cases} 0 & j \in \mathcal{A} \\ 0 & j \in \mathcal{B} \\ 0 & \text{else} \end{cases},$$

By applying Corollary 3.3, it can be verified that $\mathbf{v}, \mathbf{u}$ are the first two vectors in a Jordan chain of $K(F)$ for $\lambda = -2$. Therefore, the eigenvalue $\lambda = -2$ is defective, as desired. ∎

4.2. Crustacean family. Let G be one of the graphs shown in Fig. 4 and define gluing candidates for either graph to be the vertices labeled $*$. Both of these graphs can serve as the graph G in Corollary 4.2, as we see below. It is interesting to note that graph (a) can be constructed from graph (b) by gluing the top $*$ vertex (adjacent to 1 and 2) to one of the other $*$ vertices.

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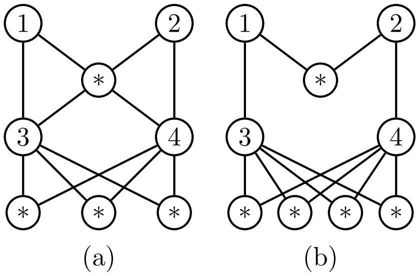

FIGURE 4. *Two different base graphs with candidate gluing set* $\{*\}$.

PROPOSITION 4.4. *Let G be either of the graphs in Fig. 4 with gluing set $X \subseteq \{*\}$ and let H be a graph with $Y \subseteq V(H)$ such that $|Y| = |X|$. Then $K(G_X \circ_Y H)$ has the defective eigenvalues $\lambda = \pm\sqrt{2}i$.*

Proof. Let G be either of the graphs in Fig. 4 and let $\lambda = \pm\sqrt{2}i$. Let the vectors $\mathbf{v}$ and $\mathbf{u}$ be defined as follows:

$$\mathbf{v}_j = \begin{cases} 1 & j=1 \\ -1 & j=2 \\ -\lambda/2 & j=3 \\ \lambda/2 & j=4 \\ 0 & \text{else} \end{cases} \qquad \mathbf{v}_{n'+j} = \begin{cases} \lambda/2 & j=1 \\ -\lambda/2 & j=2 \\ 1/2 & j=3 \\ -1/2 & j=4 \\ 0 & \text{else} \end{cases} \qquad \mathbf{u}_j = \begin{cases} \lambda & j=1 \\ -\lambda & j=2 \\ -1/2 & j=3 \\ 1/2 & j=4 \\ 0 & \text{else} \end{cases} \qquad \mathbf{u}_{n'+j} = \begin{cases} -3/2 & j=1 \\ 3/2 & j=2 \\ 0 & j=3 \\ 0 & j=4 \\ 0 & \text{else.} \end{cases}$$

By applying Corollary 3.3, it can be verified that $\mathbf{v}, \mathbf{u}$ are the first two vectors in a Jordan chain of $K(G)$ for $\lambda = \pm\sqrt{2}i$. Furthermore, applying Corollary 4.2 extends the same result to $K(G_X \circ_Y H)$. Therefore, the eigenvalues $\lambda = \pm\sqrt{2}i$ are defective for $K(G_X \circ_Y H)$, as desired. ∎

4.3. Restricted diamonds family.

Our last family explains two of the three defective graphs on seven vertices (the third being the cycle C_7) and again makes use of Corollary 4.2.

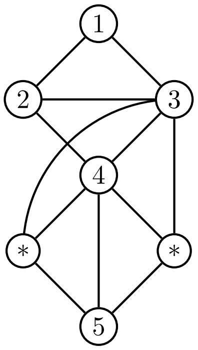

FIGURE 5. *Restricted diamonds base graph with gluing set* $\{*\}$.

PROPOSITION 4.5. *Let G the graph in Fig. 5 with gluing set X and let H be a graph with $Y \subseteq V(H)$ such that $|Y| = |X|$. Then $K(G_X \circ_Y H)$ has the defective eigenvalues $\lambda = \frac{-1\pm\sqrt{7}i}{2}$.*

Proof. Let G be either of the graphs in Fig. 5 with gluing set X and let $\lambda = \frac{-1\pm\sqrt{7}i}{2}$. Let the vectors $\mathbf{v}$ and $\mathbf{u}$ be defined as follows:

$$\mathbf{v}_j = \begin{cases} \lambda & j=1 \\ 2/\lambda & j=2 \\ 0 & j=3 \\ 1 & j=4 \\ -1 & j=5 \\ 0 & \text{else} \end{cases} \qquad \mathbf{v}_{n'+j} = \begin{cases} -1 & j=1 \\ -2/\lambda^2 & j=2 \\ 0 & j=3 \\ -1/\lambda & j=4 \\ 1/\lambda & j=5 \\ 0 & \text{else} \end{cases},$$

$$\mathbf{u}_j = \begin{cases} 1 & j=1 \\ (5\lambda-3)/2 & j=2 \\ (3-\lambda)/2 & j=3 \\ -7(\lambda+1)/2 & j=4 \\ 4\lambda+2 & j=5 \\ 0 & \text{else} \end{cases} \quad \mathbf{u}_{n'+j} = \begin{cases} 0 & j=1 \\ -(\lambda+5)/2 & j=2 \\ (3\lambda+5)/4 & j=3 \\ 3(1-\lambda)/2 & j=4 \\ (3\lambda-11)/4 & j=5 \\ 0 & \text{else.} \end{cases}$$

Using Corollary 3.3 and the fact that λ satisfies $x^2+x+2=0$, it can be verified that $\mathbf{v}, \mathbf{u}$ are the first two vectors in a Jordan chain of $K(G)$ for $\lambda = \frac{-1\pm\sqrt{7}i}{2}$. Furthermore, applying Corollary 4.2 extends the same result to $K(G_X \circ_Y H)$. Therefore, the eigenvalues $\lambda = \frac{-1\pm\sqrt{7}i}{2}$ are defective for $K(G_X \circ_Y H)$, as desired. $\square$

4.4. Computational results. We now provide some computational results for defective graphs for K with minimum degree at least two. Table 1 shows that the existence of defective eigenvalues for these graphs is relatively rare. Recall that the cycle is the only graph which is degenerate for K and not B, so the count of defective graphs for B is one less than shown.

TABLE 1

The number of connected graphs with minimum degree two on $n = 7, 8, 9, 10$ vertices and the number of those graphs which are defective for K

Number of vertices	7	8	9	10
Defective	3	39	484	7280
Total	507	7442	197,772	9,808,209

Table 2 shows all values that manifest as defective eigenvalues for a graph on ten or fewer vertices, along with the number of defective graphs. We also give partial data for graphs on ten vertices for those same eigenvalues. This table omits 116 different graphs on ten vertices with other defective eigenvalues.

TABLE 2

The number of graphs on n vertices with λ as a defective eigenvalue for K, where the minimum degree of the graph is at least two. Note that some graphs have multiple defective eigenvalues and appear more than once

$n \rightarrow$ $\downarrow \lambda$	7	8	9	10
1	1	1	1	1
-1		1		1
$(-1\pm\sqrt{7}i)/2$	2	16	156	1918
$\pm\sqrt{2}i$		22	324	5063
$\pm\sqrt{3}i$			3	183
-2			1	5

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It is interesting to note that of the graphs listed in Table 2, only one graph has a Jordan block of size larger than two (see Fig. 6). Defective eigenvalues of ±1 are the cycle graphs of that order, as explained by Theorem 2.11. Six different graphs are counted in Table 2 twice: cycle graphs C_8 and C_{10} (eigenvalues ± 1) and four graphs on 10 vertices (eigenvalues: $\pm\sqrt{2}i$ and $(-1\pm\sqrt{7}i)/2$).

Finally, with regard to the constructions discussed in this section, we note the following:

- Of the graphs with defective eigenvalue -2, the graph on nine vertices and four of the five graphs on ten vertices are constructed from Bipartite Base Family.
- Of the graphs with defective eigenvalues $\pm\sqrt{2}i$, 20 of the 22 graphs on eight vertices and 250 of the 324 graphs on nine vertices (160 from Graph (a), 90 from Graph (b)) are constructed from the Crustacean Family.
- Of the graphs with defective eigenvalues $(-1\pm\sqrt{7}i)/2$, both graphs on seven vertices are constructed by Restricted Diamonds Family, as well as two of the sixteen graphs on eight and ten of the 156 graphs on nine vertices.

SageMath code used to generate these results is available upon request.

5. Conclusion. In this paper, we showed that exploring the Jordan form of the non-backtracking matrix is equivalent to exploring the Jordan form of the matrix K for graphs with at least two cycles. Moreover, for other graphs, the differences between B and K can be easily qualified. As such, we can reduce the often larger matrix B into a more tractable matrix K. We also have built algebraic techniques to find Jordan chains and constructed three graph families which will have defective eigenvalues. Finally, we disproved a conjecture of Torres by constructing infinite families of graphs with defective eigenvalues of modulus greater than one.

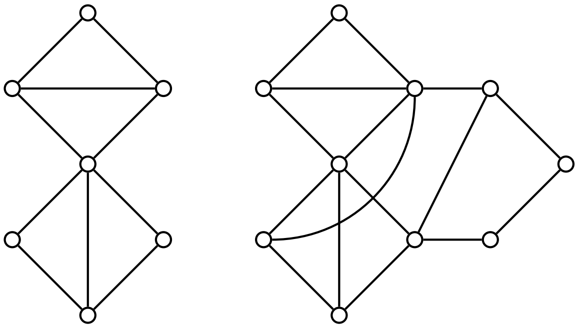
(a) (b)

FIGURE 6. *(a) The diamonds graph and (b) a graph on* 10 *vertices with* $\lambda = \frac{-1\pm\sqrt{7}i}{2}$ *as a defective eigenvalue with Jordan block size of* 3. *This is the only graph with such a block of order ten or less.*

A future direction we propose is considering the diamonds graph shown in Fig. 6(a). This graph on seven vertices appears as a subgraph in both defective graphs of order seven and fifteen of the sixteen defective graphs on order eight. We also note that this subgraph appears in the only graph on ten or fewer vertices with a Jordan block of size three or more (see Fig. 6(b)). It is interesting to note that the diamonds graph is not itself defective, but does have $(-1\pm\sqrt{7}i)/2$ as eigenvalue (with algebraic and geometric multiplicity two) and the eigenvector in Proposition 4.5 is in the eigenspace. We are interested in a more comprehensive explanation of graphs with this diamonds subgraph, in particular graphs that include larger Jordan blocks.

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