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MINIMAL RANK WEIGHTED WEAK DRAZIN INVERSES*

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Abstract. The concept of a minimal rank weak Drazin inverse for square matrices is extended to rectangular matrices. Precisely, a minimal rank weighted weak Drazin inverse is introduced and its properties are investigated. Some known generalized inverses such as the weighted Drazin inverse, the weighted core-EP inverse, and the weighted p-WGI are particular cases of a minimal rank weighted weak Drazin inverse. Thus, a wider class of generalized inverses is proposed. General representation forms of a minimal rank weighted weak Drazin inverse are presented as well as its canonical form. Applying the minimal rank weighted weak Drazin inverse, corresponding systems of linear matrix equations are solved and their solutions are expressed. As consequences of our results, new properties of minimal rank weak Drazin inverse are obtained.

Key words. Minimal rank weak Drazin inverse, Weighted weak Drazin inverse, Weighted Drazin inverse, Index.

AMS subject classifications. 15A24, 15A09, 15A10, 15A21, 65F05.

1. Introduction.

Through this work, the symbols A^*, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $rank(A)$, respectively, denote the conjugate-transpose, range, null space, and rank of $A \in \mathbb{C}^{m\times n}$, where $\mathbb{C}^{m\times n}$ is stand for the set of all $m \times n$ complex matrices.

The theory of generalized inverses attracts attentions of many researches because of its theoretical and applied significant [2]. Some useful kinds of generalized inverses are given at the beginning.

For a rectangular matrix $A \in \mathbb{C}^{m\times n}$, its Moore–Penrose inverse is the unique solution $A^\dagger = X \in \mathbb{C}^{n\times m}$ to the system of equations $AXA = A$, $XAX = X$, $AX = (AX)^*$, $XA = (XA)^*$ [2]. When only $AXA = A$ (or $XAX = X$) holds, X is an inner (or outer) inverse of A. The set of all inner (or outer) inverses of A is marked by $A\{1\}$ (or $A\{2\}$). Denote by $A\{2\}_{D,*}$ (or $A\{2\}_{*,E}$) the set of all outer inverses X of A such that $\mathcal{R}(X) = D$ (or $\mathcal{N}(X) = E$). Recall that $A^{(2)}_{D,E}$ is the unique outer inverse X of A satisfying $\mathcal{R}(X) = D$ and $\mathcal{N}(X) = E$ [2].

The symbol $\text{ind}(A)$ represents the index of a square matrix $A \in \mathbb{C}^{n\times n}$, that is, the smallest nonnegative integer k such that $rank(A^k) = rank(A^{k+1})$. If $A \in \mathbb{C}^{n\times n}$ and $k = \text{ind}(A)$, the Drazin inverse of A is the unique solution $A^D = X \in \mathbb{C}^{n\times n}$ to the system $XAX = X$, $XA = AX$, $A^{k+1}X = A^k$ [2]. In particular, when $\text{ind}(A) \leq 1$, $A^\# = A^D$ is the group inverse of A.

The Drazin inverse has applications in the control theory, Markov chains, numerical analysis, and differential equations [2, 3]. Recent results about Drazin inverse were developed in [17, 19].

As an extension of the Drazin inverse to rectangular matrices, the weighted Drazin inverse was presented in [5]. Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}\backslash\{0\}$ and $k = \max\{\text{ind}(AW), \text{ind}(WA)\}$. The W-weighted Drazin inverse

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of A is the unique solution $A^{D,W} = X \in \mathbb{C}^{m\times n}$ to

$$X = XWAWX, \quad AWX = XWA \quad \text{and} \quad (AW)^{k+1}XW = (AW)^k.$$

It is know that

$$A^{D,W} = [(AW)^D]^2 A = A[(WA)^D]^2.$$

Clearly, if $m = n$ and $W = I$, $A^{D,W}$ reduces to A^D.

For $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}\backslash\{0\}$ and $k = \max\{\text{ind}(AW), \text{ind}(WA)\}$, the W-weighted core-EP inverse $A^{\oplus,W}$ of A is uniquely determined solution to

$$WAWX = (WA)^k[(WA)^k]^\dagger \quad \text{and} \quad \mathcal{R}((AW)^k) = \mathcal{R}(X).$$

When $m = n$ and $W = I$, $A^{\oplus,I}$ is equal to the core-EP inverse $A^{\oplus}$ [13]. In particular, $A^{\oplus}$ becomes the core inverse $A^{\#\!\!\!\bigcirc}$ [1] when $\text{ind}(A) \leq 1$. Recall that $A^{\oplus,W} = A[(WA)^{\oplus}]^2 = A^{\oplus,W}WAWA^{\oplus,W} = AWA^{\oplus,W}WA^{\oplus,W}$ [7, 10].

If $p \in \mathbb{N}$, $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}\backslash\{0\}$ and $k = \max\{\text{ind}(AW), \text{ind}(WA)\}$, the W-weighted p-WGI (or W-p-WGI) $A^{ⓦ_p,W}$ of A is the unique solution to the system [11]:

$$AWX = (A^{\oplus,W}W)^p(AW)^{p-1}A \quad \text{and} \quad AWXWX = X,$$

which is expressed as:

$$A^{ⓦ_p,W} = (A^{\oplus,W}W)^{p+1}(AW)^{p-1}A.$$

In particular, for $m = n$ and $W = I$, $A^{ⓦ_p,I}$ coincides with the p-WGI $A^{ⓦ_p} = (A^{\oplus})^{p+1}A^p$ [12, 18]. If $p = 1$, $A^{ⓦ_p}$ is equal to the weak group inverse (or WGI) $A^{ⓦ} = (A^{\oplus})^2A$ [15].

The notion of the weak Drazin inverse was defined in [4] as a generalization of the Drazin inverse. For $A \in \mathbb{C}^{n\times n}$ with $k = \text{ind}(A)$, its weak Drazin inverse is a matrix $X \in \mathbb{C}^{n\times n}$ satisfying $XA^{k+1} = A^k$ [4]. The matrix X is a minimal rank weak Drazin inverse of A [4] if

$$XA^{k+1} = A^k \quad \text{and} \quad rank(X) = rank(A^D).$$

The Drazin inverse of A presents the unique minimal rank weak Drazin inverse of A which commutes with A. More significant generalized inverses (e.g., the DMP inverse [8], core-EP inverse [13], and weak group inverse [15]) are particular cases of the minimal rank weak Drazin inverse [16].

The weak Drazin inverse is not unique in general, but we can easier compute it than the Drazin inverse and apply instead of the Drazin inverse in the theory of differential equations, Markov chains, and so on.

As a dual version of weak Drazin inverse, a right weak Drazin inverse of $A \in \mathbb{C}^{n\times n}$ with $k = \text{ind}(A)$ is a solution to the equation $A^{k+1}X = A^k$ [4]. Also, a minimal rank right weak Drazin inverse of A [4] satisfies

$$A^{k+1}X = A^k \quad \text{and} \quad rank(X) = rank(A^D).$$

Note that $rank(A^D) = rank(A^k)$.

Inspired by the definition of the W-weighted Drazin inverse for rectangular matrices, weighted weak Drazin inverses were introduced in [14]. In the case that $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}\backslash\{0\}$ and $k = \text{ind}(AW)$, a W-weighted weak Drazin inverse of A is a matrix $X \in \mathbb{C}^{m\times n}$ such that $XW(AW)^{k+1} = (AW)^k$. Remark that $rank((AW)^k) \leq rank(X)$.

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The concepts of the minimal rank weak Drazin inverse and the W-weighted weak Drazin inverse motivated us to present the notion of a minimal rank W-weighted weak Drazin inverse for rectangular matrices. So, we extend the definition of the minimal rank weak Drazin inverse and further investigate this topic. We will observe that a minimal rank W-weighted weak Drazin inverse has important role for solving systems of linear matrix equations. In details, we present our goals.

(1) First, we introduce the concept of a minimal rank W-weighted weak Drazin inverse for rectangular matrices. In the case that $W \in \mathbb{C}^{n\times m}\backslash\{0\}$, $A \in \mathbb{C}^{m\times n}$ and $k = \text{ind}(AW)$, a solution $X \in \mathbb{C}^{m\times n}$ to the system

$$XW(AW)^{k+1} = (AW)^k \quad \text{and} \quad rank(X) = rank((AW)^k),$$

is called a minimal rank W-weighted weak Drazin inverse of A, because it has the minimal rank between all W-weighted weak Drazin inverses of A. Especially, if $m = n$ and $W = I$, a minimal rank W-weighted weak Drazin inverse of A becomes a minimal rank weak Drazin inverse of A. Thus, the notion of the minimal rank weak Drazin inverse is extended to a rectangular matrix.

(2) As dual version of a minimal rank W-weighted weak Drazin inverse of A, we define a minimal rank W-weighted right weak Drazin inverse of A which is a solution to

$$W(AW)^{k+1}X = (WA)^k \quad \text{and} \quad rank(X) = rank((WA)^k).$$

(3) We prove several characterizations of a minimal rank W-weighted (right) weak Drazin inverse.

(4) We show that the W-weighted Drazin inverse, the W-weighted core-EP inverse, and the W-weighted p-WGI are particular cases of a minimal rank W-weighted weak Drazin inverse.

(5) The general representation forms of the minimal rank W-weighted (right) weak Drazin inverse are proposed.

(6) Relations between a minimal rank W-weighted weak Drazin inverse of A and minimal rank weak Drazin inverses of AW and WA are developed.

(7) The canonical form of a minimal rank W-weighted weak Drazin inverse is given.

(8) Applications of the minimal rank W-weighted (right) weak Drazin inverse in solving some systems of linear matrix equations and expressing their general solution forms are presented.

Our work is organized in the following way. Section 2 contains characterizations of the minimal rank W-weighted (right) weak Drazin inverse as well as of its special cases. General representations forms of the minimal rank W-weighted (right) weak Drazin inverse are established in Section 3. Section 4 gives properties of minimal rank W-weighted weak Drazin inverse of A related to minimal rank weak Drazin inverses of AW and WA as well as the canonical form of minimal rank W-weighted weak Drazin inverse. In Section 5, we solve certain systems of linear matrix equations applying minimal rank W-weighted (right) weak Drazin inverse. These results are illustrated by numerical examples. As consequences of our results, we obtain new properties of minimal rank weak Drazin inverse.

2. Minimal rank weighted weak Drazin inverses. In this section, we develop characterizations of a minimal rank W-weighted weak Drazin inverse and a minimal rank W-weighted right weak Drazin inverse.

Assume that $W \in \mathbb{C}^{n\times m}\backslash\{0\}$ and $A \in \mathbb{C}^{m\times n}$ in this paper. First, we develop equivalent conditions for a matrix X to be a minimal rank W-weighted weak Drazin inverse of A.

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THEOREM 2.1. *For $k = \operatorname{ind}(AW)$ and $X \in \mathbb{C}^{m\times n}$, the next statements are equivalent:*

(i) $XW(AW)^{k+1} = (AW)^k$ *and* $rank(X) = rank((AW)^k)$, *that is, X is a minimal rank W-weighted weak Drazin inverse of A;*
(ii) $XW(AW)^{k+1} = (AW)^k$ *and* $\mathcal{R}(X) = \mathcal{R}((AW)^k)$*;*
(iii) $XWAWX = X$ *and* $\mathcal{R}(X) = \mathcal{R}((AW)^k)$, *i.e.* $X \in A\{2\}_{\mathcal{R}((AW)^k),*}$*;*
(iv) $X = (AW)^k[(AW)^k]^\dagger X$ *and* $XW(AW)^{k+1} = (AW)^k$*;*
(v) $XWAWX = X$, $X = (AW)^k[(AW)^k]^\dagger X$ *and* $XW(AW)^{k+1} = (AW)^k$*;*
(vi) $AWXWX = X$ *and* $XW(AW)^{k+1} = (AW)^k$*;*
(vii) $X = (AW)^D AWX$ *and* $XW(AW)^{k+1} = (AW)^k$.

Proof. (i) $\Rightarrow$ (ii): The hypothesis $XW(AW)^{k+1} = (AW)^k$ implies $\mathcal{R}((AW)^k) \subseteq \mathcal{R}(X)$. Since $rank(X) = rank((AW)^k)$, we deduce that $\mathcal{R}(X) = \mathcal{R}((AW)^k)$.

(ii) $\Rightarrow$ (iii): Because $\mathcal{R}(X) = \mathcal{R}((AW)^k)$, then $X = (AW)^k U$, for some $U \in \mathbb{C}^{m\times n}$. Hence, by $XW(AW)^{k+1} = (AW)^k$, we get $XWAWX = (XWAW(AW)^k)U = (AW)^k U = X$.

(iii) $\Rightarrow$ (iv): Using $\mathcal{R}(X) = \mathcal{R}((AW)^k)$ and $XWAWX = X$, we observe, for some $U \in \mathbb{C}^{m\times n}$ and $V \in \mathbb{C}^{n\times m}$, that

$$X = (AW)^k U = (AW)^k[(AW)^k]^\dagger((AW)^k U) = (AW)^k[(AW)^k]^\dagger X.$$

and

$$(AW)^k = XV = XWAW(XV) = XWAW(AW)^k = XW(AW)^{k+1}.$$

(iv) $\Rightarrow$ (v): From $X = (AW)^k[(AW)^k]^\dagger X$ and $XW(AW)^{k+1} = (AW)^k$, we have

$$XWAWX = (XWAW(AW)^k)[(AW)^k]^\dagger X = (AW)^k[(AW)^k]^\dagger X = X.$$

(v) $\Rightarrow$ (vi): The assumptions $X = (AW)^k[(AW)^k]^\dagger X$ and $XW(AW)^{k+1} = (AW)^k$ yield

$$\begin{aligned} AWXWX &= AWXW(AW)^k[(AW)^k]^\dagger X = AW(XW(AW)^{k+1})(AW)^D[(AW)^k]^\dagger X \\ &= AW(AW)^k(AW)^D[(AW)^k]^\dagger X = (AW)^k[(AW)^k]^\dagger X = X. \end{aligned}$$

(vi) $\Rightarrow$ (i): Clearly, by $AWXWX = X$, it follows

$$\begin{aligned} X &= AWXWX = AWAWXWXWX = (AW)^2X(WX)^2 \\ &= (AW)^3X(WX)^3 = \cdots = (AW)^kX(WX)^k. \end{aligned} \tag{2.1}$$

Now, $X = (AW)^kX(WX)^k$ and $XW(AW)^{k+1} = (AW)^k$ yield $rank(X) = rank((AW)^k)$.

(iii) $\Rightarrow$ (vii) $\Rightarrow$ (v): These implications follow as (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). □

The equalities $XW(AW)^{k+1} = (AW)^k$, $X = (AW)^k[(AW)^k]^\dagger X$, and $X = (AW)^D AWX$ given in Theorem 2.1 can be replaced with some of the following equivalent conditions.

COROLLARY 2.2. *Let $k = \operatorname{ind}(AW)$ and $X \in \mathbb{C}^{m\times n}$.*

(a) *The next statements are equivalent:*

(i) $XW(AW)^{k+1} = (AW)^k$*;*

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 40, pp. 714-728, October 2024.

(ii) $XW(AW)^2(AW)^D = AW(AW)^D$;
(iii) $XWAW(AW)^D = (AW)^D$.

(b) *If $XWAWX = X$, the next statements are equivalent:*

(i) $XW(AW)^{k+1} = (AW)^k$;
(ii) $\mathcal{R}((AW)^k) \subseteq \mathcal{R}(X)$.

(c) *If $XWAWX = X$, the next statements are equivalent:*

(i) $X = (AW)^k[(AW)^k]^\dagger X$;
(ii) $XWAW = (AW)^k[(AW)^k]^\dagger XWAW$;
(iii) $\mathcal{R}(X) \subseteq \mathcal{R}((AW)^k)$.
(iv) $X = (AW)^D AWX$;
(v) $XWAW = (AW)^D AWXWAW$.

In the case that $m = n$ and $W = I$ in Theorem 2.1, we recover [16, Theorem 2.1] about characterizations of a minimal rank weak Drazin inverse and we present new characterizations (iv)–(v) and (vii) in Corollary 2.3.

Corollary 2.3. *For $A, X \in \mathbb{C}^{n\times n}$ and $k = \mathrm{ind}(A)$, the next statements are equivalent:*

(i) $XA^{k+1} = A^k$ *and* $rank(X) = rank(A^k)$, *that is, X is a minimal rank weak Drazin inverse of A;*
(ii) $XA^{k+1} = A^k$ *and* $\mathcal{R}(X) = \mathcal{R}(A^k)$;
(iii) $XAX = X$ *and* $\mathcal{R}(X) = \mathcal{R}(A^k)$, *i.e.* $X \in A\{2\}_{\mathcal{R}(A^k),*}$;
(iv) $X = A^k(A^k)^\dagger X$ *and* $XA^{k+1} = A^k$;
(v) $XAX = X$, $X = A^k(A^k)^\dagger X$ *and* $XA^{k+1} = A^k$;
(vi) $AX^2 = X$ *and* $XA^{k+1} = A^k$;
(vii) $X = A^D AX$ *and* $XA^{k+1} = A^k$.

Now, we study when a minimal rank W-weighted weak Drazin inverse of A coincides with the W-weighted Drazin inverse of A.

Corollary 2.4. *For $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$ and $X \in \mathbb{C}^{m\times n}$, the next statements are equivalent:*

(i) $X = A^{D,W}$;
(ii) *X is a minimal rank W-weighted weak Drazin inverse of A and* $XWA = AWX$;
(iii) *X is a minimal rank W-weighted weak Drazin inverse of A and* $\mathcal{N}(X) = \mathcal{N}((WA)^k)$;
(iv) *X is a minimal rank W-weighted weak Drazin inverse of A and* $\mathcal{N}(X) \subseteq \mathcal{N}((WA)^k)$;
(v) *X is a minimal rank W-weighted weak Drazin inverse of A and* $\mathcal{N}((WA)^k) \subseteq \mathcal{N}(X)$;
(vi) *X is a W-weighted weak Drazin inverse of A and* $\mathcal{N}((WA)^k) \subseteq \mathcal{N}(X)$.

Proof. (i) $\Leftrightarrow$ (ii): This equivalence is clear by Theorem 2.1 and the definition of $A^{D,W}$.

(i) $\Rightarrow$ (iii): Since $A^{D,W} = A[(WA)^D]^2$ and $(WA)^D = WA^{D,W}$, we conclude that $\mathcal{N}(A^{D,W}) = \mathcal{N}((WA)^D) = \mathcal{N}((WA)^k)$.

(iii) $\Rightarrow$ (iv): It is evident.

(iv) $\Rightarrow$ (i): For some $G \in \mathbb{C}^{n\times m}$, we observe that $\mathcal{N}(X) \subseteq \mathcal{N}((WA)^k)$ yields $(WA)^k = GX$. Now, by

$$(WA)^k = (GX)WAWX = (WA)^k WAWX = (WA)^{k+1}WX,$$

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Volume 40, pp. 714-728, October 2024.

we have

$$\begin{aligned}(WA)^{2k+1}WX(WX)^k &= (WA)^k((WA)^{k+1}WX)(WX)^k = (WA)^k(WA)^k(WX)^k \\ &= (WA)^{k-1}((WA)^{k+1}WX)(WX)^{k-1} \\ &= (WA)^{k-1}(WA)^k(WX)^{k-1} = \dots \\ &= WA(WA)^kWX = (WA)^k.\end{aligned}$$

Because X is a minimal rank W-weighted weak Drazin inverse of A, it follows that (2.1) holds and so

$$\begin{aligned}X &= (AW)^kX(WX)^k = [(AW)^D]^{k+2}A((WA)^{2k+1}WX(WX)^k) \\ &= [(AW)^D]^{k+2}A(WA)^k = [(AW)^D]^2A = A^{D,W}.\end{aligned}$$

(iii) $\Rightarrow$ (v) $\Rightarrow$ (vi): These implications are trivial.

(vi) $\Rightarrow$ (i): The hypothesis $\mathcal{N}((WA)^k) \subseteq \mathcal{N}(X)$ implies that $X = H(WA)^k$, for some $H \in \mathbb{C}^{m\times n}$. Since $XW(AW)^{k+1} = (AW)^k$, it follows

$$\begin{aligned}X &= (H(WA)^k)WA(WA)^D = (XW(AW)^{k+1})A[(WA)^D]^{k+2} \\ &= (AW)^kA[(WA)^D]^{k+2} = A[(WA)^D]^2 = A^{D,W}.\end{aligned}$$

Corollary 2.5 implies that the W-weighted core-EP inverse is a particular case of a minimal rank W-weighted weak Drazin inverse.

Corollary 2.5. *For $k = \max\{\operatorname{ind}(AW), \operatorname{ind}(WA)\}$ and $X \in \mathbb{C}^{m\times n}$, the next statements are equivalent:*

(i) $X = A^{\oplus,W}$;
(ii) *X is a minimal rank W-weighted weak Drazin inverse of A and $\mathcal{N}(X) = \mathcal{N}([(WA)^k]^*)$;*
(iii) *X is a minimal rank W-weighted weak Drazin inverse of A and $\mathcal{N}(X) \subseteq \mathcal{N}([(WA)^k]^*)$;*
(iv) *X is a minimal rank W-weighted weak Drazin inverse of A and $\mathcal{N}([(WA)^k]^*) \subseteq \mathcal{N}(X)$;*
(v) *X is a W-weighted weak Drazin inverse of A and $\mathcal{N}([(WA)^k]^*) \subseteq \mathcal{N}(X)$.*

Proof. (i) $\Rightarrow$ (ii): Recall that $A^{\oplus,W} = A[(WA)^{\oplus}]^2 = A[(WA)^D]^2(WA)^k[(WA)^k]^\dagger$, which gives, for $X = A^{\oplus,W}$, $XW(AW)^{k+1} = (AW)^k$, $\mathcal{R}(X) = \mathcal{R}((AW)^k)$ and $\mathcal{N}(X) = \mathcal{N}([(WA)^k]^*)$. Using Theorem 2.1, we deduce that (ii) holds.

(ii) $\Rightarrow$ (iii): This part is clear.

(iii) $\Rightarrow$ (iv): The condition $\mathcal{N}(X) \subseteq \mathcal{N}([(WA)^k]^*)$, for some $E \in \mathbb{C}^{n\times m}$, implies

$$[(WA)^k]^* = EX = (EX)WAWX = [(WA)^k]^*WAWX,$$

i.e. $(WA)^k = (WAWX)^*(WA)^k$ and thus, in conjunction with (2.1),

$$\begin{aligned}WAWX &= W(AW)^{k+1}X(WX)^k = (WA)^kWA(WX)^{k+1} \\ &= (WAWX)^*[(WA)^kWA(WX)^{k+1}] = (WAWX)^*WAWX.\end{aligned}$$

Hence, $WAWX = (WAWX)^*$ which gives

$$WAWX = [(WA)^kWA(WX)^{k+1}]^* = [WA(WX)^{k+1}]^*[(WA)^k]^*,$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 40, pp. 714-728, October 2024.

and

$$X = XWAWX = X[WA(WX)^{k+1}]^*[(WA)^k]^*,$$

that is, $\mathcal{N}([(WA)^k]^*) \subseteq \mathcal{N}(X)$.

(iv) ⇒ (v): Evidently.

(v) ⇒ (i): Since $\mathcal{N}([(WA)^k]^*) \subseteq \mathcal{N}(X)$, we have

$$X = F[(WA)^k]^* = (F[(WA)^k]^*)(WA)^k[(WA)^k]^\dagger = X(WA)^k[(WA)^k]^\dagger,$$

for some $F \in \mathbb{C}^{m\times n}$. Applying $XW(AW)^{k+1} = (AW)^k$, we get

$$\begin{aligned} X &= X(WA)^k[(WA)^k]^\dagger = (XW(AW)^{k+1})A[(WA)^D]^2[(WA)^k]^\dagger \\ &= (AW)^k A[(WA)^D]^2[(WA)^k]^\dagger = A[(WA)^D]^2(WA)^k[(WA)^k]^\dagger = A^{\textcircled{\dagger},W}. \end{aligned}$$

Using a minimal rank W-weighted weak Drazin inverse of A, we also characterize the W-p-WGI of A.

Corollary 2.6. *For $p \in \mathbb{N}$, $k = \max\{\operatorname{ind}(AW), \operatorname{ind}(WA)\}$ and $X \in \mathbb{C}^{m\times n}$, the next statements are equivalent:*

(i) $X = A^{\textcircled{w}_p,W}$;
(ii) *X is a minimal rank W-weighted weak Drazin inverse of A and $\mathcal{N}(X) = \mathcal{N}([(WA)^k]^*)(WA)^p)$;*
(iii) *X is a minimal rank W-weighted weak Drazin inverse of A and $\mathcal{N}(X) \subseteq \mathcal{N}([(WA)^k]^*(WA)^p)$;*
(iv) *X is a minimal rank W-weighted weak Drazin inverse of A and $\mathcal{N}([(WA)^k]^*(WA)^p) \subseteq \mathcal{N}(X)$;*
(v) *X is a W-weighted weak Drazin inverse of A and $\mathcal{N}([(WA)^k]^*(WA)^p) \subseteq \mathcal{N}(X)$.*

Proof. (i) ⇒ (ii): By [11, Theorem 2.2 and Lemma 2.2], this implication follows.

(ii) ⇒ (iii): This part is clear.

(iii) ⇒ (i): From [11, Lemma 2.1], we know that $A^{\textcircled{w}_p,W} = A[(WA)^D]^{p+2}(WA)^k[(WA)^k]^\dagger(WA)^p$. By $\mathcal{N}(X) \subseteq \mathcal{N}([(WA)^k]^*(WA)^p)$, we have, for some $D \in \mathbb{C}^{n\times m}$,

$$[(WA)^k]^*(WA)^p = DX = (DX)WAWX = [(WA)^k]^*(WA)^{p+1}WX.$$

Applying (2.1), we observe that

$$\begin{aligned} X &= (AW)^k X(WX)^k = [(AW)^D]^{p+2}(AW)^{p+2}(AW)^k X(WX)^k \\ &= A[(WA)^D]^{p+2}(WA)^{k+p+1}WX(WX)^k \\ &= A[(WA)^D]^{p+2}(WA)^k[(WA)^k]^\dagger(WA)^{k+p+1}WX(WX)^k \\ &= A[(WA)^D]^{p+2}(WA)^k[(WA)^k]^\dagger(WA)^{p+1}W((AW)^k X(WX)^k) \\ &= A[(WA)^D]^{p+2}([(WA)^k]^\dagger)^*([(WA)^k]^*(WA)^{p+1}WX) \\ &= A[(WA)^D]^{p+2}([(WA)^k]^\dagger)^*[(WA)^k]^*(WA)^p \\ &= A[(WA)^D]^{p+2}(WA)^k[(WA)^k]^\dagger(WA)^p \\ &= A^{\textcircled{w}_p,W}. \end{aligned}$$

(ii) ⇒ (iv) ⇒ (v): It is obvious.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 40, pp. 714-728, October 2024.

(v) ⇒ (i): Using [11, Lemma 2.2], we deduce that $\mathcal{N}(A^{\circledS_p,W}) = \mathcal{N}([(WA)^k]^*)(WA)^p) \subseteq \mathcal{N}(X)$ and so, for some $C \in \mathbb{C}^{m\times m}$,

$$X = CA^{\circledS_p,W} = (CA^{\circledS_p,W})WAWA^{\circledS_p,W} = XWAWA^{\circledS_p,W}.$$

According to [11, Theorem 2.1], $A^{\circledS_p,W} = AWA^{\circledS_p,W}WA^{\circledS_p,W} = (AW)^k A^{\circledS_p,W}(WA^{\circledS_p,W})^k$, which yields in conjunction with $XW(AW)^{k+1} = (AW)^k$ that

$$\begin{aligned} X &= XWAWA^{\circledS_p,W} = (XW(AW)^{k+1})A^{\circledS_p,W}(WA^{\circledS_p,W})^k \\ &= (AW)^k A^{\circledS_p,W}(WA^{\circledS_p,W})^k = A^{\circledS_p,W}. \end{aligned}$$

Corollary 2.6 implies characterizations of p-WGI by a minimal rank weak Drazin inverse.

Corollary 2.7. *For $p \in \mathbb{N}$, $A, X \in \mathbb{C}^{n\times n}$ and $k = \operatorname{ind}(A)$, the next statements are equivalent:*

(i) $X = A^{\circledS_p}$;
(ii) *X is a minimal rank weak Drazin inverse of A and* $\mathcal{N}(X) = \mathcal{N}((A^k)^*A^p)$;
(iii) *X is a minimal rank weak Drazin inverse of A and* $\mathcal{N}(X) \subseteq \mathcal{N}((A^k)^*A^p)$;
(iv) *X is a minimal rank weak Drazin inverse of A and* $\mathcal{N}((A^k)^*A^p) \subseteq \mathcal{N}(X)$;
(v) *X is a weak Drazin inverse of A and* $\mathcal{N}((A^k)^*A^p) \subseteq \mathcal{N}(X)$.

As Theorem 2.1, we can verify necessary and sufficient conditions for a matrix X to be a minimal rank W-weighted right weak Drazin inverse.

Theorem 2.8. *For $k = \operatorname{ind}(WA)$ and $X \in \mathbb{C}^{m\times n}$, the next statements are equivalent:*

(i) $W(AW)^{k+1}X = (WA)^k$ *and* $rank(X) = rank((WA)^k)$, *that is, X is a minimal rank W-weighted right weak Drazin inverse of A;*
(ii) $W(AW)^{k+1}X = (WA)^k$ *and* $\mathcal{N}(X) = \mathcal{N}((WA)^k)$;
(iii) $XWAWX = X$ *and* $\mathcal{N}(X) = \mathcal{N}((WA)^k)$, *i.e.* $X \in A\{2\}_{*,\mathcal{N}((WA)^k)}$;
(iv) $X = X[(WA)^k]^\dagger(WA)^k$ *and* $W(AW)^{k+1}X = (WA)^k$;
(v) $XWAWX = X$, $X = X[(WA)^k]^\dagger(WA)^k$ *and* $W(AW)^{k+1}X = (WA)^k$;
(vi) $XWXWA = X$ *and* $W(AW)^{k+1}X = (WA)^k$;
(vii) $X = X(WA)^DWA$ *and* $W(AW)^{k+1}X = (WA)^k$.

Example 2.9. Consider matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$AW = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad WA = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix},$$

with $\operatorname{ind}(AW) = 2$ and $\operatorname{ind}(WA) = 1$. Using Theorem 2.1(vi) and Theorem 2.8(vi), we show that a minimal rank W-weighted weak Drazin inverse X of A and a minimal rank W-weighted right weak Drazin inverse X' of A are represented by:

$$X = \begin{bmatrix} \frac{1}{9} & x \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad X' = \begin{bmatrix} \frac{1}{9} & 0 \\ y & 0 \\ z & 0 \end{bmatrix},$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 714-728, October 2024.

where $x, y, z \in \mathbb{C}$ are arbitrary. Especially, for $m \in \mathbb{N}$, note that

$$A^{D,W} = \begin{bmatrix} \frac{1}{9} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = A^{\oplus,W} = A^{\circledS_m,W}.$$

3. General representation forms. The general representation forms of a minimal rank W-weighted (right) weak Drazin inverse are established in this section. The following lemmas will be useful.

LEMMA 3.1. [2, p. 52] *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$ and $C \in \mathbb{C}^{m\times q}$. Then the equation $AYB = C$ has a solution $Y \in \mathbb{C}^{n\times p}$ if and only if $AA^-CB^-B = C$ holds for some $A^- \in A\{1\}$ and $B^- \in B\{1\}$. In this case, the general solution Y is given as $Y = A^-CB^- + Z - A^-AZBB^-$ for arbitrary $Z \in \mathbb{C}^{n\times p}$.*

LEMMA 3.2. (a) *For $k = \operatorname{ind}(AW)$, the general solution to $XW(AW)^{k+1} = (AW)^k$ is represented by:*

$$X = (AW)^D W^- + Z(I - WAW(AW)^D W^-), \tag{3.2}$$

where $Z \in \mathbb{C}^{m\times n}$ is arbitrary and $W^- \in W\{1\}$ is arbitrary but fixed.

(b) *For $k = \max\{\operatorname{ind}(AW), \operatorname{ind}(WA)\}$, the general solution to $W(AW)^{k+1}X = (WA)^k$ is represented by*

$$X = [(AW)^D]^2 A + (I - (AW)^D AW)Z = A^{D,W} + (I - (AW)^D AW)Z, \tag{3.3}$$

where $Z \in \mathbb{C}^{m\times n}$ is arbitrary.

Proof. (a) This part is proved in [14, Theorem 1].

(b) Notice that $[(AW)^D]^{k+1}W^- \in (W(AW)^{k+1})\{1\}$, where $W^- \in W\{1\}$ is arbitrary but fixed. Because

$$\begin{aligned} W(AW)^{k+1}[(AW)^D]^{k+1}W^-(WA)^k &= W(AW)^D AWW^-WA(WA)^D(WA)^k \\ &= (WA)^D WAWA(WA)^D(WA)^k \\ &= (WA)^k, \end{aligned}$$

using Lemma 3.1, we obtain that the general solution to $W(AW)^{k+1}X = (WA)^k$ is

$$\begin{aligned} X &= [(AW)^D]^{k+1}W^-(WA)^k + (I - [(AW)^D]^{k+1}W^-W(AW)^{k+1})Z \\ &= [(AW)^D]^{k+2}AWW^-WA(WA)^D(WA)^k + (I - [(AW)^D]^{k+2}AWW^-W(AW)^{k+1})Z \\ &= [(AW)^D]^{k+2}AW(AW)^D(AW)^k A + (I - [(AW)^D]^{k+2}AW(AW)^{k+1})Z \\ &= [(AW)^D]^2 A + (I - (AW)^D AW)Z = A^{D,W} + (I - (AW)^D AW)Z. \end{aligned}$$

THEOREM 3.3. (a) *For $k = \operatorname{ind}(AW)$, a minimal rank W-weighted weak Drazin inverse X of A is represented by*

$$\begin{aligned} X &= (AW)^D W^- + (AW)^k[(AW)^k]^\dagger Z(I - WAW(AW)^D W^-) && (3.4) \\ &= (AW)^D W^- + (AW)^D AWZ(I - WAW(AW)^D W^-), && (3.5) \end{aligned}$$

where $Z \in \mathbb{C}^{m\times n}$ is arbitrary and $W^- \in W\{1\}$ is arbitrary but fixed.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 714-728, October 2024.

(b) *For* $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$, *a minimal rank* W*-weighted right weak Drazin inverse* X *of* A *is represented by*

$$\begin{aligned} X &= [(AW)^D]^2A + (I - (AW)^D AW)Z[(WA)^k]^\dagger (WA)^k \\ &= A^{D,W} + (I - (AW)^D AW)Z[(WA)^k]^\dagger (WA)^k \\ &= [(AW)^D]^2A + (I - (AW)^D AW)ZWA(WA)^D \\ &= A^{D,W} + (I - (AW)^D AW)ZWA(WA)^D, \end{aligned}$$

where $Z \in \mathbb{C}^{m\times n}$ *is arbitrary.*

Proof. (a) By Lemma 3.2, the general solution to $XW(AW)^{k+1} = (AW)^k$ is given as in (3.2). Substituting (3.2) into $X = (AW)^k[(AW)^k]^\dagger X$, we get

$$(I - (AW)^k[(AW)^k]^\dagger)Z(I - WAW(AW)^D W^-) = 0. \tag{3.6}$$

Applying Lemma 3.1, $I - (AW)^k[(AW)^k]^\dagger \in (I - (AW)^k[(AW)^k]^\dagger)\{1\}$ and $I - WAW(AW)^D W^- \in (I - WAW(AW)^D W^-)\{1\}$, the general solution to (3.6) is

$$Z = Y - (I - (AW)^k[(AW)^k]^\dagger)Y(I - WAW(AW)^D W^-). \tag{3.7}$$

From (3.2) and (3.7), by Theorem 2.1(iv), we deduce that a minimal rank W-weighted weak Drazin inverse X of A has the form (3.4).

Using Theorem 2.1(vii), we obtain (3.5) in a same way.

(b) Similarly, we verify this part. ∎

According to Theorem 3.3, we obtain general representations of the minimal rank W-weighted weak Drazin inverse and minimal rank W-weighted right weak Drazin inverse.

Corollary 3.4. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $k = \mathrm{ind}(A)$.

(a) *A minimal rank weak Drazin inverse* X *of* A *is represented by*

$$\begin{aligned} X &= A^D + A^k(A^k)^\dagger Z(I - AA^D) \\ &= A^D + A^D AZ(I - AA^D), \end{aligned} \tag{3.8}$$

where $Z \in \mathbb{C}^{n\times n}$ *is arbitrary.*

(b) *A minimal rank right weak Drazin inverse* X *of* A *is represented by*

$$\begin{aligned} X &= A^D + (I - A^D A)Z(A^k)^\dagger A^k \\ &= A^D + (I - A^D A)ZAA^D, \end{aligned}$$

where $Z \in \mathbb{C}^{m\times n}$ *is arbitrary.*

Remark that the representation (3.8) appeared in [4, Theorem 2(ii)].

4. Properties of minimal rank W-weighted weak Drazin inverse. The first aim of this section is to consider relations between a minimal rank W-weighted weak Drazin inverse of A and minimal rank weak Drazin inverses of AW and WA.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 714-728, October 2024.

THEOREM 4.1. *If $k = \max\{\text{ind}(AW), \text{ind}(WA)\}$ and X is a minimal rank W-weighted weak Drazin inverse of A, then*

(i) *XW is a minimal rank weak Drazin inverse of AW;*
(ii) *WX is a minimal rank weak Drazin inverse of WA.*

Proof. (i) Beside $XW(AW)^{k+1} = (AW)^k$ and $rank(X) = rank((AW)^k)$, by Theorem 2.1, we know that $XWAWX = X$. Hence, $rank(XW) = rank(X) = rank((AW)^k)$, which implies that XW is a minimal rank weak Drazin inverse of AW.

(ii) It follows in a similar manner. ◻

By Theorem 4.1 and [16, Proposition 2.9], we can verify the next result.

COROLLARY 4.2. *If $k = \max\{\text{ind}(AW), \text{ind}(WA)\}$ and X is a minimal rank W-weighted weak Drazin inverse of A, then*

(i) *$(XW)^{\#}$ exists and $(XW)^{\#} = AWAWXW$;*
(ii) *$(WX)^{\#}$ exists and $(WX)^{\#} = WAWAWX$;*
(iii) *$(AWXWAW)^{\#}$ exists and $(AWXWAW)^{\#} = XWXWAW$;*
(iv) *$(WAWXWA)^{\#}$ exists and $(WAWXWA)^{\#} = WXWXWA$.*

We now study the converse of Theorem 4.1.

THEOREM 4.3. (i) *If $k = \text{ind}(AW)$ and Y is a minimal rank weak Drazin inverse of AW, then $X = Y^2A$ is a minimal rank W-weighted weak Drazin inverse of A.*

(ii) *If $k = \text{ind}(WA)$ and Y is a minimal rank weak Drazin inverse of WA, then $X = AY^2$ is a minimal rank W-weighted right weak Drazin inverse of A.*

Proof. (i) Since Y is a minimal rank weak Drazin inverse of AW, we have $Y(AW)^{k+1} = (AW)^k$ and $rank(Y) = rank((AW)^k)$. For $X = Y^2A$, it follows

$$XW(AW)^{k+1} = Y^2AW(AW)^{k+1} = Y(AW)^{k+1} = (AW)^k$$

and so $rank((AW)^k) \leq rank(X)$. From $X = Y^2A$, we deduce that $rank(X) \leq rank(Y) = rank((AW)^k)$. Thus, $rank(X) = rank((AW)^k)$ and, by Theorem 2.1, X is a minimal rank W-weighted weak Drazin inverse of A.

(ii) Analogously, we prove this part. ◻

To get canonical form of a minimal rank W-weighted weak Drazin inverse, we need the next decompositions of A and W presented in [7].

LEMMA 4.4. *If $k = \max\{\text{ind}(AW), \text{ind}(WA)\}$ and $\text{rank}((AW)^k) = p$, it follows*

$$A = U \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} V^* \quad \text{and} \quad W = V \begin{bmatrix} W_1 & W_2 \\ 0 & W_3 \end{bmatrix} U^*, \tag{4.9}$$

where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary, $A_1, W_1 \in \mathbb{C}^{p\times p}$ are invertible, $A_3 \in \mathbb{C}^{(m-p)\times(n-p)}$ and $W_3 \in \mathbb{C}^{(n-p)\times(m-p)}$ such that A_3W_3 and W_3A_3 are nilpotent of indices $\text{ind}(AW)$ and $\text{ind}(WA)$, respectively.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 714-728, October 2024.

THEOREM 4.5. *If A and W are represented as in (4.9), then a minimal rank W-weighted weak Drazin inverse X of A is given by*

$$X = U \begin{bmatrix} (W_1A_1W_1)^{-1} & X_2 \\ 0 & 0 \end{bmatrix} V^*, \tag{4.10}$$

where X_2 is arbitrary.

Proof. Notice that, by (4.9),

$$(AW)^k = U \begin{bmatrix} (A_1W_1)^k & S \\ 0 & 0 \end{bmatrix} U^*, \tag{4.11}$$

where $S = \sum_{i=0}^{k-1}(A_1W_1)^{k-i}(A_1W_2 + A_2W_3)(A_3W_3)^i$. Using [6], we have

$$[(AW)^k]^\dagger = U \begin{bmatrix} [(A_1W_1)^k]^*\triangle & 0 \\ S^*\triangle & 0 \end{bmatrix} U^*,$$

where $\triangle = ((A_1W_1)^k[(A_1W_1)^k]^* + SS^*)^{-1}$. Set $X = U \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} V^*$, where $X_1 \in \mathbb{C}^{p\times p}$. Since $(AW)^{k+1} = U \begin{bmatrix} (A_1W_1)^{k+1} & A_1W_1S \\ 0 & 0 \end{bmatrix} U^*$, we obtain

$$XW(AW)^{k+1} = U \begin{bmatrix} X_1W_1(A_1W_1)^{k+1} & X_1W_1A_1W_1S \\ X_3W_1(A_1W_1)^{k+1} & X_3W_1A_1W_1S \end{bmatrix} U^*.$$

Now, $XW(AW)^{k+1} = (AW)^k$ gives $X_1 = (W_1A_1W_1)^{-1}$ and $X_3 = 0$. Further,

$$(AW)^k[(AW)^k]^\dagger X = U \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (W_1A_1W_1)^{-1} & X_2 \\ 0 & X_4 \end{bmatrix} V^* = U \begin{bmatrix} (W_1A_1W_1)^{-1} & X_2 \\ 0 & 0 \end{bmatrix} V^*$$

and $X = (AW)^k[(AW)^k]^\dagger X$ yield $X_4 = 0$. Therefore, a minimal rank W-weighted weak Drazin inverse X of A is expressed as in (4.10). ∎

5. Applications. We consider applications of a minimal rank W-weighted weak Drazin inverse and a minimal rank W-weighted right weak Drazin inverse for solving corresponding systems of linear matrix equations.

THEOREM 5.1. *For $k = \mathrm{ind}(AW)$, $B \in \mathbb{C}^{p\times m}$ and a minimal rank W-weighted weak Drazin inverse X of A, the general solution of the equation*

$$YW(AW)^{k+1} = B(AW)^k \tag{5.12}$$

is given by

$$Y = BX + Z(I - WAWX), \tag{5.13}$$

for arbitrary $Z \in \mathbb{C}^{p\times n}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 714-728, October 2024.

Proof. If Y is represented as in (5.13), by $XW(AW)^{k+1} = (AW)^k$, then $YW(AW)^{k+1} = B(AW)^k$, that is, Y is a solution to (5.12).

Since $X = (AW)^k[(AW)^k]^\dagger X$, for a solution Y of (5.12), we get

$$\begin{aligned} BX &= (B(AW)^k)[(AW)^k]^\dagger X = YW(AW)^{k+1}[(AW)^k]^\dagger X \\ &= YWAW((AW)^k[(AW)^k]^\dagger X) = YWAWX. \end{aligned}$$

Now, $Y = BX + Y - YWAWX = BX + Y(I - WAWX)$, which implies that Y has the form (5.13). ∎

Remark that the equation (5.12) is equivalent to

$$YWAW(AW)^D = B(AW)^D,$$

and thus these equations have the same general solution form.

By Theorem 5.1, when $m = n$ and $W = I$, we obtain solvability of the following equation using a minimal rank weak Drazin inverse.

Corollary 5.2. *For $B \in \mathbb{C}^{p\times n}$, $A \in \mathbb{C}^{n\times n}$, $k = \text{ind}(A)$, and a minimal rank weak Drazin inverse X of A, the general solution of the equation*

$$YA^{k+1} = BA^k,$$

is given by

$$Y = BX + Z(I - AX),$$

for arbitrary $Z \in \mathbb{C}^{p\times n}$.

Similarly as Theorem 5.1, we can obtain the next result.

Theorem 5.3. *For $k = \text{ind}(AW)$, $B \in \mathbb{C}^{n\times p}$ and a minimal rank W-weighted weak Drazin inverse X of A, the general solution of the equation*

(i) $(AW)^k[(AW)^k]^\dagger Y = XB$ *is given by* $Y = XB + (I - (AW)^k[(AW)^k]^\dagger)Z$,
(ii) $(AW)^D AWY = XB$ *is given by* $Y = XB + (I - (AW)^D AW)Z$,

for arbitrary $Z \in \mathbb{C}^{m\times p}$.

Theorem 5.3 gives the general solution form of the next equations.

Corollary 5.4. *For $B \in \mathbb{C}^{n\times p}$, $A \in \mathbb{C}^{n\times n}$, $k = \text{ind}(A)$, and a minimal rank weak Drazin inverse X of A, the general solution of the equation*

(i) $A^k(A^k)^\dagger Y = XB$ *is given by* $Y = XB + (I - A^k(A^k)^\dagger)Z$,
(ii) $A^D AY = XB$ *is given by* $Y = XB + (I - A^D A)Z$,

for arbitrary $Z \in \mathbb{C}^{n\times p}$.

Also, we can verify solvability of the following equations using a minimal rank W-weighted right weak Drazin inverse.

Theorem 5.5. *For $k = \text{ind}(WA)$, $B \in \mathbb{C}^{n\times p}$ and a minimal rank W-weighted right weak Drazin inverse X of A, the general solution of the equation*

$$W(AW)^{k+1}Y = (WA)^k B,$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 714-728, October 2024.

is given by

$$Y = XB + (I - XWAW)Z,$$

for arbitrary $Z \in \mathbb{C}^{m\times p}$.

COROLLARY 5.6. *For* $B \in \mathbb{C}^{n\times p}$, $A \in \mathbb{C}^{n\times n}$, $k = \operatorname{ind}(A)$ *and a minimal rank right weak Drazin inverse* X *of* A*, the general solution of the equation*

$$A^{k+1}Y = A^kB,$$

is given by

$$Y = XB + (I - XA)Z,$$

for arbitrary $Z \in \mathbb{C}^{n\times p}$.

THEOREM 5.7. *For* $k = \operatorname{ind}(WA)$, $B \in \mathbb{C}^{p\times n}$ *and a minimal rank* W*-weighted right weak Drazin inverse* X *of* A*, the general solution of the equation*

(i) $Y[(WA)^k]^\dagger(WA)^k = BX$ *is given by* $Y = BX + Z(I - [(WA)^k]^\dagger(WA)^k)$,
(ii) $YWA(WA)^D = BX$ *is given by* $Y = BX + Z(I - WA(WA)^D)$,

for arbitrary $Z \in \mathbb{C}^{p\times m}$.

EXAMPLE 5.8. Let A and W be given as in Example 2.9,

$$B = \begin{bmatrix} 9 & 0 & 0 \\ 9 & 0 & 0 \\ 9 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \\ z_5 & z_6 \end{bmatrix}.$$

Since

$$Y = BX + Z(I - WAWX) = \begin{bmatrix} 1 & 9x(1-z_1) + z_2 \\ 1 & 9x(1-z_3) + z_4 \\ 1 & 9x(1-z_5) + z_6 \end{bmatrix},$$

we confirm Theorem 5.1 by

$$YW(AW)^3 = \begin{bmatrix} 27 & 0 & 0 \\ 27 & 0 & 0 \\ 27 & 0 & 0 \end{bmatrix} = B(AW)^2.$$

For $B' = \begin{bmatrix} 9 & 0 \\ 0 & 0 \end{bmatrix}$, we have

$$Y = X'B' + (I - X'WAW)Z = \begin{bmatrix} 1 & 0 \\ 9y(1-z_1) + z_3 & -9yz_2 + z_4 \\ 9z(1-z_1) + z_5 & -9zz_2 + z_6 \end{bmatrix},$$

which implies confirmation of Theorem 5.5:

$$W(AW)^2Y = \begin{bmatrix} 27 & 0 \\ 0 & 0 \end{bmatrix} = WAB'.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 714-728, October 2024.

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