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CONCAVE FUNCTION INEQUALITIES FOR SUM OF MATRICES*

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Abstract. In this paper, we present some norm inequalities for concave functions, which generalize the main results in [Y. Zhang. *Linear Algebra Appl.*, 574:60–66, 2019].

Key words. Positive semidefinite matrix, Norm, Concave function.

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1. Introduction. Let M_n be the space of n square matrices. We use $\lambda_i(A)$ and $s_i(A)$ to present the eigenvalues and singular values of matrix A and arranged in decreasing order. Let $||.||$ be any unitarily invariant norm. For A, B in M_n. It is known in [1] that $||A|| \le ||B|| \iff \sum_{i=1}^{k} s_i(A) \le \sum_{i=1}^{k} s_i(B)$ for $k = 1, 2, \ldots, n$. We write $A \ge B$ to mean the eigenvalues of Hermitian matrix $A - B$ are nonnegative.

The study of concave function inequalities is an important research topic in linear algebra and matrix analysis, and a large number of concave function inequalities were listed in [2].

Let A, $B \in M_n$ and let $f(t)$ be a nonnegative concave function on $[0, \infty)$. The following inequality

$$\|f(|A+B|)\|_1 \le \|f(|A|)\|_1 + \|f(|B|)\|_1, \tag{1.1}$$

was established in [3], where $||A||_1 = tr(|A|)$.

Uchiyama extended Rotfel'd result to

$$\|f(|A+B|)\| \le \|f(|A|)\| + \|f(|B|)\|, \tag{1.2}$$

in [4].

Motivated by Uchiyama's work, Zhang in [5] obtained

$$\|f(|A+B|)\| \le \left\| f\left(\frac{1}{2}\begin{pmatrix} |A|+|B| & A^*+B^* \\ A+B & |A^*|+|B^*| \end{pmatrix}\right)\right\| \le \|f(|A|)\| + \|f(|B|)\|. \tag{1.3}$$

Inequality (1.3) is a special case of the following result:

$$\left\| f\left(\left|\sum_{i=1}^{m} A_i\right|\right)\right\| \le \left\| f\left(\frac{1}{2}\begin{pmatrix} \sum_{i=1}^{m}|A_i| & \sum_{i=1}^{m}A_i^* \\ \sum_{i=1}^{m}A_i & \sum_{i=1}^{m}|A_i^*| \end{pmatrix}\right)\right\| \le \sum_{i=1}^{m}\|f(|A_i|)\|, \tag{1.4}$$

by letting $m = 2$.

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Since we usually consider norm inequalities for the sum of several matrices more than 2, it is meaningful to study whether inequality (1.4) holds or not. We observe the method used in [5] is not suitable to obtain inequality (1.4). Therefore, in this paper, we give a new approach to deal with this result.

2. Main results. Before we start our discussion, we list the following lemmas. The first lemma is a singular value inequality between 2×2 block matrices and its off-diagonal block matrix.

Lemma 1. [6] *Let* $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \geq 0$. *Then*

$$2s_j \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \leq s_j \begin{bmatrix} A & X^* \\ X & B \end{bmatrix}.$$

for any j.

The next lemma is a key tool for us to obtain Theorem 4, which could be found on page 173 of [2].

Lemma 2. *Let* f *be a nonnegative and concave function on* $[0, \infty)$. *Then* $0 \leq A \leq B$ *implies that* $\|f(A)\| \leq \|f(B)\|$.

The last lemma is a norm inequality related to concave functions, which is a consequence of (1.2).

Lemma 3. [7] *Let* $A_i \geq 0$ $(1 \leq i \leq m)$ *and let* f *be a nonnegative concave function on* $[0, +\infty)$. *Then*

$$\left\| f\left(\sum_{i=1}^{m} A_i\right) \right\| \leq \sum_{i=1}^{m} \| f(A_i) \|.$$

In the rest of this paper, we suppose that $f : [0, +\infty) \to [0, +\infty)$ is a concave function. Next, we present the first Theorem.

Theorem 4. *Let* $A_i \in M_n$ $(1 \leq i \leq m)$. *Then*

$$\left\| f\left(\left| \sum_{i=1}^{m} A_i \right| \right) \right\| \leq \left\| f\left(\frac{1}{2} \begin{pmatrix} \sum_{i=1}^{m} |A_i| & \sum_{i=1}^{m} A_i^* \\ \sum_{i=1}^{m} A_i & \sum_{i=1}^{m} |A_i^*| \end{pmatrix} \right) \right\|.$$

Proof. An application of polar decomposition of A_i $(1 \leq i \leq m)$ shows that

$$\begin{pmatrix} |A_i| & A_i^* \\ A_i & |A_i^*| \end{pmatrix} \geq 0.$$

Hence,

$$\begin{pmatrix} \sum_{i=1}^{m} |A_i| & \sum_{i=1}^{m} A_i^* \\ \sum_{i=1}^{m} A_i & \sum_{i=1}^{m} |A_i^*| \end{pmatrix} \geq 0.$$

Therefore, according to lemma 1, we get

$$s_j \begin{pmatrix} |\sum_{i=1}^{m} A_i| & 0 \\ 0 & 0 \end{pmatrix} \leq s_j \left(\frac{1}{2} \begin{pmatrix} \sum_{i=1}^{m} |A_i| & \sum_{i=1}^{m} A_i^* \\ \sum_{i=1}^{m} A_i & \sum_{i=1}^{m} |A_i^*| \end{pmatrix} \right),$$

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which implies that

$$\begin{pmatrix} \left|\sum_{i=1}^m A_i\right| & 0 \\ 0 & 0 \end{pmatrix} \leq \frac{1}{2} K \begin{pmatrix} \sum_{i=1}^m |A_i| & \sum_{i=1}^m A_i^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*| \end{pmatrix} K^*,$$

for K, where K is a unitary matrix.

Due to lemma 2, we obtain

$$\begin{aligned} \left\| \begin{pmatrix} f\left(\left|\sum_{i=1}^m A_i\right|\right) & 0 \\ 0 & 0 \end{pmatrix} \right\| &\leq \left\| f\left(\begin{pmatrix} \left|\sum_{i=1}^m A_i\right| & 0 \\ 0 & 0 \end{pmatrix} \right) \right\| \\ &\leq \left\| f\left(\frac{1}{2} \begin{pmatrix} \sum_{i=1}^m |A_i| & \sum_{i=1}^m A_i^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*| \end{pmatrix} \right) \right\|. \end{aligned}$$

■

Now, we obtain the second Theorem.

Theorem 5. *Let* $A_i \in M_n$ $(1 \leq i \leq m)$*. Then*

$$\left\| f\left(\frac{1}{2} \begin{pmatrix} \sum_{i=1}^m |A_i| & \sum_{i=1}^m A_i^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*| \end{pmatrix} \right) \right\| \leq \sum_{i=1}^m \left\| f\left(|A_i|\right) \right\|.$$

Proof. First, we assume that $f(0) = 0$. By lemma 3, we get

$$\begin{aligned} &\left\| f\left(\frac{1}{2} \begin{pmatrix} \sum_{i=1}^m |A_i| & \sum_{i=1}^m A_i^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*| \end{pmatrix} \right) \right\| \\ &\leq \sum_{i=1}^m \left\| f\left(\frac{1}{2} \begin{pmatrix} |A_i| & A_i^* \\ A_i & |A_i^*| \end{pmatrix} \right) \right\| \\ &= \sum_{i=1}^m \left\| f\left(W_i \begin{pmatrix} |A_i| & 0 \\ 0 & 0 \end{pmatrix} W_i^* \right) \right\| \\ &= \sum_{i=1}^m \left\| f\left(\begin{pmatrix} |A_i| & 0 \\ 0 & 0 \end{pmatrix} \right) \right\|, \end{aligned} \tag{2.5}$$

for $W_i = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ U_i & U_i \end{pmatrix}$, where I is the identity matrix and $A_i = U_i |A_i|$ $(1 \leq i \leq m)$.

Hence,

$$\left\| f\left(\frac{1}{2} \begin{pmatrix} \sum_{i=1}^m |A_i| & \sum_{i=1}^m A_i^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*| \end{pmatrix} \right) \right\| \leq \sum_{i=1}^m \left\| f\left(|A_i|\right) \right\|, \tag{2.6}$$

due to inequality (2.5) and $f(0) = 0$.

For the general case $f(0) > 0$. We suppose that $g(x) = f(x) - f(0)$. Then $g(x)$ is a nonnegative and concave function with $g(0) = 0$.

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It is clear that

$$\sum_{j=1}^{k} s_j \left(f \left(\frac{1}{2} \begin{pmatrix} \sum_{i=1}^m |A_i| & \sum_{i=1}^m A_i^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*| \end{pmatrix} \right) \right) = \sum_{j=1}^{k} s_j \left(g \left(\frac{1}{2} \begin{pmatrix} \sum_{i=1}^m |A_i| & \sum_{i=1}^m A_i^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*| \end{pmatrix} \right) \right) + kf(0),$$

and

$$\sum_{j=1}^{k} s_j \left(f \left(|A_i| \right) \right) = \sum_{j=1}^{k} s_j \left(g \left(|A_i| \right) \right) + kf(0).$$

Using inequality (2.6), we obtain

$$\sum_{j=1}^{k} s_j \left(g \left(\frac{1}{2} \begin{pmatrix} \sum_{i=1}^m |A_i| & \sum_{i=1}^m A_i^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*| \end{pmatrix} \right) \right) \leq \sum_{i=1}^{m} \left(\sum_{j=1}^{k} s_j \left(g \left(|A_i| \right) \right) \right),$$

which implies that

$$\begin{aligned} \sum_{j=1}^{k} s_j \left(f \left(\frac{1}{2} \begin{pmatrix} \sum_{i=1}^m |A_i| & \sum_{i=1}^m A_i^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*| \end{pmatrix} \right) \right) &\leq \sum_{i=1}^{m} \left(\sum_{j=1}^{k} s_j \left(f \left(|A_i| \right) \right) \right) - (m-1)kf(0) \\ &\leq \sum_{i=1}^{m} \left(\sum_{j=1}^{k} s_j \left(f \left(|A_i| \right) \right) \right). \end{aligned}$$

Therefore,

$$\left\| f \left(\frac{1}{2} \begin{pmatrix} \sum_{i=1}^m |A_i| & \sum_{i=1}^m A_i^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*| \end{pmatrix} \right) \right\| \leq \sum_{i=1}^{m} \left\| f \left(|A_i| \right) \right\|,$$

for $f(0) > 0$. ■

Basing on Theorem 4 and Theorem 5, we have

Theorem 6. *Let* $A_i \in M_n$ $(1 \leq i \leq m)$*. Then*

$$\left\| f \left(\left| \sum_{i=1}^{m} A_i \right| \right) \right\| \leq \left\| f \left(\frac{1}{2} \begin{pmatrix} \sum_{i=1}^m |A_i| & \sum_{i=1}^m A_i^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*| \end{pmatrix} \right) \right\| \leq \sum_{i=1}^{m} \left\| f \left(|A_i| \right) \right\|.$$

Remark 7. *The Theorem* 10 *in* [5] *follows from* 6 *by taking* $A_1 = A$*,* $A_2 = B$ *and* $A_3 = \cdots = A_m = 0$.

REFERENCES

[1] R. Bhatia. *Matrix Analysis*. Springer, 1997.
[2] M. Ghaemi, N. Gharakhanlu, T. Rassias, R. Saadati. *Advances in Matrix Inequalities*. Springer, 2021.
[3] S. Ju. Rotfel'd. The singular values of the sum of completely continuous operators. *Prob. Math. Phys. Spectr. Theory*, 3:81–87, 1969.
[4] M. Uchiyama. Subadditivity inequality of eigenvalue sums. *Proc. Amer. Math. Soc.*, 134:1405–1412, 2005.
[5] Y. Zhang. Interpolating the Rotfel'd inequality for unitarily invariant norms. *Linear Algebra Appl.*, 574:60–66, 2019.
[6] Y. Tao. More results on singular value inequalities of matrices. *Linear Algebra Appl.*, 416:724–729, 2006.
[7] J. C. Bourin. A concavity inequality for symmetric norms. *Linear Algebra Appl.*, 413:212–217, 2006.