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AFFINE SUBSPACES OF MATRICES WITH RANK IN A RANGE*

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Abstract. The problem of finding the maximal dimension of linear or affine subspaces of matrices whose rank is constant, or bounded below, or bounded above, has attracted many mathematicians from the sixties to the present day. The problem has caught also the attention of algebraic geometers, since vector spaces of matrices of constant rank r give rise to vector bundle maps whose images are vector bundles of rank r. Moreover, there is a link with the so-called "rank metric codes," since a constant rank r subspace of $K^{n\times n}$ can be viewed as a constant weight r rank metric code; it can be interesting to study also the maximal dimension of the subspaces of $K^{n\times n}$ whose elements have rank in a range $[s, r]$, since such subspaces obviously give rank metric codes with weights in $[s, r]$. In this paper, with the main purpose to get an organic result including the ones on spaces of matrices with constant rank, the ones on spaces of matrices with rank bounded below and the ones on spaces of matrices with rank bounded above and to generalize a previous result on real matrices with constant rank to matrices on a more general field, we study the maximal dimension of affine subspaces of matrices whose rank is between two numbers under mild assumptions on the field. We get also a result on antisymmetric matrices and on matrices in row echelon form.

Key words. Affine subspaces, Matrices, Rank, Range.

AMS subject classifications. 15A30, 15A03.

1. Introduction.

For every $m, n \in \mathbb{N}$ and every field K, let $K^{m\times n}$ be the vector space of the $(m \times n)$-matrices over K, let $K_s^{n\times n}$ be the vector space of the symmetric $(n \times n)$-matrices over K and, finally, let $K_a^{n\times n}$ be the vector space of the antisymmetric $(n \times n)$-matrices over K.

We say that an affine subspace S of $K^{m\times n}$ has constant rank r if every matrix of S has rank r, and we say that a linear subspace S of $K^{m\times n}$ has constant rank r if every nonzero matrix of S has rank r.

The problem of finding the maximal dimension of linear or affine subspaces of matrices whose rank is constant or bounded below or bounded above has been studied from the sixties to the present day. It has some interest in algebraic geometry, since a $(l + 1)$-vector space of $(m \times n)$-matrices of constant rank r gives rise to a vector bundle map $\mathcal{O}^n_{\mathbb{P}^l}(-1) \to \mathcal{O}^m_{\mathbb{P}^l}$ whose image is a vector bundle of rank r; furthermore, there is a connection with dual varieties; the reader interested in the link between spaces of matrices with constant rank and algebraic geometry can read for instance the paper [10]. Moreover, there is a link with the so-called "rank metric codes," introduced by Delsarte in [5], which are subsets of $K^{n\times n}$ together with the weight function given by the metric d defined by $d(A, B) := rk(A - B)$ for any $A, B \in K^{n\times n}$; a constant rank r subspace of $K^{n\times n}$ can be viewed as a constant weight r rank metric code, and it can be interesting to study also the maximal dimension of a subspace of $K^{n\times n}$ whose elements have rank in a range $[s, r]$, since they obviously give rank metric codes with weights in $[s, r]$.

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To quote some of the main results on the maximal dimension of affine or linear subspaces of matrices with constant or bounded rank, we need some notation. Define

$$\mathcal{A}^K(m \times n; r) = \{S \mid S \text{ affine subspace of } K^{m\times n} \text{ of constant rank } r\}$$

$$\mathcal{A}^K_{sym}(n; r) = \{S \mid S \text{ affine subspace of } K^{n\times n}_s \text{ of constant rank } r\}$$

$$\mathcal{A}^K_{ant}(n; r) = \{S \mid S \text{ affine subspace of } K^{n\times n}_a \text{ of constant rank } r\}$$

$$\mathcal{A}^K(m \times n; s, r) = \left\{ \begin{array}{ll} S \mid & S \text{ affine subspace of } K^{m\times n} \text{ such that} \\ & s = \min\{rk(A) | \; A \in S\}, \; r = \max\{rk(A) | \; A \in S\} \end{array} \right\}$$

$$\mathcal{A}^K_{ant}(n; s, r) = \left\{ \begin{array}{ll} S \mid & S \text{ affine subspace of } K^{n\times n}_a \text{ such that} \\ & s = \min\{rk(A) | \; A \in S\}, \; r = \max\{rk(A) | \; A \in S\} \end{array} \right\}$$

$$\mathcal{A}^K_{ech}(m \times n; s, r) = \left\{ \begin{array}{ll} S \mid & S \text{ affine subspace of } K^{m\times n} \text{ such that} \\ & A \text{ is in row echelon form } \forall A \in S, \\ & s = \min\{rk(A) | \; A \in S\}, \; r = \max\{rk(A) | \; A \in S\} \end{array} \right\}$$

$$\mathcal{L}^K(m \times n; r) = \{S \mid S \text{ linear subspace of } K^{m\times n} \text{ of constant rank } r\}$$

$$\mathcal{L}^K_{sym}(n; r) = \{S \mid S \text{ linear subspace of } K^{n\times n}_s \text{ of constant rank } r\},$$

where rk denotes obviously the rank. Let

$$a^{\cdot}_{\cdot}(\cdot; \cdot) = \max\{\dim S \mid S \in \mathcal{A}^{\cdot}_{\cdot}(\cdot; \cdot))\},$$

and

$$l^{\cdot}_{\cdot}(\cdot; \cdot) = \max\{\dim S \mid S \in \mathcal{L}^{\cdot}_{\cdot}(\cdot; \cdot))\},$$

for example,

$$a^K_{ant}(n; r) = \max\{\dim S \mid S \in \mathcal{A}^K_{ant}(n, r)\}.$$

From the wide literature on the maximal dimension of linear subspaces of matrices with constant rank, we quote in particular the following theorems :

THEOREM 1 (Westwick, [14]). *For $2 \le r \le m \le n$, we have*

$$n - r + 1 \le l^{\mathbb{C}}(m \times n; r) \le m + n - 2r + 1.$$

THEOREM 2 (Ilic-Landsberg, [10]). *If r is even and greater than or equal to 2, then*

$$l^{\mathbb{C}}_{sym}(n; r) = n - r + 1.$$

In case r odd, the following result holds, see [10], [7], [8]:

THEOREM 3. *If r is odd, then*

$$l^{\mathbb{C}}_{sym}(n; r) = 1.$$

We mention also that, in 1962, Flanders proved the following result:

THEOREM 4 (Flanders, [6]). *If $r \le \min\{m, n\}$, a linear subspace of $\mathbb{C}^{m\times n}$ such that every of its elements has rank less than or equal to r has dimension less than or equal to $r \max\{m, n\}$.*

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We quote also Atkinson's work on the classification of linear space with bounded above rank using primitive spaces of matrices, see [1]. Flanders' theorem was generalized by de Seguins Pazzis in 2010:

Theorem 5 (de Seguins Pazzis, [4]). *Let K be a field. If $r \leq \min\{m,n\}$, an affine subspace of $K^{m\times n}$ such that every of its elements has rank less than or equal to r has dimension less than or equal to $r\max\{m,n\}$.*

In [11] we proved the following theorems:

Theorem 6 (Rubei). *Let $n, r \in \mathbb{N}$ with $r \leq n$. Then*

$$a^{\mathbb{R}}_{sym}(n;r) \leq \left\lfloor \frac{r}{2} \right\rfloor \left(n - \left\lfloor \frac{r}{2} \right\rfloor \right).$$

Theorem 7 (Rubei). *Let $m, n, r \in \mathbb{N}$ with $r \leq m \leq n$. Then*

$$a^{\mathbb{R}}(m\times n;r) = rn - \frac{r(r+1)}{2}.$$

We proved also a statement on the maximal dimension of affine subspaces with constant signature in $\mathbb{R}^{n\times n}_s$ and one on the maximal dimension of affine subspaces of constant rank in the space on $\mathbb{R}$ of the hermitian matrices.

In [12], we studied the case of antisymmetric matrices and we proved:

Theorem 8 (Rubei). *For $n \geq 2r+2$:*

$$a^{\mathbb{R}}_{ant}(n;2r) = (n-r-1)r.$$

For $n = 2r$

$$a^{\mathbb{R}}_{ant}(n;2r) = r(r-1).$$

For $n = 2r+1$

$$a^{\mathbb{R}}_{ant}(n;2r) = r(r+1).$$

The result was generalized in [3] by de Seguins Pazzis for fields K such that $|K| \geq \max\{2r-1, r+2\}$.

Moreover, de Seguins Pazzis proved the following theorems on subspaces of matrices with rank bounded below (see Theorem 8 in [2] and Theorem 4 in [3], respectively):

Theorem 9 (de Seguins Pazzis, [2]). *Let $s, m, n \in \mathbb{N}$ with $s \leq \min\{m,n\}$ and let K be a field. The maximal dimension of an affine subspace S in $K^{m\times n}$ such that $rk(A) \geq s$ for any $A \in S$ is $mn - \binom{s+1}{2}$.*

Theorem 10 (de Seguins Pazzis, [3]). *Let $s, n \in \mathbb{N}$ with s even and $s \leq n$ and let K be a field such that $|K| \geq n-1$ if n is even and $|K| \geq n-2$ if n is odd. The maximal dimension of an affine subspace S in $K^{n\times n}_a$ such that $rk(A) \geq s$ for any $A \in S$ is*

$$\frac{n(n-1)}{2} - \frac{s^2}{4}.$$

Obviously, we can ask which are the maximal dimension of affine or linear subspaces with other characteristics, which are not a specified rank: in [13], we proved that the maximal dimension of an affine subspace in the set of the nilpotent $n\times n$ matrices over a field K is $\frac{n(n-1)}{2}$ and, if the characteristic of the field is

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zero, an affine not linear subspace in such a set has dimension less than or equal to $\frac{n(n-1)}{2}-1$. Moreover, we proved that the maximal dimension of an affine subspace in the set of the normal $n\times n$ matrices is n, the maximal dimension of a linear subspace in the set of the $(n\times n)$-matrices over $\mathbb{R}$ which are diagonalizable over $\mathbb{R}$ is $\frac{n(n+1)}{2}$, while the maximal dimension of an affine not linear subspace the set of the $(n\times n)$-matrices over $\mathbb{R}$ which are diagonalizable over $\mathbb{R}$ is $\frac{n(n+1)}{2}-1$.

In this paper, we try to answer to the natural question "which is the maximal dimension of affine subspaces of matrices whose rank is bounded both below and above?" and we prove the following results.

Theorem 11. *Let $r,s,m,n\in\mathbb{N}$ with $s\le r\le\min\{m,n\}$ and let K be a field with cardinality greater than or equal to $r+2$ and characteristic different form 2; then*

$$a^K(m\times n;s,r)=r\max\{m,n\}-\binom{s+1}{2}.$$

Observe that the theorem above generalizes Theorem 7 in two directions: both because the field we consider is not only $\mathbb{R}$ and because the rank is not constant but in a range. Moreover, observe that, if we take $r=\min\{m,n\}$ in the theorem above, we get Theorem 9, while, if we take $s=0$, we get Theorem 5 with more strict assumptions on the field.

Theorem 12. *Let $r,s,n\in\mathbb{N}$ with r,s even and $s\le r\le n$ and let K be a field with cardinality greater than or equal to $r+2$ and characteristic different form 2; then*

$$a^K_{ant}(n;s,r)\le(n-1)\frac{r}{2}-\frac{s^2}{4}.$$

Observe that if we take $r=n$ in the theorem above, we get the bound of Theorem 10.

Theorem 13. *Let $s,r,m,n\in\mathbb{N}$ with $s\le r\le\min\{m,n\}$ and let K be a field with $|K|\ge r+1$; then*

$$a^K_{ech}(m\times n;s,r)=rn-\tfrac{r(r+1)}{2}\quad \textit{if}\ \ s=r,$$

$$sn-\tfrac{s(s+1)}{2}+n-s\le a^K_{ech}(m\times n;s,r)\le rn-\tfrac{r(r+1)}{2}+1\quad\textit{if}\ \ s<r.$$

Finally, we point out that, as to the techniques we use in this paper, both because of the generalization from $\mathbb{R}$ to a more general field and because of the generalization from constant rank to rank in a range, we had to use more advanced techniques with respect to the ones used in the previous papers (for instance, in the paper [10], we used that the eigenvalues of real symmetric matrices are real and the nonnegativity of a sum of squares of real numbers). Moreover, we emphasize that we cannot deduce the results of this paper from the previous ones, since the bound we get on the maximal dimension of affine subspaces of matrices with rank in a range is not the minimum of the bound on the maximal dimension of affine subspaces of matrices with rank bounded below and of the bound on the maximal dimension of affine subspaces of matrices with rank bounded above.

2. Proof of the theorems.

Notation 14. *Let $m,n\in\mathbb{N}-\{0\}$ and K be a field.*

We denote the $n\times n$ identity matrix over K by I^K_n and the $m\times n$ null matrix over K by $0^K_{m\times n}$. Moreover, we denote $E^{n,K}_{i,j}$ the $n\times n$ matrix over K such that

$$(E^{n,K}_{i,j})_{x,y}=\begin{cases}1 & \text{if } (x,y)=(i,j)\\ 0 & \text{otherwise}\end{cases}.$$

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We omit the superscripts when it is clear from the context.

We write $diag(d_1, \dots, d_n)$ for the diagonal matrix whose diagonal entries are $d_1, \dots, d_n$.

For any $A \in K^{m \times n}$, the submatrix of A given by the rows $i_1, , \dots, i_k$ and the columns $j_1, \dots, j_s$ will be denoted by $A^{(j_1,\dots,j_s)}_{(i_1,\dots,i_k)}$.

We define J to be the (2×2)-matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and we denote by $\overline{J}_{2n}$ the $(2n \times 2n)$ block diagonal matrix whose diagonal blocks are equal to J. We omit the subscript when it is clear from the context.

Let A and B be two subsets of $\mathbb{N}$; we write $A < B$ if $a < b$ for any $a \in A$ and $b \in B$.

We recall a lemma from [12]; in [12] it was stated only for $K = \mathbb{R}$, but it can be proved on any field with the same proofs.

Lemma 15. *Let $n_1, \dots, n_k, q_1, \dots, q_k, m, r \in \mathbb{N}$ and let K be a field. Let $h = 3m + n_1 + \dots + n_k$.*

Let $\pi_1 : K^h \to K^{2m}$ be the projection onto the first $2m$ coordinates.

Let $\pi_2 : K^h \to K^{2m}$ be the projection onto the coordinates $m+1, \dots, 3m$.

Let $\pi_3 : K^h \to K^{2m}$ be the projection onto the coordinates $1, \dots, m, 2m+1, \dots 3m$.

Finally, let $p_1 : K^h \to K^{n_1}$ be the projection onto the coordinates $3m+1, \dots, 3m+n_1$, let $p_2 : K^h \to K^{n_2}$ be the projection onto the coordinates $3m + n_1 + 1, \dots, 3m + n_1 + n_2$ and so on.

Let V be a vector subspace of K^h such that $\dim(\pi_i(V)) \leq 2r$ for $i = 1, 2, 3$ and $\dim(p_j(V)) \leq q_j$ for $j = 1, \dots, k$; then

$$\dim(V) \leq \sum_{j=1,\dots,k} q_j + 3r.$$

Finally, we recall Schur's Lemma:

Lemma 16 (Schur's Lemma). *Let K be a field and $m, n \in \mathbb{N} - \{0\}$. Let $A \in K^{n \times n}$, $D \in K^{m \times m}$, $C \in K^{m \times n}$, $B \in K^{n \times m}$; if A is invertible, then*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B);$$

if D is invertible, then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D) \det(A - BD^{-1}C).$$

Proof of Theorem 11. Let us suppose $m \leq n$. To prove the inequality

$$a^K(m \times n; s, r) \geq rn - \binom{s+1}{2},$$

consider the following affine subspace:

$$S = \left\{ C \in K^{m \times n} \mid \begin{array}{l} C_{i,j} = 0 \text{ for } i, j \in \{1, \dots, s\} \text{ with } i > j \\ C_{i,i} = 1 \text{ for } i = 1, \dots, s, \;\; C_{i,j} = 0 \text{ for } i > r \end{array} \right\}.$$

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We can see easily that s is the minimum of $\{rk(C)|\ C \in S\}$ and r is the maximum and that the dimension of S is $rn - \binom{s+1}{2}$.

Now let us prove the other inequality. Let $S \in \mathcal{A}^K(m \times n; s, r)$; we prove that $\dim(S) \leq rn - \binom{s+1}{2}$.

We can write $S = G + V$, where $rk(G) = r$ and V is a linear subspace of $K^{m\times n}$. We can suppose $G = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \in K^{m\times n}$. Let

$$P = \{C \in K^{m\times n}|\ C_{i,j} = 0 \text{ if } i > r \text{ and } j > r\}.$$

First let us prove that

$$V \subseteq P. \tag{1}$$

Let $v \in V$. Let $i \in \{r+1, \ldots, m\}$ and $j \in \{r+1, \ldots, n\}$. We have that

$$\det\left((G + tv)^{(1,\ldots,r,j)}_{(1,\ldots,r,i)}\right),$$

is a polynomial in t with coefficient of the term of degree 1 equal to $v_{i,j}$ and degree at most $r+1$; since $|K| \geq r+2$, we must have $v_{i,j} = 0$ (otherwise there would exist t such that $\det\left((G + tv)^{(1,\ldots,r,j)}_{(1,\ldots,r,i)}\right) \neq 0$, which is contrary to our assumption on S). Hence, we have proved (1).

Let

$$Q = \{C \in P|\ C_{i,j} = 0 \text{ if } \ i \leq r \text{ and } j \leq r\}.$$

Obviously $\dim(Q) = r(m-r) + r(n-r)$. Let

$$\pi : P \longrightarrow Q,$$

be the map

$$\begin{pmatrix} A & B & X \\ C & 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & B & X \\ C & 0 & 0 \end{pmatrix},$$

where A is $r \times r$ and B is $r \times (m-r)$.

For any $i, j \in \{1, \ldots, m-r\}$, let

$$\pi_{i,j} : P \longrightarrow K^{2r},$$

be the map

$$\begin{pmatrix} A & B & X \\ C & 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} B^{(j)} \\ {}^t(C_{(i)}) \end{pmatrix},$$

where A is $r \times r$ and B is $r \times (m-r)$.

Finally, if $n > m$, let

$$p : P \longrightarrow K^{r\times(n-m)},$$

be the map

$$\begin{pmatrix} A & B & X \\ C & 0 & 0 \end{pmatrix} \longmapsto X,$$

where, again, A is $r \times r$ and B is $r \times (m-r)$.

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Since, for any $\begin{pmatrix} A & B & X \\ C & 0 & 0 \end{pmatrix} \in V$, we have that $\det \begin{pmatrix} I_r + tA & tB^{(j)} \\ tC_{(i)} & 0 \end{pmatrix}$ is a polynomial in t with coefficient of the term of degree 2 equal to $-C_{(i)}B^{(j)}$ (in fact the term of degree 2 is equal to $\det \begin{pmatrix} I_r & tB^{(j)} \\ tC_{(i)} & 0 \end{pmatrix}$) and degree at most $r+1$ and since $|K| \geq r+2$, we must have that, for any $i = 1, \ldots, m-r$ and for any $j = 1, \ldots, m-r$, every element of the subspace $\pi_{i,j}(V) = \pi_{i,j}(\pi(V))$ must be isotropic with respect to the nondegenerate quadratic form on K^{2r} defined by $\sum_{i=1,\ldots,r} x_i x_{r+i}$, hence $\dim(\pi_{i,j}(\pi(V))) \leq r$.

Since

$$\pi(V) \subset \pi_{1,1}(\pi(V)) + \ldots + \pi_{m-r,m-r}(\pi(V)) + p(\pi(V)),$$

we get

$$\dim(\pi(V)) \leq \underbrace{r + \ldots + r}_{m-r} + (n-m)r = (m-r)r + (n-m)r = (n-r)r.$$

Hence, there exists a $(m-r)r$-dimensional subspace Z in Q such that

$$\pi(V) \cap Z = \{0\}.$$

Let

$$W = \{C \in K^{m \times n} |\ C_{i,j} = 0 \text{ if } \ i \geq r+1 \text{ or } j \geq r+1\}, \tag{2}$$

and let $\tilde{W}$ be a subspace of W such that

$$W = (W \cap V) \oplus \tilde{W};$$

hence $V \cap \tilde{W} = \{0\}$. We state that

$$V \cap (Z \oplus \tilde{W}) = \{0\}. \tag{3}$$

□

Let $\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \tilde{W}$ and $\begin{pmatrix} 0 & B & X \\ C & 0 & 0 \end{pmatrix} \in Z$ such that $\begin{pmatrix} A & B & X \\ C & 0 & 0 \end{pmatrix} \in V$, so $\begin{pmatrix} A & B & X \\ C & 0 & 0 \end{pmatrix} \in V \cap (Z \oplus \tilde{W})$; therefore,

$$\begin{pmatrix} 0 & B & X \\ C & 0 & 0 \end{pmatrix} = \pi \begin{pmatrix} A & B & X \\ C & 0 & 0 \end{pmatrix} \in \pi(V) \cap Z = \{0\},$$

so $B = 0$, $X = 0$, $C = 0$; hence, $\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in V \cap \tilde{W} = \{0\}$, so $A = 0$.

From (1) and (3), we get

$$\begin{aligned} &\dim(V) \leq \dim(P) - \dim(Z) - \dim(\tilde{W}) = \\ &= \dim(P) - \dim(Z) - \dim(W) + \dim(W \cap V) = \\ &= mn - (m-r)(n-r) - r(m-r) - r^2 + \dim(W \cap V) \leq \\ &\overset{*}{\leq} mn - (m-r)(n-r) - r(m-r) - r^2 + r^2 - \binom{s+1}{2} = \\ &= rn - \binom{s+1}{2}, \end{aligned}$$

where the inequality $(*)$ holds by what follows:

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let

$$U = \left\{ A \in K^{r \times r} | \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in W \cap V \right\}$$

$$\dim(W \cap V) = \dim(U) = \dim(I_r + U) \leq r^2 - \binom{s+1}{2},$$

where the inequality holds by Theorem 9.

Proof of Theorem 12. Let $S \in \mathcal{A}^K_{ant}(n; s, r)$; we prove that $\dim(S) \leq (n-1)\frac{r}{2} - \frac{s^2}{4}$.

We can write S as $M + V$ where $M \in K_a^{n \times n}$, $rk(M) = r$ and V is a linear subspace of $K_a^{n \times n}$. Let H be an invertible matrix such that ${}^tHMH$ is a matrix G of the kind

$$diag(d_1, d_1, d_2, d_2, \ldots, d_{\frac{r}{2}}, d_{\frac{r}{2}}, 1, \ldots, 1) \begin{pmatrix} \overline{J}_r & 0 \\ 0 & 0 \end{pmatrix},$$

with $d_i \in K - \{0\}$ for every i. Define $\gamma = diag(d_1, d_1, d_2, d_2, \ldots, d_{\frac{r}{2}}, d_{\frac{r}{2}})\overline{J}_r$.

Let

$$V' = {}^tHVH,$$

and

$$S' = {}^tHSH = G + V'.$$

Obviously, $S' \in \mathcal{A}^K_{ant}(n; s, r)$; moreover, $\dim(S') = \dim(S)$, so, to prove that $\dim(S) \leq (n-1)\frac{r}{2} - \frac{s^2}{4}$, it is sufficient to prove that $\dim(S') \leq (n-1)\frac{r}{2} - \frac{s^2}{4}$. We rename S' by S and V' by V.

Let

$$P = \{C \in K_a^{n \times n} | \, C_{i,j} = 0 \text{ if } i > r \text{ and } j > r\}.$$

First let us prove that

$$V \subseteq P. \tag{4}$$

Let $v \in V$. Let $i \in \{r+1, \ldots, n\}$ and $j \in \{r+1, \ldots, n\}$. We have that

$$\det\left((G + tv)^{(1,\ldots,r,j)}_{(1,\ldots,r,i)} \right),$$

is a polynomial in t with coefficient of the term of degree 1 equal to $d_1^2 \ldots d_{\frac{r}{2}}^2 v_{i,j}$ and degree at most $r+1$; since $|K| \geq r+2$ we must have $v_{i,j} = 0$ (otherwise, there would exist t such that $\det\left((G + tv)^{(1,\ldots,r,j)}_{(1,\ldots,r,i)} \right) \neq 0$, which is contrary to our assumption on S). Hence, we have proved (4).

Let

$$Q = \{C \in P | \, C_{i,j} = 0 \text{ if } \; i \leq r \text{ and } j \leq r\}.$$

Let

$$\pi : P \longrightarrow Q,$$

be the map

$$\begin{pmatrix} A & B \\ -{}^tB & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & B \\ -{}^tB & 0 \end{pmatrix},$$

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where A is $r \times r$.

For any $i, j \in \{1, \dots, n-r\}$, let

$$\pi_{i,j} : P \longrightarrow K^{2r},$$

be the map

$$\begin{pmatrix} A & B \\ -{}^tB & 0 \end{pmatrix} \longmapsto \begin{pmatrix} B^{(i)} \\ B^{(j)} \end{pmatrix},$$

where A is $r \times r$.

Obviously, $\det \begin{pmatrix} \gamma + hA & hB^{(j)} \\ -h({}^tB)_{(i)} & 0 \end{pmatrix}$ must be 0 for any $\begin{pmatrix} A & B \\ -{}^tB & 0 \end{pmatrix} \in V$ and for any $h \in K$; moreover $\det \begin{pmatrix} \gamma + hA & hB^{(j)} \\ -h({}^tB)_{(i)} & 0 \end{pmatrix}$ is a polynomial in h of degree less than or equal to $r+1$ and term of degree 2 equal to

$$d_1^2 \dots d_{\frac{r}{2}}^2 \sum_{l=1,\dots,r} \frac{1}{d_l}(b_{2l-1,i}b_{2l,j} - b_{2l-1,j}b_{2l,i}), \tag{5}$$

(in fact the term of degree 2 is equal to $\det \begin{pmatrix} \gamma & hB^{(j)} \\ -h({}^tB)_{(i)} & 0 \end{pmatrix}$, which can be calculated by Schur's Lemma, see Lemma 16). Therefore, since $|K| \geq r+2$, we must have that, for any $i, j \in \{1, \dots, n-r\}$, every element of the subspace $\pi_{i,j}(V) = \pi_{i,j}(\pi(V))$ in K^{2r} must be isotropic with respect to the nondegenerate quadratic form given by (5). Hence,

$$\dim(\pi_{i,j}(\pi(V))) \leq r. \tag{6}$$

If $n-r$ is even, consider the projections $\pi_{1,2}, \pi_{3,4}, \dots, \pi_{n-r-1,n-r}$. If $n-r$ is odd, consider the projections $\pi_{1,2}, \pi_{1,3}, \pi_{2,3}, \pi_{4,5}, \dots, \pi_{n-r-1,n-r}$.

By Lemma 15 and from (6), we get that

$$\dim(\pi(V)) \leq \frac{r}{2}(n-r).$$

Hence, there exists a $\frac{r}{2}(n-r)$-dimensional vector subspace Z in Q such that

$$\pi(V) \cap Z = \{0\}. \tag{7}$$

Let

$$W = \{C \in K_a^{n \times n} | \; C_{i,j} = 0 \text{ if } \; i \geq r+1 \text{ or } j \geq r+1\},$$

and let $\tilde{W}$ be a subspace of W such that

$$W = (W \cap V) \oplus \tilde{W}.$$

We state that

$$V \cap (Z \oplus \tilde{W}) = \{0\}. \tag{8}$$

□

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In fact, let $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in \tilde{W}$ and $\begin{pmatrix} 0 & B \\ -{}^tB & 0 \end{pmatrix} \in Z$ such that $\begin{pmatrix} A & B \\ -{}^tB & 0 \end{pmatrix} \in V$, so $\begin{pmatrix} A & B \\ -{}^tB & 0 \end{pmatrix} \in V \cap (Z \oplus \tilde{W})$; therefore,

$$\begin{pmatrix} 0 & B \\ -{}^tB & 0 \end{pmatrix} = \pi \begin{pmatrix} A & B \\ -{}^tB & 0 \end{pmatrix} \in \pi(V) \cap Z = \{0\},$$

so $B = 0$; hence, $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in V \cap W \cap \tilde{W} = \{0\}$, so $A = 0$.

From (4) and (8), we get

$$\begin{aligned} &\dim(V) \le \dim(P) - \dim(Z) - \dim(\tilde{W}) = \\ &= \dim(P) - \dim(Z) - \dim(W) + \dim(W \cap V) \\ &= \tfrac{n(n-1)}{2} - \tfrac{(n-r)(n-r-1)}{2} - \tfrac{r(n-r)}{2} - \tfrac{r(r-1)}{2} + \dim(W \cap V) \le \\ &\overset{*}{\le} \tfrac{n(n-1)}{2} - \tfrac{(n-r)(n-r-1)}{2} - \tfrac{r(n-r)}{2} - \tfrac{r(r-1)}{2} + \tfrac{r(r-1)}{2} - \tfrac{s^2}{4} = \\ &= \tfrac{r}{2}(n-1) - \tfrac{s^2}{4}, \end{aligned}$$

where the inequality $(*)$ holds by what follows: let

$$U = \left\{ A \in K_a^{r \times r} | \ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in W \cap V \right\}$$

$$\dim(W \cap V) = \dim(U) = \dim(\gamma + U) \le \frac{r(r-1)}{2} - \frac{s^2}{4},$$

where the inequality holds by Theorem 10.

To prove Theorem 13, we need the following proposition.

Proposition 17. *Let $m, n \in \mathbb{N} - \{0\}$ and K be a field. Let S be an affine subspace in $K^{m \times n}$ such that A is in row echelon form for every $A \in S$. Let*

$$Z(S, i) = \{j \in \{1, \ldots, n\} | \ A_{i,j} \neq 0 \text{ for some } A \in S\} \quad \forall i = 1, \ldots, n,$$

$$P := \{i \in \{1, \ldots, m\} | \ Z(S, i) \neq \emptyset\},$$

and

$$j_i = \min Z(S, i) \quad \forall i \in P.$$

Let $r = \max\{rk(A) | \ A \in S\}$. Then $P = \{1, \ldots, r\}$. Moreover, if $|K| \ge r+1$, there exists $A \in S$ such that $A_{i,j_i} \neq 0$ for every $i \in P$.

Proof. We can suppose that S contains a nonzero matrix. Obviously, P is a set of the kind $\{1, \ldots, l\}$ for some $l \in \{1, \ldots, m\}$. Since in S there exists a matrix of rank r, necessarily $l \ge r$. Moreover, there exists a matrix A in S such that $A_{(l)} \neq 0$, but A is in row echelon form, so if l were greater than r, we would have $rk(A) \ge l > r$, which is absurd; hence, $l = r$. Then $P = \{1, \ldots, r\}$.

We prove by induction on k that there exists $A \in S$ such that $A_{i,j_i} \neq 0$ for $i = 1, \ldots, k$ for $k \le r$. For $k = 1$, the statement is obvious.

Let us us prove the induction step $k \implies k+1$. Let $A \in S$ such that $A_{i,j_i} \neq 0$ for $i = 1, \ldots, k$. If $A_{k+1,j_{k+1}} \neq 0$, there is nothing to prove. Suppose $A_{k+1,j_{k+1}} = 0$; let $A' \in S$ be such that $A'_{k+1,j_{k+1}} \neq 0$. We search for $\lambda \in K$ such that, for $i = 1, \ldots, k+1$,

$$(\lambda A + (1-\lambda)A')_{i,j_i} \neq 0,$$

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that is

$$\lambda(A_{i,j_i} - A'_{i,j_i}) \neq -A'_{i,j_i};$$

observe that

- for $i = 1, \ldots, k$, if $A_{i,j_i} = A'_{i,j_i}$, then the inequality is verified for any λ because $A'_{i,j_i} = A_{i,j_i} \neq 0$, while if $A_{i,j_i} \neq A'_{i,j_i}$, then the inequality is obviously equivalent to

$$\lambda \neq -\frac{A'_{i,j_i}}{A_{i,j_i} - A'_{i,j_i}};$$

- for $i = k+1$ the inequality is obviously equivalent to $\lambda \neq 1$.

Since $|K| \geq r+1$, we can find λ as requested.

The matrix $\lambda A + (1-\lambda)A'$ satisfies the conditions we wanted. □

Now we are ready to prove Theorem 13.

Proof of Theorem 13. First consider the case $\underline{r = s}$.

To prove the inequality $a^K_{ech}(m \times n; r, r) \geq rn - \frac{r(r+1)}{2}$, consider the affine subspace:

$$\left\{ A \in K^{m\times n} \middle| \begin{array}{l} A_{i,i} = 1 \text{ for } i = 1, \ldots, r, \\ A_{i,j} = 0 \text{ for } i = r+1, \ldots, m, \ j = 1, \ldots, n \\ A_{i,j} = 0 \text{ for } i > j \end{array} \right\}.$$

It is in $\mathcal{A}^K_{ech}(m \times n; r, r)$ and its dimension is $rn - \frac{r(r+1)}{2}$.

Now let us prove the other inequality. Let $S \in \mathcal{A}^K_{ech}(m \times n; r, r)$. Let V be the direction of S. Let

$$P := \{i \in \{1, \ldots, n\} | \ Z(S,i) \neq \emptyset\}, \qquad P' := \{i \in \{1, \ldots, n\} | \ Z(S,i) = \emptyset\},$$

where we use the same notation as in Proposition 17. By Proposition 17, we have that $P = \{1, \ldots, r\}$ and there exists $H \in S$ such that $H_{i,j_i} \neq 0$ for every $i \in P$.

By simultaneous elementary column operations on every element of S, precisely by operations of the kind "to add a multiple of a column to a following column," we can suppose that the only nonzero entries of H are the pivots. Observe that a matrix in row echelon form remains in row echelon form if we add a multiple of a column to a following column, so the elements of S are still in row echelon form.

Let

$$L = \{(i,j) \in \{1, \ldots, m\} \times \{1, \ldots, n\} | \ H_{i,j} \neq 0\} = \{(1, j_1), \ldots, (r, j_r)\}.$$

Let

$$W = \left\{ A \in K^{m\times n} \middle| \begin{array}{ll} A_{i,j} = 0 & \text{if } i = r+1, \ldots, m \\ A_{i,j} = 0 & \text{if } i = 1, \ldots, r \text{ and } j < j_i \end{array} \right\}.$$

Obviously,

$$\dim(W) \leq rn - \frac{r(r-1)}{2}.$$

Let

$$D = \{A \in K^{m\times n} | \ A_{i,j} = 0 \ \forall (i,j) \notin L\}.$$

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We have that

$$V \cap D = \{0\},$$

otherwise, there would exist in S an element of rank less than r.

So by Grassmann's formula, we get

$$0 = \dim(V \cap D) = \dim(V) + \dim(D) - \dim(V + D),$$

hence,

$$\begin{aligned} &\dim(V) = -\dim(D) + \dim(V + D) = \\ &= -r + \dim(V + D) \le -r + \dim(W) \le rn - \tfrac{r(r-1)}{2} - r = rn - \tfrac{r(r+1)}{2}. \end{aligned}$$

Consider now the <u>case $s < r$.</u>

To prove the inequality $a^K_{ech}(m \times n; s, r) \ge sn - \frac{s(s+1)}{2} + n - s$, consider the affine subspace

$$\left\{ A \in K^{m \times n} | \begin{array}{l} A_{i,i} = 1 \text{ for } i = 1, \dots, s, \\ A_{i,j} = 0 \text{ for } i = r+1, \dots, m, \ j = 1, \dots, n \\ A_{i,j} = 0 \text{ for } i > j \\ A_{s+1,j} = A_{s+1+1,j+1} = \dots = A_{s+1+r-s,j+r-s} \text{ for } j = s+1, \dots, n-1 \end{array} \right\}.$$

It is in $\mathcal{A}^K_{ech}(m \times n; s, r)$ and its dimension is $sn - \frac{s(s+1)}{2} + n - s$.

Now let us prove the inequality $a^K_{ech}(m \times n; s, r) \le rn - \frac{r(r+1)}{2} + 1$.

Let $S \in \mathcal{A}^K_{ech}(m \times n; s, r)$, let V be the direction of S, P, P' and W be defined as in the case $r = s$. By Proposition 17, we have that $P = \{1, \dots, r\}$ and there exists $H \in S$ such that $H_{i,j_i} \neq 0$ for every $i \in P$. As in the case $r = s$, we can suppose that the only nonzero entries of H are the pivots.

Define

$$U = \langle E_{1,j_1}, \dots, E_{r-1,j_{r-1}} \rangle.$$

Obviously, $\dim(U) = r - 1$ and $V \cap U = \{0\}$, in fact: if there existed $\lambda_1, \dots, \lambda_{r-1}$ not all zero, for instance such that $\lambda_{\bar{\imath}} \neq 0$, such that

$$\lambda_1 E_{1,j_1} + \dots + \lambda_{r-1} E_{r-1.j_{r-1}} \in V,$$

then

$$H - \frac{H_{\bar{\imath},j_{\bar{\imath}}}}{\lambda_{\bar{\imath}}} (\lambda_1 E_{1,j_1} + \dots + \lambda_{r-1} E_{r-1,j_{r-1}}),$$

would be an element of S not in row echelon form.

So by Grassmann's formula, we get

$$0 = \dim(V \cap U) = \dim(V) + \dim(U) - \dim(V + U),$$

hence,

$$\begin{aligned} &\dim(V) = -\dim(U) + \dim(V + U) = -r + 1 + \dim(V + U) \le \\ &\le -r + 1 + \dim(W) \le rn - \tfrac{r(r-1)}{2} - r + 1 = rn - \tfrac{r(r+1)}{2} + 1. \end{aligned}$$

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Remark 18. Let $r, s, m, n \in \mathbb{N}$ with $s \leq r \leq m \leq n$ and let K be a field. It is natural to wonder what we can say about $a^K(m \times n; s, r)$ if we have not the assumptions on K, we have in Theorem 11.

It is easy to see that, without the assumptions on the field, the statement of Theorem 11 does not hold any more: consider for instance $K = \mathbb{Z}/3$, $m = n = 3$, $r = s = 2$ and the affine subspace

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a+1 & d \\ 0 & 0 & a+2 \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}/3 \right\};$$

every element of S has rank 2, the dimension of S is 4, while $rn - \binom{s+1}{2} = 3$.

Obviously, if the cardinality of K is less than $r+2$ and $p(x)$ is a polynomial on K of degree $r+1$, we cannot say that in K there is an element which is not a root of p and it seems difficult to bypass this argument we use in the proof of the theorems of this paper. Anyway, we can easily prove that

$$a^K(m \times n; s, r) \leq mn - \binom{s+1}{2},$$

and, if the characteristic of K is different from 2, we can prove also that

$$a^K(m \times n; s, r) \leq mn - (m-r)r.$$

The proof is the following.

Let $S \in \mathcal{A}^K(m \times n; s, r)$. We can suppose $S = G + V$, where $G = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \in K^{m \times n}$ and V is a vector subspace of $K^{m \times n}$. Let

$$W = \{C \in K^{m \times n} \mid C_{i,j} = 0 \text{ if } \ i \geq r+1 \text{ or } j \geq r+1\}, \tag{9}$$

and let $\tilde{W}$ be a subspace of W such that

$$W = (W \cap V) \oplus \tilde{W};$$

hence $V \cap \tilde{W} = \{0\}$. So,

$$\dim(V) \leq mn - \dim(\tilde{W}) = mn - \dim(W) + \dim(V \cap W) \leq mn - r^2 + r^2 - \binom{s+1}{2},$$

where the last inequality holds by Theorem 9 as in the proof of Theorem 11.

Moreover, let

$$U = \{C \in K^{m \times n} \mid C_{i,j} = 0 \text{ if } \ i, j \leq r \text{ or } i, j \geq r+1\}, \tag{10}$$

and let $\tilde{U}$ be a subspace of W such that

$$U = (U \cap V) \oplus \tilde{U};$$

hence, $V \cap \tilde{U} = \{0\}$. For any $i, j \in \{1, \ldots, m-r\}$, let

$$\pi_{i,j} : U \longrightarrow K^{2r},$$

be the map

$$\begin{pmatrix} 0_{r\times r} & B \\ C & 0 \end{pmatrix} \longmapsto \begin{pmatrix} B^{(j)} \\ {}^t(C_{(i)}) \end{pmatrix},$$

and, if $n > m$, let

$$p : U \longrightarrow K^{r\times(n-m)},$$

be the map

$$\begin{pmatrix} 0_{r\times r} & B \\ C & 0 \end{pmatrix} \longmapsto B^{(r+1,\dots,n)}.$$

Since for every $\begin{pmatrix} 0_{r\times r} & B \\ C & 0 \end{pmatrix} \in V \cap U$ and for any $i, j \in \{1, \dots, m-r\}$, we have that $\det \begin{pmatrix} I_r & B^{(j)} \\ C_{(i)} & 0 \end{pmatrix} = 0$, every element of the subspace $\pi_{i,j}(V \cap U)$ must be isotropic with respect to the nondegenerate quadratic form on K^{2r} defined by $\sum_{i=1,\dots,r} x_i x_{r+i}$, hence $\dim(\pi_{i,j}(V \cap U)) \leq r$.

Since

$$V \cap U \subset \pi_{1,1}(V \cap U) + \dots + \pi_{m-r,m-r}(V \cap U) + p(V \cap U),$$

we get

$$\dim(V \cap U) \leq \underbrace{r + \dots + r}_{m-r} + (n-m)r = (m-r)r + (n-m)r = (n-r)r.$$

Hence,

$$\dim(V) \leq mn - \dim(\tilde{U}) = mn - \dim(U) + \dim(V \cap U) \leq mn - (m-r)r.$$

Note. In the paper, C. De Seguins Pazzis "On affine spaces of rectangular matrices with constant rank" arXiv:2405.02689 appeared on arXiv in the same period as this paper, the author obtains a result analogous to Theorem 11 in the case $r = s$; the two results have been obtained independently.

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