Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 74-85, January 2025.

SCHUR STABILITY OF MATRIX SEGMENT VIA BIALTERNATE PRODUCT[*]

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Abstract. In this study, the problem of robust Schur stability of $n \times n$ dimensional matrix segments by using the bialternate product of matrices is considered. It is shown that the problem can be reduced to the existence of negative eigenvalues of two of three specially constructed matrices and the existence of eigenvalues belonging to the interval $[1, \infty)$ of the third matrix. A necessary and sufficient condition is given for the convex combinations of two stable matrices with rank one difference to be robust Schur stable. It is shown that the robust stability of the convex hull of a finite number of matrices, where the difference between any two matrices has a rank of 1, is equivalent to the robust stability of the segments formed by these matrices. Examples of applying the obtained results are given.

Key words. Matrix segment, Schur stability, Bialternate product, Matrix polytope.

AMS subject classifications. 15A18, 93D09.

1. Introduction.

A real $n \times n$ matrix $A \in \mathbb{R}^{n\times n}$ is said to be Schur stable if all eigenvalues of A are contained in the open unit disk in the complex plane. This property is essential in the stability theory for discrete-time dynamical systems (see [2, 3]). From now the term stable will mean Schur stable.

The robust stability analysis of uncertain systems, which can be modeled using matrix segments or matrix polytopes, has attracted significant attention in recent years due to its wide range of applications in control theory. A polytope of matrices, which is the convex hulls of a finite number of matrices, are established as one of the standard representations of uncertainties involved in state-space models of control systems [2, 3, 5, 13]. When a polytope of matrices formulates the system matrices of uncertain systems, a stability problem of the polytope naturally arises. The problem checking whether all convex combinations of finite matrices are stable is NP-hard (see [10]). One generally cannot expect extreme point or edge results on the stability of polytope of matrices. Polytopes of matrices appear in stability problems of linear switched systems as well.

Some more general problems covering this topic concern parameter-dependent matrices. In Ref. [4], necessary and sufficient conditions are formulated for the eigenvalues of a parameter-dependent matrix to belong to a certain region D of the complex plane. The stability problem of a matrix polytope revolves around identifying the robust stability conditions for its extreme matrices. While the existence of a common quadratic Lyapunov function (CQLF) serves as a sufficient condition for stability [1], robust stability for discrete-time switched systems with polytopic uncertainties often depends on additional conditions involving the convex hull of subsystem matrices and their interactions [16]. For two discrete-time systems, Ref. [12] explores the conditions for the existence or nonexistence of a CQLF and establishes corresponding theorems for stable matrices.

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[*] Received by the editors on July 2, 2024. Accepted for publication on December 23, 2024. Handling Editor: Dario Fasino. Corresponding Author: Şerife Yılmaz.

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In Ref. [7], using the notion of a block P-matrix, the condition for stability of a matrix polytope is given. In Ref. [13], for the stability of a matrix polytope, an affine Lyapunov function with a common additional variable is constructed. This condition is expressed as a set of linear matrix inequalities.

In Ref. [14], the stability problem of a matrix segment is considered by using the Kronecker product of matrices. In Ref. [15] for stable companion matrices, the box coefficients are defined, and using these coefficients, an inner approximation of the stability region by polytopes is given. Due to the Kronecker product, $n^2 \times n^2$ dimensional matrices arise in this approach. In this paper, we consider the stability problem of a polytope by using the bialternate product of matrices. In Section 2, we give a necessary and sufficient condition for the stability of a matrix segment in terms of the bialternate product. In comparison with Ref. [14], we use $d \times d$ matrices, where $d = n(n-1)/2$. A sufficient condition for 2×2 Metzler matrix segment is given at the end of the section. In Section 3, a matrix polytope with extreme matrices whose differences are of rank 1 is considered. It is shown that for this polytope, the Edge Theorem is true, and by using the results of Section 3, a necessary and sufficient condition for stability is given.

Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be stable matrices. If each matrix from the segment

$$[A_1, A_2] = \{C(\alpha) = \alpha A_1 + (1-\alpha)A_2 :\ \alpha \in [0,1]\},$$

is stable, the segment $[A_1, A_2]$ is called (robust) stable segment.

2. Stability of all convex combinations of two matrices.

In this section, we give necessary and sufficient condition for robust stability of a matrix segment in terms of the bialternate product of matrices.

For $A \in \mathbb{R}^{n\times n}$, if λ is an eigenvalue of A, then $\lambda + \alpha$ is an eigenvalue of $A + \alpha I$, where α is scalar and I is the identity matrix. If A is stable matrix, then $(I - A)$ and $(I + A)$ are nonsingular.

The boundary of the stability region is the unit circle $\partial D = \{z \in \mathbb{C} :\ |z| = 1\}$. If there is an unstable matrix on a matrix segment with stable endpoints, then according to continuous root dependence [2, p. 52] there exists a matrix in this segment with an eigenvalue on the unit circle. For this reason, we will express the following three lemmas regarding the absence of an eigenvalue on the unit circle. The first two of these are related to whether there are matrices from the segment with an eigenvalue of ±1. The third lemma is about the existence of matrices from the segment with complex eigenvalues on $\Theta = \{z \in \mathbb{C} : |z| = 1,\ z \neq \pm 1\}$.

LEMMA 2.1. *Assume that* $A_1, A_2 \in \mathbb{R}^{n\times n}$ *and* A_2 *is stable. For all* $\alpha \in [0,1]$, $(-1)^n \det[C(\alpha) - I] > 0$ *if and only if* $(I - A_1)(I - A_2)^{-1}$ *has no negative real eigenvalue.*

Proof. $(\Leftarrow)$: Assume that $(I - A_1)(I - A_2)^{-1}$ has no negative real eigenvalue and $\alpha \in (0,1]$, then

$$\begin{aligned}(-1)^n \det[C(\alpha) - I] &= \det[I - C(\alpha)]\\ &= \det[I - (\alpha A_1 + (1-\alpha)A_2)]\\ &= \det[\alpha I + (1-\alpha)I - \alpha A_1 - (1-\alpha)A_2]\\ &= \det[\alpha(I - A_1) + (1-\alpha)(I - A_2)]\\ &= \det[\alpha\left((I - A_1)(I - A_2)^{-1} + \tfrac{1-\alpha}{\alpha} I\right)(I - A_2)]\\ &= \alpha^n . \det[I - A_2] . \det[(I - A_1)(I - A_2)^{-1} + \tfrac{1-\alpha}{\alpha} I].\end{aligned}$$

Here $\det[I - A_2] \neq 0$ because the matrix A_2 is stable. Since $(I - A_1)(I - A_2)^{-1}$ has no negative eigenvalue, $\det[(I - A_1)(I - A_2)^{-1} + \frac{1-\alpha}{\alpha} I] \neq 0$ for each $\alpha \in (0,1]$ $((1-\alpha)/\alpha \in (0,\infty) \Leftrightarrow \alpha \in (0,1])$. As a result,

$$(-1)^n \det(C(\alpha) - I) \neq 0 \text{ for all } \alpha \in (0,1]. \tag{2.1}$$

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Since A_2 is stable, the matrix $A_2 - I$ is Hurwitz stable (all eigenvalues lie in the open left half of the complex plane). The Routh–Hurwitz stability criteria lead to the positivity of the coefficients of the characteristic polynomial of $A_2 - I$ [2, p. 51]. Here $\det[C(0) - I] = \det[A_2 - I]$.

The constant term of the characteristic polynomial of $A_2 - I$ is $(-1)^n \det[A_2 - I]$. Then

$$(-1)^n \det[C(0) - I] = (-1)^n \det[A_2 - I] > 0. \tag{2.2}$$

By (2.1), (2.2) the function $\alpha \to f(\alpha) = (-1)^n \det[C(\alpha) - I]$ satisfies $f(0) > 0$, $f(\alpha) \neq 0$ for all $\alpha \in (0,1)$. By continuity $f(\alpha) > 0$ for all $\alpha \in [0,1]$.

$(\Rightarrow)$: Assume that $(-1)^n \det[C(\alpha) - I] > 0$. Take arbitrary $\alpha \in (0,1]$. Then

$$\begin{aligned} 0 < (-1)^n \det[C(\alpha) - I] &= \det[I - C(\alpha)] \\ &= \alpha^n \det[I - A_2] \det[(I - A_1)(I - A_2)^{-1} - \beta I], \end{aligned} \tag{2.3}$$

where $\beta = -(1-\alpha)/\alpha$. Since the mapping $-(1-\alpha)/\alpha : (0,1] \to (-\infty, 0)$ is onto, from (2.3) it follows that for all $\beta < 0$

$$\det[(I - A_1)(I - A_2)^{-1} - \beta I] \neq 0.$$

Therefore, $(I - A_1)(I - A_2)^{-1}$ has no negative eigenvalue. □

Corollary 2.2. *Assume that $A_1, A_2 \in \mathbb{R}^{n\times n}$ and A_2 is stable. $C(\alpha)$ has no eigenvalue $\lambda = 1 \Leftrightarrow (I - A_1)(I - A_2)^{-1}$ has no negative real eigenvalue.*

Proof. $(\Leftarrow)$: If $(I - A_1)(I - A_2)^{-1}$ has no negative real eigenvalue, then by Lemma 2.1 $(-1)^n \det[C(\alpha) - I] > 0$, consequently, $\det[C(\alpha) - I] \neq 0$ and $\lambda = 1$ is not eigenvalue of $C(\alpha)$ for all $\alpha \in [0,1]$.

$(\Rightarrow)$: If $C(\alpha)$ has no eigenvalue $\lambda = 1$ then $\det[C(\alpha) - I] \neq 0$ and by the proof of Lemma 2.1 (see the part of Lemma 2.1 starting from formula (2.1)) $(-1)^n \det[C(\alpha) - I] > 0$ and by Lemma 2.1 $(I - A_1)(I - A_2)^{-1}$ has no negative real eigenvalue. □

Lemma 2.3. *Assume that $A_1, A_2 \in \mathbb{R}^{n\times n}$ and A_2 is stable. For all $\alpha \in [0,1]$, $\det[C(\alpha) + I] > 0$ if and only if $(I + A_1)(I + A_2)^{-1}$ has no negative real eigenvalue.*

Proof. Define $B_1 = -A_1$, $B_2 = -A_2$ and in view of A is stable $\Leftrightarrow -A$ is stable by Lemma 2.1 we have

$$(I - B_1)(I - B_2)^{-1} \text{ has no negative real eigenvalue if and only if}$$

$$(-1)^n \det[\alpha B_1 + (1-\alpha) B_2 - I] > 0,$$

or

$$(I + A_1)(I + A_2)^{-1} \text{ has no negative real eigenvalue if and only if}$$

$$(-1)^{2n} \det[\alpha A_1 + (1-\alpha) A_2 + I] = \det[C(\alpha) + I] > 0.$$

□

Corollary 2.4. *Assume that $A_1, A_2 \in \mathbb{R}^{n\times n}$ and A_2 is stable. $C(\alpha)$ has no eigenvalue $\lambda = -1 \Leftrightarrow (I + A_1)(I + A_2)^{-1}$ has no negative real eigenvalue.*

In the following, we obtain condition on nonexistence of eigenvalue of $C(\alpha)$ in the set $\Theta = \{z \in \mathbb{C} : |z| = 1,\ z \neq \pm 1\}$. This condition is given in terms of the bialternate product of matrices.

Definition 2.5. *[6] The bialternate product of matrices $A = [a_{ij}]$, $B = [b_{ij}] \in \mathbb{R}^{n\times n}$ is defined to be the matrix $F = A \cdot B$ where the entries of F are given by*

$$f_{ij,kl} = \frac{1}{2}\left(\begin{vmatrix} a_{ik} & a_{il} \\ b_{jk} & b_{jl} \end{vmatrix} + \begin{vmatrix} b_{ik} & b_{il} \\ a_{jk} & a_{jl} \end{vmatrix}\right),$$

where (i, j), $(k, l) \in Q_{2,n} = \{(p, q) : \ p < q\}$.

The dimension of $A \cdot B$ is $d \times d$, where $d = n(n-1)/2$. If the eigenvalues of A are $\lambda_1, \lambda_2, \dots, \lambda_n$, then the eigenvalues of $A \cdot A$ are written $\lambda_i\lambda_j$, where $i = 1, 2, \dots, n-1$ and $j = i+1, i+2, \dots, n$ (see [6, 8, 9]). For example, if a matrix $A \in \mathbb{R}^{3\times 3}$ has three eigenvalues λ_1, λ_2, λ_3, then the eigenvalues of $A \cdot A$ are $\lambda_1\lambda_2$, $\lambda_1\lambda_3$, $\lambda_2\lambda_3$.

Lemma 2.6. *A matris $A \in \mathbb{R}^{n\times n}$ has an eigenvalue in Θ if and only if $\nu(A) := \det[I - A \cdot A] = 0$.*

Proof. If $\lambda \in \Theta$ is an eigenvalue of A, then $\bar{\lambda} \neq \lambda$ and $\bar{\lambda}$ is an eigenvalue of A as well and $\lambda \cdot \bar{\lambda} = |\lambda|^2 = 1$ is an eigenvalue of $A \cdot A$. Therefore, $\nu(A) = 0$. □

We have

$$\begin{aligned} I - C(\alpha)\cdot C(\alpha) &= I - (\alpha A_1 + (1-\alpha)A_2)\cdot(\alpha A_1 + (1-\alpha)A_2) \\ &= I - A_2 \cdot A_2 - 2\alpha(A_1 \cdot A_2 - A_2 \cdot A_2) \\ &\quad -\alpha^2(A_1 \cdot A_1 + A_2 \cdot A_2 - 2A_2 \cdot A_1). \end{aligned}$$

(We have used $(A + B)\cdot(C + D) = A\cdot C + A\cdot D + B\cdot C + B\cdot D$ and $A\cdot B = B\cdot A$). Denote

$$\begin{aligned} F_0 &= I - A_2 \cdot A_2, \\ F_1 &= -2(A_1 \cdot A_2 - A_2 \cdot A_2), \\ F_2 &= -(A_1 \cdot A_1 + A_2 \cdot A_2 - 2A_2 \cdot A_1). \end{aligned} \tag{2.4}$$

Then

$$I - C(\alpha)\cdot C(\alpha) = F_0 + \alpha F_1 + \alpha^2 F_2.$$

Lemma 2.7. *Assume that $A_1, A_2 \in \mathbb{R}^{n\times n}$ and A_2 is stable. $C(\alpha)$ has no eigenvalue in Θ for all $\alpha \in [0, 1]$ if and only if the $(2d \times 2d)$ dimensional matrix*

$$M = \begin{bmatrix} 0 & I \\ -F_0^{-1}F_2 & -F_0^{-1}F_1 \end{bmatrix},$$

has no real eigenvalue in $[1, \infty)$. Here I is $d \times d$ identity matrix.

Proof. Assume that $C(\alpha)$ has no eigenvalue in Θ for all $\alpha \in [0, 1]$. We write

$$\begin{aligned} \nu(C(\alpha)) &= \det[I - C(\alpha)\cdot C(\alpha)] \\ &= \det[F_0 + \alpha F_1 + \alpha^2 F_2] \\ &= \det[F_0^{-1}].\det[I + \alpha F_0^{-1}F_1 + \alpha^2 F_0^{-1}F_2]. \end{aligned}$$

Therefore, $\nu(C(\alpha)) \neq 0$ for all $\alpha \in (0, 1]$ if and only if $\det[I + \alpha F_0^{-1}F_1 + \alpha^2 F_0^{-1}F_2] \neq 0$ for all $\alpha \in (0, 1]$.

We rewrite the above second determinant as follows

$$\det[I + \alpha F_0^{-1}F_1 + \alpha^2 F_0^{-1}F_2] = \alpha^{2d}\det[1/\alpha^2 I + 1/\alpha F_0^{-1}F_1 + F_0^{-1}F_2].$$

Take $\mu := 1/\alpha$, then

$$\det[I + \alpha F_0^{-1} F_1 + \alpha^2 F_0^{-1} F_2] = \mu^{-2d} \det[\mu^2 I + \mu F_0^{-1} F_1 + F_0^{-1} F_2].$$

On the other hand, the characteristic polynomial of the matrix $Z = \begin{bmatrix} 0 & I \\ -X & -Y \end{bmatrix}$ is $\det[\lambda I - Z] = \det[\lambda^2 I + \lambda Y + X]$ (see [2, p. 309]). Therefore,

$$\det[\mu I - M] = \det[\mu^2 I + \mu F_0^{-1} F_1 + F_0^{-1} F_2].$$

Thus, $\det[I + \alpha F_0^{-1} F_1 + \alpha^2 F_0^{-1} F_2] \neq 0$ for all $\alpha \in (0, 1]$ if and only if $\det[\mu I - M] \neq 0$ for all $\mu \in [1, \infty)$ (recall that $\mu = 1/\alpha$ and $\alpha \in (0, 1] \Leftrightarrow \mu \in [1, \infty)$). This means that the matrix M has no real eigenvalue in $[1, \infty)$. ∎

Using the above lemmas, we derive the following result.

THEOREM 2.8. *Assume that* $A_1, A_2 \in \mathbb{R}^{n \times n}$ *and* A_2 *is stable.* $C(\alpha)$ *is stable for all* $\alpha \in [0, 1]$ *if and only if*

(i) $(I - A_1)(I - A_2)^{-1}$ *and* $(I + A_1)(I + A_2)^{-1}$ *have no negative real eigenvalue,*
(ii) M *has no real eigenvalue in* $[1, \infty)$.

Proof. $(\Rightarrow)$: Assume that $C(\alpha)$ is stable for all $\alpha \in [0, 1]$. Therefore, the matrix $C(\alpha)$ has no eigenvalue in $\Theta \cup \{-1, 1\}$. Then by Corollary 2.2, 2.4 and Lemma 2.7, (i) and (ii) are satisfied.

$(\Leftarrow)$: By the theorem of continuity eigenvalues on a parameter (see [2, p. 52]), there exists continuous functions $\lambda_i : [0, 1] \to \mathbb{C}$ $(i = 1, 2, \ldots, n)$ such that $\lambda_1(\alpha)$, $\lambda_2(\alpha)$, $\ldots$, $\lambda_n(\alpha)$ are eigenvalues of $C(\alpha)$. Here $|\lambda_i(0)| < 1$ $(i = 1, 2, \ldots, n)$, since the matrix $C(0)$ is stable. Suppose that $C(\alpha_*)$ is not stable for some $\alpha_* \in (0, 1)$. Therefore, there exists an index $i_0 \in \{1, 2, \ldots, n\}$ such that $|\lambda_{i_0}(\alpha_*)| \geq 1$. In view of the continuity of $\lambda_{i_0}(\alpha)$ with respect to α, there must exists $\tilde{\alpha} \in (0, 1)$ such that $|\lambda_{i_0}(\tilde{\alpha})| = 1$. The matrix $C(\tilde{\alpha})$ has an eigenvalue which lies on the unit circle. If the real eigenvalue is 1, this contradicts Corollary 2.2 and (i). If the real eigenvalue is -1, this contradicts Corollary 2.4 and (i). If the eigenvalue is complex, this contradicts (ii). These contradictions show that $C(\alpha)$ is stable for all $\alpha \in [0, 1]$. ∎

EXAMPLE 2.9. *Consider the matrix segment* $[A_1, A_2]$ *with stable matrices*

$$A_1 = \begin{bmatrix} 0.1 & -0.2 & 0.4 \\ -0.2 & 0.3 & 0.6 \\ -0.3 & 0.2 & 0.1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.6 & 0.1 & -0.6 \\ -0.3 & -0.2 & 0.4 \end{bmatrix}.$$

The eigenvalues of the matrices $(I - A_1)(I - A_2)^{-1}$ *and* $(I + A_1)(I + A_2)^{-1}$ *are* 13.4, 0.2, 1.2 *and* 2.5, 0.8, 0.4, *respectively. From* (2.4), *we have*

$$F_0 = \begin{bmatrix} 1.27 & 0.3 & 0.32 \\ -0.09 & 0.82 & -0.24 \\ 0.09 & -0.06 & 1.08 \end{bmatrix}, \quad F_1 = \begin{bmatrix} -0.86 & -0.52 & -0.96 \\ 0.05 & 0.11 & 0.47 \\ -0.46 & 0.14 & -0.53 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.6 & 0.08 & 0.88 \\ 0.08 & -0.06 & -0.13 \\ 0.32 & -0.24 & 0.54 \end{bmatrix}.$$

The eigenvalues of matrix

$$M=\begin{bmatrix}0 & I\\ -F_0^{-1}F_2 & -F_0^{-1}F_1\end{bmatrix}=\begin{bmatrix}0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\-0.35&-0.15&-0.56&0.55&0.48&0.74\\-0.21&0.12&-0.03&0.11&-0.13&-0.37\\-0.27&0.24&-0.45&0.38&-0.17&0.40\end{bmatrix},$$

are -0.81*,* 0.33*,* $0.54\pm0.75j$*,* $0.10\pm0.39j$*. Hence,* M *has no real eigenvalue in* $[1,\infty)$*. By Theorem 2.8, the segment* $[A_1,A_2]$ *is robust stable.*

Example 2.10. *For the stable matrices*

$$A_1=\begin{bmatrix}-42.912&-57.078&-53.082\\25.164&33.516&31.104\\5.616&7.254&7.326\end{bmatrix}\quad\text{and } A_2=\begin{bmatrix}-0.342&-1.638&-4.212\\2.7&4.5&4.5\\-4.158&-5.562&-2.088\end{bmatrix},$$

we consider the matrix segment $[A_1,A_2]$*. The eigenvalues of the matrix* $(I-A_1)(I-A_2)^{-1}$ *are calculated as* -10.3584*,* -0.4883 *and* 7.6502*. Since the matrix has at least one negative eigenvalue,* $C(\alpha)$ *has eigenvalue* $\lambda=1$ *for some* $\alpha\in(0,1)$ *by Corollary 2.2. Similarly, the eigenvalues of matrix* $(I+A_1)(I+A_2)^{-1}$ *are* -4.5996*,* -0.01153*,* 0.4870 *and* $C(\alpha)$ *has eigenvalue* $\lambda=-1$ *for some* $\alpha\in(0,1)$ *by Corollary 2.4 (see Fig. 1(a)). As a result, there is an unstable matrix in the segment* $[A_1,A_2]$ *by Theorem 2.8 (see Fig. 1(b)).*

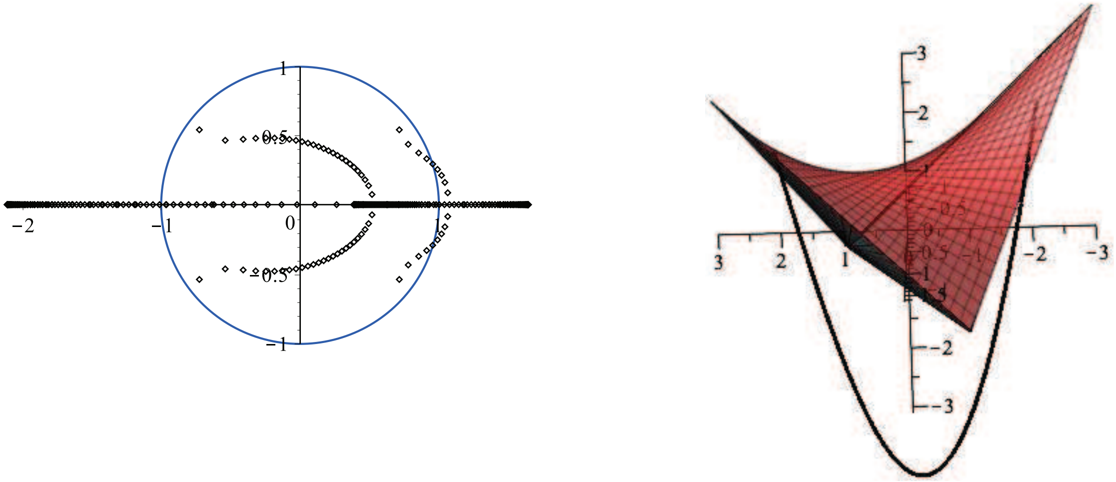

Figure 1: (a) Eigenvalues of the $C(\alpha)$ ($\alpha\in[0,1]$) for Example 2.10 and (b) The set of coefficient vectors of third-order monic stable polynomials and the curve corresponding to the coefficient vectors of the characteristic polynomials of $C(\alpha)$ for $\alpha\in[0,1]$.

Let A be a Metzler matrix that is all off-diagonal elements of A are nonnegative. Then there exists $\alpha\in\mathbb{R}$ and a nonnegative matrix P such that $A=P-\alpha I$. By Perron's theorem, the spectral radius $\rho(P)$ is

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an eigenvalue of P; therefore, $\rho(P) - \alpha$ is an eigenvalue of A. Consequently, if A or $-A$ is Metzler matrix, then it has a real eigenvalue.

If λ is a complex eigenvalue of A, then the complex conjugate $\bar{\lambda}$ is also an eigenvalue. Consequently, if A or $-A$ is 2×2 Metzler matrix, then it has no complex eigenvalues. Additionally, if A_1 and A_2 are 2×2 Metzler, then $C(\alpha)$ are Metzler for all $\alpha \in [0, 1]$.

Summarizing above we have

Theorem 2.11. *Let A_1 and A_2 are 2×2 Metzler stable (or $-A_1$ and $-A_2$ are 2×2 Metzler stable) matrices. The convex combination $C(\alpha)$ is stable for all $\alpha \in [0, 1]$ if and only if $(I - A_1)(I - A_2)^{-1}$ and $(I + A_1)(I + A_2)^{-1}$ have no negative real eigenvalue.*

Proof. By the above, condition (ii) in Theorem 2.8 is redundant. ∎

Theorem 2.11 is not true for the case $n \geq 3$. Consider the following

Example 2.12. *Consider 3×3 Metzler stable matrices*

$$A_1 = \begin{bmatrix} -0.9 & 0.1 & 0 \\ 0.9 & -0.5 & 0.9 \\ 1.6 & 0.1 & -0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & 0.8 & 0.1 \\ -0.8 & -0.5 & 0.9 \\ 0.7 & 0 & -0.4 \end{bmatrix}.$$

Here, the eigenvalues of the matrices $(I - A_1)(I - A_2)^{-1}$ and $(I + A_1)(I + A_2)^{-1}$ are given as follows:

$$(I - A_1)(I - A_2)^{-1} \text{ has eigenvalues } \quad 0.954, \quad 0.803 \pm 1.047j,$$

$$(I + A_1)(I + A_2)^{-1} \text{ has eigenvalues } \quad 0.120, \quad 0.592, \quad 1.234.$$

Both matrices have no negative real eigenvalues, the matrix $\frac{1}{2}A_1 + \frac{1}{2}A_2$ is not stable (see Fig. 2) with eigenvalues

$$\lambda_1 = 0.433, \quad \lambda_{2,3} = -0.816 \pm 0.623j, \quad |\lambda_{2,3}| = 1.027 > 1.$$

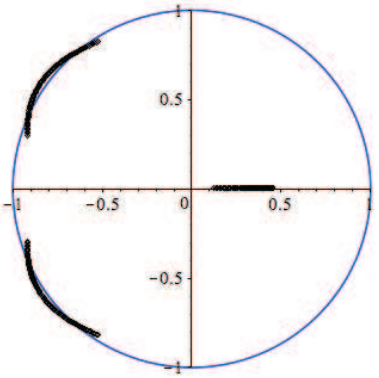

Figure 2: Eigenvalues of $C(\alpha)$ ($\alpha \in [0, 1]$) for Example 2.12.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Consider the stability region $D = \{(x,y) \in \mathbb{R}^2 : (x-\delta)^2 + y^2 < r^2\}$, where $\delta > 0$, $r > 0$. Denote

$$\begin{aligned} \tilde{F}_0 &= I - A_2 \cdot A_2 + 2\delta A_2 \cdot I - \delta^2 I \cdot I, \\ \tilde{F}_1 &= -2(A_1 \cdot A_2 - A_2 \cdot A_2) + 2\delta(A_1 \cdot I - A_2 \cdot I), \\ \tilde{F}_2 &= -(A_1 \cdot A_1 + A_2 \cdot A_2 - 2A_2 \cdot A_1). \end{aligned} \tag{2.5}$$

Theorem 2.13. *Assume that $A_1, A_2 \in \mathbb{R}^{n\times n}$ and A_2 is D-stable (all eigenvalues lie in D). $C(\alpha)$ is D-stable for all $\alpha \in [0,1]$ if and only if*

(i) $[(\delta+r)I - A_1]\,[(\delta+r)I - A_2]^{-1}$ *and* $[(\delta-r)I + A_1]\,[(\delta-r)I + A_2]^{-1}$ *have no negative real eigenvalue,*

(ii) $M = \begin{bmatrix} 0 & I \\ -\tilde{F}_0^{-1}\tilde{F}_2 & -\tilde{F}_0^{-1}\tilde{F}_1 \end{bmatrix}$ *has no real eigenvalue in* $[1,\infty)$.

The proof is immediate.

3. Robust stability of a one matrix polytope.

In this section using results of Section 2, we obtain condition for robust stability of a matrix polytope with rank one uncertainty, namely consider a polytope

$$\mathcal{A} = \operatorname{co}\{A_1, A_2, \ldots, A_N\}, \quad A_i = B_0 + B_i \quad (i = 1,2,\ldots,N), \tag{3.6}$$

where $B_0 \in \mathbb{R}^{n\times n}$, $\operatorname{rank}(B_i) = 1$. For $A \in \mathbb{R}^{n\times n}$, it is well known that

$$\operatorname{rank}(A) = 1 \quad \Leftrightarrow \quad A = bc^T \text{ for some nonzero column vectors } b, c \in \mathbb{R}^n.$$

Therefore, we will assume that

$$B_i = bc_i^T \text{ or } B_i = b_i c^T \; (i = 1,2,\ldots,N).$$

Lemma 3.1. *For $A \in \mathbb{R}^{n\times n}$, if $\operatorname{rank}(A) = 1$ then the eigenvalues of A are zero with multiplicity $(n-1)$ and $\operatorname{trace}(A)$ with multiplicity 1.*

Proof. Assume that $A = uv^T$ for nonzero $u, v \in \mathbb{R}^n$. The equation $v^T x = 0$ has $(n-1)$ linear independent solutions $x^1, x^2, \ldots, x^{n-1} \in \mathbb{R}^n$. Then $Ax^i = 0.x^i$ $(i = 1,2,\ldots,n-1)$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$. On the other hand from the well-known equality

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \operatorname{trace}(A),$$

it follows that $\lambda_n = \operatorname{trace}(A)$. ∎

Lemma 3.2. *If $A = uv^T$ $(u, v \in \mathbb{R}^n)$, then $A \cdot A = 0$.*

Proof.

$$A = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & \cdots & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nv_1 & u_nv_2 & \cdots & u_nv_n \end{bmatrix}.$$

The entries of the matrix $A \cdot A$ are

$$\begin{aligned} f_{ij,kl} &= \frac{1}{2}\left(\begin{vmatrix} u_iv_k & u_iv_l \\ u_jv_k & u_jv_l \end{vmatrix} + \begin{vmatrix} u_iv_k & u_iv_l \\ u_jv_k & u_jv_l \end{vmatrix}\right) = \begin{vmatrix} u_iv_k & u_iv_l \\ u_jv_k & u_jv_l \end{vmatrix} \\ &= u_iu_j \begin{vmatrix} v_k & v_l \\ v_k & v_l \end{vmatrix} = 0. \end{aligned}$$

Therefore, $A \cdot A = 0$. ∎

Lemma 3.3. *Let a segment $[A_1, A_2]$ be given, where A_1 and A_2 are stable,* $\operatorname{rank}(A_1 - A_2) = 1$ *with $A_1 = A_0 + ba^T$, $A_2 = A_0 + bc^T$. Then $[A_1, A_2]$ is robustly stable if and only if*

(i) the matrices $(I - A_1)(I - A_2)^{-1}$ and $(I + A_1)(I + A_2)^{-1}$ have no real negative eigenvalue,
(ii) $-F_0^{-1}F_1$ has no real eigenvalue in $[1, \infty)$, where F_0 and F_1 are defined in (2.4).

Proof.

$$\begin{aligned} A_1 \cdot A_1 &= A_0 \cdot A_0 + 2A_0 \cdot (ba^T) + (ba^T) \cdot (ba^T), \\ A_2 \cdot A_2 &= A_0 \cdot A_0 + 2A_0 \cdot (bc^T) + (bc^T) \cdot (bc^T), \\ A_1 \cdot A_2 &= A_0 \cdot A_0 + A_0 \cdot (bc^T) + A_0 \cdot (ba^T) + (ba^T) \cdot (bc^T). \end{aligned}$$

From (2.4),

$$\begin{aligned} -F_2 &= A_1 \cdot A_1 + A_2 \cdot A_2 - 2A_2 \cdot A_1 \\ &= (ba^T) \cdot (ba^T) - 2(ba^T) \cdot (bc^T) + (bc^T) \cdot (bc^T) \\ &= (ba^T - bc^T) \cdot (ba^T - bc^T) \\ &= [b(a^T - c^T)] \cdot [b(a^T - c^T)] = 0, \end{aligned}$$

by Lemma 3.2. Therefore, the matrix M in Lemma 2.7 becomes

$$M = \begin{bmatrix} 0 & I \\ 0 & -F_0^{-1}F_1 \end{bmatrix}.$$

The eigenvalues of M consist of 0 and the eigenvalues of the matrix $-F_0^{-1}F_1$. Then the necessary and sufficient condition for stability of $\mathcal{A}$ follows from Theorem 2.8. ∎

Lemma 3.4. *If $A = uv^T$ $(u, v \in \mathbb{R}^n)$ then*

$$\operatorname{trace}(A) = 1 - \det(I - A).$$

Proof. The eigenvalues of A are $\lambda_1 = \cdots = \lambda_{n-1} = 0$, $\lambda_n = \operatorname{trace}(A)$ (Lemma 3.1). The eigenvalues of $I - A$ are $\mu_1 = \cdots = \mu_{n-1} = 1$, $\mu_n = 1 - \operatorname{trace}(A)$. The determinant of any matrix equals to the product of the eigenvalues, therefore $\det(I - A) = 1 - \operatorname{trace}(A)$. ∎

Lemma 3.5. *Consider the polytope $\mathcal{A}$ defined by (3.6) and let A be any matrix from $\mathcal{A}$:*

$$A = \alpha_1 A_1 + \cdots + \alpha_N A_N, \ \sum_{i=1}^{N} \alpha_i = 1, \ \alpha_i \geq 0.$$

Then

$$p_A(s) = \alpha_1 p_{A_1}(s) + \alpha_2 p_{A_2}(s) + \cdots + \alpha_N p_{A_N}(s),$$

where $p_A(s)$ is the characteristic polynomial of A.

Proof. $A = B_0 + (\alpha_1 B_1 + \alpha_2 B_2 + \cdots + \alpha_N B_N)$,

$$\begin{aligned} p_A(s) &= \det[sI - B_0 - (\alpha_1 B_1 + \alpha_2 B_2 + \cdots + \alpha_N B_N)] \\ &= \det\Big[(sI - B_0)(I - (sI - B_0)^{-1})(\textstyle\sum_{i=1}^N \alpha_i b c_i^T)\Big] \\ &= \det[sI - B_0]\det[I - \tilde{b}(\tilde{c})^T], \ \left(\tilde{b} = (sI - B_0)^{-1} b, \ \tilde{c} = \left(\textstyle\sum_{i=1}^N \alpha_i c_i\right)^T\right). \end{aligned}$$

Using Lemma 3.4

$$\begin{aligned} p_A(s) &= \det[(sI - B_0)(1 - \operatorname{trace}\tilde{b}(\tilde{c})^T)] \\ &= \det[(sI - B_0)(1 - \sum_{i=1}^N \alpha_i \operatorname{trace}\tilde{b}\tilde{c_i}^T)] \\ &= \det[(sI - B_0)(1 - \sum_{i=1}^N \alpha_i (1 - \det[I - \tilde{b} c_i^T]))] \\ &= \det[(sI - B_0)(1 - 1 + \sum_{i=1}^N \alpha_i \det[I - \tilde{b} c_i^T])] \\ &= \sum_{i=1}^N \alpha_i \det[sI - B_0]\det[I - (sI - B_0)^{-1} b c_i^T] \\ &= \sum_{i=1}^N \alpha_i \det\left[(sI - B_0)(I - (sI - B_0)^{-1} b c_i^T)\right] \\ &= \sum_{i=1}^N \alpha_i \det[sI - B_0 - b c_i^T] \\ &= \sum_{i=1}^N \alpha_i p_{A_i}(s). \end{aligned}$$

∎

Now we arrive at the main result of this section.

THEOREM 3.6. *Let the family (3.6) be given with stable generators A_i. This family is robustly stable if and only if*

(i) $(I - A_i)(I - A_j)^{-1}$ and $(I + A_i)(I + A_j)^{-1}$ have no negative real eigenvalues $(i, j = 1, 2, \dots, N;$ $i < j)$,

(ii) $-\left(F_0^{ij}\right)^{-1} F_1^{ij}$ have no real eigenvalue in $[1, \infty)$ $(i, j = 1, 2, \dots, N;\ i < j)$, where F_0^{ij} and F_1^{ij} are defined by (2.4) with replacing A_1 by A_i and A_2 by A_j, respectively.

Proof. By Lemma 3.5, the family of the characteristic polynomials of the matrix family $\mathcal{A}$ (3.6) is the polynomial polytope

$$\mathcal{P} = \operatorname{co}\{p_{A_1}(s), \dots, p_{A_N}(s)\},$$

and $\mathcal{A}$ is robustly stable if and only if $\mathcal{P}$ is robustly stable. By the Edge Theorem ([2, p. 153]), the polytope $\mathcal{P}$ is robustly stable if and only if all the edges

$$\begin{aligned} [p_{A_i}(s), p_{A_j}(s)] &= \{\alpha p_{A_i}(s) + (1 - \alpha) p_{A_j}(s) : \ \alpha \in [0, 1]\} \\ &= \operatorname{co}\{p_{A_i}(s), p_{A_j}(s)\} \quad (i, j = 1, 2, \dots, N; \ i < j), \end{aligned}$$

are stable. The polynomial segment $[p_{A_i}(s), p_{A_j}(s)]$ is the set of the characteristic polynomials of the matrix segment $[A_i, A_j]$. By Lemma 3.3, the segment $[A_i, A_j]$ is stable if and only if (i) and (ii) are satisfied.

Summarizing, $\mathcal{A}$ is stable $\Leftrightarrow$ $\mathcal{P}$ is stable $\Leftrightarrow$ All edges $[p_{A_i}(s), p_{A_j}(s)]$ are stable $\Leftrightarrow$ All matrix segment $[A_i, A_j]$ are stable $\Leftrightarrow$ (i) and (ii) are satisfied. $\square$

Example 3.7. *For the vectors* $b = (1, -1, 1)^T$, $c_1 = (0.5, 0.5, -0.5)^T$, $c_2 = (-0.25, 0.5, 0.5)^T$, $c_3 = (0.1, -0.1, -0.1)^T$ *and matrix*

$$B_0 = \begin{bmatrix} -0.3 & 0.3 & -0.3 \\ -0.1 & 0.1 & -0.1 \\ 0.2 & -0.2 & 0.2 \end{bmatrix},$$

consider the polytope $\mathcal{A} = \mathrm{co}\{A_1, A_2, A_3\}$ *where*

$$A_1 = B_0 + b\,c_1^T = \begin{bmatrix} 0.2 & 0.8 & -0.8 \\ -0.6 & -0.4 & 0.4 \\ 0.7 & 0.3 & -0.3 \end{bmatrix},$$

$$A_2 = B_0 + b\,c_2^T = \begin{bmatrix} -0.55 & 0.8 & 0.2 \\ 0.15 & -0.4 & -0.6 \\ -0.05 & 0.3 & 0.7 \end{bmatrix},$$

$$A_3 = B_0 + b\,c_3^T = \begin{bmatrix} -0.2 & 0.2 & -0.4 \\ -0.2 & 0.2 & 0 \\ 0.3 & -0.3 & 0.1 \end{bmatrix},$$

are stable matrices. The matrices $(I - A_i)(I - A_j)^{-1}$ *and* $(I + A_i)(I + A_j)^{-1}$ *have no negative real eigenvalue* $(i, j = 1, 2, 3,\ i < j)$. *We calculate*

$$F_0^{12} = \begin{bmatrix} 0.9 & -0.3 & 0.4 \\ 0.125 & 1.375 & -0.5 \\ -0.025 & -0.075 & 1.1 \end{bmatrix}, \quad F_1^{12} = \begin{bmatrix} -0.3 & 0.7 & -0.4 \\ 0.375 & -0.875 & 0.5 \\ -0.075 & 0.175 & -0.1 \end{bmatrix},$$

$$F_0^{13} = \begin{bmatrix} 1 & 0.08 & -0.08 \\ 0 & 0.9 & 0.1 \\ 0 & 0.02 & 0.98 \end{bmatrix}, \quad F_1^{13} = \begin{bmatrix} -0.4 & 0.32 & 0.08 \\ 0.5 & -0.4 & -0.1 \\ -0.1 & 0.08 & 0.02 \end{bmatrix},$$

$$F_0^{23} = \begin{bmatrix} 1 & 0.08 & -0.08 \\ 0 & 0.9 & 0.1 \\ 0 & 0.02 & 0.98 \end{bmatrix}, \quad F_1^{23} = \begin{bmatrix} -0.1 & -0.38 & 0.48 \\ 0.125 & 0.475 & -0.6 \\ -0.025 & -0.095 & 0.12 \end{bmatrix}.$$

The eigenvalues of $-(F_0^{ij})^{-1}F_1^{ij}$ *are not in* $[1, \infty)$ $(i, j = 1, 2, 3,\ i < j)$. *By Theorem 3.6, all matrices in the family* $\mathcal{A}$ *are stable.*

4. Conclusion. In this study, we have investigated the robust Schur stability of $n \times n$ dimensional matrix segments using the bialternate product of matrices. We have shown that the problem can be reduced to checking the existence of negative eigenvalues in two out of three specially constructed matrices and the presence of eigenvalues in the interval $[1, \infty)$ for the third matrix.

We have provided a necessary and sufficient condition for the convex combinations of two stable matrices with rank one difference to be robustly Schur stable. Furthermore, we have demonstrated that the robust stability of the convex hull of a finite number of matrices, whose pairwise differences are of rank 1, is equivalent to the robust stability of the segments formed by these matrices.

The obtained results have been illustrated through examples, showcasing their applicability in analyzing the stability of matrix polytopes with rank one uncertainty. These findings can be particularly useful in the stability analysis of linear systems subject to polytopic uncertainties, which are commonly encountered in control systems.

Acknowledgement. The author thanks the reviewers for their constructive comments in improving this paper.

Contribution of individual authors to the creation of a scientific article

The entire research, including the formulation of the problem, development of the theoretical framework, implementation of the algorithms, and writing of the manuscript, was conducted by Şerife Yılmaz.

Sources of funding for research presented in a scientific article or scientific article itself

No funding was received for conducting this study.

Conflicts of interest

The author has no conflicts of interest to declare that are relevant to the content of this article.

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