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# LYAPUNOV-LIKE TRANSFORMATIONS ON THE TENSOR PRODUCT OF NUCLEAR PAIRS OF PROPER CONES*

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**Abstract.** Lyapunov-like transformation/matrix on a cone appears in the theory of dynamical systems and linear complementarity problems. The set of all Lyapunov-like transformations on a proper cone in a finite dimensional inner product space is the Lie algebra of the automorphism group of that cone. The dimension of this Lie algebra is called the Lyapunov rank. A pair of proper cones is said to be a nuclear pair if one of them is simplicial. In this paper, we find the Lyapunov rank and Lyapunov-like transformations on the tensor product of nuclear pairs of cones. Further, we prove that the space of Lyapunov-like transformations on the tensor product of a nuclear pair is the tensor product of the spaces of Lyapunov-like transformations on the individual cones. As a consequence, given a nuclear pair $(K_1, K_2)$, we describe the space of Lyapunov-like transformations on the cone of positive operators between $K_1$ and $K_2$.

**Key words.** Proper cones, Lyapunov-like transformation, Bilinearity rank, Lyapunov rank, Lie algebra, Automorphism group, Tensor product of proper cones, Positive operators, Nuclear pair.

**AMS subject classifications.** 15B48, 17B40, 22E60, 47L07, 52A20, 90C46.

**1. Introduction.** In [14], Rudolf et al. studied proper cones $K$ in a finite dimensional real inner product space $V$, where complementarity conditions on $K$ can be expressed using bilinear relations on $K$. They illustrated this by expressing the complementarity condition on well-known cones (like nonnegative orthant, positive semidefinite cone, second-order cone etc.,) as a system of linearly independent bilinear relations. However, expressing the complementarity condition as independent bilinear relations is not always possible. For a proper cone $K \subseteq V$, a linear transformation $L : V \to V$ is a bilinear complementarity relation on $K$, if the following condition holds:

$$[x \in K, a \in K^* \text{ with } \langle x, a\rangle = 0] \Rightarrow \langle x, L(a)\rangle = 0.$$

In this context, Rudolf et al. defined bilinearity rank of $K$ as the number of linearly independent bilinear relations on the cone $K$. When the dimension of the ambient space $V$ is $n$, the study of bilinearity rank is necessary to understand the cones whose bilinearity rank is greater than or equal to $n$. They also computed the bilinearity ranks of polyhedral cones and some other classes of proper cones. For a proper cone $K \subseteq V$, a linear transformation $L : V \to V$ is called a Lyapunov-like transformation on the cone $K$ if the following condition holds:

$$[x \in K, a \in K^* \text{ with } \langle x, a\rangle = 0] \Rightarrow \langle L(x), a\rangle = 0.$$

In [5], Gowda and Tao observed a relation between bilinear complementarity relations and Lyapunov-like transformations/matrices. They showed that $L$ is a bilinear complementarity relation on $K$ if and only if $L^T$ is Lyapunov-like with respect to the cone $K$. In view of this, Gowda and Tao called the bilinearity rank as the Lyapunov rank. They also connected the Lyapunov-like transformations on a proper cone $K$ to the Lie algebra of the automorphism group of the cone $K$. Lyapunov-like transformations appear naturally

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in the theory of dynamical systems and linear complementarity problems (LCP). Gowda and Tao studied the Lyapunov rank of proper polyhedral cones, symmetric cones, and completely positive cones. They also found an upper bound for the Lyapunov rank of proper cones in $\mathbb{R}^n$. In [12], Gowda and Orlitzky gave an improved bound for the Lyapunov rank of proper cones in $\mathbb{R}^n$. In [11], Orlitzky gave a tight bound for the Lyapunov rank of proper cones in $\mathbb{R}^n$. Orlitzky also introduced the Lyapunov rank of improper cones in [9]. He extended some of the results of proper cones to improper cones. Further studies on the Lyapunov rank of many special cones appeared in [6], [8], [10], and [17].

Given a linear transformation $L : V \to V$, a proper cone $K \subseteq V$ and a vector $q \in V$, the linear complementarity problem $\mathrm{LCP}(L, K, q)$ is to find an $x \in K$ such that

$$z = Lx + q \in K^* \text{ and } \langle x, z\rangle = 0.$$

In [4], Gowda and Tao introduced $Z$-transformations on proper cones as a generalization of $Z$-matrices (a square matrix whose off diagonal entries are nonpositive) from the theory of standard linear complementarity problems $\mathrm{LCP}(M, \mathbb{R}^n_+, q)$. Here, $M$ is an $n \times n$ real matrix, $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$ and $q \in \mathbb{R}^n$. $Z$-transformations are widely studied in LCP, matrix theory, dynamical systems, etc. A linear transformation $L : V \to V$ is said to be a $Z$-transformation on $K$ if

$$[x \in K, y \in K^* \text{ with } \langle x, y\rangle = 0] \Rightarrow \langle L(x), y\rangle \leq 0.$$

It is easy to observe that a linear transformation $L : V \to V$ is Lyapunov-like on $K$ if and only if $L$ and $-L$ are $Z$-transformations on $K$. In [7], Gowda and Ravindran introduced linear games and found the game theoretic value of some special $Z$-transformations over the cone of $n \times n$ real symmetric positive semidefinite matrices. In [3], Gokulraj and Chandrashekaran generalized the classical von Neumann symmetrization of two-person zero-sum games to general linear games. In their work, for proper cones $K_1$ and $K_2$ in finite dimensional real inner product spaces $U$ and $V$, respectively, the tensor product $K_1 \otimes K_2$ of the cones $K_1$ and $K_2$ is defined as

$$K_1 \otimes K_2 := \left\{ \sum_{i=1}^{n} (x_i \otimes y_i) \mid x_i \in K_1, y_i \in K_2, \ \forall\, 1 \leq i \leq n \right\}.$$

They proved that for proper cones $K_1$ and $K_2$, the tensor product cone $K_1 \otimes K_2$ is again a proper cone in $U \otimes V$.

For closed convex cones $K_1$ and $K_2$ in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, the cone of positive operators from $K_1$ to $K_2$ is defined as $\pi(K_1, K_2) := \{A \in \mathbb{R}^{m\times n} \mid AK_1 \subseteq K_2\}$. In [2], Berman and Gaiha proved that $\pi(K_2, (K_1)^*) = (K_1 \otimes K_2)^*$. This motivates us to study the bilinearity/Lyapunov rank of the tensor product of proper cones. Therefore, the results of this paper essentially answer the following questions: 1. Is the Lyapunov rank of the tensor product cone the product of the Lyapunov ranks of the individual cones? 2. What are all the Lyapunov-like transformations on the tensor product cone $K_1 \otimes K_2$?

The tensor product of two cones $K_1 \subseteq U$ and $K_2 \subseteq V$ can be defined in the following ways.

- Projective tensor product cone or minimal tensor product of the cones $K_1$ and $K_2$ is defined as $K_1 \otimes_\pi K_2 := \{\sum_{i=1}^{p} x_i \otimes y_i \mid x_i \in K_1, y_i \in K_2, \text{and } p \in \mathbb{N}\}$.
- Injective tensor product cone or maximal tensor product of the cones $K_1$ and $K_2$ is defined as $K_1 \otimes_\epsilon K_2 := \{u \in U \otimes V \mid \langle u, f \otimes g\rangle \geq 0 \ \forall f \in (K_1)^*, g \in (K_2)^*\}$. Here, $(K_1)^*$ and $(K_2)^*$ are dual of the cones $K_1$ and $K_2$, respectively.

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From the definition of projective tensor product cone and injective tensor product cone, one could observe that $(K_1 \otimes_\pi K_2)^* = (K_1)^* \otimes_\epsilon (K_2)^*$ and $(K_1 \otimes_\epsilon K_2)^* = (K_1)^* \otimes_\pi (K_2)^*$ refer [1]. De Bruyn in [18], studied many properties of projective tensor product cones and injective tensor product cones. It is easy to observe that $K_1 \otimes_\pi K_2 \subseteq K_1 \otimes_\epsilon K_2$. Aubrun et al. in [1], called the pair of cones $(K_1, K_2)$, *nuclear* if $K_1 \otimes_\pi K_2 = K_1 \otimes_\epsilon K_2$ and *entangleable* if $K_1 \otimes_\pi K_2 \subsetneq K_1 \otimes_\epsilon K_2$. They proved that a pair $(K_1, K_2)$ is nuclear if and only if either $K_1$ or $K_2$ is isomorphic to $\mathbb{R}^n_+$ (simplicial). The dual of the nuclear pair $(K_1, K_2)$ is given as $(K_1 \otimes_\pi K_2)^* = K_1^* \otimes_\pi K_2^*$. It can be observed that there exist proper polyhedral cones $K_1$ and $K_2$ such that the pair $(K_1, K_2)$ is not a nuclear pair. For cones of positive semidefinite matrices, the fact that the minimal and maximal tensor products do not coincide is connected to the concept of quantum entanglement [1].

Here in this paper, we find the Lyapunov rank and describe Lyapunov-like transformations on nuclear pair of cones. We also describe the Lyapunov-like transformations of projective tensor product cone of proper polyhedral cones. By definition, the dual of the injective tensor product of cones is the projective tensor product of the dual of the corresponding cones. Therefore, the Lyapunov rank of injective tensor product of proper polyhedral cones $K_1 \otimes_\epsilon K_2$ and injective tensor product of nuclear pair of cones $K_1 \otimes_\epsilon K_2$ ($K_1$ is simplicial) is also known. That is, for proper polyhedral cones $K_1$ and $K_2$ or nuclear pair of cones $(K_1, K_2)$,

$$\beta(K_1 \otimes_\epsilon K_2) = \beta((K_1 \otimes_\epsilon K_2)^*) = \beta((K_1)^* \otimes_\pi (K_2)^*) = \beta((K_1)^*)\beta((K_2)^*) = \beta(K_1)\beta(K_2).$$

Here, $\beta((K_1)^* \otimes_\pi (K_2)^*) = \beta((K_1)^*)\beta((K_2)^*)$ which follows from our main result. In this paper, we discuss the Lyapunov rank of projective tensor product cones. Hereinafter, $K_1 \otimes K_2$ denotes the projective tensor product cone of proper cones $K_1$ and $K_2$. From Proposition 9 of Rudolf et al. [14], the Lyapunov rank of the Cartesian product of proper polyhedral cones $K_1$ and $K_2$ is the sum of the corresponding Lyapunov ranks, that is, $\beta(K_1 \times K_2) = \beta(K_1) + \beta(K_2)$. In contrast, we prove in Theorem 3.6, the Lyapunov rank of the tensor product of polyhedral cones $K_1$ and $K_2$ is the product of the corresponding Lyapunov ranks, that is, $\beta(K_1 \otimes K_2) = \beta(K_1)\beta(K_2)$. This result is also true for any pair of proper cones $K_1$ and $K_2$, where either $K_1$ or $K_2$ is simplicial. In both the above cases, we also describe the Lyapunov-like transformations on the tensor product cone $K_1 \otimes K_2$. In [10], Orlitzky has found the Lyapunov rank and Lyapunov-like transformations on the cone of positive operators $\pi(K_1, K_2)$ from $K_1$ to $K_2$, where $K_1$ and $K_2$ are proper polyhedral cones. We present the relationship between the cone of positive operators and the tensor product cones in Section 3.1.

We organize the paper as follows. Section 2 deals with the preliminaries, notations, and a few definitions required for our main results. In Section 3, we prove that $LL(K_1 \otimes K_2) = LL(K_1) \otimes LL(K_2)$, where $K_1$ and $K_2$ are proper polyhedral cones in $\mathbb{R}^n$. In Section 4, we determine the Lyapunov rank of the nuclear pair of cones $(K_1, K_2)$ as $\beta(K_1 \otimes K_2) = \beta(K_1)\beta(K_2)$ and prove that $LL(K_1 \otimes K_2) = LL(K_1) \otimes LL(K_2)$. As a consequence, we deduce the Lyapunov-like transformations and the Lyapunov rank of the cone $\pi(K_1, K_2)$ when $(K_1, K_2)$ is a nuclear pair. That is, $LL(\pi(K_1, K_2)) = LL(K_2) \otimes LL((K_1)^*)$ and $\beta(\pi(K_1, K_2)) = \beta(K_1)\beta(K_2)$, which is an extension of the results in [10].

**2. Preliminaries.** A subset $K$ of a finite dimensional real inner product space $V$ is said to be a cone if $x \in K$ implies $\alpha x \in K$ for all $\alpha \geq 0$ and a convex cone if $x, y \in K$ implies $\alpha x + \beta y \in K$ for all $\alpha, \beta \geq 0$. A cone $K$ is said to be closed if it is topologically closed. The dual of the cone $K$ is defined as $K^* := \{y \in V \mid \langle x, y\rangle \geq 0, \ \forall\ x \in K\}$. A cone $K$ is said to be self-dual if $K = K^*$. A cone $K$ is pointed if $K \cap -K = \{0\}$. The cone $K$ is a spanning cone or generating cone of $V$ if $K - K = V$. A closed convex cone $K$ is said to be proper if it is pointed and spanning. The complementarity set of a cone $K$ is given by

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$$C(K) := \{(x,a) \mid x \in K, a \in K^*, \langle x,a\rangle = 0\}\,. \tag{2.1}$$

A linear transformation $L : V \to V$ is called Lyapunov-like on the cone $K$ if $\langle L(x), a\rangle = 0$ for all $(x, a) \in C(K)$. The set of all Lyapunov-like transformations forms a vector space and is denoted by $LL(K)$. That is,

$$LL(K) := \{T : V \to V \mid \text{T is Lyapunov-like on } K\}. \tag{2.2}$$

The dimension of $LL(K)$ is called the bilinearity rank or the Lyapunov rank of the cone $K$ and is denoted by $\beta(K)$. It is well known that Lyapunov-like matrices on the cone $\mathbb{R}^n_+$ are diagonal matrices, see [5].

Let $K$ be a convex cone. A nonzero vector $x \in K$ is called an extreme vector of $K$ if, whenever $x = y+z$ for $y, z \in K$, then $y, z \in \{\alpha x|\alpha \geq 0\}$. A set consisting of all nonproportional extreme vectors of the cone $K$ is denoted as $\text{ext}(K)$. Two cones $K_1 \subseteq U$ and $K_2 \subseteq V$ are isomorphic if there exists an invertible linear map $\phi : U \to V$ such that $\phi(K_1) = K_2$. For a nonempty set $S$ in $V$, $\text{span}(S)$ denotes the subspace generated by $S$ in $V$.

**2.1. Tensor product.** Let $U$ and $V$ be two real finite dimensional inner product spaces. For $u \in U$, $v \in V$ the elementary tensor $u \otimes v$ is a bilinear functional $u \otimes v : U \times V \to \mathbb{R}$ defined as $u \otimes v(a, b) := \langle u, a\rangle_U \langle v, b\rangle_V$. The tensor product of the vector spaces $U$ and $V$ is denoted as $U \otimes V$ and is defined as

$$U \otimes V = \left\{ \sum_{i=1}^{k} u_i \otimes v_i \mid u_i \in U, v_i \in V \text{ and } k \in \mathbb{N} \right\}.$$

It is easy to observe that if $\mathbf{B}_U = \{u_i \mid 1 \leq i \leq n\}$ and $\mathbf{B}_V = \{v_j \mid 1 \leq j \leq m\}$ are bases of $U$ and $V$, respectively, then $\mathbf{B}_{U\otimes V} = \{u_i \otimes v_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis of $U \otimes V$. Hence, $\dim(U \otimes V) = \dim(U)\dim(V)$. The tensor product space $U \otimes V$ of the inner product spaces $U$ and $V$ is an inner product space, and the inner product is given by

$$\left\langle \sum_{i=1}^{n_1} u_i \otimes v_i, \sum_{j=1}^{n_2} a_j \otimes b_j \right\rangle := \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} \langle u_i, a_j\rangle_U \langle v_i, b_j\rangle_V, \tag{2.3}$$

for every $\sum_{i=1}^{n_1} u_i \otimes v_i$ and $\sum_{j=1}^{n_2} a_j \otimes b_j$ in $U \otimes V$. The norm in this space $U \otimes V$ is induced by the inner product given in (2.3). The set of all linear transformations from $U$ to $V$ is denoted by $\mathcal{B}(U, V)$. The set of all linear operators on $V$ is denoted by $\mathcal{B}(V)$. Given a linear transformation $T \in \mathcal{B}(U, V)$, the adjoint transformation $T^* \in \mathcal{B}(V, U)$ is defined by $\langle T(u), v\rangle_V = \langle u, T^*(v)\rangle_U$ for all $u \in U$ and $v \in V$. For $L, M \in \mathcal{B}(U, V)$, the inner product on $\mathcal{B}(U, V)$ is defined as $\langle L, M\rangle := \text{trace}(LM^*)$, where trace is the sum of eigenvalues of $LM^*$. The tensor product of two linear operators $T \in \mathcal{B}(U)$ and $S \in \mathcal{B}(V)$ is the linear operator $T \otimes S : U \otimes V \to U \otimes V$ defined as $(T \otimes S)(\sum_{i=1}^{p} u_i \otimes v_i) = \sum_{i=1}^{p} T(u_i) \otimes S(v_i)$.

The following are some properties of the tensor product.

LEMMA 2.1. *For $\alpha \in \mathbb{R}, u, u_1, u_2 \in U, v, v_1, v_2 \in V$, the following are true,*

*(i)* $\alpha(u \otimes v) = (\alpha u \otimes v) = (u \otimes \alpha v)$,
*(ii)* $(u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v$,
*(iii)* $u \otimes (v_1 + v_2) = u \otimes v_1 + u \otimes v_2$.

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**2.2. Tensor product of cones.** Let $K_1$ and $K_2$ be two cones in finite dimensional real inner product spaces $U$ and $V$, respectively. The tensor product of the cones $K_1$ and $K_2$ is defined as

$$K_1 \otimes K_2 := \left\{ \sum_{i=1}^{n} (x_i \otimes y_i) \mid x_i \in K_1, y_i \in K_2, \ \forall\ 1 \leq i \leq n \right\}.$$

Here, $K_1 \otimes K_2 \subseteq U \otimes V$ is the projective tensor product cone, and we use $\otimes$ instead of $\otimes_\pi$ from this point throughout the paper. It is known that if $K_1$ and $K_2$ are proper cones, then the cone $K_1 \otimes K_2$ is also a proper cone, see [3]. Given a self-dual cone $K$ in $V$, $K \otimes K$ is not necessarily self-dual cone in $V \otimes V$. For example, when $K = S_n^+$, the positive semidefinite cone in the space $S_n$ of all $n \times n$ real symmetric matrices, $K \otimes K$ is not a self-dual cone in $S_n \otimes S_n$, see [1].

The complementarity set of $K_1 \otimes K_2$ and the space of all Lyapunov-like transformations on $K_1 \otimes K_2$ are defined as in equation (2.1) and equation (2.2), respectively.

**2.3. Notation.** Given any vector $x \in \mathbb{R}^n$, $x$ is a column vector of order $n \times 1$ by convention. The corresponding row vector is denoted by $x^T = (x^1, x^2, \ldots, x^n)$. The standard basis of $\mathbb{R}^n$ is denoted as $e_1, e_2, \ldots, e_n$, where $e_i$ has one in the $i$th coordinate and zeros elsewhere. The tensor product of vectors $x, y \in \mathbb{R}^n$ is defined as $x \otimes y := xy^T$. This definition coincides with the above definition of elementary tensor in the following way, that is, $xy^T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined as $xy^T(u, v) = \langle x, u\rangle\langle y, v\rangle$. For any linear map $L : \mathbb{R}^n \to \mathbb{R}^m$, the linear equation $L(x) = y$ holds true if and only if $[M]_L(x) = y$, where $[M]_L$ is the matrix representation of the linear map $L$ with respect to appropriate bases of some fixed order for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

In this paper, for a set $S \subseteq \mathbb{R}^n$, we denote the convex cone generated by $S$ as

$$\text{cone}(S) := \left\{ \sum_{i=1}^{m} \alpha_i s_i | \alpha_i \geq 0, s_i \in S \text{ for } m \in \mathbb{N} \right\}.$$

We have the following theorem due to Gokulraj and Chandrashekaran [3], which relates the extreme vectors of $K_1 \otimes K_2$ and the extreme vectors of $K_1$ and $K_2$.

Theorem 2.2. *[3, Lemma 3.5] Let $U$ and $V$ be vector spaces equipped with proper cones $K_1$ and $K_2$, respectively. Consider the tensor product cone $K_1 \otimes K_2$ (defined above). Then, every extreme vector of $K_1 \otimes K_2$ is of the form $x \otimes y$, where $x$ and $y$ are extreme vectors of $K_1$ and $K_2$, respectively.*

The statements in the following proposition were proved by Rudolf et al., and we will use them in the upcoming sections.

Proposition 2.3. *[14]*

*(i) A proper cone and its dual have equal Lyapunov ranks.*
*(ii) Isomorphic proper cones have equal Lyapunov ranks.*
*(iii) The Lyapunov rank is additive on a direct product/sum.*

**3. Main results.** In this section, we describe the Lyapunov-like transformations on $K_1 \otimes K_2$, where $K_1$ and $K_2$ are proper polyhedral cones in $\mathbb{R}^n$. The following known theorems will be used in this paper.

Theorem 3.1. *[5, Theorem 2] Suppose $K$ is a proper polyhedral cone in $V$. Then, a linear transformation $L$ is Lyapunov-like on $K$ if and only if every extreme vector of $K$ is an eigenvector of $L$.*

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Using the above theorem, one can easily observe that for $L \in LL(\mathbb{R}^n_+)$, $e_1, e_2, \ldots, e_n$ are the eigenvectors of $L$ and so $L$ is a diagonal matrix.

THEOREM 3.2. *[5, Theorem 3] The following statements hold:*

*(i) For every proper polyhedral cone $K$ in $\mathbb{R}^n$, $1 \leq \beta(K) \leq n, \beta(K) \neq n-1$.*
*(ii) For every natural number $m$ with $1 \leq m \leq n$, $m \neq n-1$, there is a proper polyhedral cone $K$ in $\mathbb{R}^n$, with $\beta(K) = m$.*

**3.1. Relating tensor product of cones with the cone of positive operators.** For closed convex cones $K_1$ and $K_2$ in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, the set of all positive operators from $K_1$ to $K_2$ is denoted by $\pi(K_1, K_2)$ and is defined as $\pi(K_1, K_2) := \{A \in \mathbb{R}^{m\times n} \mid AK_1 \subseteq K_2\}$. It is easy to observe that $\pi(K_1, K_2)$ is a closed convex cone in $\mathbb{R}^{m\times n}$. In particular, when $K_1 = K = K_2$, we denote the set of all positive operators on $K$ as $\pi(K)$. For proper polyhedral cones $K_1$ and $K_2$, $\pi(K_1, K_2)$ is called the cone of polyhedral positive operators. Now by the definition used by Berman and Gaiha in [2], and our definition of the tensor product of cones in Section 2.2, we have the following theorem.

THEOREM 3.3. *[2, Theorem 3.1]. For closed convex cones $K_1$ and $K_2$,*

*(i) $\pi(K_2, (K_1)^*) = (K_1 \otimes K_2)^*$.*
*(ii) $(\pi(K_2, (K_1)^*)^* = K_1 \otimes K_2$.*

For any general cone $K$, the set $\pi(K)$ is difficult to characterize. In particular, when $K = S^+_n$, $\pi(S^+_n)$ is not known. Therefore, the dual $(S^+_n \otimes S^+_n)^*$ is also not known. In [16], the authors characterized the membership of an element in the dual $(S^+_n \otimes S^+_n)^*$ of the cone $S^+_n \otimes S^+_n$.

Let $K_1$ and $K_2$ be proper polyhedral cones in a finite dimensional real inner product space $V$, the Lyapunov rank of proper polyhedral cone $K_1 \otimes K_2$ can also be established using the following theorems.

PROPOSITION 3.4. *[15, Lemma 9] If $K_1$ and $K_2$ are proper (polyhedral) cones in finite dimensional real inner product spaces $U$ and $V$, respectively, then $\pi(K_1, K_2)$ is a proper (polyhedral) cone in $\mathcal{B}(U, V)$.*

The Lyapunov rank of the cone of polyhedral positive operators was proved by Orlitzky in the following theorem.

THEOREM 3.5. *[10, Theorem 2] If $K_1$ and $K_2$ are proper polyhedral cones in finite dimensional real Hilbert spaces $U$ and $V$, respectively, then $\beta(\pi(K_1, K_2)) = \beta(K_1)\beta(K_2)$.*

The multiplicative property of the Lyapunov rank of the tensor product of proper polyhedral cones $K_1$ and $K_2$ can be proved as consequences of Theorems 3.3 and 3.5.

THEOREM 3.6. *If $K_1$ and $K_2$ are two proper polyhedral cones in $\mathbb{R}^n$, then $\beta(K_1 \otimes K_2) = \beta(K_1)\beta(K_2)$.*

*Proof.* Since $K_1$ and $K_2$ are proper polyhedral cones, $K_1 \otimes K_2$ is also a proper polyhedral cone. Now by Theorem 3.3, $K_1 \otimes K_2 = \pi(K_2, (K_1)^*)^*$ and $\pi(K_2, (K_1)^*)^*$ is a polyhedral proper cone and so is $\pi(K_2, (K_1)^*)$. By Theorem 3.5 and Proposition 2.3, the Lyapunov rank of $\pi(K_2, (K_1)^*)$ is $\beta(\pi(K_2, (K_1)^*)) = \beta(K_2)\beta((K_1)^*) = \beta(K_2)\beta(K_1)$. Thus, $\beta(K_1 \otimes K_2) = \beta(K_1)\beta(K_2)$. ∎

The Lyapunov-like transformations on the cone of polyhedral positive operators were proved by Orlitzky in the following theorem.

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Theorem 3.7. *[10, Theorem 4] If $K_1$ and $K_2$ are proper polyhedral cones in finite dimensional real Hilbert spaces $U$ and $V$, respectively, then $LL(\pi(K_1, K_2)) = LL(K_2) \otimes LL((K_1)^*)$.*

Thus, from Theorems 3.3 and 3.7, we have the following description for the Lyapunov-like transformations on the cone $K_1 \otimes K_2$, when both $K_1$ and $K_2$ are proper polyhedral cones.

Corollary 3.8. *If $K_1$ and $K_2$ are proper polyhedral cones in $U$ and $V$, respectively, then $LL(K_1 \otimes K_2)^* = LL(K_2) \otimes LL((K_1)^*)$.*

For $1 \leq r \leq n$ and $r \neq n-1$, by Item (ii) of Theorem 3.2, we can find proper polyhedral cones $K_1$ and $K_2$ such that $\beta(K_1) = r$ and $\beta(K_2) = 1$. Therefore, we have $\beta(K_1 \otimes K_2) = r$. Now we ask, given any $r \in \mathbb{N}$ such that $n < r \leq n^2$ and $r \neq n^2 - 1$, does there exist proper polyhedral cones $K_1$ and $K_2$ such that $\beta(K_1 \otimes K_2) = r$? The following proposition answers this negatively.

Proposition 3.9. *Let $n \geq 3$. There is a prime $p$ such that $n \leq p \neq n^2 - 1 < n^2$ and $p \neq \beta(K_1 \otimes K_2)$ for any proper polyhedral cones $K_1$ and $K_2$ in $\mathbb{R}^n$.*

*Proof.* Using Bertrand–Chebyshev's theorem (see, [13]), choose a prime number $p$, such that $n < p \leq 2n \leq n^2$. Suppose $K_1$ and $K_2$ are proper polyhedral cones in $\mathbb{R}^n$ such that $\beta(K_1 \otimes K_2) = p$. Then, from Theorem 3.6 we have $p = \beta(K_1 \otimes K_2) = \beta(K_1)\beta(K_2)$. Therefore, $\beta(K_i) = 1$ or $p$ for $i = 1, 2$. That is, there exists $j \in \{1, 2\}$ such that $\beta(K_j) = p$. However, from item (ii) of Theorem 3.2, we have $\beta(K_i) \leq n < p$ for $i = 1, 2$, which is a contradiction. ∎

**4. Lyapunov-like transformations on the tensor product of a nuclear pair** $(K_1, K_2)$**.** In this section, we study the Lyapunov-like transformations on the cone $K_1 \otimes K_2$, where $K_1$ is simplicial, and $K_2$ is any proper cone in $\mathbb{R}^m$, that is, the pair $(K_1, K_2)$ is nuclear.

Theorem 4.1. *Let $K$ be any proper cone in $\mathbb{R}^m$. Then $\beta(\mathbb{R}^n_+ \otimes K) = n\beta(K)$.*

*Proof.* Consider $a \otimes b \in \mathbb{R}^n \otimes \mathbb{R}^m$. Since $\{e_1, e_2, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$, we have $a = a^1 e_1 + a^2 e_2 + \cdots + a^n e_n$ and so $a \otimes b = \sum_{k=1}^n e_k \otimes a^k b$. Furthermore, when $a \in \mathbb{R}^n_+$ and $b \in K$, we have $a^k b \in K$. More generally, for any element in $\mathbb{R}^n \otimes \mathbb{R}^m$ (in $\mathbb{R}^n_+ \otimes K$) can be represented as $\sum_{k=1}^n e_k \otimes y_k$, where $y_k \in \mathbb{R}^m$ (respectively, in $K$) for all $1 \leq k \leq n$. It is easy to verify that such a representation is unique. Let $\phi : \mathbb{R}^n \otimes \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m$ ($n$-times) defined by

$$\phi\left(\sum_{k=1}^n e_k \otimes y_k\right) = (y_1, y_2, \ldots, y_n).$$

This $\phi$ is an isomorphism. Now consider the proper cone

$$\mathbb{R}^n_+ \otimes K = \left\{\sum_{i=1}^l w_i \otimes x_i \mid w_i \in \mathbb{R}^n_+, x_i \in K\right\} = \left\{\sum_{k=1}^n e_k \otimes y_k | e_k \in \mathbb{R}^n_+, y_k \in K\right\}.$$

Under this isomorphism, the image of the proper cone $\mathbb{R}^n_+ \otimes K$ is given by $\phi(\mathbb{R}^n_+ \otimes K) = K \times K \times \cdots \times K$ ($n$-times) and is defined as

$$\phi\left(\sum_{i=1}^l w_i \otimes x_i\right) = \phi\left(\sum_{k=1}^n e_k \otimes y_k\right) = (y_1, y_2, \ldots, y_n).$$

Thus, the cones $\mathbb{R}^n_+ \otimes K$ and $K \times K \times \cdots \times K$($n$-times) are isomorphic. Therefore, by Item (i) of Proposition 2.3, $\beta(\mathbb{R}^n_+ \otimes K) = \beta(K \times K \times \cdots \times K)$. Now by Item (iii) of Proposition 2.3, $\beta(K \times K \times \cdots \times K) = n\beta(K)$ so we have $\beta(\mathbb{R}^n_+ \otimes K) = n\beta(K)$. ∎

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Corollary 4.2. *If $K_1$ and $K_2$ are proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, such that $(K_1, K_2)$ is a nuclear pair (assume $K_1$ is simplicial), then $\beta(K_1 \otimes K_2) = n\beta(K_2)$.*

*Proof.* Let $K_1$ be a simplicial cone and $T$ be an isomorphism between $K_1$ and $\mathbb{R}^n_+$. Consider the isomorphism $T \otimes I$ between $K_1 \otimes K_2$ and $\mathbb{R}^n_+ \otimes K_2$. So by the above theorem, we have $\beta(K_1 \otimes K_2) = n\beta(K_2)$ as the cones are isomorphic. ∎

The following theorem gives the description of elements of $LL(K_1 \otimes K_2)$ for a nuclear pair $(K_1, K_2)$.

Theorem 4.3. *Let $K_1$ and $K_2$ be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, such that $(K_1, K_2)$ is a nuclear pair. Then $LL(K_1 \otimes K_2) = LL(K_1) \otimes LL(K_2)$.*

*Proof.* Let $\sum_{k=1}^{q} T_k \otimes S_k \in LL(K_1) \otimes LL(K_2)$. We claim that $\sum_{k=1}^{q} T_k \otimes S_k \in LL(K_1 \otimes K_2)$. Let $z \in K_1 \otimes K_2$ and $w \in (K_1 \otimes K_2)^*$ with $\langle z, w\rangle = 0$. Since $(K_1, K_2)$ is a nuclear pair, we have $(K_1 \otimes K_2)^* = K_1^* \otimes K_2^*$ and $\text{ext}((K_1 \otimes K_2)^*) = \text{ext}(K_1^* \otimes K_2^*)$. From Theorem 2.2, $\text{ext}(K_1 \otimes K_2) = \text{ext}(K_1) \otimes \text{ext}(K_2)$ and $\text{ext}((K_1 \otimes K_2)^*) = \text{ext}(K_1^* \otimes K_2^*) = \text{ext}((K_1)^* \otimes (K_2)^*)$. Since any vector in a proper cone is a nonnegative combination of its extreme vectors, we have $z = \sum_{i=1}^{r} a_i \otimes b_i$, where $a_i \in \text{ext}(K_1)$ and $b_i \in \text{ext}(K_2)$ for $1 \le i \le r$ and $w = \sum_{j=1}^{s} c_j \otimes d_j$, where $c_j \in \text{ext}(K_1^*)$ and $d_j \in \text{ext}(K_2^*)$ for $1 \le j \le s$. Since $\langle z, w\rangle = 0$, we have

$$\left\langle \sum_{i=1}^{r} a_i \otimes b_i, \sum_{j=1}^{s} c_j \otimes d_j \right\rangle = \sum_{i=1}^{r}\sum_{j=1}^{s} \langle a_i \otimes b_i, c_j \otimes d_j\rangle = \sum_{i=1}^{r}\sum_{j=1}^{s} \langle a_i, c_j\rangle\langle b_i, d_j\rangle = 0.$$

It is easy to observe that for all $1 \le i \le r$ and $1 \le j \le s$, either $\langle a_i, c_j\rangle = 0$ or $\langle b_i, d_j\rangle = 0$. Thus, either $(a_i, c_j) \in C(K_1)$ or $(b_i, d_j) \in C(K_2)$. Now

$$\begin{aligned}
\left\langle \sum_{k=1}^{q} T_k \otimes S_k(z), w \right\rangle &= \left\langle \sum_{k=1}^{q} T_k \otimes S_k \left( \sum_{i=1}^{r} a_i \otimes b_i \right), \sum_{j=1}^{s} c_j \otimes d_j \right\rangle \\
&= \sum_{k=1}^{q}\sum_{i=1}^{r}\sum_{j=1}^{s} \langle T_k \otimes S_k(a_i \otimes b_i), c_j \otimes d_j\rangle \\
&= \sum_{k=1}^{q}\sum_{i=1}^{r}\sum_{j=1}^{s} \langle T_k(a_i) \otimes S_k(b_i), c_j \otimes d_j\rangle \\
&= \sum_{k=1}^{q}\sum_{i=1}^{r}\sum_{j=1}^{s} \langle T_k(a_i), c_j\rangle\langle S_k(b_i), d_j\rangle.
\end{aligned}$$

Since $T_k \in LL(K_1)$ and $S_k \in LL(K_2)$ for all $1 \le k \le q$, we have either $\langle T_k(a_i), c_j\rangle = 0$ or $\langle S_k(b_i), d_j\rangle = 0$ for all $1 \le i \le r$ and $1 \le j \le s$. Thus,

$$\sum_{k=1}^{q}\sum_{i=1}^{r}\sum_{j=1}^{s} \langle T_k(a_i), c_j\rangle\langle S_k(b_i), d_j\rangle = 0.$$

Therefore, $\sum_{k=1}^{q} T_k \otimes S_k \in LL(K_1 \otimes K_2)$. So $LL(K_1) \otimes LL(K_2) \subseteq LL(K_1 \otimes K_2)$. By Corollary 4.2, we have $\beta(K_1 \otimes K_2) = np$ if $\beta(K_2) = p$. Since the dimension of $LL(K_1) \otimes LL(K_2)$ is $np$, we have $LL(K_1) \otimes LL(K_2) = LL(K_1 \otimes K_2)$. ∎

Corollary 4.4. *Let $K_1$ be a simplicial cone and $K_2$ be a proper cone in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then $LL(\pi(K_1, K_2)) = LL(K_2) \otimes LL(K_1)$ and $\beta(\pi(K_1, K_2)) = n\beta(K_2)$.*

*Proof.* From Theorem 3.3, we have $\pi(K_1, K_2) = ((K_2)^* \otimes K_1)^*)$. Since $(K_1, K_2)$ is a nuclear pair, we have $((K_2)^* \otimes K_1)^*) = (K_2 \otimes (K_1)^*)$. Now from Theorem 4.3, $LL(\pi(K_1, K_2)) = LL(K_2) \otimes LL((K_1)^*) = LL(K_2) \otimes LL(K_1)$ since $K_1$ is simplicial. Thus, $\beta(\pi(K_1, K_2)) = n\beta(K_2)$. ∎

The following remark is a version of Proposition 3.9 in the case of nuclear pair $\mathbb{R}^n_+ \otimes K$.

REMARK 4.5. *Let $n \leq m$. For a prime number $p$ such that $n < p \leq n^2 \leq nm$, there does not exist a proper cone $K$ in $\mathbb{R}^m$ such that $\beta(\mathbb{R}^n_+ \otimes K) = p$.*

**Concluding remarks.** Here in this paper, we describe the Lyapunov-like transformations and Lyapunov rank of the tensor product cone $K_1 \otimes K_2$, where either $K_1$ or $K_2$ is simplicial. As a consequence, we also find the Lyapunov rank and Lyapunov-like transformations on the cone of positive operators $\pi(K_1, K_2)$, where $(K_1, K_2)$ is a nuclear pair . We also observe that, for a prime number $p$ such that $n < p \leq n^2$, there does not exist proper polyhedral cones $K_1$ and $K_2$ such that $\beta(K_1 \otimes K_2) = p$. This is also true in the case of nuclear pair of cones. In addition, the results discussed in this paper lead to the following questions.

- If $K_1$ and $K_2$ are general proper cones what can we say about $\beta(K_1 \otimes K_2)$ and Lyapunov-like transformations on $K_1 \otimes K_2$?
- The converse of Corollary 3.8 and Theorem 4.3. That is, for proper cones $K_1$ and $K_2$ with $LL(K_1 \otimes K_2) = LL(K_1) \otimes LL(K_2)$ or $\beta(K_1 \otimes K_2) = \beta(K_1)\beta(K_2)$ can we conclude either both $K_1$ and $K_2$ are proper polyhedral cones or the pair $(K_1, K_2)$ is a nuclear pair.

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REFERENCES

[1] G. Aubrun, L. Lami, C. Palazuelos, and M. Plvala. Entangleability of cones. *Geom. Funct. Anal.* 31:181–205, 2021. https://doi.org/10.1007/s00039-021-00565-5

[2] A. Berman and P. Gaiha. A generalization of irreducible monotonicity. *Linear Algebra Appl.* 5:29–38, 1972. https://doi.org/10.1016/0024-3795(72)90016-x

[3] S. Gokulraj and A. Chandrashekaran. On symmetric linear games. *Linear Algebra Appl.* 562:44–54, 2019. https://doi.org/10.1016/j.laa.2018.10.004

[4] M. Gowda and J. Tao. $Z$-transformations on proper and symmetric cones: $Z$-transformations. *Math. Program.* 117: 195–221, 2009. https://doi.org/10.1007/s10107-007-0159-8

[5] M. Gowda and J. Tao. On the bilinearity rank of a proper cone and Lyapunov-like transformations. *Math. Program.* 147:155–170, 2014. https://doi.org/10.1007/s10107-013-0715-3

[6] M. Gowda and D. Trott. On the irreducibility, Lyapunov rank, and automorphisms of special Bishop-Phelps cones. *J. Math. Anal. Appl.* 419:172–184, 2014. https://doi.org/10.1016/j.jmaa.2014.04.061

[7] M. Gowda and G. Ravindran. On the game-theoretic value of a linear transformation relative to a self-dual cone. *Linear Algebra Appl.* 469:440–463, 2015. https://doi.org/10.1016/j.laa.2014.11.032

[8] J. Jeong and M. Gowda. Permutation invariant proper polyhedral cones and their Lyapunov rank. *J. Math. Anal. Appl.* 463:377–385 2018. https://doi.org/10.1016/j.jmaa.2018.03.024

[9] M. Orlitzky. The Lyapunov rank of an improper cone. *Optim. Methods Softw.* 32:109–125, 2017. https://doi.org/10.1080/10556788.2016.1202246

[10] M. Orlitzky. Lyapunov rank of polyhedral positive operators. *Linear Multilinear Algebra* 66:992–1000, 2018. https://doi.org/10.1080/03081087.2017.1331998

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 41, pp. 142-151, February 2025.



[11] M. Orlitzky. Tight bounds on Lyapunov rank. *Optim. Lett.* 16:723–728, 2022. https://doi.org/10.1007/s11590-021-01750-z

[12] M. Orlitzky and M. Gowda. An improved bound for the Lyapunov rank of a proper cone. *Optim. Lett.* 10:11–17, 2016. https://doi.org/10.1007/s11590-015-0903-6

[13] S. Ramanujan. A proof of Bertrand's postulate [J. Indian Math. Soc. 11:181–182, 1919]. *Collected Papers of Srinivasa Ramanujan* 208–209, 2000. https://doi.org/10.1016/s0164-1212(00)00033-9

[14] G. Rudolf, N. Noyan, D. Papp, and F. Alizadeh. Bilinear optimality constraints for the cone of positive polynomials. *Math. Program.* 129:5–31, 2011. https://doi.org/10.1007/s10107-011-0458-y

[15] H. Schneider and M. Vidyasagar. Cross-positive matrices. *SIAM J. Numer. Anal.* 7:508–519, 1970. https://doi.org/10.1137/0707041

[16] A. Shanmugapriya and A. Chandrashekaran. On the dual of the tensor product of semidefinite cones. *J. Anal.* 32:19–26, 2024. https://doi.org/10.1007/s41478-023-00573-8

[17] R. Sznajder. The Lyapunov rank of extended second order cones. *J. Global Optim.* 66:585–593, 2016. https://doi.org/10.1007/s10898-016-0445-1

[18] J. van Dobben de Bruyn. *Tensor Products of Convex Cones*, arXiv:2009.11843, 2020.