

MULTIPLICATIVE DIAGONALS OF MATRIX SEMIGROUPS*

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Abstract. We consider semigroups of matrices where either the diagonal map or the diagonal product map is multiplicative, and deduce structural properties of such semigroups.

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1. Introduction. For $A = [a_{ij}]$ in $M_n(\mathbb{C})$, the set of $n \times n$ complex matrices, define $\Delta(A) = [\alpha_{ij}]$, the *diagonal* of A , to be the $n \times n$ matrix such that

$$\alpha_{ij} = \begin{cases} a_{ij}, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

For a *matrix semigroup* \mathcal{S} (a collection of matrices in $M_n(\mathbb{C})$ which is closed under matrix multiplication), the diagonals of elements of \mathcal{S} seem to have an inordinate effect on the spatial properties of the semigroup, such as *reducibility* or *decomposability*.

A matrix semigroup \mathcal{S} in $M_n(\mathbb{C})$ is *reducible* if there exists a non-trivial proper subspace of \mathbb{C}^n which is invariant for each S in \mathcal{S} . Otherwise \mathcal{S} is said to be *irreducible*. A matrix semigroup is called *triangularizable* if there exists a chain of subspaces which are invariant for \mathcal{S} and maximal as a chain in the lattice of all subspaces of \mathbb{C}^n . This is equivalent to being similar (via any invertible matrix) to a set of upper triangular matrices.

A matrix semigroup \mathcal{S} in $M_n(\mathbb{C})$ is *decomposable* if there exists a non-trivial proper subspace of \mathbb{C}^n which is invariant for \mathcal{S} and which is spanned by a subset of the standard basis vectors (the vectors with exactly one non-zero entry, and that non-zero entry is one). We call such subspaces *standard subspaces*. Otherwise \mathcal{S} is said to be *indecomposable*. A matrix semigroup is called *completely decomposable* (the phrase *has a standard triangularization* is also used) if there exists a chain of

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standard subspaces which are invariant for \mathcal{S} and maximal as a chain in the lattice of all subspaces of \mathbb{C}^n . This is equivalent to being similar, via a permutation matrix, to a set of upper triangular matrices.

There are a number of results concerning the effect that the diagonals of semigroup have on a semigroup. In [1], it is shown that a group of matrices which has non-negative diagonals is either reducible or similar to a group of positive matrices (i.e., matrices with all entries greater than or equal to zero).

The structural effect of the diagonal is most pronounced in the case where the semigroup is *positive*, that is each entry of each matrix in \mathcal{S} is a non-negative real number. For example, in [3], it is shown that a positive semigroup whose diagonal entries are binary (equal to 0 or 1), is either decomposable, or similar, via a positive diagonal similarity, to a semigroup of binary matrices.

If we let $\delta(A)$ denote the diagonal product, i.e., $\delta(A) = \prod_{i=1}^n a_{ii}$, then it is shown in [6] and [4] that under some mild additional assumptions, if δ is a submultiplicative function on a positive semigroup \mathcal{S} then \mathcal{S} is completely decomposable.

In this paper, we consider the effect of the diagonal on the structure of a semigroup without assuming positivity. While the conclusions we draw cannot be as strong as those for the positive case, a surprising amount of structural information can be obtained.

The two problems we are most interested in are:

1. If Δ is a multiplicative map on a semigroup \mathcal{S} in $M_n(\mathbb{C})$, what can we determine about the structure of \mathcal{S} ?
2. If δ is a multiplicative map on a semigroup \mathcal{S} in $M_n(\mathbb{C})$, what can we determine about the structure of \mathcal{S} ?

Since the multiplicativity of Δ obviously implies the multiplicativity of δ , it is a stronger condition so we begin there.

2. Multiplicative diagonals. A semigroup having multiplicative diagonals gives an immediate triangularization result.

THEOREM 2.1. *If the diagonal map Δ is multiplicative on a matrix semigroup \mathcal{S} , then \mathcal{S} is triangularizable.*

Proof. For A, B and C in \mathcal{S} ,

$$\begin{aligned} \text{tr}(ABC) &= \text{tr}(\Delta(ABC)) \\ &= \text{tr}(\Delta(A)\Delta(B)\Delta(C)) \\ &= \text{tr}(\Delta(A)\Delta(C)\Delta(B)) \\ &= \text{tr}(\Delta(ACB)) = \text{tr}(ACB), \end{aligned}$$

so trace is permutable on \mathcal{S} and so by Theorem 2.2.1 of [6] (page 33), \mathcal{S} is triangularizable. \square

We can obtain additional information about the invariant subspaces of a semigroup \mathcal{S} via a theorem of Sarason [7]. First note that a subspace M is *semi-invariant* for a collection \mathcal{C} of operators in $M_n(\mathbb{C})$ if there exist two subspaces $M_1 \subseteq M_2$, both invariant for \mathcal{C} , such that $M_1 \oplus M = M_2$.

THEOREM 2.2 (Sarason). *For \mathcal{A} a subalgebra of $M_n(\mathbb{C})$ and M a subspace of \mathbb{C}^n with P_M denoting the orthogonal projection onto M , the compression map $\phi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ defined by $\phi(A) = P_M A P_M$ is an algebra homomorphism if and only if M is semi-invariant for \mathcal{A} .*

The multiplicativity of Δ on a semigroup \mathcal{S} implies that the compression to the span of each standard vector is an algebra homomorphism on the linear span of \mathcal{S} . Thus, we have the following.

THEOREM 2.3. *If the diagonal map Δ is multiplicative on a matrix semigroup \mathcal{S} in $M_n(\mathbb{C})$, then for each $i = 1, 2, \dots, n$, $M_i = \{\lambda e_i : \lambda \in \mathbb{C}\}$ is a semi-invariant subspace for \mathcal{S} .*

It would seem that by combining Theorems 2.1 and 2.3, we should be able to obtain that multiplicative diagonal map Δ implies complete decomposability, but this is not the case as the following example shows.

EXAMPLE 2.4. Let

$$F_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$F_3 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 \\ -1 & 1 & 0 & 1 \end{bmatrix}.$$

Then

$$F_1 + F_2 + F_3 + F_4 = I$$

and

$$F_i F_j = 0$$

for $i \neq j$. In particular, every F_i is a rank one idempotent, and $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$ is a semigroup. Writing $|F_i|$ for the matrix obtained by taking the absolute values of the entries of F_i , we see that

$$|F_1| + |F_2| + |F_3| + |F_4| = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 \\ 2 & 2 & 0 & 1 \end{bmatrix},$$

from which it follows that the four matrices are not only not simultaneously completely decomposable, but also they do not even share a non-trivial standard invariant subspace. Yet it is easy to see that the diagonal map Δ is multiplicative on \mathcal{F} .

The above example eliminates any possibility of major decomposability theorems arising from a multiplicative Δ map. The example can be modified in many ways to provide counterexamples. For example,

$$\mathcal{G} = \left\{ \sum_{i=1}^4 \alpha_i F_i : \alpha_i \in \mathbb{T} \right\}$$

gives an example of a compact group on which Δ is multiplicative, but the group is indecomposable.

Four dimensions are necessary to obtain a counterexample to complete decomposability.

THEOREM 2.5. *If the diagonal map Δ is multiplicative on a semigroup \mathcal{S} in $M_3(\mathbb{C})$, then \mathcal{S} is completely decomposable.*

The proof is surprisingly lengthy. In the appendix, we will do similar low dimension calculations, so we will leave this proof to the reader. It turns out that many proofs regarding multiplicative diagonals (or diagonal products) in semigroups in $M_2(\mathbb{C})$ or $M_3(\mathbb{C})$ are unexpectedly nontrivial and intricate.

THEOREM 2.6. *If the diagonal map Δ is multiplicative on a semigroup \mathcal{S} in $M_n(\mathbb{C})$ and $A \in \mathcal{S}$, then the diagonal entries of A are exactly the eigenvalues of A repeated according to their algebraic multiplicities.*

Proof. We have that $\text{tr}(A^n) = \text{tr}(\Delta(A^n)) = \text{tr}(\Delta(A)^n)$, so this follows from Theorem 2.1.16 of [6]. \square

Note that since both Δ and the trace are linear maps, Theorem 2.6 is also true for A in the linear span of \mathcal{S} .

THEOREM 2.7. *If the diagonal map Δ is multiplicative on a semigroup \mathcal{S} in $M_n(\mathbb{C})$ generated by a rank-one matrix, then the semigroup is completely decomposable.*

Proof. If the generator is A , then there exists vectors $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ in \mathbb{C}^n such that $A = xy^*$ and then the trace of A equals y^*x . The condition that $\Delta(A^2) = \Delta(A)^2$ implies that for all $i = 1, 2, \dots, n$, $\text{tr}(A)x_i\overline{y_i} = (x_i\overline{y_i})^2$. So either $x_i\overline{y_i} = 0$ or $x_i\overline{y_i} = \text{tr}(A)$. However, the trace of A is the sum over all $x_i\overline{y_i}$ so we must have that there is at most one i for which $x_i\overline{y_i} \neq 0$.

In the case where there does exist i for which $x_i\overline{y_i} \neq 0$, let $S_x = \{j : x_j \neq 0, j \neq i\}$ and let m be the cardinality of this set. Apply any permutation of $\{1, 2, \dots, n\}$ that maps S_x to $\{1, 2, \dots, m\}$ and maps i to $m+1$. Then it is easy to see that the corresponding permutation matrix P is such that, with respect to the decomposition $\mathbb{C}^n = \mathbb{C}^m \oplus \mathbb{C} \oplus \mathbb{C}^{n-m-1}$,

$$P^{-1}SP \subseteq \left\{ \begin{bmatrix} 0 & ub & uv^* \\ 0 & ab & av^* \\ 0 & 0 & 0 \end{bmatrix} \right\},$$

so \mathcal{S} is completely decomposable.

In the case, where $x_i\overline{y_i} = 0$ for all i , it is even simpler. Let $S_x = \{j : x_j \neq 0\}$ and again do as above. \square

This theorem does not extend to semigroups generated by matrices of higher rank, even if complete decomposability is weakened to mere decomposability in the conclusion.

EXAMPLE 2.8. Again modifying Example 2.4, let

$$A = F_1 + 2F_3 = \begin{bmatrix} 2 & -1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -2 & 2 & 0 & 0 \\ 2 & -2 & 0 & 0 \end{bmatrix}.$$

Then A is indecomposable matrix of rank two, but the diagonal map Δ is multiplicative on the semigroup generated by A , since

$$\Delta(A^n) = \Delta(F_1 + 2^n F_3) = \begin{bmatrix} 2^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, one can check that $B = F_1 + 2F_2 + 3F_3 + 4F_4$ is an invertible matrix with the same property.

THEOREM 2.9. *If the diagonal map Δ is multiplicative on a semigroup \mathcal{S} of matrices of rank one, then \mathcal{S} is completely decomposable.*

Proof. For a matrix $S = [s_{ij}]$, define $|S| = [|s_{ij}|]$. Note that if S is a rank-one matrix $S = xy^*$, then $|S| = |x||y|^*$ is also a rank-one matrix. Thus,

$$\mathcal{S}' = \{r|S| : r \in \mathbb{R}^+, S \in \mathcal{S}\}$$

is a semigroup of positive matrices, and it is easily verified that Δ is still multiplicative on \mathcal{S}' . Thus, by Lemma 5.1.3 of [6], \mathcal{S}' is completely decomposable, from which it immediately follows that \mathcal{S} is completely decomposable as well. \square

In Theorem 2.9, the implicit hypothesis that \mathcal{S} does not contain non-trivial zero divisors is essential. Otherwise we do not even obtain decomposability, as the following example shows.

EXAMPLE 2.10. Let

$$\begin{aligned}\mathcal{X} &= \{ (a, -a, 0, 0, b, -b, 0, 0) \mid a, b \in \mathbb{C} \}, \\ \mathcal{Y} &= \{ (0, 0, c, c, 0, 0, d, d) \mid c, d \in \mathbb{C} \}, \\ \mathcal{U} &= \{ (0, 0, e, -e, 0, 0, f, -f) \mid e, f \in \mathbb{C} \}, \\ \mathcal{W} &= \{ (g, g, 0, 0, h, h, 0, 0) \mid g, h \in \mathbb{C} \},\end{aligned}$$

and let

$$\begin{aligned}\mathcal{S} &= \{ x^*y \mid x \in \mathcal{X}, y \in \mathcal{Y} \}, \\ \mathcal{T} &= \{ u^*w \mid u \in \mathcal{U}, w \in \mathcal{W} \}.\end{aligned}$$

Then $\mathcal{F} = \mathcal{S} \cup \mathcal{T}$ is an indecomposable semigroup of nilpotent matrices all of which have zero diagonal and rank at most one. In fact, $AB = 0$ for any $A, B \in \mathcal{F}$.

If zero divisors are present, then we can obtain complete decomposability in one special case.

THEOREM 2.11. *If the diagonal map Δ is multiplicative on a semigroup generated by a nilpotent matrix $N \in M_n(\mathbb{C})$ of rank $n - 1$, then the semigroup is completely decomposable.*

Proof. We use induction on n , the dimension of the underlying space. The base case is $n = 1$ and in this case $N = 0$ and so is completely decomposable. Suppose that the statement is true for n , and consider $N \in M_{n+1}(\mathbb{C})$ of rank n . Such an N must be a single Jordan block, and so $N^n \neq 0$. So there exists some standard vector e (with only one non-zero entry and that entry is 1) with $N^n e \neq 0$. By applying a permutation similarity to our semigroup, there is no loss of generality in assuming that $e = e_1$, the standard vector whose first entry is 1.

Then $\{e_1, Ne_1, N^2e_1, \dots, N^ne_1\}$ is a basis for \mathbb{C}^{n+1} . The multiplicativity of Δ implies that $\Delta(N^k) = 0$ for all $k = 1, 2, \dots, n$, so N^ke_1 is perpendicular to e_1 for all $k = 1, 2, \dots, n$. This implies that $\text{span}\{Ne_1, N^2e_1, \dots, N^ne_1\} = \{e_1\}^\perp$. Thus, the matrix N is of the form

$$N = \begin{bmatrix} 0 & 0 \\ * & N_1 \end{bmatrix},$$

where N_1 is nilpotent of rank $n - 1$. By induction, N is completely decomposable. \square

We complete our investigation of multiplicative Δ map by considering the case of a self-adjoint semigroup

THEOREM 2.12. *If \mathcal{S} is a self-adjoint semigroup and Δ is multiplicative on \mathcal{S} , then \mathcal{S} is diagonal.*

Proof. This can be proved in a number of ways. The simplest is to consider that for each A in \mathcal{S} , $\Delta(A^*A) = \Delta(A)\Delta(A^*)$. From here the result follows immediately. \square

3. Multiplicative diagonal products. We now relax our condition on the diagonals. In this section we only require that $\delta : M_n(\mathbb{C}) \rightarrow \mathbb{C}$, mapping a matrix to the product of its diagonal entries, is multiplicative on our semigroup.

THEOREM 3.1. *If $\mathcal{U} \subseteq M_n(\mathbb{C})$ is a semigroup of unitaries and $\delta : \mathcal{U} \rightarrow \mathbb{C}^n$ is multiplicative, then \mathcal{U} consists of diagonal matrices.*

Proof. First note that if $U = [u_{ij}]$ is a unitary matrix in \mathcal{U} , then there exists an increasing sequence of natural numbers $\{q_j\}_{j=1}^\infty$ with $\lim_j q_j = \infty$ and $\lim_j U^{q_j} = I$. To see this, we may assume without loss of generality that U is diagonal with eigenvalues $e^{2\pi i \alpha_1}, e^{2\pi i \alpha_2}, \dots, e^{2\pi i \alpha_n}$ and use the multivariate version of Dirichlet's Theorem on Diophantine approximation (see [2]). This theorem allows us to find, given a natural number Q , a natural number $q \leq Q$ with

$$\max\{\langle \alpha_1 q \rangle, \langle \alpha_2 q \rangle, \dots, \langle \alpha_n q \rangle\} \leq \frac{1}{Q^{1/n}},$$

where $\langle x \rangle$ is the distance from x to the nearest integer. From this it is straightforward to construct the sequence $\{q_j\}_{j=1}^\infty$ as above.

By the continuity of δ , we must have that $\lim_j \delta(U^{q_j}) = 1$. The multiplicativity of δ gives that $\delta(U)^{q_j} = (\prod u_{ii})^{q_j}$ converges to 1. Thus $|\prod u_{ii}| = 1$, and as these are entries of a unitary matrix and hence bounded by 1, we must have that $|u_{ii}| = 1$ for $i = 1, 2, \dots, n$ and thereby U is a diagonal matrix. \square

COROLLARY 3.2. *If \mathcal{S} is a bounded group of invertible matrices in $M_n(\mathbb{C})$ and δ is multiplicative on \mathcal{S} , then $|\delta(S)| = 1$ for all $S \in \mathcal{S}$.*

Proof. The group \mathcal{S} is similar to a unitary group (see Theorem 3.1.5 of [6]). Hence, similarly to the proof of Theorem 3, if $S = [s_{ij}]$ is an element of \mathcal{S} , then $|\delta(S)| = |\prod s_{ii}| = 1$. \square

THEOREM 3.3. *If \mathcal{S} is a self-adjoint semigroup of $M_n(\mathbb{C})$, and δ is multiplicative and nonzero on \mathcal{S} ($\delta(S) \neq 0$ for all $S \in \mathcal{S}$), then \mathcal{S} is a semigroup of diagonal matrices.*

Proof. If $A = [a_{ij}]$ is a matrix in \mathcal{S} with column vectors c_1, c_2, \dots, c_n , then

$$\prod_{i=1}^n \|c_i\|^2 = \delta(A^*A) = \delta(A^*)\delta(A) = \prod_{i=1}^n |a_{ii}|^2.$$

Now $\|c_i\|^2 = |a_{ii}|^2 + \sum_{j \neq i} |a_{ji}|^2$, and $\delta(S) \neq 0$ for all $S \in \mathcal{S}$, so the only possibility is that for $i \neq j$, $a_{ji} = 0$. \square

We note that the condition that $\delta(S) \neq 0$ for all $S \in \mathcal{S}$ in the above theorem, could be replaced by the condition that \mathcal{S} consists of invertible matrices, or that matrices in \mathcal{S} have no zero columns. Even without self-adjointness, these conditions are connected.

THEOREM 3.4. *Let \mathcal{S} be a semigroup of invertible matrices. If δ is multiplicative on \mathcal{S} , then $\delta(S) \neq 0$ for all $S \in \mathcal{S}$.*

Proof. The set $\mathcal{S}_0 = \{S \in \mathcal{S} : \delta(S) = 0\}$ is a semigroup ideal of \mathcal{S} . Consider $\mathcal{Z}(\mathcal{S}_0)$, the Zariski closure of \mathcal{S}_0 (see [5] for information on the Zariski closure). It is known that $\mathcal{Z}(\mathcal{S}_0)$ must contain a group which contains \mathcal{S}_0 and that polynomial conditions (like $\delta(S) = 0$) extend to the Zariski closure. This implies that δ is constantly zero on a group, which is a contradiction since $\delta(I) = 1$. \square

Note that the invertibility assumption is required, as shown by the following example. Let ω be a primitive third root of unity, then

$$\mathcal{S}_\omega = \left\{ \pm \omega^i \begin{bmatrix} 1 & 1 \\ w & w \end{bmatrix} : i = 1, 2, 3 \right\}$$

has $\delta(\mathcal{S}_\omega) \subseteq \{1, \omega, \omega^2\}$ but clearly \mathcal{S}_ω is not a semigroup of invertible matrices.

We conclude by considering the rank-one case, but also assuming matrices in our semigroup have real entries.

THEOREM 3.5. *For $n \in \mathbb{N}$ odd, if $\mathcal{S} \in M_n(\mathbb{C})$ is a semigroup of rank-one matrices such that each A in \mathcal{S} has real entries and δ is non-zero and multiplicative on \mathcal{S} , then \mathcal{S} is triangularizable.*

Proof. There are some reductions we can make immediately. By Theorem 3.3.9 of [6], it suffices to show that spectrum is submultiplicative on \mathcal{S} to obtain triangu-

larizability. The condition of submultiplicative spectrum involves only two matrices in \mathcal{S} , so if we can show doubly generated subsemigroups of \mathcal{S} are triangularizable, then the general result follows. Also, given a semigroup \mathcal{S} with δ multiplicative and non-zero, the set

$$\left\{ \frac{1}{(\delta(\mathcal{S}))^{\frac{1}{n}}} S : S \in \mathcal{S} \right\}$$

is a semigroup which clearly is triangularizable if and only if \mathcal{S} is triangularizable. So, without loss of generality, we may assume δ is constantly one on \mathcal{S} .

Finally, note that if $E = xy^*$ is rank-one, then $E^2 = (xy^*)^2 = (y^*x)xy^* = (y^*x)E$. So if E is in a semigroup which has δ constantly one, then $(y^*x)^n = 1$ and so if n is odd then E is idempotent. (If n is even, then we can just say that either E or $-E$ is idempotent).

Since n is odd, \mathcal{S} is triangularizable if and only if a related, doubly generated, semigroup of idempotents is triangularizable. But all such semigroups are triangularizable (see Theorem 2.3.5 of [6]). \square

In the case where n is even, the conditions of the above theorem do impose some structural constraints on the semigroup, but not enough to force triangularizability, as the following example shows.

EXAMPLE 3.6. If we are looking for a counterexample to triangularizability for a semigroup \mathcal{S} in $M_4(\mathbb{R})$ satisfying the conditions of Theorem 3.5, using the reductions above, we can assume that δ is constantly one on our semigroup, which is of the form

$$\mathcal{S} = \pm \{E, F, EF, FE\},$$

where E, F are idempotent, and $EFE = -E$ and $FEF = -F$ (so $-EF$ and $-FE$ are idempotent).

So, for $E = xy^*$ and $F = uv^*$,

1. that E is idempotent implies $\langle x, y \rangle = 1$,
2. that F is idempotent implies $\langle u, v \rangle = 1$,
3. $\delta(E) = 1$ implies $(\prod x_i)(\prod y_i) = 1$,
4. $\delta(F) = 1$ implies $(\prod u_i)(\prod v_i) = 1$,
5. $\delta(EF) = 1$ implies that $(\prod x_i)(\prod v_i) = 1$ and $\langle u, y \rangle = \pm 1$,
6. $\delta(FE) = 1$ implies that $(\prod y_i)(\prod u_i) = 1$ and $\langle x, v \rangle = \pm 1$, and
7. $EFE = -E$ implies that $\langle x, v \rangle \langle u, y \rangle = 1$.

By applying a diagonal similarity (which leaves the diagonal unchanged), we can

assume that $x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Then our conditions reduce to

1. $\prod y_i = \prod u_i = \prod v_i = 1$,
2. $\sum y_i = \sum u_i y_i = \sum u_i v_i = 1$ and $\sum v_i = -1$.

There are many possible solutions for this system. One is obtained by letting $\tau = \frac{1+\sqrt{5}}{2}$ and $\sigma = \frac{-1+\sqrt{5}}{2}$, which gives

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, u = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, y = \begin{bmatrix} \tau \\ -\frac{1}{\tau} \\ -1 \\ 1 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 1 \\ \sigma \\ -\frac{1}{\sigma} \end{bmatrix}.$$

It is straightforward to show that the only invariant subspaces for \mathcal{S} are those subspaces which either are contained in $\ker(\mathcal{S}) = \{y, v\}^\perp$ or contain $\text{Range}(\mathcal{S}) = \text{span}\{x, u\}$. So the two dimensional subspace in any triangularizing chain would need to be $\{y, v\}^\perp$ or $\text{span}\{x, u\}$, and these subspaces would need to intersect non-trivially to be able to extend to a triangularizing chain. It is easily verified that $\{y, v\}^\perp \cap \text{span}\{x, u\} = \{0\}$, so \mathcal{S} is not triangularizable.

4. Appendix: The 2×2 case. As we have mentioned before, deriving conditions from multiplicative δ map is complicated even in the 2 by 2 case. Since the semigroups \mathcal{S} in $M_2(\mathbb{C})$ for which δ restricted to \mathcal{S} is multiplicative can be completely classified, and since this classification may prove useful in a further study of higher dimensions, we include an appendix on this low dimensional case.

First we note that any matrix in a semigroup in $M_2(\mathbb{C})$ on which δ is multiplicative must be one of three special types.

THEOREM 4.1. *If the diagonal product δ is multiplicative on a semigroup \mathcal{S} in $M_2(\mathbb{C})$ and A is in \mathcal{S} , then one of the following is true:*

1. (Type 1) Up to a diagonal similarity A is a non-zero multiple of $\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$
 (here $\delta(A^n) = (-1)^n \det(A^n)$);
2. (Type 2) Up to a diagonal similarity, A is a non-zero multiple of $\begin{bmatrix} 1 & 1 \\ w & w \end{bmatrix}$,
 where $w \neq 1$ is a cubic root of unity;
3. (Type 3) A is triangular.

Proof. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the equation

$$\delta(A^2) = (\delta(A))^2$$

gives that

$$bc(d^2 + a^2 + bc) = 0.$$

If $bc = 0$, then A is triangular, and since the same must be true for all powers of A , δ coincides with the determinant on S .

Assume $bc \neq 0$ henceforth. Since δ is invariant under diagonal similarity, it is sufficient to assume $c = 1$. Hence,

$$-(d^2 + a^2) = b, \quad (4.1)$$

and therefore,

$$A^2 = \begin{bmatrix} a^2 + b & (a+d)b \\ (a+d) & d^2 + b \end{bmatrix} = \begin{bmatrix} -d^2 & (a+d)b \\ (a+d) & -a^2 \end{bmatrix}. \quad (4.2)$$

If A has zero trace, then

$$A = \begin{bmatrix} a & -2a^2 \\ 1 & -a \end{bmatrix} \quad \text{and} \quad A^2 = -a^2 I$$

so that $\delta(A^n) = (-1)^n a^{2n} = (-1)^n \det(A^n)$. If $a = 0$ then A is triangular. If $a \neq 0$, then

$$\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{a} \end{bmatrix} = -a \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

Assume $\text{trace}(A) \neq 0$ henceforth. The equation

$$\delta(A^4) = (\delta(A^2))^2$$

yields via (4.2)

$$a^4 + d^4 + (a+d)^2 b = 0,$$

which in conjunction with equation (4.1) leads to

$$a^4 + d^4 = (a+d)^2(a^2 + d^2),$$

and reduces to

$$2ad(a^2 + ad + d^2) = 0. \quad (4.3)$$

If $ad = 0$ (i.e., $\delta(A) = 0$) then $A^3 = -(a^3 + d^3)I$, so that

$$\delta(A^3) = (\delta(A))^3$$

gives

$$a^3 + d^3 = 0,$$

and consequently $a = d = 0$, since $ad = 0$ is assumed. This contradicts the assumption that $\text{trace}(A) \neq 0$.

The only choice left is $ad \neq 0$. In this case equations (4.1) and (4.3) yield

$$ad = -(a^2 + d^2) = b \quad (4.4)$$

(so that $\det(A) = 0$) and

$$A = \begin{bmatrix} a & ad \\ 1 & d \end{bmatrix}.$$

A simple calculation shows that

$$A^2 = (a + d)A$$

so that $A^n = (a + d)^{n-1}A$ and $\delta(A^n) = ad((a + d)^2)^{n-1} = (ad)^n$. Furthermore,

$$\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{d} \end{bmatrix} = a \begin{bmatrix} 1 & 1 \\ \frac{d}{a} & \frac{d}{a} \end{bmatrix},$$

and equation (4.4) states that $1 + \frac{d}{a} + \left(\frac{d}{a}\right)^2 = 0$ so that $\frac{d}{a}$ is a cubic root of unity distinct from 1. \square

One consequence of Theorem 4.1 is that: if the diagonal product δ is multiplicative on the semigroup \mathcal{S} in $M_2(\mathbb{C})$, then each matrix A in \mathcal{S} is either triangular or $\det(A) = 0$ or $\text{trace}(A) = 0$.

It can be shown that if δ is multiplicative on \mathcal{S} in $M_2(\mathbb{C})$, and A and B are in \mathcal{S} , then A and B have to either be of the same type (as per Theorem 4.1) or one of A or B must be scalar. In particular, you cannot have matrices of type 2 and type 3 in the same semigroup \mathcal{S} . Furthermore, if A and B are of the same type, then the multiplicativity of δ forces additional stringent conditions. These are summarized in the following three theorems.

THEOREM 4.2. *The diagonal product δ is multiplicative on a semigroup $S \subset M_2(\mathbb{C})$ containing a matrix A of type 1 if and only if every element of S is either a multiple of A or a scalar matrix.*

THEOREM 4.3. *The diagonal product δ is multiplicative on a semigroup $S \subset M_2(\mathbb{C})$ containing a matrix A of type 2 if and only if one of the following is true:*

1. *There exists a cubic root of unity $w \neq 1$ such that after a simultaneous diagonal similarity, every element of S is either a multiple of $\begin{bmatrix} 1 & 1 \\ w & w \end{bmatrix}$ or a multiple of $\begin{bmatrix} 1 & 1 \\ w^2 & w^2 \end{bmatrix}$ or a scalar matrix;*
2. *There exists a cubic root of unity $w \neq 1$ such that after a simultaneous diagonal similarity, every element of S is either a multiple of $\begin{bmatrix} 1 & 1 \\ w & w \end{bmatrix}$ or a multiple of $\begin{bmatrix} w^2 & 1 \\ 1 & w \end{bmatrix}$ or a scalar matrix.*

THEOREM 4.4. *The diagonal product δ is multiplicative on a semigroup $S \subset M_2(\mathbb{C})$ containing a non-scalar matrix A of type 3 if and only if one of the following is true:*

1. *S is completely decomposable (i.e., either S or its transpose S^T is contained in the upper triangular matrices);*
2. *either S or S^T has the property that each element in the semigroup is either scalar or has at most one non-zero row.*

The proofs of all these theorems proceed along the same lines. Besides the matrix A in the semigroup of the given type, you assume you also have a matrix B in the semigroup of one of the three types. Examine each of the three cases using the multiplicativity of δ to arrive at the conclusion. We include the proof of Theorem 4.2 and leave the other two to the reader.

Proof of Theorem 4.2. Without loss of generality, we assume that $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ is in our semigroup.

If $B = \begin{bmatrix} x & y \\ z & v \end{bmatrix}$ is also in the semigroup, then the condition

$$\delta(AB) = \delta(A)\delta(B) = \delta(BA)$$

states that

$$(x - 2z)(y - v) = -xv = -(x + y)(2z + v),$$

or equivalently

$$xy - 2zy + 2zv = 0 = yv + 2xz + 2yz. \quad (4.5)$$

If B was also of type 1, then $B = \alpha \begin{bmatrix} 1 & -2p \\ \frac{1}{p} & -1 \end{bmatrix}$ for some $\alpha, p \neq 0$. The leftmost

equality in (4.5), necessary for multiplicativity of δ , states in the present case that

$$-2p + 4 - \frac{2}{p} = 0,$$

which has a unique solution $p = 1$.

If B was of type 2, then $B = \alpha \begin{bmatrix} 1 & p \\ \frac{w}{p} & w \end{bmatrix}$ (where $\alpha \neq 0 \neq p$). For δ to be multiplicative on the semigroup generated by A and B , equalities (4.5) must hold. In other words, p and w satisfy

$$p - 2w + 2\frac{w^2}{p} = 0 = pw + 2\frac{w}{p} + 2w,$$

or equivalently

$$p^2 - 2wp + 2w^2 = 0 = p^2 + 2p + 2.$$

Hence $w(1 \pm i) = p = -1 \pm i$, and it is easy to check that cubic roots of unity can not satisfy this equation.

Finally, if B was of type 3 then, since $\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ is similar to its transpose via a diagonal similarity, and δ is invariant under transposition, we may assume $B = \begin{bmatrix} x & y \\ 0 & v \end{bmatrix}$. Equalities in (4.5) force

$$xy = 0 = yv. \quad (4.6)$$

If $y \neq 0$, then B is nilpotent, and if $y = 0$, then B is diagonal. In the first case $B = \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}$ so that

$$ABA = \begin{bmatrix} y & -y \\ y & -y \end{bmatrix}.$$

The requirement $\delta(ABA) = \delta(B)\delta(A)^2 = 0$ dictates that $y = 0$, which is a contradiction. Hence, it must be that $y = 0$ is the only option. In this case, the condition $\delta(ABA) = \delta(B)\delta(A)^2$ can be easily seen to state that $(x-v)^2 = 0$, which is a required conclusion. \square

The preceding three structure theorems give two immediate reducibility or decomposability results

COROLLARY 4.5. *If the diagonal product δ is multiplicative on a semigroup S of invertibles (in $M_2(\mathbb{C})$) with nonzero trace, then S is (completely) decomposable.*

Proof. By Theorem 4.1 all elements of S are triangular with no zeros on the diagonal. Theorem 4.4 does the rest. \square

COROLLARY 4.6. *If δ is multiplicative on a semigroup S in \mathbb{SL}_2 , then S is reducible.*

Proof. If S contains an element of type 1, then S is abelian (and hence reducible) by Theorem 4.2. By Theorem 4.1, the only alternative is that S is a semigroup of triangular matrices. In such a case, by Theorem 4.4, S is (completely) decomposable. \square

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