Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 774-787, October 2024.

DIAMETER VS. LAPLACIAN EIGENVALUE DISTRIBUTION*

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Abstract. Let G be a simple graph of order n. It is known that any Laplacian eigenvalue of G belongs to the interval $[0, n]$. For an interval $I \subseteq [0, n]$, denote by $m_G I$ the number of Laplacian eigenvalues of G in I, counted with multiplicities. Let d be the diameter of G. If $2 \leq d \leq n-4$, we show that $m_G[n-d, n] \leq n-d+2$, and it may be improved into $m_G[n-d, n] \leq n-d+1$ when $d = 2, 3, 4$. We also show that $m_G[n-2d+4, n] \leq n-2$ if $d = 2, \lfloor \frac{n+3}{2} \rfloor$, and $m_G[n-2d+4, n] \leq n-3$ if $3 \leq d \leq \lfloor \frac{n+1}{2} \rfloor$. The diameter constraint provides an insightful approach to understand how the Laplacian eigenvalues are distributed.

Key words. Diameter, Laplacian spectrum, Permutational similar.

AMS subject classifications. 05C12, 05C50, 15A48.

1. Introducion. We consider simple graphs [3]. Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$. The adjacency matrix $A(G)$ of G is the $n \times n$ matrix where the (i, j)-entry is equal to 1 if v_i and v_j are adjacent, and is otherwise equal to 0. Moreover, if $D(G)$ is the $n \times n$ diagonal matrix whose (i, i)-entry is the degree of vertex v_i for $i = 1, \ldots, n$, then $L(G) = D(G) - A(G)$ is called the Laplacian matrix of G. This is a symmetric positive semidefinite matrix and hence has n real nonnegative eigenvalues, which are said to be the Laplacian eigenvalues of G and can be arranged as

$$\mu_n(G) \leq \cdots \leq \mu_1(G),$$

counted with multiplicities. One can see that $\mu_n(G) = 0$, and $\mu_j(G)$ is the jth (largest) Laplacian eigenvalue of G for $j = 1, \ldots, n$.

For a graph G of order n, any Laplacian eigenvalue of G lies in the interval $[0, n]$ [19, 20, 21], and the multiplicity of the Laplacian eigenvalue 0 is equal to the number of the (connected) components of G [8]. The distribution of Laplacian eigenvalues of graphs is relevant to the many applications related to Laplacian matrices [9, 19, 21]. There are results on the Laplacian eigenvalues of n-vertex graphs in subintervals of $[0, n]$. For example, the number of Laplacian eigenvalues in certain intervals is related to classical graph parameters, notably the independence number [1, 6, 7], the matching number [11], the edge covering number [12], the domination number [5, 13], the chromatic number [1], and the diameter [1, 10, 26, 27], and for the cases of trees, see [4, 15, 23, 28]. However, it is not well understood how the Laplacian eigenvalues are distributed in $[0, n]$, see [15].

The diameter of a connected graph G is defined as the maximum distance over all pairs of vertices in G. Recently, progress is made on the connections between the distribution of the Laplacian eienvalues and the diameter. For an interval $I \subseteq [0, n]$, denote by $m_G I$ the number of Laplacian eigenvalues of a graph G of order n in I, counted with multiplicities. In particular, if I is degenerate, say $I = \{a\}$, then $m_G I$ is the

*Received by the editors on June 11, 2024. Accepted for publication on October 15, 2024. Handling Editor: Francesco Belardo. Corresponding Author: Bo Zhou.

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Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 774-787, October 2024.

multiplicity of a as a Laplacian eigenvalue of G. Let G be a connected graph of order n with diameter d. For $d \geq 4$, Ahanjideh et al. [1] showed that $m_G(n-d+3, n] \leq n-d-1$. For $2 \leq d \leq n-2$, Xu and Zhou showed that $m_G[n-d+2, n] \leq n-d$ in [26], which was conjectured in [1] and $m_G[n-d+1, n] \leq n-d+1$ in [27]. In this paper, we give further upper bounds for $m_G[n-d, n]$.

THEOREM 1.1. *Let G be a connected graph of order n with diameter d, where $2 \leq d \leq n-4$. Then*

$$m_G[n-d, n] \leq \begin{cases} n-d+1 & \text{if } d = 2, 3, 4, \\ n-d+2 & \text{if } d \geq 5. \end{cases}$$

We remark that the diameter condition in Theorem 1.1 is tight. It is known [2] that $\mu_j(P_n) = 4\sin^2\frac{(n-j)\pi}{2n}$ for $j = 1, \ldots, n$. Thus, we have:

(i) If $d = n-1$, then G is a path P_n, and $\mu_j(P_n) \geq 1$ if and only if $j = 1, \ldots, \lfloor \frac{2}{3}n \rfloor$, so $m_G[1, n] = \lfloor \frac{2}{3}n \rfloor$.

(ii) If $d = n-2$, then P_{n-1} is a subgraph of G, and $\mu_j(P_{n-1}) \geq 2$ if and only if $j = 1, \ldots, \lfloor \frac{1}{2}(n-1) \rfloor$, so we have by Lemma 2.3 that $\mu_{\lfloor \frac{1}{2}(n-1) \rfloor}(G) \geq \mu_{\lfloor \frac{1}{2}(n-1) \rfloor}(P_{n-1}) \geq 2$, implying $m_G[2, n] \geq \lfloor \frac{1}{2}(n-1) \rfloor$.

(iii) If $d = n-3$, then P_{n-2} is a subgraph of G, and $\mu_j(P_{n-2}) \geq 3$ if and only if $j = 1, \ldots, \lfloor \frac{1}{3}(n-2) \rfloor$, so we have by Lemma 2.3 that $m_G[3, n] \geq \lfloor \frac{1}{3}(n-2) \rfloor$.

Next, we give the second result.

THEOREM 1.2. *Let G be a connected graph of order n with diameter d. Then*

$$m_G[n-2d+4, n] \leq \begin{cases} n-2 & \text{if } d = 2, \lfloor \frac{n+3}{2} \rfloor, \\ n-3 & \text{if } 3 \leq d \leq \lfloor \frac{n+1}{2} \rfloor. \end{cases}$$

Note that the case $d = 3$ in Theorem 1.2 has been given in [27]. Also, the diameter condition is tight in Theorem 1.2, see Section 4.

Suppose that G is a connected graph of order n with diameter $d \geq 2$. Motivated by the aforementioned results in [26, 27] and Theorems 1.1 and 1.2, we propose the following conjecture.

CONJECTURE 1.1. *Let G be a connected graph of order n with diameter $d \geq 2$. If $c = 0, \ldots, d-2$ with $\max\{2, c\} \leq d \leq n-2-c$, then $m_G[n-d+2-c, n] \leq n-d+c$.*

Note that in Conjecture 1.1, as the interval $[\max\{2, c\}, n-2-c]$ becomes smaller, the bound for the number of Laplacian eigenvalues in $[n-d+2-c, n]$ becomes larger. For the general c, it seems that some different technique is needed. Anyway, it is helpful to understand how the Laplacian eigenvalues are distributed and how this distribution is related to the diameter.

2. Preliminaries. Let G be a graph of order n with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of G, the neighborhood of v, denoted by $N_G(v)$, is the set of vertices that are adjacent to v in G, and the degree of v, denoted by $\delta_G(v)$, is the number of vertices that are adjacent to v in G, i.e., $\delta_G(v) = |N_G(v)|$. The degree sequence of G is the sequence $(\delta_1(G), \ldots, \delta_n(G))$ of the degrees of the vertices in nonincreasing order. For $S \subseteq V(G)$, denote by $G[S]$ the subgraph of G induced by S if $S \neq \emptyset$ and $G - S$ denotes $G[V(G) \setminus S]$, that is, the subgraph obtained from G by deleting the vertices of S if $S \neq V(G)$. In particular, if $S = \{v\}$, then we write $G - v$ for $G - \{v\}$. For $F \subseteq E(G)$, denote by $G - F$ the subgraph of G obtained from G by

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Volume 40, pp. 774-787, October 2024.

deleting all edges in F. In particular, if $F = \{e\}$, then we write $G - e$ for $G - \{e\}$. Now suppose that G is connected. The distance between vertices v, w, denoted by $d_G(v, w)$, is the length of a shortest path between v and w in G. The diameter of G is $\max\{d_G(v, w) : v, w \in V(G)\}$. A path of G that joins a pair of vertices whose distance is equal to the diameter is called a diametral path. For vertex disjoint graphs G and H, denote by $G \cup H$ the disjoint union of them. The disjoint union of k copies of G is denoted by kG. Denote by P_n the path of order n and K_n the complete graph of order n. For undefined notation and terminology, we refer to [3].

For an $n \times n$ Hermitian matrix M, $\rho_i(M)$ denotes its ith largest eigenvalue of M. We need Weyl's inequalities [17, 25] with a characterization of the equality cases [24, Theorem 1.3].

Lemma 2.1. [24, *Theorem* 1.3] *Let A and B be Hermitian matrices of order n. For $1 \le i, j \le n$ with $i + j - 1 \le n$,*

$$\rho_{i+j-1}(A + B) \le \rho_i(A) + \rho_j(B),$$

with equality if and only if there exists a nonzero vector $\mathbf{x}$ such that $\rho_{i+j-1}(A+B)\mathbf{x} = (A+B)\mathbf{x}$, $\rho_i(A)\mathbf{x} = A\mathbf{x}$ and $\rho_j(B)\mathbf{x} = B\mathbf{x}$.

We also need two types of interlacing theorem or inclusion principle.

Lemma 2.2. [14, *Theorem* 4.3.28] *If M is a Hermitian matrix of order n and B is its principal submatrix of order p, then $\rho_{n-p+i}(M) \le \rho_i(B) \le \rho_i(M)$ for $i = 1, \dots, p$.*

Lemma 2.3. [20, *Theorem* 3.2] *If G is a graph on n vertices with $e \in E(G)$, then*

$$\mu_1(G) \ge \mu_1(G - e) \ge \mu_2(G) \ge \cdots \ge \mu_{n-1}(G - e) \ge \mu_n(G) = \mu_n(G - e) = 0.$$

For integers n, d, and t with $2 \le d \le n - 2$ and $2 \le t \le d$, let $P_{d+1} := u_1 \dots u_{d+1}$, $V = V(K_{n-d-1})$ and let $G_{n,d,t}$ be the graph obtained from the disjoint union of P_{d+1} and K_{n-d-1} by adding all edges in $\{u_i w : i = t - 1, t, t + 1, w \in V\}$.

Lemma 2.4. [26, *Lemma* 2.6] *For integers n, d and t with $2 \le d \le n - 2$ and $2 \le t \le d$, $\mu_{n-d}(G_{n,d,t}) = n - d + 2$.*

For integers n, p, and q with $2 \le p \le q \le n - 3$, let $H_{n,p,q}$ be the graph obtained from $G_{n-1,n-3,p}$ (with a diametral path $u_1 \dots u_{n-2}$ and an additional vertex u outside) by adding a vertex v and three edges connecting v and u_{q-1}, u_q and u_{q+1} if $q \ge p + 2$ and four edges connecting v and u_{q-1}, u_q, u_{q+1} and u if $q = p, p + 1$, see Figs. 1 and 2.

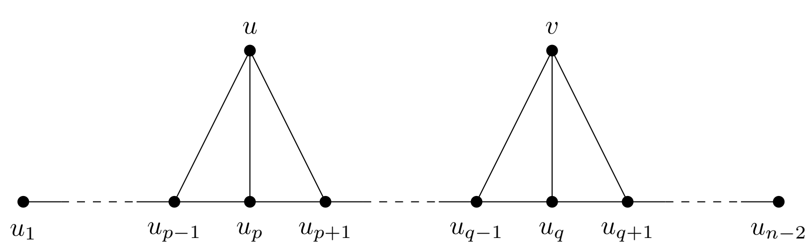

Figure 1. *The graph $H_{n,p,q}$ with $q \ge p + 2$.*

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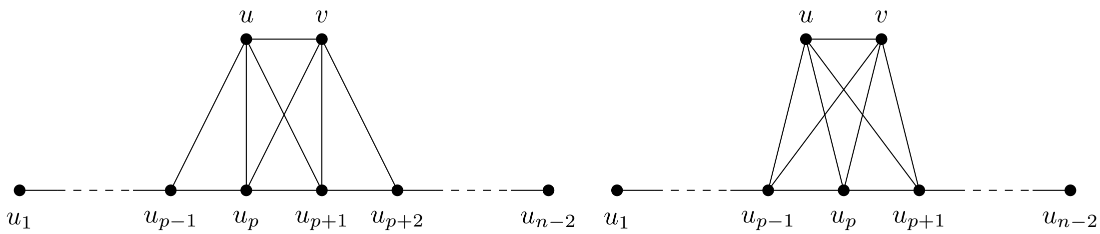

FIGURE 2. *The graph $H_{n,p,q}$ with $q = p + 1$ (left) and $q = p$ (right).*

For integers n, d, r, and a with $3 \le d \le n-2$, $2 \le r \le d-1$, and $1 \le a \le n-d-2$, let $P_{d+1} := u_1 \dots u_{d+1}$ and $V(K_{n-d-1}) = V_1 \cup V_2$ with $|V_1| = a$, and let $G_{n,d,r,a}$ be the graph obtained from the disjoint union of P_{d+1} and K_{n-d-1} by adding all edges in $\{u_i v : i = r-1, r, r+1, v \in V_1\} \cup \{u_j w : j = r, r+1, r+2, w \in V_2\}$.

LEMMA 2.5. *The following statements are true.*

(i) $\mu_1(P_n) < 4$.

(ii) $\mu_5(H_{n,p,q}) < 4$.

(iii) $\mu_5(G_{7,3,2,1}) < 4$.

Proof. Part (i) follows from the fact that $\mu_1(P_n) = 4\sin^2 \frac{(n-1)\pi}{2n}$ given in [2, p. 145].

Part (ii) follows from the proof of [26, Theorem 4].

Part (iii) follows from a direct calculation that $\mu_5(G_{7,3,2,1}) = 3.6601 < 4$. □

Given a graph G with $V(G) = \{v_1, \dots, v_n\}$, a vector $\mathbf{x} = (x_1, \dots, x_n)^\top$ can be viewed as a function defined on $V(G)$ mapping v_i to x_{v_i} i.e., $\mathbf{x}(v_i) = x_{v_i} = x_i$ for $i = 1, \dots, n$.

A pendant path $u_1 \dots u_p$ of G at u_p is an induced path of G with $\delta_G(u_1) = 1$, $\delta_G(u_p) \ge 3$, and $\delta_G(u_i) = 2$ for $i = 2, \dots, p-1$ if $p \ge 3$.

LEMMA 2.6. *Let $P := v_1 \dots v_\ell$ be a pendant path of a graph G at v_ℓ. If there is a vector $\mathbf{x}$ such that $L(G)\mathbf{x} = 4\mathbf{x}$, then for $i = 1, \dots, \ell$, $x_i = (-1)^{i-1}(2i-1)x_1$, where $x_i = x_{v_i}$.*

Proof. We prove the statement by induction on i. It is trivial for $i = 1$. From $L(G)\mathbf{x} = 4\mathbf{x}$ at v_1, we have $x_1 - x_2 = 4x_1$, i.e., $x_2 = -3x_1$, so the statement is true for $i = 2$. Suppose that $2 \le i \le \ell - 1$ and $x_j = (-1)^{j-1}(2j-1)x_1$ for each $j \le i$. From $L(G)\mathbf{x} = 4\mathbf{x}$ at v_i, we have

$$2x_i - x_{i-1} - x_{i+1} = 4x_i,$$

so

$$x_{i+1} = -2x_i - x_{i-1} = (-1)^i(2i+1)x_1,$$

proving the lemma. □

Denote by $\overline{G}$ the complement of G.

LEMMA 2.7. [20, *Theorem* 3.6] *Let G be a graph of order n. Then $\mu_i(G) + \mu_{n-i}(\overline{G}) = n$ for $i = 1, \dots, n-1$.*

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The following lemma comes from [9, Corollary 3.2] and subsequent remark there.

LEMMA 2.8. *Let G be a graph of order n with at least one edge. Then $\mu_1(G) \geq \delta_1(G)+1$ with equality when G is connected if and only if $\delta_1(G) = n-1$.*

LEMMA 2.9. [18, *Theorem* 4], [22, *Theorems* 2.5 *and* 2.6]. *Let G be a connected graph of order n with $n \geq 3$. Then $\mu_2(G) \geq \delta_2(G)$ with equality only if, under reordering the vertices so that $\delta_G(v_i) = \delta_i(G)$ for $i = 1, \ldots, n$, G satisfies one of the following conditions:*

(i) $v_1v_2 \notin E(G)$ *and* $N_G(v_1) = N_G(v_2)$,

(ii) *G is not a star,* $v_1v_2 \in E(G)$, $\delta_1(G) = \delta_2(G) = \frac{n}{2}$ *and* $N_G(v_1) \cap N_G(v_2) = \emptyset$.

From Lemmas 2.7 and 2.9, we have

COROLLARY 2.10. *Let G be a connected graph of order n such that $\overline{G}$ is connected. Then $\mu_{n-2}(G) \leq \delta_{n-1}(G)+1$ with equality only if, under reordering the vertices so that $\delta_G(v_i) = \delta_i(G)$ for $i = 1, \ldots, n$, G satisfies one of the following conditions:*

(i) $v_{n-1}v_n \in E(G)$ *and* $N_G(v_{n-1}) \setminus \{v_n\} = N_G(v_n) \setminus \{v_{n-1}\}$,

(ii) $v_{n-1}v_n \notin E(G)$, $\delta_{n-1}(G) = \delta_n(G) = \frac{n-2}{2}$ *and* $N_G(v_{n-1}) \cap N_G(v_n) = \emptyset$.

Let G be a graph of order n. Denote by $\kappa(G)$ the connectivity of G. By the well-known Whitney's inequality, $\kappa(G) \leq \delta_n(G)$. For two vertex disjoint graphs G_1 and G_2, their join is the graph $G_1 \cup G_2 + \{uv : u \in V(G_1), v \in V(G_2)\}$.

LEMMA 2.11. [8, *Result* 4.1], [16, *Theorem* 2.1] *Let G be a connected graph of order n that is not complete. Then $\mu_{n-1}(G) \leq \kappa(G)$ with equality if and only if G is a join of two graphs G_1 and G_2, where G_1 is a disconnected graph of order $n-\kappa(G)$ and G_2 is a graph of order $\kappa(G)$ with $\mu_{\kappa(G)-1}(G_2) \geq 2\kappa(G) - n$.*

Let G be a connected graph and P be a diametral path of G. For vertex z of G outside P, we denote by $\Gamma_{G,P}(z)$ the set of neighbors of z on P, that is, $\Gamma_{G,P}(z) = N_G(z) \cap V(P)$.

We say two matrices A and B are permutational similar if $A = QBQ^\top$ for some permutation matrix Q.

3. Proof of Theorem 1.1.

Theorem 1.1 follows from Theorems 3.1 and 3.2.

THEOREM 3.1. *Let G be a connected graph of order n with diameter d, where $d \leq n-4$. If $d = 2, 3, 4$, then $m_G[n-d, n] \leq n-d+1$.*

Proof. The result for $d = 2$ is trivial as $\mu_n(G) = 0$.

Suppose that $d = 3$. Let $P := v_1 \ldots v_4$ be a diametral path of G. Then $v_1v_3, v_1v_4 \in E(\overline{G})$, so $\delta_1(\overline{G}) \geq 2$. By Lemma 2.8, $\mu_1(\overline{G}) > \delta_1(\overline{G}) + 1 \geq 3$. So by Lemma 2.7, $\mu_{n-1}(G) = n - \mu_1(\overline{G}) < n-3$. Thus, $m_G[n-3, n] \leq n-2$.

Now suppose that $d = 4$. It suffices to show that $\mu_{n-2}(G) < n-4$, or $\mu_2(\overline{G}) > 4$ by Lemma 2.7.

Let $P := v_1v_2v_3v_4v_5$ be a diametral path of G. Then, $\overline{G}[\{v_1, v_2, v_3, v_4, v_5\}]$ is H_0 in Fig. 3.

As $n \geq 8$, there are at least three vertices outside P in G. Let u, v and w be three such vertices.

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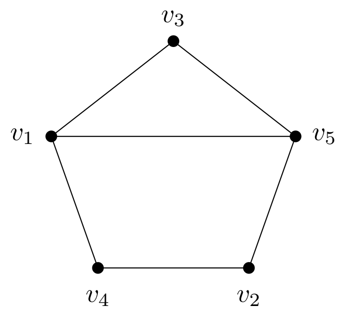

FIGURE 3. *The graph H_0.*

Suppose first that u, v, and w are all adjacent to v_1 in G. As P is a diametral path of G, none of u, v, and w is adjacent to v_4 or v_5 in G, so all of them are adjacent to both v_4 and v_5 in $\overline{G}$. Thus, $\delta_1(\overline{G}) \geq \delta_{\overline{G}}(v_5) \geq 6$ and $\delta_2(\overline{G}) \geq \delta_{\overline{G}}(v_4) \geq 5$. By Lemma 2.9, $\mu_2(\overline{G}) \geq \delta_2(\overline{G}) \geq 5 > 4$.

Suppose next that exactly two of u, v, and w, say u and v, are adjacent to v_1 in G. Then $uv_5, vv_5 \notin E(G)$, so $uv_5, vv_5, wv_1 \in E(\overline{G})$, implying that $\delta_{\overline{G}}(v_5) \geq 5$ and $\delta_{\overline{G}}(v_1) \geq 4$. Thus, $\delta_1(\overline{G}) \geq 5$ and $\delta_2(\overline{G}) \geq 4$. By Lemma 2.9, $\mu_2(\overline{G}) \geq \delta_2(\overline{G}) \geq 4$. Suppose that $\mu_2(\overline{G}) = 4$. Then $\delta_{\overline{G}}(v_5) = \delta_1(\overline{G}) \geq 5$ and $\delta_{\overline{G}}(v_1) = \delta_2(\overline{G}) = 4$. Note that v_1 and v_5 are adjacent in $\overline{G}$ with a common neighbor v_3. By Lemma 2.9, this is impossible. It thus follows that $\mu_2(\overline{G}) > 4$.

Now, suppose that exactly one of u, v and w, say u, is adjacent to v_1 in G. Then $uv_5 \notin E(G)$, so $uv_5, vv_1, wv_1 \in E(\overline{G})$. Thus, $\delta_1(\overline{G}) \geq \delta_{\overline{G}}(v_1) \geq 5$ and $\delta_2(\overline{G}) \geq \delta_{\overline{G}}(v_5) \geq 4$. As v_1 and v_5 are adjacent in $\overline{G}$ and with a common neighbor v_3, one gets $\mu_2(\overline{G}) > \delta_2(\overline{G}) \geq 4$ by Lemma 2.9.

Finally, suppose that none of u, v, and w is adjacent to v_1 in G. If two of them, say u and v, are adjacent to v_2 in G, then $uv_5, vv_5 \in E(\overline{G})$, implying that $\delta_2(\overline{G}) \geq 5$, so we have by Lemma 2.9 that $\mu_2(\overline{G}) > 4$. If at most one of u, v, and w is adjacent to v_2 in G, then we may assume that $uv_2, vv_2 \notin E(G)$, i.e., $uv_2, vv_2 \in E(\overline{G})$, so $\delta_2(\overline{G}) \geq 4$ and we have $\mu_2(\overline{G}) > 4$ by Lemma 2.9. ◻

It is evident that $m_{K_n - e}[n, n] = n - 2$. Note that $G_{n,3,2}$ ($G_{n,4,3}$, respectively) is an n-vertex graph with diameter 3 (4, respectively). As $K_{n-1} - e$ is a subgraph of $G_{n,3,2}$, we have $\mu_{n-2}(G_{n,3,2}) \geq \mu_{n-2}(K_{n-1} - e) = n - 3$, so by Theorem 3.1, $m_{G_{n,3,2}}[n-3, n] = n - 2$. From [27, Proposition 1], $\mu_{n-3}(G_{n,4,3}) > n - 3$, so by Theorem 3.1 again, $m_{G_{n,4,3}}[n-4, n] = n - 3$. Thus, the bound in Theorem 3.1 is tight.

THEOREM 3.2. *Let G be a connected graph of order n with diameter d, where $5 \leq d \leq n - 4$. Then $m_G[n-d, n] \leq n - d + 2$.*

Proof. Let $P := v_1 \ldots v_{d+1}$ be a diametral path of G. As $d \leq n - 4$, there are at least three vertices lying outside P. Assume that u, v, and w are three such vertices. Let G' be the subgraph of G induced by $V(P) \cup \{u, v, w\}$ and B the principal submatrix of $L(G)$ corresponding to vertices of G'. Denote by M the diagonal matrix whose diagonal entry corresponding to vertex z is $\delta_G(z) - \delta_{G'}(z)$ for $z \in V(G')$. Then $B = L(G') + M$. By Lemma 2.2,

$$\mu_{n-d+3}(G) = \rho_{n-(d+4)+7}(L(G)) \leq \rho_7(B).$$

By Lemma 2.1,

$$\rho_7(B) \leq \mu_7(G') + \rho_1(M).$$

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Thus, $\mu_{n-d+3}(G) \le \mu_7(G') + \rho_1(M)$. Obviously, $\rho_1(M) \le n - |V(G')| = n-d-4$. If $\mu_7(G') < 4$, then $\mu_{n-d+3}(G) < n-d$, so $m_G[n-d,n] \le n-d+2$. Thus, it suffices to show that $\mu_7(G') < 4$.

As P is a diametral path of G, any vertex outside P has at most three consecutive neighbors on P. For $z = u, v, w$, let $n_z = |\Gamma_{G,P}(z)|$. If $n_z < 3$ for some $z = u, v, w$, then there exist $3 - n_z$ vertices on P so that P remains to be a diametral path of the graph obtained by adding edges between z and the $3 - n_z$ vertices. By Lemma 2.3, we can assume that $\Gamma_{G,P}(u) = \{v_{p-1}, v_p, v_{p+1}\}$, $\Gamma_{G,P}(v) = \{v_{q-1}, v_q, v_{q+1}\}$ and $\Gamma_{G,P}(w) = \{v_{r-1}, v_r, v_{r+1}\}$, where $2 \le p, q, r \le d$. Assume that $p \le q \le r$. If $q - p > r - q$, then we relabel the vertices of G by setting $v'_i = v_{d+2-i}$ for $i = 1, \dots, d+1$, $u' = u$, $v' = v$, and $w' = w$, so we have $p' \le q' \le r'$ and $q' - p' \le r' - q'$, where $p' = d+2-r$, $q' = d+2-q$, and $r' = d+2-p$. So, we assume furthermore that $q - p \le r - q$.

Case 1. $r \ge q + 2$.

Note that $uw, vw \notin E(G)$. It is easy to see that $G' - \{v_{r-1}v_r, wv_{r+1}\} \cong H_{d+4,p,q}$ or $H_{d+4,p,q} - uv$, where u and v are the two vertices outside the diametral path P. By Lemmas 2.3 and 2.5, one gets

$$\mu_7(G') \le \mu_5(H_{d+4,p,q}) < 4,$$

as desired.

Case 2. $r = q + 1$.

By assumption, we have $p \le q \le p + 1$.

Case 2.1. $q = p + 1$.

It is possible that v is adjacent to u or w. Assume that $uv, vw \in E(G)$ by Lemma 2.3. Let $u_i = v_i$ for $i = 1, \dots, p-1$, $u_p = u$, $u_{i+1} = v_i$ for $i = p, p+1$, $u_{p+3} = w$, $u_{i+2} = v_i$ for $i = p+2, \dots, d+1$ and $u_{d+4} = v$. Under this new labeling,

$$G' - \{u_{p-1}u_{p+1}, u_pu_{p+2}, u_{p+2}u_{p+4}, u_{p+3}u_{p+5}, u_pu_{d+4}, u_{p+4}u_{d+4}\},$$

is a copy of $G_{d+4,d+2,p+2}$. So

$$L(G') = L(G_{d+4,d+2,p+2}) + R,$$

where $R = (r_{ij})_{(d+4)\times(d+4)}$ with

$$r_{ij} = \begin{cases} 1 & \text{if } i = j \in \{p-1, p+1, p+3, p+5\}, \\ 2 & \text{if } i = j \in \{p, p+2, p+4, d+4\}, \\ -1 & \text{if } \{i,j\} \in \{\{p-1,p+1\}, \{p,p+2\}, \{p,d+4\}\}, \\ -1 & \text{if } \{i,j\} \in \{\{p+2,p+4\}, \{p+3,p+5\}, \{p+4,d+4\}\}, \\ 0 & \text{otherwise.} \end{cases}$$

As R is permutational similar to $L(2P_2 \cup C_4 \cup (d-4)K_1)$, we have $\rho_6(R) = 0$. So by Lemmas 2.1 and 2.4, we have

$$\mu_7(G') \le \mu_2(G_{d+4,d+2,p+2}) + \rho_6(R) = 4.$$

Suppose that $\mu_7(G') = 4$. By Lemma 2.1, there exists a nonzero vector $\mathbf{x}$ such that $R\mathbf{x} = \mathbf{0}$ and $L(G_{d+4,d+2,p+2})\mathbf{x} = 4\mathbf{x}$. Let $x_i = x_{u_i}$ for $i = 1, \dots, d+4$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 774-787, October 2024.

From $R\mathbf{x} = \mathbf{0}$, we have $L(C_4)(x_p, x_{p+2}, x_{p+4}, x_{d+4})^\top = \mathbf{0}$, so $x_p = x_{p+2} = x_{p+4} = x_{d+4}$.

From $R\mathbf{x} = \mathbf{0}$ at u_{p-1} and u_{p+3}, respectively, we have $x_{p-1} = x_{p+1}$ and $x_{p+3} = x_{p+5}$.

From $L(G_{d+4,d+2,p+2})\mathbf{x} = 4\mathbf{x}$ at u_p, we have

$$2x_p - x_{p-1} - x_{p+1} = 4x_p,$$

so $x_{p-1} = -x_p$.

As $u_1 \dots u_{p+1}$ is a pendant path of $G_{d+4,d+2,p+2}$ at u_{p+1}, we have by Lemma 2.6 that

$$x_i = (-1)^{i-1}(2i-1)x_1 \text{ for } i = 1, \dots, p+1.$$

From $L(G_{d+4,d+2,p+2})\mathbf{x} = 4\mathbf{x}$ at u_{p+1}, we have $3x_{p+1} - x_p - x_{p+2} - x_{d+4} = 4x_{p+1}$, so $x_{p+1} = -3x_p$. It hence follows that

$$(-1)^p(2p+1)x_1 = x_{p+1} = -3x_p = -3(-1)^{p-1}(2p-1)x_1,$$

i.e.,

$$(2p+1)x_1 = 3(2p-1)x_1,$$

equivalently, $x_1 = 0$. So $x_i = 0$ for $i = 1, \dots, p+2, p+4, d+4$. From $L(G_{d+4,d+2,p+2})\mathbf{x} = 4\mathbf{x}$ at u_{d+4}, we have

$$3x_{d+4} - x_{p+1} - x_{p+2} - x_{p+3} = 4x_{d+4}.$$

As $x_{d+4} = x_{p+1} = x_{p+2} = x_{p+4} = 0$, one gets $x_{p+3} = 0$. So $x_{p+5} = x_{p+3} = 0$. It follows that $x_i = 0$ for $i = 1, \dots, p+5$. Now from $L(G_{d+4,d+2,p+2})\mathbf{x} = 4\mathbf{x}$ at u_i for $i = p+5, \dots, d+2$, we have $x_{i+1} = 0$. Thus, $\mathbf{x}$ is a zero vector, a contradiction. Therefore, $\mu_7(G') < 4$.

Case 2.2. $q = p$.

By Lemma 2.3, we assume that $uv, vw, uw \in E(G')$.

If $3 \le p \le d-2$, then $G' - \{v_{p-2}v_{p-1}, v_{p+2}v_{p+3}\} \cong G_{7,3,2,1} \cup P_{p-2} \cup P_{d-p-1}$, so we have by Lemmas 2.3 and 2.5 that

$$\mu_7(G') \le \mu_5(G' - \{v_{p-2}v_{p-1}, v_{p+2}v_{p+3}\}) \le \max\{\mu_5(G_{7,3,2,1}), \mu_1(P_{p-2}), \mu_1(P_{d-p-1})\} < 4.$$

If $p = 2$, then $G' - v_{p+2}v_{p+3} \cong G_{7,3,2,1} \cup P_{d-p-1}$, so we have by Lemmas 2.3 and 2.5 that

$$\mu_7(G') \le \mu_6(G' - v_{p+2}v_{p+3}) \le \max\{\mu_5(G_{7,3,2,1}), \mu_1(P_{d-p-1})\} < 4.$$

If $p = d-1$, then $G' - v_{p-2}v_{p-1} \cong G_{7,3,2,1} \cup P_{p-2}$ and so by Lemmas 2.3 and 2.5,

$$\mu_7(G') \le \mu_6(G' - v_{p-2}v_{p-1}) \le \max\{\mu_5(G_{7,3,2,1}), \mu_1(P_{d-p-1})\} < 4.$$

Case 3. $r = q$.

In this case, $p = q = r$. By Lemma 2.3, we assume that $uv, vw, uw \in E(G')$. Let $u_i = v_i$ for $i = 1, \dots, p-1$, $u_p = u$, $u_{p+1} = v_p$, $u_{p+2} = v$, $u_{i+2} = v_i$ for $i = p+1, \dots, d+1$ and $u_{d+4} = w$. Under this new labeling,

$$G' - \{u_{p-1}u_{p+1}, u_{p-1}u_{p+2}, u_pu_{p+2}, u_pu_{p+3}, u_{p+1}u_{p+3}, u_{p+2}u_{d+4}, u_{p+3}u_{d+4}\},$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 774-787, October 2024.

is a copy of $G_{d+4,d+2,p}$. So

$$L(G') = L(G_{d+4,d+2,p}) + R,$$

where $R = (r_{ij})_{(d+4)\times(d+4)}$ with

$$r_{ij} = \begin{cases} 2 & \text{if } i = j \in \{p-1, p, p+1, d+4\}, \\ 3 & \text{if } i = j \in \{p+2, p+3\}, \\ -1 & \text{if } \{i, j\} \in \{\{p-1, p+1\}, \{p-1, p+2\}, \{p, p+2\}, \{p, p+3\}\}, \\ -1 & \text{if } \{i, j\} \in \{\{p+1, p+3\}, \{p+2, d+4\}, \{p+3, d+4\}\}, \\ 0 & \text{otherwise.} \end{cases}$$

As R is permutational similar to $L(H \cup (d-2)K_1)$ where H is a graph on 6 vertices consisting of a cycle $u_{p-1}u_{p+1}u_{p+3}u_pu_{p+2}u_{p-1}$ and additional two edges $u_{p+2}u_{d+4}$ and $u_{p+3}u_{d+4}$, we have $\rho_6(R) = 0$. So by Lemmas 2.1 and 2.4, we have

$$\mu_7(G') \le \mu_2(G_{d+4,d+2,p}) + \rho_6(R) = 4.$$

Suppose that $\mu_7(G') = 4$. By Lemma 2.1, there exists a nonzero vector $\mathbf{x}$ such that $R\mathbf{x} = \mathbf{0}$ and $L(G_{d+4,d+2,p})\mathbf{x} = 4\mathbf{x}$. As earlier, let $x_i = x_{u_i}$ for $i = 1, \dots, d+4$. From $R\mathbf{x} = \mathbf{0}$, we have

$$L(H)(x_{p-1}, x_{p+1}, x_{p+2}, x_p, x_{p+3}, x_{d+4})^\top = \mathbf{0},$$

so $x_{p-1} = \cdots = x_{p+3} = x_{d+4}$.

Suppose first that $p \ge 3$. From $L(G_{d+4,d+2,p})\mathbf{x} = 4\mathbf{x}$ at u_{p-1}, we have

$$3x_{p-1} - x_{p-2} - x_p - x_{d+4} = 4x_{p-1},$$

so $x_{p-2} = -3x_{p-1}$. As $u_1 \dots u_{p-1}$ is a pendant path of G' at u_{p-1}, we have by Lemma 2.6 that

$$x_i = (-1)^{i-1}(2i-1)x_1 \text{ for } i = 1, \dots, p-1.$$

Then

$$x_{p-2} = (-1)^{p-3}(2(p-2)-1)x_1 = -3 \cdot (-1)^{p-2}(2(p-1)-1)x_1,$$

i.e.,

$$(2p-5)x_1 = 3(2p-3)x_1,$$

equivalently, $x_1 = 0$, so $x_i = 0$ for $i = 1, \dots, p+3, d+4$. If $p = 2$, this follows from $L(G_{d+4,d+2,p})\mathbf{x} = 4\mathbf{x}$ at u_p.

Now from $L(G_{d+4,d+2,p})\mathbf{x} = 4\mathbf{x}$ at u_i with $i = p+3, \dots, d+2$, we have $x_{i+1} = 0$. Thus $\mathbf{x} = \mathbf{0}$, a contradiction. Therefore, $\mu_7(G') < 4$. ∎

The bound in Theorem 1.1 can be improved under certain conditions.

Proposition 3.3. *Let G be an n-vertex connected graph with diametral path $P := v_1 \dots v_{d+1}$, where $5 \le d \le n-4$. If there exist three vertices u, v, w outside P such that $|\Gamma_{G,P}(u)| + |\Gamma_{G,P}(v)| + |\Gamma_{G,P}(w)| \le 5$, then $m_G[n-d, n] \le n-d+1$.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 774-787, October 2024.

Proof. Let $H = G[V(P) \cup \{u, v, w\}]$ and E' be the set of edges between vertices on P and u, v, w. As $|\Gamma_{G,P}(u)| + |\Gamma_{G,P}(v)| + |\Gamma_{G,P}(w)| \leq 5$, $|E'| \leq 5$. Note that $H - E' \cong P_{d+1} \cup G[\{u, v, w\}]$. We have by Lemmas 2.3 and 2.5 that

$$\mu_6(H) \leq \mu_1(H - E') = \max\{\mu_1(P_{d+1}), \mu_1(G[\{u, v, w\}])\} < 4.$$

Let B be the principal submatrix of $L(G)$ corresponding to vertices of H and M be the diagonal matrix whose diagonal entry corresponding to vertex z is $\delta_G(z) - \delta_H(z)$ for $z \in V(H)$. Then, by Lemmas 2.2 and 2.1,

$$\mu_{n-d+2}(G) = \rho_{n-(d+4)+6}(L(G)) \leq \rho_6(B) \leq \mu_6(H) + \rho_1(M) < n - d,$$

as desired. ∎

4. Proof of Theorem 1.2. Theorem 1.2 follows from Theorems 4.1 and 4.2.

THEOREM 4.1. *Let G be a connected graph of order n with diameter d. If $2 \leq d \leq \lfloor \frac{n+3}{2} \rfloor$, then $m_G[n - 2d + 4, n] \leq n - 2$.*

Proof. It suffices to show that $\mu_{n-1}(G) < n - 2d + 4$. If $d = 2$, then G is a spanning subgraph of $K_n - e$ for some $e \in E(K_n)$, so we have by Lemma 2.3 that $\mu_{n-1}(G) \leq n - 2 < n = n - 2d + 4$, as desired. Suppose that $d \geq 3$ and $P := v_1 \ldots v_{d+1}$ is a diametral path of G. For $i = 2, \ldots, d - 1$, let V_i be the set of vertices of G such that the distance to v_1 is $i - 1$. Let V_d be the set of vertices of G such that the distance to v_1 is $d - 1$ and the neighbors of v_{d+1}. Evidently, $v_i \in V_i$ and V_i is a cut set of G for each $i = 2, \ldots, d$. As P is a diametral path, $V_i \cap V_j = \emptyset$ if $i \neq j$ and there is no edge between V_i and V_j if $|j - i| \geq 2$. If $\kappa(G) \geq n - 2d + 5$, then

$$2 + (n - 2d + 5)(d - 2) \leq |\{v_1, v_{d+1}\}| + \sum_{i=2}^{d} |V_i| \leq n,$$

i.e., $2d^2 - (n + 9)d + 3n + 8 \geq 0$, so $d < 3$ or $d > \frac{n+3}{2}$, a contradiction. So $\kappa(G) \leq n - 2d + 4$. As $d \geq 3$, G is not a join, so we have by Lemma 2.11 that $\mu_{n-1}(G) < \kappa(G) \leq n - 2d + 4$. ∎

We give examples for which the bound in Theorem 4.1 is attained. If $G = K_n - e$, then $d = 2$ and $m_{K_n - e}[n, n] = n - 2$. Let R_1 (R_2, respectively) be the graph on 8 vertices (7 vertices, respectively) with diameter 5 in Fig. 4. By a direct calculation, we have $\mu_6(R_1) = 2$ and $\mu_5(R_2) = 1$. By Theorem 4.1, $m_{R_1}[2, 8] = 6$ and $m_{R_2}[1, 7] = 5$, agreeing the bound in Theorem 4.1 for $d = \frac{n+2}{2} = 5$ and $d = \frac{n+3}{2} = 5$, respectively.

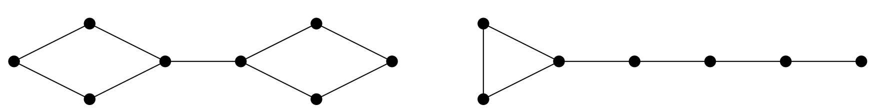

FIGURE 4. *The graph R_1 (left) and R_2 (right).*

If $3 \leq d \leq \lfloor \frac{n+1}{2} \rfloor$, Theorem 4.1 may be improved as follows.

THEOREM 4.2. *Let G be a connected graph of order n with diameter d. If $3 \leq d \leq \lfloor \frac{n+1}{2} \rfloor$, then $m_G[n - 2d + 4, n] \leq n - 3$.*

Proof. The case for $d = 3$ is known from [27, Theorem 6], and the case for $d = 4$ follows from Theorem 1.1. Suppose in the following that $d \geq 5$. It suffices to show that $\mu_{n-2}(G) < n - 2d + 4$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 774-787, October 2024.

Let $P := v_1 \dots v_{d+1}$ be a diametral path of G. For $i = 1, \dots, d-1$, let V_i be the set of vertices of G such that the distance to v_1 is $i-1$. Let V_d be the set of vertices of G except v_{d+1} such that the distance to v_1 is $d-1$ or d. Let $V_{d+1} = \{v_{d+1}\}$. By Lemma 2.3, we assume that $G[V_i \cup V_{i+1}]$ is complete for each $i = 2, \dots, d$. Note that V_i is a cut set of G for each $i = 2, \dots, d$ and that v is a cut vertex of G if and only if $v = v_i$ and $|V_i| = 1$ for some $i = 2, \dots, d$. Let s be the number of sets $V_2, \dots, V_d$ with cardinality 1.

We divide the proof into two cases.

Case 1. $\delta_{n-1}(G) \geq n - 2d + 4$.

Note that $\max\{|V_2|, |V_d|\} = \max\{\delta_G(v_1), \delta_G(v_{d+1})\} \geq \delta_{n-1}(G) \geq n - 2d + 4$. Assume that $|V_2| \geq n - 2d + 4 > 2$ and $|V_j| = \min\{|V_i| : i = 3, \dots, d\}$. Then $|V_i| \geq 2$ for $i = 3, \dots, d$ if $s = 0$, and $|V_i| \geq 2$ for $i = 3, \dots, d$ with $i \neq j$ if $s = 1$. Thus, if $s = 0, 1$, then

$$n + 1 = 2 + 1 + (n - 2d + 4) + 2(d - 3) \leq |\{v_1, v_{d+1}\}| + |V_j| + |V_2| + \sum_{\substack{i=3 \\ i \neq j}}^{d} |V_i| \leq n,$$

a contradiction. So $s \geq 2$. Assume that $|V_\ell| = 1$. Then v_ℓ is one cut vertex of G. Suppose that there is a component G_0 of $G - v_\ell$ such that G_0 has a cut vertex. Then $\kappa(G_0) = 1$ and by Lemma 2.11, $\mu_{|V(G_0)|-1}(G_0) \leq \kappa(G_0) = 1$. Let B be the principal submatrix of $L(G)$ by deleting the row and column corresponding to vertex v_ℓ. By Lemma 2.1, $\rho_{n-3}(B) \leq \mu_{n-3}(G - v_\ell) + \rho_1(B - L(G - v_\ell)) = \mu_{n-3}(G - v_\ell) + 1$. Then, by Lemma 2.2, we have

$$\mu_{n-2}(G) \leq \rho_{n-3}(B) \leq \mu_{n-3}(G - v_\ell) + 1 \leq \mu_{|V(G_0)|-1}(G_0) + 1 \leq 2 < n - 2d + 4,$$

as desired. Suppose that there is no cut vertices of any component of $G - v_\ell$. Then $s = 2$ and either $|V_{\ell-1}| = 1$ or $|V_{\ell+1}| = 1$, say $|V_{\ell+1}| = 1$. As

$$n = 4 + n - 2d + 4 + 2(d - 4) \leq |\{v_1, v_{d+1}, v_\ell, v_{\ell+1}\}| + |V_2| + \sum_{\substack{i=3 \\ i \neq \ell, \ell+1}}^{d} |V_i| \leq n,$$

we have $|V_2| = n - 2d + 4$ and $|V_i| = 2$ for $i = 3, \dots, d$ with $i \neq \ell, \ell + 1$, where $3 \leq \ell \leq d - 1$. If $\ell = d - 1$, then $\delta_G(v_{d+1}) = 1$ and $\delta_G(v_{n-1}) = 2$, so $n - 2d + 4 \leq \delta_{n-1}(G) \leq 2$, which is a contradiction. So $\ell \leq d - 2$ and $|V_d| = 2$. Let B' be the principal submatrix of $L(G)$ by deleting the rows and columns corresponding to vertices in V_d. Let $G_1 = G - V_d - v_{d+1}$. By Lemma 2.1, $\rho_{n-4}(B') \leq \mu_{n-4}(G - V_d) + \rho_1(B' - L(G - V_d)) = \mu_{n-4}(G_1) + 2$. Note that G_1 is not a join with a cut vertex v_ℓ. By Lemma 2.11, $\mu_{n-4}(G_1) < \kappa(G_1) = 1$. Therefore, by Lemma 2.2,

$$\mu_{n-2}(G) \leq \rho_{n-4}(B') \leq \mu_{n-4}(G_1) + 2 < \kappa(G_1) + 2 = 3 \leq n - 2d + 4,$$

as desired.

Case 2. $\delta_{n-1}(G) \leq n - 2d + 3$.

By Corollary 2.10, $\mu_{n-2}(G) \leq \delta_{n-1}(G) + 1 \leq n - 2d + 4$. Suppose by contradiction that $\mu_{n-2}(G) = n - 2d + 4$. Then $\mu_{n-2}(G) = \delta_{n-1}(G) + 1$ and $\delta_{n-1}(G) = n - 2d + 3$. Let u_1 and u_2 be two vertices of degree $\delta_n(G)$ and $\delta_{n-1}(G)$ in G, respectively. By the diameter condition, $\overline{G}$ is connected. By Corollary 2.10 and the fact that $\mu_{n-2}(G) = \delta_{n-1}(G) + 1$, we have the following two cases.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 774-787, October 2024.

Case 2.1. $u_1u_2 \notin E(G)$, $\delta_{n-1}(G) = \delta_n(G) = \frac{n-2}{2}$ and $N_G(u_1) \cap N_G(u_2) = \emptyset$.

Note that $V(G) = \{u_1, u_2\} \cup N_G(u_1) \cup N_G(u_2)$. Let $U_i = N_G(u_i)$ for $i = 1, 2$. As G is connected, there is a vertex $w_i \in U_i$ with $i = 1, 2$ such that $w_1w_2 \in E(G)$. The distance between any vertex pair of vertices in $\{u_1, u_2\} \cup U_i$ with $i = 1, 2$ is at most three. Let $z_1 \in U_1 \setminus \{w_1\}$. If $z_1w_1 \in E(G)$, then the distance between z_1 and any vertex in U_2 is at most three. If $z_1w_1 \notin E(G)$, then as $\delta_G(z_1) \geq \delta_n(G) = \frac{n-2}{2} = |U_1|$, we have $z_1z_2 \in E(G)$ for some $z_2 \in U_2$, so the distance between z_1 and any vertex in U_2 is at most three. This shows that $d \leq 3$, a contradiction.

Case 2.2. $u_1u_2 \in E(G)$ and $N_G(u_1) \setminus \{u_2\} = N_G(u_2) \setminus \{u_1\}$.

Note that $\delta_n(G) = \delta_G(u_1) = \delta_G(u_2) = \delta_{n-1}(G) = n - 2d + 3$. Then $|V_2|, |V_d| \geq \delta_n(G) = n - 2d + 3 \geq 2$. Let $|V_j| = \min\{|V_i| : i = 3, \dots, d-1\}$. If $|V_j| \geq 2$, then

$$2 + (n - 2d + 3) \cdot 2 + 2(d - 3) \leq |\{v_1, v_{d+1}\}| + |V_2| + |V_d| + \sum_{i=3}^{d-1} |V_i| \leq n,$$

i.e., $n \leq 2d - 2$, which is a contradiction. So $|V_\ell| = 1$ for some ℓ with $3 \leq \ell \leq d - 1$. Denote by B the principal submatrix of $L(G)$ by deleting the row and column corresponding to vertex v_ℓ.

Suppose that there is a component G_0 of $G - v_\ell$ such that $\kappa(G_0) = 1$. It then follows from Lemmas 2.2, 2.1 and 2.11 that

$$\mu_{n-2}(G) \leq \rho_{n-3}(B) \leq \mu_{n-3}(G - v_\ell) + 1 \leq \mu_{|V(G_0)|-1}(G_0) + 1 \leq \kappa(G_0) + 1 = 2 < n - 2d + 4,$$

a contradiction. So there is no cut vertices of any component of $G - v_\ell$, $s = 1, 2$, and if $s = 2$, then one of $v_{\ell-1}$ and $v_{\ell+1}$, say $v_{\ell+1}$, is a cut vertex of G. Thus, $G - v_\ell$ consists of two components, say H and F, with $v_1, \dots, v_{\ell-1} \in V(H)$ and and $v_{\ell+1}, \dots, v_{d+1} \in V(F)$.

If H and F are both complete, then $d \leq 4$, which is a contradiction to the assumption that $d \geq 5$. Assume that H is not complete. Let $p = |V(H)|$.

If F is not complete, then as one of u_1 and u_2 lies in $G - v_\ell = H \cup F$ and $\delta_G(u_1) = \delta_G(u_2) = n - 2d + 3$, we have $\min\{\delta_p(H), \delta_{n-p}(F)\} \leq n - 2d + 3$, so we assume that $\delta_p(H) \leq n - 2d + 3$ (if $\delta_{n-p}(F) \leq n - 2d + 3$, then we exchange the roles of H and F). If F is complete, then $\delta_p(H) \leq n - 2d + 3$, as otherwise, we have $|V_d| \geq 2$, $s = 1$, $\ell = d - 1$, and so

$$3 + (n - 2d + 4) + 2(d - 4) + (n - 2d + 3) \leq |\{v_1, v_{d+1}, v_{d-1}\}| + \sum_{i=2}^{d} |V_i| \leq n,$$

i.e., $n \leq 2d - 2$, which is a contradiction. It then follows that $\kappa(H) \leq \delta_p(H) \leq n - 2d + 3$. By Lemma 2.11, $\mu_{p-1}(H) \leq \kappa(H) \leq n - 2d + 3$. Now, by Lemmas 2.2 and 2.1, we have

$$n - 2d + 4 = \mu_{n-2}(G) \leq \rho_{n-3}(B) \leq \mu_{n-3}(G - v) + 1 \leq \mu_{p-1}(H) + 1 \leq n - 2d + 4,$$

so $\mu_{p-1}(H) = n - 2d + 3 = \kappa(H) = \delta_p(H)$. By Lemma 2.11, H is a join, say $H = H_1 \vee H_2$, and one of H_1 and H_2, say H_1, is disconnected and the other H_2 has order $n - 2d + 3$, so $\{v_1\} \cup V_3 \subseteq V(H_1)$ and $\ell = 4$.

Suppose that $s = 1$. Then we have

$$3 + (n - 2d + 3) + 2(d - 4) + (n - 2d + 3) \leq |\{v_1, v_{d+1}, v_4\}| + \sum_{\substack{i=2 \\ i \neq 4}}^{d} |V_i| \leq n,$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 774-787, October 2024.

i.e., $n \le 2d-1$, so $n = 2d-1$ and $|V_i| = 2$ for $i = 2, \dots, d$ with $i \ne 4$. This is impossible because there are no vertices u_1 and u_2 such that $u_1u_2 \in E(G)$ and $\delta_G(u_1) = \delta_G(u_2) = 2$.

Suppose that $s = 2$. Then $4 = \ell \le d-2$. Then $H' = G[V(H) \cup \{v_4\}]$ is a component of $G - v_5$ and it is not a join. Note that $\delta_{p+1}(H') \le \delta_p(H) \le n-2d+3$. So $\kappa(H') \le n-2d+3$. By Lemmas 2.2, 2.1, and 2.11, we have

$$n-2d+4 = \mu_{n-2}(G) \le \rho_{n-3}(B') \le \mu_{n-3}(G-v_5)+1 \le \mu_p(H')+1 < \kappa(H')+1 \le n-2d+4,$$

a contradiction. ∎

Acknowledgments. The authors thank Professors Domingos M. Cardoso, Vilmar Trevisan, Saieed Akbari, Elismar R. Oliveira, and Stephen T. Hedetniemi for helpful comments and suggestions on an early version of this paper. The authors would like to thank the reviewers for constructive comments and suggestions. This work was supported by the National Natural Science Foundation of China (No. 12071158).

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