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PSEUDOMONOTONICITY AND RELATED PROPERTIES IN EUCLIDEAN JORDAN ALGEBRAS*

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Abstract. In this paper, we extend the concept of pseudomonotonicity from R^n to the setting of Euclidean Jordan algebras. We study interconnections between pseudomonotonicity, monotonicity, and the Z-property.

Key words. Euclidean Jordan algebra, Pseudomonotone, Positive subdefinite (PSBD), Copositive, Copositive star, Automorphism, Z-property.

AMS subject classifications. 90C33, 17C20, 17C55.

1. Introduction. Given a convex set K in R^n, a map $f : K \to R^n$ is said to be pseudomonotone on K if

$$x,\, y \in K, \quad \langle f(x), y - x\rangle \geq 0 \Rightarrow \langle f(y), y - x\rangle \geq 0.$$

This concept is a generalization of monotonicity defined by $\langle f(x)-f(y), x-y\rangle \geq 0$ for any $x,\, y \in R^n$. There is an extensive literature associated with this property covering theory and applications, see e.g., [11], [13], [14].

Gowda (see [4], [5], [6]), Crouzeix et al. (see [1]), and Hassouni et al. (see [12]) studied pseudomonotone matrices on R^n_+ and investigated some properties of the linear complementarity under the condition of pseudomonotonicity.

Motivated by their results, in this paper, we extend the concept of pseudomonotonicity from R^n to the setting of Euclidean Jordan algebras. Specifically, we give a characterization of pseudomonotonicity for a linear transformation and a matrix-induced transformation defined on a Euclidean Jordan algebra. We show that pseudomonotonicity and monotonicity coincide under the condition of the Z-property. Moreover, we present the invariance of pseudomonotonicity under the algebra and cone automorphisms and describe interconnections between pseudomonotonicity of a linear transformation and its principal subtransformations. We note that symmetric cones (see Section 2 for definition) are, in general, nonpolyhedral. Therefore, these generalizations presented in this paper are not routine generalizations.

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Here is an outline of the paper. In Section 2, we cover the basic material dealing with Euclidean Jordan algebras and pseudomonotonicity. In Section 3, we present some general results for pseudomonotone transformations. In Section 4, we investigate the relation between pseudomonotonicity of a linear transformation and its principal subtransformations. In Section 5, we study pseudomonotonicity for some special linear transformations. Specifically, we show that pseudomonotone linear transformations are invariant under the algebra and cone automorphisms, we specialize pseudomonotonicity of a linear transformation with having the Z-property, and we give a characterization of pseudomonotonicity for a matrix-induced transformation.

2. Preliminaries.

2.1. Euclidean Jordan algebras. In this subsection, we briefly recall some concepts, properties, and results from Euclidean Jordan algebras. Most of these can be found in [3], [9].

Let V be a Euclidean Jordan algebra and K be a symmetric cone in V. A Jordan product is denoted by $x \circ y$ for any two elements x and y in V. In addition, an element $e \in V$ is called the *unit* element if $x \circ e = x$ for all $x \in V$. We define

$$z^+ := \Pi_K(z) \qquad \text{and} \qquad z^- := z^+ - z,$$

where $\Pi_K(z)$ denotes the (orthogonal) projection of z onto K.

For an element $z \in V$, we write

$$z \geq 0 \quad (z > 0) \quad \text{if and only if} \quad z \in K \quad (z \in K^\circ (= \text{interior}(K))),$$

and $z \leq 0$ $(z < 0)$ when $-z \geq 0$ $(-z > 0)$.

An element $c \in V$ such that $c^2 = c$ is called an *idempotent* in V; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say that a finite set $\{e_1, e_2, \ldots, e_r\}$ of primitive idempotents in V is a *Jordan frame* if $e_i \circ e_j = 0$ if $i \neq j$ and $\sum_1^r e_i = e$.

Given $x \in V$, there exists a Jordan frame $\{e_1, \ldots, e_r\}$ and real numbers $\lambda_1, \ldots, \lambda_r$ such that

$$x = \lambda_1 e_1 + \cdots + \lambda_r e_r. \tag{2.1}$$

The numbers λ_i are called the *eigenvalues* of x, and the representation (2.1) is called the *spectral decomposition (or the spectral expansion)* of x.

Given (2.1), we have

$$x = \sum_1^r {\lambda_i}^+ e_i - \sum_1^r {\lambda_i}^- e_i \qquad \text{and} \qquad \langle \sum_1^r {\lambda_i}^+ e_i, \sum_1^r {\lambda_i}^- e_i \rangle = 0,$$

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where for a real number α, $\alpha^+ := \max\{0, \alpha\}$ and $\alpha^- := (\alpha)^+ - \alpha$.

From this we easily verify that $x^+ = \sum_1^r {\lambda_i}^+ e_i$ and $x^- = \sum_1^r {\lambda_i}^- e_i$, and so $x = x^+ - x^-$ with $\langle x^+, x^- \rangle = 0$.

For an $x \in V$, a linear transformation $L_x : V \to V$ is defined by $L_x(z) = x \circ z$, for all $z \in V$. We say that two elements x and y *operator commute* if $L_x L_y = L_y L_x$.

It is known that x and y operator commute if and only if x and y have their spectral decompositions with respect to a common Jordan frame (Lemma X.2.2, Faraut and Korányi [3]).

Here are some standard examples.

Example 2.1. R^n is a Euclidean Jordan algebra with inner product and Jordan product defined respectively by $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ and $x \circ y = x * y$. Here R^n_+ is the corresponding symmetric cone.

Example 2.2. $\mathcal{S}^n$, the set of all $n \times n$ real symmetric matrices, is a Euclidean Jordan algebra with the inner and Jordan product given by $\langle X, Y \rangle := \mathrm{trace}(XY)$ and $X \circ Y := \frac{1}{2}(XY + YX)$. In this setting, the symmetric cone $\mathcal{S}^n_+$ is the set of all positive semidefinite matrices in $\mathcal{S}^n$. Also, X and Y operator commute if and only if $XY = YX$.

Example 2.3. Consider R^n $(n > 1)$ where any element x is written as

$$x = \begin{bmatrix} x_0 \\ \overline{x} \end{bmatrix}$$

with $x_0 \in R$ and $\overline{x} \in R^{n-1}$. The inner product in R^n is the usual inner product. The Jordan product $x \circ y$ in R^n is defined by

$$x \circ y = \begin{bmatrix} x_0 \\ \overline{x} \end{bmatrix} \circ \begin{bmatrix} y_0 \\ \overline{y} \end{bmatrix} := \begin{bmatrix} \langle x, y \rangle \\ x_0 \overline{y} + y_0 \overline{x} \end{bmatrix}.$$

We shall denote this Euclidean Jordan algebra $(R^n, \circ, \langle \cdot, \cdot \rangle)$ by $\mathcal{L}^n$. In this algebra, the cone of squares, denoted by $\mathcal{L}^n_+$, is called the *Lorentz cone* (or the second-order cone). It is given by $\mathcal{L}^n_+ = \{x : ||\overline{x}|| \leq x_0\}$.

The unit element in $\mathcal{L}^n$ is $e = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We note the spectral decomposition of any x with $\overline{x} \neq 0$: $x = \lambda_1 e_1 + \lambda_2 e_2$, where $\lambda_1 := x_0 + ||\overline{x}||$, $\lambda_2 := x_0 - ||\overline{x}||$, and

$$e_1 := \frac{1}{2} \begin{bmatrix} 1 \\ \frac{\overline{x}}{||\overline{x}||} \end{bmatrix}, \text{ and } e_2 := \frac{1}{2} \begin{bmatrix} 1 \\ -\frac{\overline{x}}{||\overline{x}||} \end{bmatrix}.$$

In this setting, x and y operator commute if and only if either $\overline{y}$ is a multiple of $\overline{x}$ or $\overline{x}$ is a multiple of $\overline{y}$.

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Peirce decomposition. Fix a Jordan frame $\{e_1, e_2, \dots, e_r\}$ in a Euclidean Jordan algebra V. For $i, j \in \{1, 2, \dots, r\}$, define the eigenspaces

$$V_{ii} := \{x \in V : x \circ e_i = x\} = R\, e_i \text{ (where R is the set of all real numbers)}$$

and when $i \neq j$, $V_{ij} := \{x \in V : x \circ e_i = \frac{1}{2}x = x \circ e_j\}$. Then, we have the following theorem.

THEOREM 2.4. (Theorem IV.2.1, [3]) *The space V is the orthogonal direct sum of the spaces V_{ij} $(i \leq j)$. Furthermore,*

$$\begin{array}{l} V_{ij} \circ V_{ij} \subset V_{ii} + V_{jj} \\ V_{ij} \circ V_{jk} \subset V_{ik} \ \textit{if}\ i \neq k \\ V_{ij} \circ V_{kl} = \{0\} \ \textit{if}\ \{i, j\} \cap \{k, l\} = \emptyset. \end{array}$$

Thus, given any Jordan frame $\{e_1, e_2, \dots, e_r\}$, we can write any element $x \in V$ as $x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}$ where $x_i \in R$ and $x_{ij} \in V_{ij}$.

A Euclidean Jordan algebra is said to be *simple* if it is not the direct sum of two Euclidean Jordan algebras. It is well known that any Euclidean Jordan algebra is product of simple Euclidean Jordan algebras and every simple Euclidean Jordan algebra is isomorphic to the Jordan spin algebra $\mathcal{L}^n$ or to the algebra of all $n \times n$ real symmetric matrices $\mathcal{S}^n$ or to $n \times n$ complex Hermitian matrices $\mathcal{H}_n$ or to $n \times n$ quaternion Hermitian matrices $\mathcal{Q}_n$ or to the algebra of all 3×3 octonion Hermitian matrices $\mathcal{O}_3$.

2.2. Pseudomonotone and positive subdefiniteness concepts. Throughout this paper, we assume that V is a Euclidean Jordan algebra with the corresponding symmetric cone K and $L : V \to V$ is a linear transformation.

We say that L is:

(a) copositive on K if $\langle L(x), x\rangle \geq 0$ for all $x \in K$;
(b) copositive star on K if L is copositive on K, and

$$[x \geq 0,\ \ L(x) \geq 0,\ \ \textit{and}\ \ \langle L(x), x\rangle = 0] \Rightarrow L^T(x) \leq 0.$$

DEFINITION 2.5. Let S be a set in V. L is said to be pseudomonotone on S if

$$x \in S,\ y \in S,\ \langle L(x), y - x\rangle \geq 0 \Rightarrow \langle L(y), y - x\rangle \geq 0.$$

DEFINITION 2.6. Let S be a set in V. L is said to be positive subdefinite (PSBD) on S if

$$x \in V,\ \ \langle L(x), x\rangle < 0 \ \ \Rightarrow \ \ - L^T(x) \in S \ \text{ or } \ L^T(x) \in S.$$

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We note that every monotone linear transformation is vacuously PSBD.

Definition 2.7. L is said to be merely positive subdefinite (MPSBD) if L is PSBD but not monotone.

The following definition is a generalization of (Moore-Penrose) pseudo-inverse of a square matrix.

Definition 2.8. The pseudo-inverse of L is the uniquely defined linear transformation $L^\dagger$ which satisfies the following conditions:

$$LL^\dagger L = L,\ \ L^\dagger LL^\dagger = L^\dagger,\ \ (LL^\dagger)^T = LL^\dagger,\ \text{ and }\ (L^\dagger L)^T = L^\dagger L.$$

We define $L^s = L^T(L + L^T)^\dagger L$. We note that L^s is always self-adjoint.

Given a self-adjoint linear transformation B on V, its inertia is defined by

$$\mathrm{In}(B) := (\pi(B), \nu(B), \delta(B)),$$

where $\pi(B)$, $\nu(B)$, and $\delta(B)$ are, respectively, the number of eigenvalues of B with positive, negative, and zero real parts, counting multiplicities. Note that

$$\pi(B) + \nu(B) + \delta(B) = \dim(V).$$

3. General results for pseudomonotone transformations.

In this section, we present some general results for pseudomonotone transformations on V.

Specializing Gowda's result (Proposition 1, [4]) to symmetric cone, we have Lemma 3.1.

Lemma 3.1. *If L is pseudomonotone on K, then L is copositive star on K.*

The following lemma is a modification of Theorem 5.2 in [15]. We omit its proof.

Lemma 3.2. *L is pseudomonotone on K° if and only if*

$$x \in K^\circ,\ \ v \in V,\ \ \langle v, L(x)\rangle = 0 \Rightarrow \langle v, L(v)\rangle \geq 0.$$

Throughout the paper, pseudomonotone, PSBD, copositive, and copositive star concepts are defined relative/on K, in which case, we drop a phrase on K.

Lemma 3.3. *If L is pseudomonotone, then L is PSBD.*

Proof. Suppose that L is pseudomonotone but not PSBD. Then there exists x_0, such that $\langle L(x_0), x_0\rangle < 0$, $L^T(x_0) \not\geq 0$, and $L^T(x_0) \not\leq 0$. Thus, in view of the spectral decomposition of $L^T(x_0)$, there exists $w > 0$ such that $\langle L(w), x_0\rangle = \langle L^T(x_0), w\rangle = 0$. Hence $\langle L(x_0), x_0\rangle \geq 0$ by Lemma 3.2. This is a contradiction. Therefore, L is PSBD. ∎

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The following is analogous to Proposition 3.1 in [1].

THEOREM 3.4. *L is pseudomonotone if and only if L is PSBD and*

$$\langle L(z), z\rangle < 0, \quad L^T(z) \leq 0 \Rightarrow \langle z, L(z^-)\rangle < 0. \tag{3.1}$$

Proof. "Only if" part. Suppose that L is pseudomonotone. Then L is PSBD by Lemma 3.3. Now, suppose that $\langle L(z), z\rangle < 0$ and $L^T(z) \leq 0$. Since $\langle z, L(z^-)\rangle = \langle L^T(z), z^-\rangle \leq 0$, we claim that $\langle z, L(z^-)\rangle \neq 0$. Suppose not, so that

$$\langle z, L(z^-)\rangle = 0. \tag{3.2}$$

Then $\langle z^+ - z^-, L(z^-)\rangle = 0 \Rightarrow \langle z^+ - z^-, L(z^+)\rangle \geq 0$ by pseudomonotonicity of L. Hence,

$$\langle z, L(z^+)\rangle \geq 0. \tag{3.3}$$

Now, from (3.2) and (3.3), we have $\langle z, L(z^+ - z^-)\rangle \geq 0$, i.e., $\langle L(z), z\rangle \geq 0$, this contradicts $\langle L(z), z\rangle < 0$. Thus, the claim is true. Therefore, (3.1) holds.

"If" part. Suppose that L is PSBD and (3.1) holds. The condition

$$x, y \geq 0,\ \langle L(x), y - x\rangle \geq 0 \Rightarrow \langle L(y), y - x\rangle \geq 0$$

is equivalent to

$$u \geq 0,\ z \in V,\ \langle L(z^- + u), z\rangle \geq 0 \Rightarrow \langle L(z^+ + u), z\rangle \geq 0,$$

which is the same as

$$u \geq 0,\ z \in V,\ \langle u, L^T(z)\rangle \geq -\langle L(z^-), z\rangle \Rightarrow \langle u, L^T(z)\rangle \geq -\langle L(z^+), z\rangle. \tag{3.4}$$

Now, let $u \geq 0$ with $\langle u, L^T(z)\rangle \geq -\langle L(z^-), z\rangle$. We verify the rightmost inequality in (3.4). Without loss of generality, let $L^T(z) \neq 0$.

Case 1: $L^T(z) \leq 0$. Then, $0 \geq \langle u, L^T(z)\rangle \geq -\langle L(z^-), z\rangle = -\langle z^-, L^T(z)\rangle \geq 0$ shows that $\langle L(z^-), z\rangle = 0$. By (3.1), $\langle L(z), z\rangle \geq 0$. Now, $\langle L(z), z\rangle \geq 0 \Rightarrow -\langle L(z^-), z\rangle \geq -\langle L(z^+), z\rangle$, thus, $\langle u, L^T(z)\rangle \geq -\langle L(z^+), z\rangle$.

Case 2: $L^T(z) \geq 0$. Then, $\langle u, L^T(z)\rangle \geq 0 \geq -\langle z^+, L^T(z)\rangle = -\langle L(z^+), z\rangle$.

Case 3: $L^T(z) \ngeq 0$ and $L^T(z) \nleq 0$. Then $\langle L(z), z\rangle \geq 0$ (otherwise, if $\langle L(z), z\rangle < 0$, then either $L^T(z) \leq 0$ or $L^T(z) \geq 0$, because L is PSBD). Again, $\langle L(z), z\rangle \geq 0 \Rightarrow -\langle L(z^-), z\rangle \geq -\langle L(z^+), z\rangle$, thus, $\langle u, L^T(z)\rangle \geq -\langle L(z^+), z\rangle$ by (3.4).

Therefore, L is pseudomonotone. ∎

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The following Lemma is similar to Lemma 3.1 in [1]; we give a proof for completeness.

LEMMA 3.5. *L is pseudomonotone if and only if L is PSBD and*

$$\langle L(z), z\rangle < 0,\ L^T(z) \le 0 \Rightarrow L(z^-) \neq 0. \tag{3.5}$$

Proof. "Only if" part follows from Lemma 3.3 and Theorem 3.4.

"If" part. Assume that (3.5) holds, L is PSBD and not pseudomonotone. Then, by Theorem 3.4, there exists z such that

$$\langle L(z), z\rangle < 0,\ L^T(z) \le 0 \text{ and } \langle z, L(z^-)\rangle \ge 0.$$

From $L^T(z) \le 0$, it follows that $\langle z, L(z^-)\rangle = \langle L^T(z), z^-\rangle \le 0$. Hence, for such a z, $\langle z, L(z^-)\rangle = 0$. Now, for a $v \in V$, there exists $\delta > 0$ such that $\langle L(z+tv), z+tv\rangle < 0$, $\forall t \in [-\delta, \delta]$. Since L is PSBD, we have either $L^T(z+tz) \ge 0$ or $L^T(z+tz) \le 0$. If $L^T(z+tz) \ge 0$, then $0 \le \langle L^T(z+tv), z^-\rangle = t\langle L(z^-), v\rangle$, $\forall t \in [-\delta, \delta]$. Hence $\langle L(z^-), v\rangle = 0$ for all $v \in V$ and therefore $L(z^-) = 0$, which is a contradiction. Similarly, we can reach a contradiction if $L^T(z+tz) \le 0$. Therefore, (3.5) holds. ∎

LEMMA 3.6. *Let L be copositive star. Then*

$$\langle L(z), z\rangle < 0,\ L^T(z) \le 0 \Rightarrow L(z^-) \neq 0. \tag{3.6}$$

Proof. Suppose that $\langle L(z), z\rangle < 0$, $L^T(z) \le 0$ and $L(z^-) = 0$. Then $\langle L(z^-), z^-\rangle = 0$. Since L is copositive star, we have that $L^T(z^-) \le 0$. Then

$$0 > \langle L(z), z\rangle = \langle z^+ - z^-, L(z^+)\rangle = \langle z^+, L(z^+)\rangle - \langle L^T(z^-), z^+\rangle \ge 0.$$

This is a contradiction. Therefore, (3.6) holds. ∎

Theorem 3.4, Lemmas 3.1, 3.5, and 3.6 result in the following theorem.

THEOREM 3.7. *L is pseudomonotone if and only if L is PSBD and copositive star.*

The following theorem is a characterization of MPSBD.

THEOREM 3.8. *Suppose that* $\mathrm{Rank}(L) \ge 2$. *Then L is MPSBD if and only if* $\delta(L + L^T) = 1$, $L(V) = L^T(V) = (L + L^T)(V)$, *and* $-L^s$ *is copositive.*

The proof of the above result is based on several lemmas.

LEMMA 3.9. *Assume that C is a closed convex cone and* $C \cap (-C) = \{0\}$. *If* $a \in C$, $b \in -C$, *and* $ta + (1-t)b \in C \cup -C$ *for all* $t \in [0, 1]$, *then there exists* $t_0 \in (0, 1)$ *such that* $t_0 a + (1 - t_0)b = 0$.

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Proof. Suppose that this is not true. Then the set $\{ta+(1-t)b : t \in [0,1]\} \subseteq A \cup B$, where $A := C \setminus \{0\}$ and $B := -C \setminus \{0\}$. Since $\overline{A} \cap B = \emptyset = A \cap \overline{B}$, we have a separation of the connected set $\{ta + (1-t)b : t \in [0,1]\}$, we reach a contradiction. Hence the conclusion. $\square$

Corollary 3.10. *Assume that C is a closed convex cone and $C \cap (-C) = \{0\}$. If $a + t_1 b \in C$, $a + t_2 b \in -C$ for some $t_1,\ t_2 \in R$, and $a + tb \in C \cup -C$ for all $t \in R$, then there exists $t_0 \in R$ such that $a + t_0 b = 0$.*

The following lemma and its proof are similar to Proposition 2.2 and its proof in [1]; we give a proof for completeness.

Lemma 3.11. *Suppose that L is MPSBD. Then*

(i) $\nu(L + L^T) = 1$;
(ii) $(L + L^T)z = 0 \Rightarrow L(z) = L^T(z) = 0$.

Proof. Let $B := L + L^T$.

(i) B has at least one negative eigenvalue since L is not monotone. Assume, for contradiction, that there exist $\lambda_1 \leq \lambda_2 < 0$, z_1, and z_2 such that

$$B(z_1) = \lambda_1 z_1,\ \ B(z_2) = \lambda_2 z_2,\ \ ||z_1||^2 = ||z_2||^2 = 1,\ \text{ and }\ \langle z_1, z_2\rangle = 0. \tag{3.7}$$

Then $\langle B(z_1), z_1\rangle < 0$ and $\langle B(z_2), z_2\rangle < 0$, and hence $\langle L(z_1), z_1\rangle < 0$ and $\langle L(z_2), z_2\rangle < 0$. Since L is MPSBD, without loss of generality, we assume that $L^T(z_1) \leq 0$ and $L^T(z_2) \geq 0$. Now, we define $z(t) = tz_1 + (1-t)z_2$ for $t \in [0,1]$. Then $\langle B(z(t)), z(t)\rangle = t^2\lambda_1 + (1-t)^2\lambda_2 < 0$. Hence $\langle L(z(t)), z(t)\rangle < 0$ for all $t \in [0,1]$. This implies that either $L^T(z(t)) \leq 0$ or $L^T(z(t)) \geq 0$, therefore, $L^T(z(t)) = tL^T(z_1) + (1-t)L^T(z_2) \in K \cup -K$. Since $L^T(z(0)) = L^T(z_1) \leq 0$ and $L^T(z(1)) = L^T(z_2) \geq 0$, there exists $t_0 \in (0,1)$ such that $L^T(z(t_0)) = 0$ by the above lemma; this is a contradiction. Thus, (i) holds.

(ii) Let z_1, λ_1 be defined as in (3.7); let z be such that $B(z) = 0$. Then we have $\langle L(z_1), z_1\rangle = \frac{1}{2}\langle B(z_1), z_1\rangle < 0$, and hence $L^T(z_1) \neq 0$. Now, for $t \in R$, we define $w(t) = z_1 + tz$. Then we have $\langle B(w(t)), w(t)\rangle = \lambda_1 < 0$. Hence $\langle L(w(t)), w(t)\rangle < 0$ for all $t \in R$. This implies that either $L^T(w(t)) \neq 0$ and $L^T(w(t)) \leq 0$ or $L^T(w(t)) \neq 0$ and $L^T(w(t)) \geq 0$. Therefore, for all $t \in R$, $0 \neq L^T(w(t)) = L^T(z_1) + tL^T(z) \in K \cup -K$. We claim that $L^T(z) = 0$. First, we show that either $L^T(w(t)) \in K$ for all $t \in R$ or $L^T(w(t)) \in -K$ for all $t \in R$. Suppose not. Then there exist $t_1,\ t_2 \in R$ such that $L^T(z_1) + t_1L^T(z) \in K$ and $L^T(z_1) + t_2L^T(z) \in -K$. Thus, there exists $t_0 \in R$ such that $L^T(z_1) + t_0L^T(z) = 0$ by Corollary 3.10. This is a contradiction. Now, we consider the following two cases:

Case 1: $L^T(z_1) + tL^T(z) \in K$ for all $t \in R$. Then $\frac{1}{t}L^T(z_1) + L^T(z) = \frac{1}{t}(L^T(z_1) + tL^T(z)) \in K$ for a large $t > 0$. This implies that $L^T(z) \in K$ as $t \to \infty$ since K

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is closed. Similarly, $\frac{1}{t}L^T(z_1) + L^T(z) \in -K$ for a large $-t > 0$. This implies that $L^T(z) \in -K$ as $t \to \infty$ since K is closed. Thus, $L^T(z) = 0$.

Case 2: $L^T(z_1) + tL^T(z) \in -K$ for all $t \in R$. The proof is similar to Case 1.

Therefore, the claim holds. Hence $L(z) = 0$ from $B(z) = 0$. ◻

REMARK 3.12. Suppose M is an $n \times n$ matrix. Then the following are equivalent (see [1]):

(i) $(M + M^T)z = 0 \Rightarrow Mz = M^T z = 0$;
(ii) $M(R^n) \subseteq (M + M^T)(R^n)$.

Now, given $L : V \to V$ linear transformation, by identifying V with some R^k and L with a matrix, we get the following equivalent statements:

(i) $(L + L^T)(z) = 0 \Rightarrow L(z) = L^T(z) = 0$;
(ii) $L(V) \subseteq (L + L^T)(V)$.

Using the same argument as in the above remark, Lemma 2 in [2] and Proposition 2.3 in [1] reduce to the following two lemmas.

LEMMA 3.13. *Suppose that* $L(V) \subseteq (L + L^T)(V)$. *Then*

$$\begin{aligned}\pi(L^s) &= \pi(L + L^T) + \delta(L + L^T) - k,\\ \nu(L^s) &= \nu(L + L^T) + \delta(L + L^T) - k,\\ \delta(L^s) &= 2k - \delta(L + L^T),\end{aligned}$$

where k *is the dimension of the kernel of* L.

LEMMA 3.14. *Suppose that* B *is self-adjoint linear transformation defined on* V *and* $\nu(B) = 1$. *Then there exists a closed convex cone* T *such that*

$$T \cup -T = \{z : \langle B(z), z\rangle \le 0\} \text{ and } \operatorname{int}(T) \cup -\operatorname{int}(T) = \{z : \langle B(z), z\rangle < 0\}.$$

Furthermore,

$$T^{\triangle} \cap -T^{\triangle} = \{0\} \text{ and } T^{\triangle} \cup -T^{\triangle} = \{y : \langle B^{\dagger}(y), y\rangle \le 0\} \cap B(V).$$

Where $T^{\triangle}$ *is the polar cone of* T *defined by* $T^{\triangle} = \{x^\star : \langle x^\star, x\rangle \le 0 \ \forall x \in T\}$.

The following lemma gives a characterization of MPSBD in terms of such a closed convex cone as in the above lemma.

LEMMA 3.15. *Given* L *on* V. *The following are equivalent:*

(i) L *is MPSBD.*
(ii) *There exists a closed convex cone* T *defined in the above lemma such that either* $T \subseteq \{z : L^T(z) \le 0\}$ *or* $-T \subseteq \{z : L^T(z) \le 0\}$.

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(iii) *For T in (ii), either $L(K) \subseteq T^{\triangle}$ or $L(K) \subseteq -T^{\triangle}$.*

Proof. Let $B = L + L^T$.

(ii)$\Rightarrow$ (i): Suppose (ii) holds and let $\langle L(x), x\rangle < 0$. Then $\langle B(x), x\rangle < 0$, and hence $x \in \mathrm{int}(T) \cup -\mathrm{int}(T)$ by the above lemma. Thus, either $x \in \mathrm{int}(T) \subseteq \{z : L^T(z) \leq 0\}$ or $x \in -\mathrm{int}(T) \subseteq \{z : L^T(z) \leq 0\}$. Therefore, $L^T(x) \leq 0$ or $L^T(x) \geq 0$. Hence L is MPSBD.

(i)$\Rightarrow$ (ii): Suppose (i) holds. Then $\nu(B) = 1$ by Lemma 3.11. Hence there exists a closed convex cone T defined in the above lemma. It is enough to show that $\mathrm{int}(T) \subseteq \{z : L^T(z) \leq 0\}$ or $\mathrm{int}(T) \subseteq \{z : L^T(z) \geq 0\}$. Suppose not. Then there exist u, v in $\mathrm{int}(T)$ such that $L^T(u) \not\leq 0$ and $L^T(v) \not\geq 0$. We note that $u \neq v$ since L is MPSBD. Again, since L is MPSBD, we have $L^T(u) \geq 0$ and $L^T(v) \leq 0$. Now, we define $w(t) = tu + (1-t)v$ for $t \in [0, 1]$. Then $w(t) \in \mathrm{int}(T)$ since u, v in $\mathrm{int}(T)$, and $L^T(w(t)) = tL^T(u) + (1-t)L^T(v)$. Hence $\langle B(w(t)), w(t)\rangle < 0 \Rightarrow \langle L(w(t)), w(t)\rangle < 0$. The last inequality implies that $L^T(w(t)) \leq 0$ or $L^T(w(t)) \geq 0$ since L is MPSBD. Thus, $L^T(w(t)) \in K \cup -K$. Also, $L^T(w(0)) = L^T(v) \leq 0$ and $L^T(w(1)) = L^T(u) \geq 0$. Therefore, there exists $t_0 \in (0, 1)$ such that $L^T(w(t_0)) = 0$ by Lemma 3.9. This is a contradiction.

(ii)$\Rightarrow$ (iii): For any $y \in L(K)$, there exists $x \in K$ such that $y = L(x)$. Now, we claim that either $y \in T^{\triangle}$ or $y \in -T^{\triangle}$.

Case 1: $z \in T$. Then we have $L^T(z) \leq 0$. Thus, $\langle z, y\rangle = \langle z, L(x)\rangle = \langle L^T(z), x\rangle \leq 0$. Hence $y \in T^{\triangle}$.

Case 2: $z \in -T$, Then we have $L^T(z) \leq 0$. Thus, $\langle z, y\rangle = \langle z, L(x)\rangle = \langle L^T(z), x\rangle \leq 0$. Hence $y \in -T^{\triangle}$.

Therefore, the claim holds.

(iii)$\Rightarrow$ (ii): Suppose $L(K) \subseteq T^{\triangle}$. Then $L(x) \in T^{\triangle}$ for all $x \in K$. Hence, for any $z \in T$, we have $0 \geq \langle z, L(x)\rangle = \langle L^T(z), x\rangle$. This is implies that $L^T(z) \leq 0$. Therefore, $T \subseteq \{z : L^T(z) \leq 0\}$. Similarly, if $L(K) \subseteq -T^{\triangle}$, then $-T \subseteq \{z : L^T(z) \leq 0\}$. $\square$

Proof of Theorem 3.8. Suppose that L is MPSBD. Then $\nu(L + L^T) = 1$ by Lemma 3.11, and hence there exists a closed convex cone T that satisfies the conditions in Lemma 3.14. Also, $L(K) \subseteq T^{\triangle}$ or $L(K) \subseteq -T^{\triangle}$ by Lemma 3.15. Without loss of generality, we assume that $L(K) \subseteq T^{\triangle}$. Then for any $x \geq 0$, $\langle L^s(x), x\rangle = \langle L^T(L + L^T)^{\dagger} L(x), x\rangle = \langle (L + L^T)^{\dagger} L(x), L(x)\rangle \leq 0$. Therefore, $-L^s$ is copositive. From Lemma 3.11 and Remark 3.12, we have $L(V) \subseteq (L+L^T)(V)$. Thus, Rank$(L)\leq$ Rank$(L + L^T)$. Since $L^s = L^T(L + L^T)^{\dagger}L$, we have Rank$(L^s)\leq$ Rank(L). Hence Rank$(L^s)\leq$ Rank$(L)\leq$ Rank$(L + L^T)$. Since Rank$(L^s)= \pi(L^s) + \nu(L^s) = \pi(L + L^T) + 1 + 2\delta(L + L^T) - 2k$, (the last equality is from Lemma 3.13), and Rank$(L^s)\leq$

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Rank$(L+L^T)$, we have $\pi(L+L^T)+1+2\delta(L+L^T)-2k \le \pi(L+L^T)+\nu(L+L^T)$. This implies that $k \ge \delta(L+L^T)$. Since $0 \le \nu(L^s) = \nu(L+L^T)+\delta(L+L^T)-k$ (see Lemma 3.13), we have $k \le 1+\delta(L+L^T)$. Since k is an integer, we have that either $k = \delta(L+L^T)$ or $k = 1+\delta(L+L^T)$.

Case 1: Suppose $k = 1+\delta(L+L^T)$. Then $\nu(L+L^T) = 1$ implies that $\nu(L^s) = 0$ (see Lemma 3.13) and so L^s is monotone. Since $-L^s$ is copositive, we have $\langle L^s(x), x\rangle = 0$, for all $x \in K$. We claim that $L^s = 0$. Take any $y \in V$, $y = y^+ - y^-$, since $y^+,\ y^- \in K$, we have $y^+ + y^- \in K$. Thus, $\langle L^s(y^+ + y^-), y^+ + y^-\rangle = 0$, which implies that $\langle L^s(y^+), y^-\rangle = 0$. Now, it is easy to verify that $\langle L^s(y), y\rangle = \langle L^s(y^+ - y^-), y^+ - y^-\rangle = 0$. Thus, $L^s(y) = 0$ for all $y \in V$. Hence $L^s = 0$. Therefore, $\dim(V) = \delta(L^s) = 2k - \delta(L+L^T) = 2k-(k-1) = k+1$. Since $\dim(V) = \text{Rank}(L)+k$, we have Rank$(L)$= 1. This is a contradiction.

Case 2: Suppose $k = \delta(L+L^T)$. Then from Lemma 3.13, $\delta(L^s) = \delta(L+L^T)$, $\nu(L^s) = \nu(L+L^T)$, and $\pi(L^s) = \pi(L+L^T)$, i.e., $L+L^T$ and L^s have the same inertia. Therefore, Rank$(L+L^T)$= Rank(L^s).

Subcase 2.1: Suppose $L+L^T$ is invertible. Then $(L+L^T)(V) = V$. Since Rank$(L+L^T)$= Rank(L^s), we have that L^s is invertible. Since $L^s = L^T(L+L^T)^\dagger L$, we have that L and L^T are invertible. Thus $L(V) = V$ and $L^T(V) = V$. Therefore, $L(V) = L^T(V) = (L+L^T)(V)$.

Subcase 2.2: Suppose $L+L^T$ is not invertible. Since $L(V) \subseteq (L+L^T)(V)$, we have Rank$(L)\le$ Rank$(L+L^T)$. Since Rank$(L+L^T)$= Rank$(L^s)\le$ Rank$(L)\le$ Rank$(L+L^T)$, we have Rank$(L+L^T)$= Rank(L). Again, since $L(V) \subseteq (L+L^T)(V)$, we have $L(V) = (L+L^T)(V)$. By symmetry, we have $L^T(V) = (L+L^T)(V)$. This completes the "Only if" part of Theorem 3.3.

Conversely, suppose that $\nu(L+L^T) = 1$, $L(V) = L^T(V) = (L+L^T)(V)$, and $-L^s$ is copositive. Then there exists a closed convex cone T that satisfies the conditions in Lemma 3.14. We claim that

$$L(K) \subseteq T^\triangle \cup -T^\triangle = \{y : \langle (L+L^T)^\dagger(y), y\rangle \le 0\} \cap (L+L^T)(V).$$

Now, for any $y \in L(K)$, there exists $x \in K$ such that $y = L(x)$.

Then $y = L(x) \in L(V) = (L+L^T)(V)$. Hence

$$\langle (L+L^T)^\dagger(y), y\rangle = \langle (L+L^T)^\dagger L(x), L(x)\rangle = \langle L^T(L+L^T)^\dagger L(x), x\rangle = \langle L^s(x), x\rangle \le 0.$$

Thus, $y \in T^\triangle \cup -T^\triangle$, and hence $L(K) \subseteq T^\triangle \cup -T^\triangle$. Therefore, the claim holds. Now, it is enough to show that the condition

$$L(K) \subseteq T^\triangle \cup -T^\triangle \tag{3.8}$$

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is equivalent to the condition (iii) in Lemma 3.15. Suppose that the condition (iii) holds. Then it is obvious that the condition (3.8) holds. Conversely, Suppose that the condition (3.8) holds, but the condition (iii) does not hold.

Then $L(K)\setminus\{0\} \subseteq T^{\triangle}\setminus\{0\} \cup -T^{\triangle}\setminus\{0\}$. It is clear that $T^{\triangle}\setminus\{0\} \cup -T^{\triangle}\setminus\{0\}$ is a separation of the set $L(K)\setminus\{0\}$. Now, we show that $L(K)\setminus\{0\}$ is connected set. We note that $L(K)$ is a convex cone, and $\dim L(K) = \dim L(K-K) = \dim L(V) =$ Rank(L) ≥ 2. We consider the following cases:

Case 1: Suppose that 0 is on the boundary of $L(K)$. Then $L(K)\setminus\{0\}$ is connected set.

Case 2: Suppose that 0 is in the relative interior of $L(K)$. Then $L(K)$ is a subspace. Hence $L(K)\setminus\{0\}$ is (path) connected because of $\mathrm{Rank}(L(K)) \geq 2$.

This is a contradiction. Thus, the condition (iii) holds, and hence L is MPSBD by Lemma 3.15. □

Theorem 3.8 immediately yields the following.

LEMMA 3.16. *Suppose that* $\mathrm{Rank}(L) \geq 2$. *Then L is MPSBD if and only if L^T is MPSBD.*

When $\mathrm{Rank}(L) = 1$, the above lemma is not true, as the following example shows.

EXAMPLE 3.17. Let

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \mathcal{L}^3 \to \mathcal{L}^3.$$

Then it is easy to verify that L is MPSBD but L^T is not.

THEOREM 3.18. *Suppose that L is pseudomonotone with* $\mathrm{Rank}(L) \geq 2$. *Then L^T is pseudomonotone.*

Proof. Assume without loss of generality that L is not monotone. Hence, by Theorem 3.7, L is PSBD and copositive. Thus, L^T is copositive and by the above lemma, L^T is PSBD. Now, suppose that there exists x such that $\langle L^T(x), x\rangle < 0$, $L(x) \leq 0$, and $L^T(x^-) = 0$. Since L is PSBD, by Theorem 3.8, $L(V) = L^T(V)$, and hence Ker(L)=Ker(L^T). Thus, $L(x^-) = 0$ and

$$\begin{aligned} 0 > \langle L^T(x), x\rangle = \langle L(x), x\rangle &= \langle L(x^+ - x^-), x^+ - x^-\rangle \\ &= \langle L(x^+), x^+ - x^-\rangle \\ &= \langle L(x^+), x^+\rangle - \langle x^+, L^T(x^-)\rangle \\ &= \langle L(x^+), x^+\rangle \geq 0. \end{aligned}$$

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The last inequality comes from copositivity of L. This is a contradiction. Using Lemma 3.5 shows that L^T is pseudomonotone. ▯

When Rank(L)=1, the above theorem is not true, as the following example shows.

Example 3.19. Let

$$L = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} : R^2 \to R^2.$$

Then it is easy to verify that L is pseudomonotone, but L^T is not.

In the matrix case, Gowda ([6]) conjectured that pseudomonotonicity of L implies pseudomonotonicity of L^T when L is normal ($LL^T = L^T L$). In what follows, we give a positive answer in the setting of Euclidean Jordan algebras.

First, we prove the following lemma.

Lemma 3.20. *Suppose L is pseudomonotone defined by $L(x) = \langle a, x\rangle b$ with $0 \neq a \geq 0$ and $0 \neq b \in V$. Then $\langle a, x\rangle\langle b, x\rangle \geq 0$ for all $x \geq 0$ implies $b \geq 0$.*

Proof. Since $L(x) = \langle a, x\rangle b$, we have $L^T(x) = \langle b, x\rangle a$. Since L is pseudomonotone, L is PSBD and copositive star by Theorem 3.7. Since $\langle a, x\rangle \geq 0$ for all $x \geq 0$, we consider the following cases:

Case 1: If $\langle a, x\rangle > 0$ for all $x \geq 0$, then $\langle a, x\rangle\langle b, x\rangle \geq 0$ for all $x \geq 0$ implies $\langle b, x\rangle \geq 0$ for all $x \geq 0$. Hence, $b \geq 0$.
Case 2: If $\langle a, x_0\rangle = 0$ for some $x_0 \geq 0$, then $L(x_0) = 0$. By copositive star property of L, $L^T(x_0) \leq 0$, so $\langle b, x_0\rangle a \leq 0$. If $\langle b, x_0\rangle < 0$, there exist a $u > 0$ and $\epsilon > 0$, such that $\langle b, x_0 + \epsilon u\rangle < 0$. Since $x_0 + \epsilon u \geq 0$, we have $\langle a, x_0 + \epsilon u\rangle\langle b, x_0 + \epsilon u\rangle \geq 0$. Since $\langle a, x_0 + \epsilon u\rangle = \epsilon\langle a, u\rangle > 0$, we have $\langle a, x_0 + \epsilon u\rangle\langle b, x_0 + \epsilon u\rangle < 0$. This is a contradiction. Thus, $\langle b, x_0\rangle = 0$.
Now, combining cases 1 and 2, we get that $\langle a, x\rangle\langle b, x\rangle \geq 0$ for all $x \geq 0$, which implies $\langle b, x\rangle \geq 0$. Hence, $b \geq 0$. ▯

Remark 3.21. If $0 \neq a \leq 0$ in the above lemma, then replacing a by $-a$ and repeating the argument in the proof, we have $b \leq 0$.

Theorem 3.22. *If L is a normal and pseudomonotone, then L^T is pseudomonotone.*

Proof. When Rank(L)=2, the implication follows from Theorem 3.18.

When Rank(L)=0, it is obvious. We will show that it is true for Rank(L)=1 case. Since Rank(L)=1, there exist nonzero a, $b \in V$, such that $L(x) = \langle a, x\rangle b$. Then we have $L^T(x) = \langle b, x\rangle a$. Since L is pseudomonotone, L is PSBD and copositive star by Theorem 3.7. First, we show that L^T is PSBD. Suppose that $\langle L^T(x), x\rangle =$

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$\langle a, x\rangle\langle b, x\rangle < 0$, which implies $\langle L(x), x\rangle < 0$. Then we have that either $L^T(x) = \langle b, x\rangle a \geq 0$ or $L^T(x) = \langle b, x\rangle a \leq 0$ by PSBD of L. Since $\langle b, x\rangle \neq 0$, we have that either $a \geq 0$ or $a \leq 0$. Since L is copositive, we have $\langle a, y\rangle\langle b, y\rangle \geq 0$ for all $y \geq 0$, which implies that either $a \geq 0$ and $b \geq 0$ or $a \leq 0$ and $b \leq 0$ by the above lemma and remark. This implies that either $L(x) \geq 0$ or $L(x) \leq 0$. Therefore, L^T is PSBD. Now, suppose that $\langle L^T(x), x\rangle < 0$, $L(x) \leq 0$ and $L^T(x^-) = 0$. Then $\langle L(x^-), L(x^-)\rangle = \langle x^-, L^T L(x^-)\rangle = \langle x^-, LL^T(x^-)\rangle = 0$, which implies that $L(x^-) = 0$. Then, as in the proof of Theorem 3.18,

$$\begin{aligned} 0 > \langle L^T(x), x\rangle = \langle L(x), x\rangle &= \langle L(x^+ - x^-), x^+ - x^-\rangle \\ &= \langle L(x^+), x^+ - x^-\rangle \\ &= \langle L(x^+), x^+\rangle - \langle x^+, L^T(x^-)\rangle \\ &= \langle L(x^+), x^+\rangle \geq 0. \end{aligned}$$

The last inequality comes from copositivity of L. This is a contradiction. By Lemma 3.5, L^T is pseudomonotone. $\square$

In the examples below, we describe matrices on $\mathcal{L}^3$ of rank 1, 2, and 3 that are pseudomonotone, but not monotone. The proofs are given in Appendix.

Example 3.23. Let

$$L = \begin{bmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix} : \mathcal{L}^3 \to \mathcal{L}^3$$

be such that $a \geq \sqrt{b^2 + c^2}$. Then L is pseudomonotone.

Remark 3.24. By putting $a = b = 1$ and $c = 0$, we get that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \mathcal{L}^3 \to \mathcal{L}^3$$

is pseudomonotone, but not monotone.

Example 3.25. Let

$$L = \begin{bmatrix} 1 & c & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \mathcal{L}^3 \to \mathcal{L}^3$$

be such that

(a) $0 \leq b + c \leq 1$,
(b) $c > 0$,

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(c) $b < 0$.

Then L is pseudomonotone.

REMARK 3.26. By putting $b = -\frac{1}{4}$ and $c = \frac{1}{2}$, we get that

$$L = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \mathcal{L}^3 \to \mathcal{L}^3$$

is pseudomonotone, but not monotone.

EXAMPLE 3.27. Let

$$L = \begin{bmatrix} 1 & c & 0 \\ b & 0 & 0 \\ 0 & 0 & \epsilon \end{bmatrix} : \mathcal{L}^3 \to \mathcal{L}^3$$

be such that

(a) $0 < b + c < 1$,
(b) $0 < c < \sqrt{2}$,
(c) $b < 0$,
(d) $0 < \epsilon \leq 1$,
(e) $(3b + c)^2 - 4b^2(2 - c^2) < 0$.

Then L is pseudomonotone.

REMARK 3.28. By putting $b = -\frac{1}{4}$, $c = \frac{1}{2}$ and $\epsilon = 1$, we get that

$$L = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} : \mathcal{L}^3 \to \mathcal{L}^3$$

is pseudomonotone, but not monotone.

THEOREM 3.29. *Suppose that L is self-adjoint and copositive. Then L is PSBD if and only if L is monotone.*

Proof. As the "If" part is obvious, we prove the "Only if" part. Since L is self-adjoint, every eigenvalue of L is real. Assume there exists an eigenvalue $\lambda < 0$, and nonzero $x \in V$ such that $L(x) = \lambda x$. Then we have $\langle L(x), x\rangle = \langle \lambda x, x\rangle = \lambda||x||^2 < 0$; it follows that $L^T(x) = L(x) = \lambda x \geq 0$ or $L^T(x) = L(x) = \lambda x \leq 0$ since L is PSBD. If $L(x) = \lambda x \leq 0$, then $x \geq 0$, because $\lambda < 0$. Thus, $\langle L(x), x\rangle \geq 0$, because L is copositive. This is a contradiction. If $L(x) = \lambda x \geq 0$, then $x \leq 0$, because $\lambda < 0$. Thus, $\langle L(x), x\rangle = \langle L(-x), -x\rangle \geq 0$, because L is copositive. Again, this is a contradiction. Therefore, each eigenvalue of L is nonnegative. Hence, L is monotone. $\square$

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Corollary 3.30. *Suppose that L is self-adjoint. Then L is pseudomonotone if and only if L is monotone.*

4. Pseudomonotonicity and principal subtransformations. In the matrix case, Mohan, Neogy and Das (see [16]) showed that if L is PSBD, then its principal subtransformations are also PSBD. In this section, we study the relation between pseudomonotonicity of L and its principal subtransformations.

First, we recall the notion of "principal subtransformations" of a given linear transformation on V.

Given a Jordan frame $\{e_1, \ldots, e_r\}$ in a Euclidean Jordan algebra V, we define

$$V^{(l)} := V(e_1 + e_2 + \cdots + e_l, 1) := \{x \in V : x \circ (e_1 + e_2 + \cdots + e_l) = x\}$$

for $1 \leq l \leq r$. Corresponding to $V^{(l)}$, we consider the orthogonal projection $P^{(l)} : V \mapsto V^{(l)}$. For a given linear transformation $L : V \mapsto V$, the transformation $P^{(l)} \circ L : V^{(l)} \mapsto V^{(l)}$ is, by definition, a *principal subtransformation* of L corresponding to $\{e_1, \ldots, e_l\}$, and is denoted by $L_{\{e_1,\ldots,e_l\}}$ (the symbol "$\circ$" means here the composition rather than Jordan multiplication).

We note that for a given Jordan frame $\{e_1, \ldots, e_r\}$, we can permute the objects and select the first l objects (for any $1 \leq l \leq r$). Thus, there are $2^r - 1$ principal subtransformations corresponding to a Jordan frame. Of course, by taking other Jordan frames, we generate other principal subtransformations.

Proposition 4.1. *Fix a Jordan frame $\{e_1, \ldots, e_r\}$. Suppose that L is PSBD. Then $P^{(l)} \circ L$ is PSBD for any l, $1 \leq l \leq r$.*

Proof. Let $\widehat{L} = P^{(l)} \circ L : V^{(l)} \to V^{(l)}$. Then for $x \in V^{(l)}$, $0 > \langle x, \widehat{L}(x)\rangle = \langle x, P^{(l)} \circ L(x)\rangle = \langle x, L(x)\rangle$, hence, $L^T(x) \leq 0$ or $L^T(x) \geq 0$, because L is PSBD. Suppose $L^T(x) \leq 0$. Then for any $u \geq 0$ in $V^{(l)}$, $\langle u, \widehat{L}^T(x)\rangle = \langle \widehat{L}(u), x\rangle = \langle P^{(l)} \circ L(u), x\rangle = \langle L(u), P^{(l)}(x)\rangle = \langle L(u), x\rangle = \langle u, L^T(x)\rangle \leq 0$. Hence, $\widehat{L}^T(x) \leq 0$ in $V^{(l)}$. Similarly, we show that $\widehat{L}^T(x) \geq 0$ in $V^{(l)}$ when $L^T(x) \geq 0$. Therefore, $P^{(l)} \circ L$ is PSBD for any l, $1 \leq l \leq r$. ∎

Lemma 4.2. *Fix a Jordan frame $\{e_1, \ldots, e_r\}$. If L is copositive on V, then $P^{(l)} \circ L$ is copositive on $V^{(l)}$ for any l, $1 \leq l \leq r$.*

Proof. Suppose $x \geq 0$ in $V^{(l)}$. Then $\langle x, P^{(l)} \circ L(x)\rangle = \langle P^{(l)}(x), L(x)\rangle = \langle x, L(x)\rangle \geq 0$. ∎

Theorem 4.3. *Fix a Jordan frame $\{e_1, \ldots, e_r\}$. If L is pseudomonotone and* Rank*$(P^{(l)} \circ L) \geq 2$ for $1 < l \leq r$, then $P^{(l)} \circ L$ is pseudomonotone for any l, $1 < l \leq r$.*

Proof. Assume without loss of generality that L is not monotone. To use Lemma

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3.5, suppose that L is pseudomonotone. Then L is PSBD and copositive star by Theorem 3.7. Hence $P^{(l)} \circ L$ is PSBD for $1 \leq l \leq r$ by Proposition 4.1 and copositive by the above lemma. Suppose there exists l_0 $(1 < l_0 \leq r)$ such that $\widehat{L_0} := P^{(l_0)} \circ L$ is not pseudomonotone. Then there exists $x \in V^{(l_0)}$ such that $\langle x, \widehat{L_0}(x)\rangle < 0$, $\widehat{L}_0^T(x) \leq 0$ and $\widehat{L}_0(x^-) = 0$. Since $\widehat{L_0}$ is PSBD and Rank$(\widehat{L_0}) \geq 2$, we have $\widehat{L_0}(V^{(l_0)}) = \widehat{L}_0^T(V^{(l_0)})$ by Theorem 3.8; thus Ker$(\widehat{L_0})$=Ker$(\widehat{L}_0^T)$. Hence $\widehat{L}_0^T(x^-) = 0$. Repeating the argument given in the proof of Theorem 3.18 and 3.22, we come to a contradiction, so $\widehat{L}_0(x^-) \neq 0$. Thus, $P^{(l)} \circ L$ is pseudomonotone for any l, $1 < l \leq r$. ▯

5. Pseudomonotonicity of some specialized transformations.

5.1. Quadratic representations. We now characterize the pseudomonotonicity for quadratic representations.

Given any element a in V, the *quadratic representation* of a is defined by

$$P_a(x) := 2a \circ (a \circ x) - a^2 \circ x.$$

Recall $P_a(K) \subseteq K$, so P_a is copositive.

THEOREM 5.1. *For $a \in V$, the following are equivalent:*

(a) P_a is monotone on V.
(b) P_a is pseudomonotone.
If, in addition, V is simple, then the above conditions are further equivalent to
(c) $\pm a \in K$.

Proof. Since P_a is self-adjoint and copositive, we only need to show (a) is equivalent to (c), when V is simple. For a given $a \in V$, there exits a Jordan frame $\{e_1, e_2, \ldots, e_r\}$ such that

$$a = a_1 e_1 + a_2 e_2 + \cdots + a_r e_r.$$

For any $x \in V$, we have the Peirce decomposition of x with respect to this Jordan frame

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}$$

(with $x_i \in R$ and $x_{ij} \in V_{ij}$). It can be easily verified that

$$P_a(x) = \sum_{i=1}^{r} {a_i}^2 x_i e_i + \sum_{i<j} a_i a_j x_{ij}.$$

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When V is simple, V_{ij} is nonzero for each $i \leq j$ (see Corollary IV.2.4 in [3]), so we have

$$\begin{aligned} 0 \leq \langle x, P_a(x) \rangle &= \sum_{i=1}^{r} {a_i}^2 {x_i}^2 ||e_i||^2 + \sum_{i<j} a_i a_j ||x_{ij}||^2 \ (\forall x \in V) \\ &\Leftrightarrow a_i a_j \geq 0 \ (i \leq j) \\ &\Leftrightarrow a_i \geq 0 \text{ or } a_i \leq 0, \forall i. \end{aligned}$$

Hence, when V is simple, P_a is monotone on V if and only if $\pm a \in K$. □

Remark 5.2. When $V = \mathcal{S}^n$, for a real $n \times n$ matrix A, the *two sided multiplicative transformation* is defined by

$$M_A(X) := AXA^T.$$

Clearly, M_A is copositive. If we specialize P_a on $\mathcal{S}^n$, then $P_A(X) = AXA$ for $A \in \mathcal{S}^n$. Thus for M_A, the following are equivalent when A is a real symmetric square matrix.

(a) A is either positive semidefinite or negative semidefinite.
(b) M_A is monotone.
(c) M_A is pseudomonotone.

When A is any nonsingular matrix, we claim that M_A is PSBD if and only if M_A is monotone. We only need to show that M_A is monotone if M_A is PSBD. Suppose there exists $X \in \mathcal{S}^n$, such that $\langle X, M_A(X) \rangle < 0$. Then ${M_A}^T(X) = M_{A^T}(X) \succeq 0$ or $M_{A^T}(X) \preceq 0$. Let $0 \preceq Y := M_{A^T}(X) = A^T X A$. Then $X = (A^{-1})^T Y A^{-1} \succeq 0$. As M_A is copositive, this contradicts $\langle X, M_A(X) \rangle < 0$. Similarly, we can get the contradiction when $Y \preceq 0$. Since monotonicity implies pseudomonotonicity and pseudomonotonicity implies PSBD, we have M_A is pseudomonotone if and only if M_A is monotone when A is any nonsingular matrix. This need not be true when A is a singular matrix, as the following example shows.

Example 5.3. Let

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Then, it is easy to verify that M_A is PSBD, but not monotone.

5.2. Automorphisms. In [16], Mohan, Neogy and Das showed that PSBD matrices are invariant under principal rearrangement. In this section, we show that PSBD property is invariant for cone invariant transformations and pseudomonotonicity is invariant under algebra and cone automorphisms.

A linear transformation $\Lambda : V \to V$ is said to be an *algebra automorphism* if Λ is invertible and $\Lambda(x \circ y) = \Lambda(x) \circ \Lambda(y)$ for all x, $y \in V$. The set of all automorphisms of V is denoted by $\mathrm{Aut}(V)$.

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A linear transformation $\Gamma : V \to V$ is said to be a *cone automorphism* if $\Gamma(K) = K$. Because $K^\circ \neq \emptyset$, such a transformation is necessarily invertible. We denote the set of all automorphisms of K by $\mathrm{Aut}(K)$. It is immediate that $\mathrm{Aut}(V) \subseteq \mathrm{Aut}(K)$.

We use $\Pi(K)$ to denote the set of all linear transformations on V that leave K invariant, i.e., $L(K) \subseteq K$ for any $L \in \Pi(K)$.

We note that $\mathrm{Aut}(V) \subseteq \mathrm{Aut}(K) \subseteq \Pi(K)$.

To illustrate these concepts, we recall the following examples from [9].

Example 5.4. Consider $V = R^n$. In this case, the permutation matrices are the automorphisms of R^n and any automorphism of R^n_+ is a product of a positive definite diagonal matrix and a permutation matrix.

Example 5.5. (see [9]) Consider $V = \mathcal{S}^n$. In this case, for any $\Gamma \in \mathrm{Aut}(\mathcal{S}^n_+)$, there exists an invertible matrix $Q \in R^{n\times n}$ such that $\Gamma(Z) = QZQ^T$, $\forall Z \in \mathcal{S}^n$. In particular, for $\Lambda \in \mathrm{Aut}(\mathcal{S}^n)$, there exists a real orthogonal matrix U such that $\Lambda(Z) = UZU^T$, $\forall Z \in \mathcal{S}^n$.

Example 5.6. (see [9]) Consider $V = \mathcal{L}^n$. In this case, an $n \times n$ matrix $A \in \mathrm{Aut}(\mathcal{L}^n_+)$ if and only if there exists a $\mu > 0$ such that $A^T J_n A = \mu J_n$, where $J_n = diag(1, -1, \ldots, -1)$. In particular, for $A \in \mathrm{Aut}(\mathcal{L}^n)$, $A = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}$, where $D : R^{n-1} \to R^{n-1}$ is an orthogonal matrix.

Theorem 5.7. *If L is PSBD, then PLP^T is PSBD for all $P \in \Pi(K)$.*

Proof. Suppose $\langle x, PLP^T(x)\rangle < 0$. Then we have $\langle P^T(x), LP^T(x)\rangle < 0$. Let $y = P^T(x)$. Then $\langle y, L(y)\rangle < 0$, which implies either $L^T(y) \geq 0$ or $L^T(y) \leq 0$, because L is PSBD. Since $P \in \Pi(K)$, we have either $PL^T(y) \geq 0$ or $PL^T(y) \leq 0$, i.e., $PL^TP^T(x) \geq 0$ or $PL^TP^T(x) \leq 0$. Hence, PLP^T is PSBD. □

Recall the following proposition from [8].

Proposition 5.8. (Proposition 4.1, [8]) *Let $\Gamma \in Aut(K)$. Then Γ^{-1} and $\Gamma^T \in Aut(K)$.*

Theorem 5.9. *Let* $\Gamma \in$ Aut(K). *Then L is pseudomonotone if and only if $\Gamma L \Gamma^T$ is pseudomonotone.*

Proof. To use Theorem 3.7, first, we show that L is PSBD if and only if $\Gamma L\Gamma^T$ is PSBD. Suppose that L is PSBD. The implication of that $\Gamma L\Gamma^T$ is PSBD follows from Theorem 5.7 because $\Gamma \in$ Aut(K) $\subseteq \Pi$(K). The converse follows from the fact that Γ^{-1}, $\Gamma^T \in$ Aut(K) $\subseteq \Pi$(K). Now, we prove that L is copositive star if and only if $\Gamma L\Gamma^T$ is copositive star. Suppose that L is copositive star, $x \geq 0$, $\Gamma L\Gamma^T(x) \geq 0$,

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and $\langle x, \Gamma L \Gamma^T(x)\rangle = 0$. Then, we have $\Gamma^T(x) \geq 0$, because $\Gamma^T \in \text{Aut(K)} \subseteq \Pi(\text{K})$; $L\Gamma^T(x) \geq 0$, because $\Gamma^{-1} \in \text{Aut(K)} \subseteq \Pi(\text{K})$ and $\langle \Gamma^T(x), L\Gamma^T(x)\rangle = 0$. Since L is copositive star, we have $L^T\Gamma^T(x) \leq 0$, so $\Gamma L^T \Gamma^T(x) \leq 0$. Therefore, $\Gamma L \Gamma^T$ is copositive star. Again, the converse follows from the fact that Γ^{-1}, $\Gamma^T \in \text{Aut(K)} \subseteq \Pi(\text{K})$. ∎

We note that the above theorem holds if Γ is replaced by Λ because $\text{Aut}(V) \subseteq \text{Aut}(K)$.

5.3. Z transformations. We say that L has the Z-property on V if

$$x, y \in K, \quad \text{and } \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leq 0.$$

Recently, Gowda and Tao (see [10]) introduced and studied the properties of such transformations.

Theorem 5.10. *Suppose that L has the Z and copositive properties. Then L is monotone.*

Proof. Assume that L has the Z and copositive properties. Then L^T has the Z and copositive properties. So $A := L + L^T$ has the Z and copositive properties. Let λ be the minimum eigenvalue of A. Then a corresponding eigenvector u is in K (see Theorem 6, [17]). It follows that $0 \leq \langle A(u), u\rangle = \lambda ||u||^2$, so $\lambda \geq 0$. Thus A is monotone. Hence L is monotone. ∎

This theorem and Lemma 3.1 immediately yield the following result.

Corollary 5.11. *Suppose that L has the Z-property and L is pseudomonotone. Then L is monotone.*

Example 5.12. When $V = \mathcal{S}^n$, for a real $n \times n$ matrix A, the *Lyapunov transformation* is defined by

$$L_A(X) := AX + XA^T.$$

Then L_A has the Z-property (see [10]). It is easy to verify that $\langle L_A(c), c\rangle \geq 0$ for all primitive idempotents c in $\mathcal{S}^n_+$ if and only if A is positive semidefinite. Since $\langle L_A(c), c\rangle \geq 0$ for all primitive idempotents c in $\mathcal{S}^n_+$ if and only if L_A is monotone (see the proof of Theorem 7.1 in [7]), we have that L_A is pseudomonotone if and only if A is positive semidefinite by the above theorem.

5.4. Relaxation transformations. In this section, we apply the ideas of the previous sections to study the transformation $R_A : V \to V$ that arises from a matrix $A \in R^{r \times r}$.

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Suppose we are given a Jordan frame $\{e_1, \ldots, e_r\}$ in V and $A \in R^{r\times r}$. We define $R_A : V \to V$ as follows. For any $x \in V$, we write the Peirce decomposition

$$x = \sum_1^r x_i e_i + \sum_{i<j} x_{ij}.$$

Then

$$R_A(x) := \sum_1^r y_i e_i + \sum_{i<j} x_{ij},$$

where

$$[y_1 \ldots y_r]^T = A\left([x_1 \ldots x_r]^T\right).$$

Our objective in this section is to study some interconnections between the properties of A and the properties of R_A. Such a study has found to be quite interesting and useful in the context of matrix based linear transformations on $V = \mathcal{S}^n$ and $V = \mathcal{L}^n$. In particular, it will provide examples to study complementarity problems on V (see [18]). In what follows, we will study the relationship between a matrix A and a transformation R_A, when R_A is pseudomonotone.

Through this discussion, we denote $D := diag(||e_1||^2, \ldots, ||e_r||^2)$.

THEOREM 5.13. *A is pseudomonotone on R^r_+ if and only if $R_{D^{-1}A+I} - I'$ is pseudomonotone on K, where I is an identity matrix on R^r, and $I' : V \to V$ is an identity transformation on V.*

Proof. "Only if" part. Let $B := D^{-1}A + I$. Then $A = D(B - I)$. For all x, $y \geq 0$, let

$$x = \sum_1^r x_i e_i + \sum_{i<j} x_{ij} \text{ and } y = \sum_1^r y_i e_i + \sum_{i<j} y_{ij}.$$

Then $x_i \geq 0$ and $y_i \geq 0$ for all $i = 1, \ldots, r$ and

$$R_B(x) := \sum_1^r \bar{x}_i e_i + \sum_{i<j} x_{ij},$$

where

$$[\bar{x}_1 \ldots \bar{x}_r]^T = B\left([x_1 \ldots x_r]^T\right).$$

Then we have

$$\langle R_B(x), y - x\rangle = \sum_1^r \bar{x}_i (y_i - x_i)||e_i||^2 + \sum_{i<j} \langle x_{ij}, y_{ij} - x_{ij}\rangle.$$

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We note that

$$\langle x, y-x\rangle = \sum_1^r x_i(y_i - x_i)||e_i||^2 + \sum_{i<j}\langle x_{ij}, y_{ij} - x_{ij}\rangle.$$

We define $\hat{x} := [x_1 \dots x_r]^T$ and $\hat{y} := [y_1 \dots y_r]^T$. Then we have

$$\begin{aligned}\langle (R_B - I')(x), y-x\rangle &= \sum_1^r (\bar{x}_i - x_i)(y_i - x_i)||e_i||^2 \\ &= \langle (B-I)\hat{x}, D(\hat{y}-\hat{x})\rangle \\ &= \langle D(B-I)\hat{x}, (\hat{y}-\hat{x})\rangle \\ &= \langle A\hat{x}, \hat{y}-\hat{x}\rangle.\end{aligned}$$

Similarly, $\langle (R_B - I')(y), y-x\rangle = \langle A\hat{y}, \hat{y}-\hat{x}\rangle$. Assume that $\langle (R_B - I')(x), y-x\rangle \geq 0$, which means $\langle A\hat{x}, \hat{y}-\hat{x}\rangle \geq 0$. Since A is pseudomonotone, we have $\langle A\hat{y}, \hat{y}-\hat{x}\rangle \geq 0$, and hence $\langle (R_B - I')(y), y-x\rangle \geq 0$. Therefore, $R_B - I' = R_{D^{-1}A+I} - I'$ is pseudomonotone on K.

"If" part. Let $\hat{x} = [x_1 \dots x_r]^T$ and $\hat{y} = [y_1 \dots y_r]^T$ with $x_i \geq 0$ and $y_i \geq 0$ for all $1 \leq i \leq r$. Suppose $\langle A\hat{x}, \hat{y}-\hat{x}\rangle \geq 0$. Then we define $x = \sum_1^r x_i e_i$ and $y = \sum_1^r y_i e_i$. So x, $y \geq 0$. Then that A is pseudomonotone follows from the proof of "Only if" part. □

THEOREM 5.14. *If R_A is pseudomonotone on K, then DA is pseudomonotone on R^r_+.*

Proof. Let $x = [x_1 \dots x_r]^T$ and $y = [y_1 \dots y_r]^T$ with $x_i \geq 0$ and $y_i \geq 0$ for all $1 \leq i \leq r$. Suppose $\langle (DA)x, y-x\rangle \geq 0$. Define $\bar{x} = \sum_{i=1}^r x_i e_i$ and $\bar{y} = \sum_{i=1}^r y_i e_i$. Thus, $\bar{x} \geq 0$, $\bar{y} \geq 0$, and

$$R_A(\bar{x}) := \sum_1^r z_i e_i,$$

where

$$[z_1 \dots z_r]^T = A\left([x_1 \dots x_r]^T\right).$$

Therefore,

$$\langle R_A(\bar{x}), \bar{y}-\bar{x}\rangle = \sum_1^r z_i(y_i - x_i)||e_i||^2 = \langle Ax, D(y-x)\rangle = \langle (DA)x, y-x\rangle \geq 0.$$

Since R_A is pseudomonotone, we have $\langle R_A(\bar{y}), \bar{y}-\bar{x}\rangle \geq 0$. Since $\langle R_A(\bar{y}), \bar{y}-\bar{x})\rangle = \langle (DA)y, y-x\rangle$, we have $\langle (DA)y, y-x\rangle \geq 0$. Hence, DA is pseudomonotone. □

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The following example shows that the converse of the above theorem may not be true.

EXAMPLE 5.15. Consider $R_A : \mathcal{L}^3 \to \mathcal{L}^3$, where $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

Fix a Jordan frame $\{e_1, e_2\}$, where $e_1 = \frac{1}{2}[1\,1\,0]^T$ and $e_2 = \frac{1}{2}[1\,-1\,0]^T$. Then we have $||e_1||^2 = ||e_2||^2 = \frac{1}{2}$ and $D = \frac{1}{2}I$.

It is easy to verify that DA is pseudomonotone. Now, take $x = e_1 + 2e_2 + [0\ 0\ -1]^T$ and $y = 12e_1 + e_2 + [0\ 0\ -3]^T$. It is easy to verify that $x, y \geq 0$. Then $R_A(x) = 3e_2 + [0\ 0\ -1]^T$, $R_A(y) = 13e_2 + [0\ 0\ -3]^T$ and $y - x = 11e_1 - e_2 + [0\ 0\ -2]^T$. By simple calculation, we have $\langle R_A(x), y - x\rangle = \frac{1}{2} > 0$ and $\langle R_A(y), y - x\rangle = -\frac{1}{2} < 0$.

6. Concluding remarks. In this paper, we have extended the concept of pseudomonotonicity from R^n to the setting of Euclidean Jordan algebras. Some interconnections between pseudomonotonicity, monotonicity, and the Z-property were studied. In particular, we have given a characterization of pseudomonotonicity for a linear transformation and a matrix-induced transformation defined on a Euclidean Jordan algebra. We have shown that pseudomonotonicity and monotonicity coincide under the condition of the Z-property. In addition, we have proved the invariance of pseudomonotonicity under the algebra and cone automorphisms and described interconnections between pseudomonotonicity of a linear transformation and its principal subtransformations.

7. Appendix. In this section, we use Theorem 3.7 to prove claims made in Examples 3.23, 3.25, and 3.27 from Section 3.

EXAMPLE 7.1. Let

$$L = \begin{bmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix} : \mathcal{L}^3 \to \mathcal{L}^3$$

be such that $a \geq \sqrt{b^2 + c^2}$. Then L is pseudomonotone.

Proof. First, we show that L is PSBD. Let $x = [x_0\ x_1\ x_2]^T$. Then, from $\langle L(x), x\rangle < 0$, we have $x_0(ax_0 + bx_1 + cx_2) < 0$. Thus, we have either $x_0 > 0$ and $ax_0 + bx_1 + cx_2 < 0$ or $x_0 < 0$ and $ax_0 + bx_1 + cx_2 > 0$. Hence, we have either $L^T(x) = (ax_0 + bx_1 + cx_2)e > 0$ or $L^T(x) < 0$. Therefore, L is PSBD. Next, we need to show that L is copositive star. For $x \geq 0$, we have that $x_0 \geq \sqrt{{x_1}^2 + {x_2}^2}$. Using the condition of $a \geq \sqrt{b^2 + c^2}$, we have

$$\begin{aligned} ax_0 &\geq \sqrt{{x_1}^2 + {x_2}^2}\sqrt{b^2 + c^2} \\ &= \sqrt{b^2{x_1}^2 + c^2{x_2}^2 + c^2{x_1}^2 + b^2{x_2}^2} \end{aligned}$$

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$$\begin{aligned}
&\geq \sqrt{b^2{x_1}^2 + c^2{x_2}^2 + 2bcx_1x_2} \\
&= \sqrt{(bx_1 + cx_2)^2} = |bx_1 + cx_2| \\
&\Rightarrow ax_0 + bx_1 + cx_2 \geq 0.
\end{aligned}$$

Since $\langle L(x), x\rangle = x_0(ax_0 + bx_1 + cx_2)$, we have $\langle L(x), x\rangle \geq 0$ for $x \geq 0$. Thus, L is copositive. It can be easily verify that

$$[x \geq 0,\ L(x) \geq 0,\ \langle L(x), x\rangle = 0] \Rightarrow L^T(x) = 0.$$

This shows that L is copositive star. Therefore, by Theorem 3.7, L is pseudomonotone. □

Example 7.2. Let

$$L = \begin{bmatrix} 1 & c & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \mathcal{L}^3 \to \mathcal{L}^3$$

be such that

(a) $0 \leq b + c \leq 1$,
(b) $c > 0$,
(c) $b < 0$.

Then L is pseudomonotone.

Proof. Without loss of generality, we assume that $b+c > 0$. Let $x = [x_0\ \ x_1\ \ x_2]^T$. Then

$$\begin{aligned}
\langle L(x), x\rangle &= x_0[x_0 + (b + c)x_1], \\
L(x) &= [x_0 + cx_1\ \ bx_0\ \ 0]^T, \\
\text{and}\ \ L^T(x) &= [x_0 + bx_1\ \ cx_0\ \ 0]^T.
\end{aligned}$$

First, we show that L is PSBD. Suppose that $\langle L(x), x\rangle < 0$. Then either $x_0 > 0$ and $x_0 + (b + c)x_1 < 0$ or $x_0 < 0$ and $x_0 + (b + c)x_1 > 0$.

Case 1: $x_0 > 0$ and $x_0 + (b+c)x_1 < 0$. Then $x_0 + (b+c)x_1 < 0 \Rightarrow x_1 < -\dfrac{x_0}{b+c} < 0$ by the item (a). Thus, $x_0 + bx_1 > 0$ by the item (c). Since $x_1 < -\dfrac{x_0}{b+c}$, we have $x_0 + bx_1 > x_0 - \dfrac{bx_0}{b+c} = \dfrac{cx_0}{b+c} \geq cx_0 > 0$ by the item (b). Therefore, $L^T(x) > 0$. Hence, L is PSBD.

Case 2: $x_0 < 0$ and $x_0 + (b+c)x_1 > 0$. Then $x_0 + (b+c)x_1 > 0 \Rightarrow x_1 > -\dfrac{x_0}{b+c} > 0$ by the item (a). Thus, $x_0 + bx_1 < 0$ by the item (c). Now, $-(x_0 + bx_1) = -x_0 - bx_1 > -x_0 + \dfrac{bx_0}{b+c} = -\dfrac{cx_0}{b+c} \geq -cx_0 > 0$. Therefore, $L^T(x) < 0$. Hence, L is PSBD.

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Now, we need to show that L is copositive star. For $x \geq 0$, we have

$$x_0 \geq \sqrt{{x_1}^2 + {x_2}^2} \geq \sqrt{{x_1}^2} = |x_1| \geq (b+c)|x_1| \Rightarrow x_0 + (b+c)x_1 \geq 0.$$

Thus, $\langle L(x), x\rangle \geq 0$, which implies that L is copositive. Now, suppose that $x \geq 0$, $L(x) \geq 0$, and $\langle L(x), x\rangle = x_0[x_0 + (b+c)x_1] = 0$. Then we have that either $x_0 = 0$ or $x_0 + (b+c)x_1 = 0$. If $x_0 = 0$, then $x = 0$, and hence $L^T(x) = 0$. If $x_0 + (b+c)x_1 = 0$ and $x_0 > 0$, then $x_1 = -\dfrac{x_0}{b+c}$. Thus, $x_0 + cx_1 = \dfrac{bx_0}{b+c} < 0$, which contradicts $L(x) \geq 0$. This shows that L is copositive star. Therefore, L is pseudomonotone by Theorem 3.7. ◻

Example 7.3. Let

$$L = \begin{bmatrix} 1 & c & 0 \\ b & 0 & 0 \\ 0 & 0 & \epsilon \end{bmatrix} : \mathcal{L}^3 \to \mathcal{L}^3$$

be such that

(a) $0 < b + c < 1$,
(b) $0 < c < \sqrt{2}$,
(c) $b < 0$,
(d) $0 < \epsilon \leq 1$,
(e) $(3b + c)^2 - 4b^2(2 - c^2) < 0$.

Then L is pseudomonotone.

Proof. Let $x = [x_0 \;\; x_1 \;\; x_2]^T$. Then we have $\langle L(x), x\rangle = x_0[x_0 + (b+c)x_1] + \epsilon {x_2}^2$,

$$L(x) = [x_0 + cx_1 \;\; bx_0 \;\; \epsilon x_2]^T \;\text{ and }\; L^T(x) = [x_0 + bx_1 \;\; cx_0 \;\; \epsilon x_2]^T.$$

First, we show that L is PSBD. Suppose that $\langle L(x),\, x\rangle < 0$. Then $x_0[x_0+(b+c)x_1] < -\epsilon {x_2}^2 \leq 0$. This implies that either $x_0 > 0$ and $x_0 + (b+c)x_1 < 0$, or $x_0 < 0$ and $x_0 + (b+c)x_1 > 0$.

Case 1: $x_0 > 0$ and $x_0 + (b+c)x_1 < 0$. Then we have that $x_0 + bx_1 > 0$ by the proof in the previous example. Since $\epsilon^2 {x_2}^2 \leq \epsilon {x_2}^2 < -x_0[x_0 + (b+c)x_1]$, we have $\epsilon^2 {x_2}^2 + c^2 {x_0}^2 < -x_0[x_0+(b+c)x_1] + c^2{x_0}^2$. We show that $-x_0[x_0+(b+c)x_1]+c^2{x_0}^2 \leq (x_0 + bx_1)^2$. Now,

$$-x_0[x_0 + (b+c)x_1] + c^2{x_0}^2 \leq (x_0 + bx_1)^2 \Leftrightarrow (2 - c^2){x_0}^2 + (3b + c)x_0x_1 + b^2{x_1}^2 \geq 0.$$

The last inequality holds because of the items (b) and (e). Therefore, $\epsilon^2{x_2}^2 + c^2{x_0}^2 \leq (x_0 + bx_1)^2$. Thus, $L^T(x) > 0$, and hence L is PSBD.

Case 2: $x_0 < 0$ and $x_0 + (b+c)x_1 > 0$. Then we have that $x_0 + bx_1 < 0$ by the proof in the previous example. From Case 1, we know that $\epsilon^2{x_2}^2 + c^2{x_0}^2 \leq [-(x_0 + bx_1)]^2$. Therefore, $L^T(x) < 0$. Hence, L is PSBD.

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Now, we show that L is copositive star. We know that $x_0+(b+c)x_1 \geq 0$ for $x \geq 0$ by the proof in the previous example. Thus, $\langle L(x), x\rangle \geq 0$, which implies that L is copositive. It is easy to verify that $[x \geq 0,\ L(x) \geq 0,\ \langle L(x), x\rangle = 0] \Rightarrow L^T(x) = 0$ by the proof in the previous example. This shows that L is copositive star. Therefore, by Theorem 3.7, L is pseudomonotone. $\square$

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REFERENCES

[1] J.-P. Crouzeix, A. Hassouni, A. Lahlou, and S. Schaible. Positive subdefinite matrices, generalized monotonicity and linear complementarity problems. *SIAM J. Matrix Anal. Appl.*, 22:66–85, 2000.

[2] J.-P. Crouzeix and S. Schaible. Generalized monotone affine maps. *SIAM J. Matrix Anal. Appl.*, 17:992–997, 1996.

[3] J. Faraut and A. Korányi. *Analysis on Symmetric Cones.* Clarendon Press, Oxford, 1994.

[4] M.S. Gowda. Pseudomonotone and copositive star matrices. *Linear Algebra Appl.*, 113:107–118, 1989.

[5] M.S. Gowda. Affine pseudomonotone mappings and the linear complementarity problem. *SIAM J. Matrix Anal. Appl.*, 11:373–380, 1990.

[6] M.S. Gowda. On the transpose of a pseudomonotone matrix and the LCP. *Linear Algebra Appl.*, 140:129–137, 1990.

[7] M.S. Gowda and R. Sznajder. Some global uniqueness and solvability results for linear complementarity problems over symmetric cones. *SIAM J. Optim.*, 18:461–481, 2007.

[8] M.S. Gowda and R. Sznajder. Automorphism invariance of P and GUS-properties of linear transformations on Euclidean Jordan algebras. *Math. Oper. Res.*, 31:109–123, 2006.

[9] M.S. Gowda, R. Sznajder, and J. Tao. Some P-properties for linear transformations on Euclidean Jordan algebras. *Linear Algebra Appl.*, 393:203–232, 2004.

[10] M.S. Gowda and J. Tao. Z-transformations on proper and symmetric cones. *Math. Program.*, Series B, 117:195–221, 2009.

[11] N. Hadjisavvas and S. Schaible. Quasimonotonicity and pseudomonotonicity in variational inequalities and equilibrium problems. In J.-P. Crouzeix, J.-E. Martinez-Legaz, and M. Volle, editors, *Generalized Convexity, Generalized Monotonicity: Recent Results*, Nonconvex Optim. Appl., 27, Kluwer Acad. Publ., Dordrecht, 257–275, 1998.

[12] A. Hassouni, A. Lahlou, and A. Lamghri. Existence theorems for linear complementarity problems on solid closed convex cones. *J. Optim. Theory Appl.*, 126:225–246, 2005.

[13] S. Karamardian. Complementarity problems over cones with monotone and pseudomonotone maps. *J. Optim. Theory Appl.*, 18:445–454, 1976.

[14] S. Karamardian and S. Schaible. Seven kinds of monotone maps. *J. Optim. Theory Appl.*, 66:37–46, 1990.

[15] S. Karamardian, S. Schaible and J.-P Crouzeix. Characterizations of generalized monotone maps. *J. Optim. Theory Appl.*, 76:399–413, 1993.

[16] S.R. Mohan, S.K. Neogy and A.K. Das. More on positive subdefinite matrices and the linear complementarity problem. *Linear Algebra Appl.*, 338:275–285, 2001.

[17] H. Schneider and M. Vidyasagar. Cross-positive matrices. *SIAM J. Numer. Anal.*, 7:508–519, 1970.

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[18] J. Tao. Positive principal minor property of linear transformations on Euclidean Jordan algebras. *J. Optim. Theory Appl.*, 140:131–152, 2009.