Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

CONSTRUCTION OF A SOLUTION TO THE RANK 2 HORN PROBLEM*

SARAH GIFT† AND HUGO J. WOERDEMAN†

Abstract. Given three sets of n real eigenvalues satisfying the trace equality and the Horn inequalities, we know that there are $n \times n$ real symmetric matrices A and B so that A has the first set of eigenvalues, B has the second set of eigenvalues, and $A + B$ has the last set of eigenvalues. Under the condition that B is a rank 2 matrix, we give a construction for the matrices A and B. This construction is based on performing two orthogonal rank 1 updates on A. We end with a discussion of the relationship between this rank 2 Horn problem and the following similar problem: given a set of n real eigenvalues, a set of 2 real eigenvalues, and a set of $n+2$ real eigenvalues satisfying certain conditions, find an $(n+2) \times (n+2)$ real symmetric matrix such that the top left principal submatrix has the first set of eigenvalues, the bottom right principal submatrix has the second set of eigenvalues, and the full matrix has the last set of eigenvalues.

Key words. Horn Problem, Real Symmetric Matrices, Eigenvalue Inequalities, Principal Submatrices.

AMS subject classifications. 15A18, 15A29, 15A42, 15B57.

1. Introduction.

In 1962, Alfred Horn asked the question: given three n-tuples of real numbers, $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$, $\mu = (\mu_1 \geq \cdots \geq \mu_n)$, and $\nu = (\nu_1 \geq \cdots \geq \nu_n)$, when do there exist $n \times n$ Hermitian matrices A and B such that $\sigma(A) = \lambda$, $\sigma(B) = \mu$, and $\sigma(A + B) = \nu$? Several necessary conditions were already known, but they were only sufficient in low dimensions. In particular, aside from the trace equality

$$\sum_{\ell=1}^{n} \lambda_\ell + \sum_{\ell=1}^{n} \mu_\ell = \sum_{\ell=1}^{n} \nu_\ell,$$

the eigenvalues had to satisfy certain linear inequalities of the form

$$\nu_{k_1} + \cdots + \nu_{k_r} \leq \lambda_{i_1} + \cdots + \lambda_{i_r} + \mu_{j_1} + \cdots + \mu_{j_r}, \tag{1.1}$$

where $1 \leq r < n$. Alfred Horn conjectured a complete list of necessary and sufficient inequalities, all of which were of the form given in (1.1) [9]. His conjecture was proven true in the 1990s by Klyachko [11] and Knutson and Tao [12].

Now that we know when A and B exist, the question becomes how to find these matrices. One case that is known is when B is rank 1, see e.g. [16, 2]. Additionally, Cao and Woerdeman provided an algorithm using semidefinite programming for finding 3×3 Hermitian solutions A and B [3]. The algorithm was influenced by an earlier paper's construction of a determinantal representation of a given two variable real-zero polynomial [8]. In 2018, Franks presented an iterative numerical algorithm for finding A and B; however, the theoretical probability of failure may be up to 1/3 [5, Algorithm 1], see also [6, Algorithm 5].

In this paper, we will show how to find A and B in the case when B is rank 2. The key idea behind our algorithms is splitting B up as a sum of two rank 1 matrices. Thus, we state several results pertaining to

*Received by the editors on May 16, 2024. Accepted for publication on April 24, 2025. Handling Editor: João Filipe Queirò. Corresponding Author: Sarah Gift.

†Department of Mathematics, Drexel University, Philadelphia, Pennsylvania 19104, USA (sg3664@drexel.edu, hjw27@drexel.edu). Supported by National Science Foundation Grants DMS 2000037 and DMS 2348720.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

rank 1 updates of matrices in Section 2. Then in Section 3 we extend these prior results in preparation for building our algorithms. Equipped with the necessary background, in Section 4, we address the case when all eigenvalues of A (λ_i's) are distinct from each other and from all eigenvalues of $A+B$ (ν_k's). This leads us to Algorithm 1. We discuss the implementation of the algorithm and provide an example.

Now to address the case when some eigenvalues are repeated, we detour in Section 5 to discuss the idea of reducing the sets of eigenvalues. The results developed in this section help us simplify the case when there are repeated eigenvalues. Equipped with these useful results, we address the case of repeated λ_i's in Section 6. Then we separately address the case when some λ_i equals some ν_k in Section 7. Combining the results of Sections 6 and 7, we get Algorithm 2, a fully general algorithm for the rank 2 case, presented in the Appendix.

Finally, in the last two sections we talk about generalizing our algorithm to higher rank cases and discuss the connection between the rank 2 Horn problem and another problem, found in Li and Poon's 2003 paper [14]. The problem was, given eigenvalues

$$\lambda_1 \geq \cdots \geq \lambda_n, \quad \mu_1 \geq \cdots \geq \mu_m, \quad \nu_1 \geq \cdots \geq \nu_{n+m},$$

when does there exist an $(n+m)\times(n+m)$ real symmetric matrix D such that the eigenvalues of D are the ν_k's, the eigenvalues of the top left $n \times n$ block of D are the λ_i's, and the eigenvalues of the bottom right $m \times m$ block of D are the μ_j's?

2. Background.

We first develop some notation used throughout the paper, and then we present and discuss a few known results that are relevant to our algorithms.

2.1. Notation.

Throughout this paper, we will often refer to ordered tuples of real numbers. Thus we define the notation $\lambda^{(n)}$ for an n-tuple of non-increasing values

$$\lambda^{(n)} := \{\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}.$$

Similarly, we define the notation $\lambda_>^{(n)}$ for an n-tuple of strictly decreasing values

$$\lambda_>^{(n)} := \{\lambda_1 > \lambda_2 > \cdots > \lambda_n\}.$$

Finally, we define the notation $\lambda_<^{(n)}$ for an n-tuple of strictly increasing values

$$\lambda_<^{(n)} := \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}.$$

For a positive integer m, we notate the set of integers from 1 to m as

$$[m] := \{1, 2, \ldots, m\}.$$

For a finite set S, we let $|S|$ denote the number of elements of S. For a matrix A, we notate the multi-set of eigenvalues by $\sigma(A)$. For a vector v, we let v_i denote the ith element of v and let $\|v\|$ denote the Euclidean norm of v.

In this paper, anytime we write out a matrix and there is an entry missing, that entry is implied to be zero. If there is a star in the matrix, then any value can occupy that position (assuming it satisfies the conditions of the matrix, e.g. real or symmetric). For two matrices X and Y, the block diagonal matrix is notated as

$$X \oplus Y := \left[\begin{array}{c|c} X & \\ \hline & Y \end{array}\right].$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

2.2. Horn inequalities. Alfred Horn asked the question: given any $n \times n$ Hermitian matrices A and B, if we define the real n-tuples of values $\lambda^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ by

$$\sigma(A) = \lambda^{(n)}, \quad \sigma(B) = \mu^{(n)}, \quad \text{and} \quad \sigma(A+B) = \nu^{(n)},$$

then for what sets of indices must (1.1) hold? In particular, if we define

$$T_{r,n} := \left\{ \left(i_<^{(r)}, j_<^{(r)}, k_<^{(r)}\right) \in ([n]^r)^3 : \text{ (1.1) must hold } \right\},$$

then we ask what elements are in $T_{r,n}$? We now know, thanks to Klyachko, Knutson, and Tao [12, 11], that

$$T_{1,n} = \left\{ \left(i_1, j_1, k_1\right) \in [n]^3 : i_1 + j_1 = k_1 + 1 \right\},$$

and for $r \geq 2$,

$$\left(i_<^{(r)}, j_<^{(r)}, k_<^{(r)}\right) \in T_{r,n} \iff \begin{cases} \sum_{\ell=1}^r i_\ell + \sum_{\ell=1}^r j_\ell = \sum_{\ell=1}^r k_\ell + \frac{r(r+1)}{2}, \\ \text{and} \\ \sum_{\ell=1}^s i_{a_\ell} + \sum_{\ell=1}^s j_{b_\ell} \leq \sum_{\ell=1}^s k_{c_\ell} + \frac{s(s+1)}{2}, \quad \forall s \in [r-1] \text{ and } \left(a_<^{(s)}, b_<^{(s)}, c_<^{(s)}\right) \in T_{s,r} \end{cases}.$$

Even better, Klyachko, Knutson, and Tao [12, 11], proved the following result.

Theorem 2.1. *Suppose the real n-tuples of values $\lambda^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ satisfy the trace equality, and for all*

$$\left(i_<^{(r)}, j_<^{(r)}, k_<^{(r)}\right) \in T_{r,n},$$

(1.1) holds. Then there exist real symmetric matrices A and B such that

$$\sigma(A) = \lambda^{(n)}, \quad \sigma(B) = \mu^{(n)}, \quad \text{and} \quad \sigma(A+B) = \nu^{(n)}.$$

Notice that we know there are not only Hermitian matrices A and B with the specified eigenvalues but also real symmetric matrices. Now while Horn did not prove the above results, he did conjecture them. Thus, we say that $\lambda^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ satisfy the Horn inequalities if for all $\left(i_<^{(r)}, j_<^{(r)}, k_<^{(r)}\right) \in T_{r,n}$, (1.1) holds. We will call the pair of real symmetric matrices (A, B) guaranteed by Theorem 2.1 a real Horn solution pair. While we have a condition for when the real symmetric matrices A and B exist, the question remains how to find these matrices.

2.3. Rank one updates. The goal of this paper is to provide a way of finding a solution to the rank 2 Horn problem, i.e. given eigenvalues $\lambda^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ with all but two μ_j's equal to zero, we want to find $n \times n$ matrices A and B such that

$$\sigma(A) = \lambda^{(n)}, \quad \sigma(B) = \mu^{(n)}, \quad \text{and} \quad \sigma(A+B) = \nu^{(n)}.$$

We will break this problem down into constructing two orthogonal rank 1 updates. In particular, if B is a real symmetric rank 2 matrix, then for some real orthonormal column vectors v and w, $B = \mu_\gamma vv^T + \mu_\delta ww^T$. Note that the pair of subscripts (γ, δ) can be $(1,2)$, $(1,n)$, or $(n-1,n)$, depending on the signs of the nonzero eigenvalues of B. Now we can always take A to be diagonal, so the problem becomes finding orthonormal vectors v and w such that

$$\sigma\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} + \mu_\gamma vv^T + \mu_\delta ww^T\right) = \nu^{(n)}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

Now the rank 1 Horn problem has already been solved, see e.g. [10, Theorem 4.3.26] or [16]. Thus, if we knew the eigenvalues of $A+\mu_\gamma vv^T$, we could find v. Then implementing the algorithm again with $\tilde{A} = A+\mu_\gamma vv^T$ and $\tilde{B} = \mu_\delta ww^T$, we could find w. Therefore, the problem boils down to finding the eigenvalues of $\tilde{A}$, which we will denote by $\rho^{(n)}$, such that v and w are orthogonal. To figure out the conditions on $\rho^{(n)}$ that ensure this, we need a few results from the rank 1 Horn problem.

For this and throughout this paper, we will let $\lambda^{(n)}$ be the eigenvalues of A, $\rho^{(n)}$ be the eigenvalues of the rank 1 update of A, i.e., $\tilde{A}$, and $\nu^{(n)}$ be the eigenvalues of the rank 2 update of A, i.e., $A+B$. Then we define polynomials f, g, and h as follows.

$$f(x) := \prod_{\ell=1}^{n}(x-\lambda_\ell), \qquad g(x) := \prod_{\ell=1}^{n}(x-\rho_\ell), \qquad h(x) := \prod_{\ell=1}^{n}(x-\nu_\ell). \tag{2.2}$$

Note that f, g, and h are the characteristic polynomials of A, $\tilde{A}$, and $A+B$, respectively.

Now recall the Cauchy interlacing result on the possible eigenvalues of a rank 1 update (see, e.g. [7, 19]).

Theorem 2.2. *Let $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ be a real matrix and let z be a real unit vector. Assume for some $\mu \in \mathbb{R}$, $\sigma(\Lambda + \mu zz^T) = \rho^{(n)}$.*

(a) If $\mu \geq 0$

$$\begin{cases} \lambda_1 + \mu \geq \rho_1 \geq \lambda_1, & \\ \lambda_i \geq \rho_{i+1} \geq \lambda_{i+1}, & i \in [n-1] \end{cases}.$$

(b) If $\mu \leq 0$

$$\begin{cases} \lambda_i \geq \rho_i \geq \lambda_{i+1}, & i \in [n-1] \\ \lambda_n \geq \rho_n \geq \lambda_n + \mu & \end{cases}.$$

Note that in the rank 1 case, aside from the trace equality, the Horn inequalities on $\lambda^{(n)}$, $\mu^{(n)}$, and $\rho^{(n)}$, reduce to such interlacing conditions. In particular,

$$\begin{cases} \rho_1 \geq \lambda_1 \geq \rho_2 \geq \lambda_2 \geq \cdots \geq \rho_n \geq \lambda_n, & \text{if } \mu_1 > 0 = \mu_2 = \cdots = \mu_n \\ \lambda_1 \geq \rho_1 \geq \lambda_2 \geq \rho_2 \geq \cdots \geq \lambda_n \geq \rho_n, & \text{if } \mu_1 = \cdots = \mu_{n-1} = 0 > \mu_n \end{cases}. \tag{2.3}$$

Assuming we are in the first case (i.e. $\mu_1 > 0$), we have the following result, which is a combination of Theorems 4.3.26 and 4.3.21 in [10].

Theorem 2.3. *Assume*

$$\rho_1 \geq \lambda_1 \geq \rho_2 \geq \lambda_2 \geq \cdots \geq \rho_n \geq \lambda_n.$$

Let $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and $\mu = \sum_{\ell=1}^{n} \rho_\ell - \sum_{\ell=1}^{n} \lambda_\ell$. Define polynomials f and g as in (2.2). If all the λ_ℓ's are distinct, then for $z \in \mathbb{R}^n$, $\sigma\left(\Lambda + \mu zz^T\right) = \rho^{(n)}$ if and only if for all $i \in [n]$,

$$z_i^2 = \frac{-g(\lambda_i)}{\mu f'(\lambda_i)}. \tag{2.4}$$

In general, define $0 = m_0 < m_1 < \cdots < m_k = n$ such that

$$\lambda_1 = \cdots = \lambda_{m_1} > \lambda_{m_1+1} = \cdots = \lambda_{m_2} > \cdots > \lambda_{m_{k-1}+1} = \cdots = \lambda_{m_k}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

Then $\sigma\left(\Lambda + \mu z z^T\right) = \rho^{(n)}$ *if and only if* z *satisfies*

$$\sum_{j=m_{i-1}+1}^{m_i} z_j^2 = \frac{-\prod_{\ell=0}^{k-1}(\lambda_{m_i} - \rho_{m_\ell+1})}{\mu \prod_{\ell\neq i}^{k}(\lambda_{m_i} - \lambda_{m_\ell})}. \tag{2.5}$$

Here, (2.5) comes from (2.4) by removing the repeated eigenvalues in the products. In particular, the interlacing condition guarantees $\lambda_{m_\ell+1} = \rho_{m_\ell+2} = \cdots = \rho_{m_{\ell+1}} = \lambda_{m_{\ell+1}}$, so if we remove $\lambda_{m_\ell+1}$ to $\lambda_{m_{\ell+1}-1}$, we can also remove $\rho_{m_\ell+2}$ to $\rho_{m_{\ell+1}}$.

Note that in both (2.4) and (2.5), the right-hand side is guaranteed to be nonnegative by the interlacing condition and thus z_i is indeed real. Using the definitions of f and g given in (2.2), we see

$$z_i^2 = \frac{-\prod_{\ell=1}^{n}(\lambda_i - \rho_\ell)}{\mu \prod_{\ell\neq i}^{n}(\lambda_i - \lambda_\ell)}.$$

The expression on the right-hand side appeared in a result of Fan and Pall's on Hermitian imbedding conditions [4, Theorem 1] and also in Mirsky's paper on conditions for matrices with prescribed eigenvalues and diagonal elements [15, Lemma 2]. In these papers, (2.4) appeared as a condition on bordering a diagonal matrix to get a larger matrix with prescribed eigenvalues. The relationship between this problem and the Horn problem was explicitly shown in the general case in Li and Poon's paper [14, Theorem 2.2], but the ideas date back further, see e.g. [17, Theorem 2] and [18, page 120]. The relationship in the rank 1 case is shown in Theorems 4.3.26 and 4.3.21 in [10]. In particular, take $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \geq 0$ and $\mu > 0$ (note we can always shift the eigenvalues to ensure these non-negativity conditions). For the Horn problem, we want an $n \times n$ matrix B such that $\sigma(B) = \{\mu, 0, \ldots, 0\}$ and $\sigma(\Lambda + B) = \rho^{(n)}$. Note that for some real unit vector z, $B = \mu z z^T$. On the other hand, for the bordering problem, we want a real vector y such that

$$\sigma\left(\left[\begin{array}{c|c} \Lambda & y \\ \hline y^T & \mu \end{array}\right]\right) = \{\rho^{(n)}, 0\}.$$

Then it turns out there is a relationship between the vectors z and y. In particular,

$$y = \sqrt{\mu}\Lambda^{1/2} z.$$

We see this as follows.

$$\begin{aligned}
\rho^{(n)} &= \sigma\left(\Lambda + \mu z z^T\right) \\
&= \sigma\left(\begin{bmatrix}\Lambda^{1/2} & \sqrt{\mu} z\end{bmatrix}\begin{bmatrix}\Lambda^{1/2} \\ \sqrt{\mu} z^T\end{bmatrix}\right) \\
&= \sigma\left(\begin{bmatrix}\Lambda^{1/2} \\ \sqrt{\mu} z^T\end{bmatrix}\begin{bmatrix}\Lambda^{1/2} & \sqrt{\mu} z\end{bmatrix}\right) \setminus \{0\} \\
&= \sigma\left(\begin{bmatrix}\Lambda & \sqrt{\mu}\Lambda^{1/2} z \\ \sqrt{\mu} z^T \Lambda^{1/2} & \mu\end{bmatrix}\right) \setminus \{0\}.
\end{aligned}$$

Thus we obtain $y = \sqrt{\mu}\Lambda^{1/2} z$, as desired.

Now Corollary 3.2 assumes we are updating a diagonal matrix. In our rank 2 Horn problem, we want to perform two rank 1 updates. Thus, in order to use Theorem 2.3 a second time, we will have to diagonalize

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

after the first rank 1 update. For this, it will be useful to have the following result about the eigenvectors after a rank 1 update.

THEOREM 2.4. *[1, Equation 5.1] Let $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ be a real matrix and let z be a real unit vector. Assume for some $\mu \in \mathbb{R}$, $\sigma(\Lambda + \mu zz^T) = \rho^{(n)}$. If all λ_i's and ρ_j's are distinct, then the eigenvector associated with eigenvalue ρ_i is $(\Lambda - \rho_i I_n)^{-1} z$.*

3. Extending the background. In preparation for developing the rank 2 Horn algorithm, we now aim to rephrase and extend some of the background presented in Section 2.

First, we wish to rephrase Theorem 2.3 to consider the second interlacing condition given in (2.3) (i.e. when $\mu_n < 0$). Note that in the rank 1 Horn problem, we did not need to consider such a situation since all eigenvalues could simply be negated. However, in our current case of the rank 2 Horn problem, we may have one positive and one negative eigenvalue, in which case negating all eigenvalues would not be helpful. Thus we obtain the following corollary to address the negative eigenvalue situation.

COROLLARY 3.1. *Assume*

$$\lambda_1 \geq \rho_1 \geq \lambda_2 \geq \rho_2 \geq \cdots \geq \lambda_n \geq \rho_n.$$

Let $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and $\mu = \sum_{\ell=1}^n \rho_\ell - \sum_{\ell=1}^n \lambda_\ell$. Define the polynomials f and g as in (2.2). If all the λ_ℓ's are distinct, then for $z \in \mathbb{R}^n$, $\sigma\left(\Lambda + \mu zz^T\right) = \rho^{(n)}$ if and only if for all $i \in [n]$,

$$z_i^2 = \frac{-g(\lambda_i)}{\mu f'(\lambda_i)}. \tag{3.6}$$

In general, define $0 = m_0 < m_1 < \cdots < m_k = n$ such that

$$\lambda_1 = \cdots = \lambda_{m_1} > \lambda_{m_1+1} = \cdots = \lambda_{m_2} > \cdots > \lambda_{m_{k-1}+1} = \cdots = \lambda_{m_k}.$$

Then $\sigma\left(\Lambda + \mu zz^T\right) = \rho^{(n)}$ if and only if z satisfies

$$\sum_{j=m_{i-1}+1}^{m_i} z_j^2 = \frac{-\prod_{\ell=1}^k (\lambda_{m_i} - \rho_{m_\ell})}{\mu \prod_{\ell \neq i}^k (\lambda_{m_i} - \lambda_{m_\ell})}. \tag{3.7}$$

Proof. We know $\sigma(-\Lambda - \mu zz^T) = -\rho^{(n)}$. Then

$$-\rho_n \geq -\lambda_n \geq -\rho_{n-1} \geq \cdots \geq -\rho_1 \geq -\lambda_1.$$

Also, the trace equality gives $-\mu = \sum_{\ell=1}^n -\rho_\ell - \sum_{\ell=1}^n -\lambda_\ell$. Then, using the definitions of g and f, and replacing λ_ℓ's, ρ_ℓ's and μ by their negations, Theorem 2.3 says

$$z_i^2 = \frac{-\prod_{\ell=1}^n (-\lambda_i - (-\rho_\ell))}{-\mu \prod_{\ell \neq i} (-\lambda_i - (-\lambda_\ell))} = \frac{(-1)^n g(\lambda_i)}{\mu (-1)^{n-1} f'(\lambda_i)}.$$

Thus we have shown (3.6). Now (3.7) is shown similarly and is analogous to (2.5), but this time the interlacing condition guarantees $\lambda_{m_\ell+1} = \rho_{m_\ell+1} = \cdots = \rho_{m_{\ell+1}-1} = \lambda_{m_{\ell+1}}$, so if we remove $\lambda_{m_\ell+1}$ to $\lambda_{m_{\ell+1}-1}$, we can also remove $\rho_{m_\ell+1}$ to $\rho_{m_{\ell+1}-1}$. ∎

We then easily get the following by combining this result with Theorem 2.3 in the case when all λ_ℓ's are distinct.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

Corollary 3.2. *Given real values* $\lambda^{(n)}_>$*, consider the* $n \times n$ *matrix* $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$*. If* $\sigma(\Lambda + \mu z z^T) = \rho^{(n)}$ *for some real number* $\mu \in \mathbb{R}$ *and real unit vector* $z \in \mathbb{R}^n$*, then for* f *and* g *as defined in (2.2),*

$$z_i^2 = \frac{-g(\lambda_i)}{\mu f'(\lambda_i)}.$$

For the next result, we use Theorem 2.4 to figure out the form of the unitary that can diagonalize a rank 1 updated matrix. This will be useful in the rank 2 case because after performing a rank 1 update, we want to diagonalize our result before performing the second rank 1 update.

Lemma 3.3. *Given* $\lambda^{(n)}_>$*,* $\rho^{(n)}_>$*, and* $\mu \in \mathbb{R}$*, let* $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ *and define the polynomials* f *and* g *as in (2.2). Assume for some real unit vector* z*,* $\sigma(\Lambda + \mu z z^T) = \rho^{(n)}$*. Let* U_ρ *be an* $n \times n$ *real orthogonal matrix such that*

$$U_\rho^T \left(\Lambda + \mu z z^T\right) U_\rho = \operatorname{diag}\left(\rho_1, \rho_2, \ldots, \rho_n\right).$$

Then for some signs $s_1, s_2, \ldots, s_n \in \{-1, 1\}$*, the* (i,j)*th entry of* U_ρ *is given by*

$$U_\rho(i,j) = \begin{cases} s_j, & \rho_j = \lambda_i \\ 0, & \rho_j = \lambda_k,\ k \neq i \\ s_j \frac{z_i}{(\lambda_i - \rho_j)} \cdot \sqrt{\frac{\mu f(\rho_j)}{g'(\rho_j)}}, & \rho_j \neq \lambda_\ell\ \forall \ell \in [n] \end{cases}.$$

Proof. Fix $j \in [n]$. Since $\Lambda + \mu z z^T$ is real symmetric and the ρ_ℓ's are all distinct, then the jth column of U_ρ is just a normalized eigenvector of $\Lambda + \mu z z^T$ associated with ρ_j. If $\rho_j = \lambda_\ell$ for some $\ell \in [n]$, that eigenvector is plus or minus the ℓth standard basis vector e_ℓ. Otherwise, Theorem 2.4 tells us that the normalized eigenvector for ρ_j is

$$\frac{\pm(\Lambda - \rho_j I_n)^{-1} z}{\|(\Lambda - \rho_j I_n)^{-1} z\|}.$$

It remains to show then

$$\|(\Lambda - \rho_j I_n)^{-1} z\| = \sqrt{\frac{g'(\rho_j)}{\mu f(\rho_j)}}.$$

By Corollary 3.2, we have

$$z_i^2 = \frac{-g(\lambda_i)}{\mu f'(\lambda_i)} \qquad \Longrightarrow \qquad \|(\Lambda - \rho_j I_n)^{-1} z\| = \sqrt{\sum_{i=1}^n \frac{-g(\lambda_i)}{(\lambda_i - \rho_j)^2 \mu f'(\lambda_i)}}.$$

Therefore, it suffices to show

$$\sum_{i=1}^n \frac{-g(\lambda_i)}{(\lambda_i - \rho_j)^2 f'(\lambda_i)} = \frac{g'(\rho_j)}{f(\rho_j)}. \tag{3.8}$$

For this, define a_i to be the coefficients of the polynomial $g_j(x) := \frac{g(x)}{x - \rho_j} = \prod_{\ell \neq j}(x - \rho_\ell)$, i.e.

$$g_j(x) = \prod_{\ell \neq j}(x - \rho_\ell) = \sum_{i=0}^{n-1} a_i x^i.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

Note that $g_j(\rho_j) = g'(\rho_j)$. Consider the $n \times n$ matrices

$$V = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} \rho_j & & & & a_0 \\ -1 & \rho_j & & & a_1 \\ & \ddots & \ddots & & \vdots \\ & & -1 & \rho_j & a_{n-2} \\ & & & -1 & a_{n-1} \end{bmatrix}.$$

Then as the determinant of a Vandermonde matrix, we have

$$\det(V) = \prod_{1 \le \ell < k \le n} (\lambda_k - \lambda_\ell).$$

Next, note that $a_{n-1} = 1$. Using elementary row operations to eliminate ρ_j in the second to last row, then expanding along the second to last column of C, we have

$$\det(C) = \det\left(\begin{bmatrix} \rho_j & & & & a_0 \\ -1 & \rho_j & & & a_1 \\ & \ddots & \ddots & & \vdots \\ & & -1 & 0 & \rho_j + a_{n-2} \\ & & & -1 & 1 \end{bmatrix}\right) = \det(\rho_j I_{n+1} - C_{g_j}),$$

where C_{g_j} is the companion matrix of the polynomial g_j. Thus, $\det(C) = g_j(\rho_j) = g'(\rho_j)$, so

$$\det(VC) = \left[\prod_{1 \le \ell < k \le n} (\lambda_k - \lambda_\ell)\right] g'(\rho_j). \tag{3.9}$$

Now computing the same determinant by first multiplying and then using cofactor expansion, we get

$$\det(VC) = \det\left(\begin{bmatrix} 1(\rho_j - \lambda_1) & \lambda_1(\rho_j - \lambda_1) & \dots & \lambda_1^{n-2}(\rho_j - \lambda_1) & g_j(\lambda_1) \\ 1(\rho_j - \lambda_2) & \lambda_2(\rho_j - \lambda_2) & \dots & \lambda_2^{n-2}(\rho_j - \lambda_2) & g_j(\lambda_2) \\ \vdots & \vdots & & \vdots & \vdots \\ 1(\rho_j - \lambda_n) & \lambda_n(\rho_j - \lambda_n) & \dots & \lambda_n^{n-2}(\rho_j - \lambda_n) & g_j(\lambda_n) \end{bmatrix}\right)$$

$$= \sum_{i=1}^{n} (-1)^{n+i}\, g_j(\lambda_i) \det\left(\begin{bmatrix} 1(\rho_j - \lambda_1) & \lambda_1(\rho_j - \lambda_1) & \dots & \lambda_1^{n-2}(\rho_j - \lambda_1) \\ \vdots & \vdots & & \vdots \\ 1(\rho_j - \lambda_{i-1}) & \lambda_{i-1}(\rho_j - \lambda_{i-1}) & \dots & \lambda_{i-1}^{n-2}(\rho_j - \lambda_{i-1}) \\ 1(\rho_j - \lambda_{i+1}) & \lambda_{i+1}(\rho_j - \lambda_{i+1}) & \dots & \lambda_{i+1}^{n-2}(\rho_j - \lambda_{i+1}) \\ \vdots & \vdots & & \vdots \\ 1(\rho_j - \lambda_n) & \lambda_n(\rho_j - \lambda_n) & \dots & \lambda_n^{n-2}(\rho_j - \lambda_n) \end{bmatrix}\right)$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

$$= \sum_{i=1}^{n} (-1)^{n+i}\ g_j(\lambda_i) \left[\prod_{\ell\neq i}(\rho_j - \lambda_\ell)\right] \det\left(\begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_{i-1} & \dots & \lambda_{i-1}^{n-2} \\ 1 & \lambda_{i+1} & \dots & \lambda_{i+1}^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^{n-2} \end{bmatrix}\right)$$

$$= \sum_{i=1}^{n} (-1)^{n+i}\ g_j(\lambda_i) \left[\prod_{\ell\neq i}(\rho_j - \lambda_\ell)\right] \left[\prod_{\substack{1\leq \ell<k\leq n;\\ \ell,k\neq i}} (\lambda_k - \lambda_\ell)\right].$$

Equating this result with (3.9) and using the definitions of g_j, g, and f, we get

$$g'(\rho_j) = \sum_{i=1}^{n} (-1)^{n+i}\ g_j(\lambda_i)\ \frac{\left[\prod_{\ell\neq i}(\rho_j - \lambda_\ell)\right]}{\prod_{\substack{1\leq \ell<k\leq n;\\ \ell=i \text{ or } k=i}}(\lambda_k - \lambda_\ell)}$$

$$g'(\rho_j) = \sum_{i=1}^{n} (-1)^{n+i}\ \frac{g(\lambda_i)}{\lambda_i - \rho_j} \cdot \frac{f(\rho_j)}{(\rho_j - \lambda_i)\left[\prod_{1\leq \ell<i}(\lambda_i - \lambda_\ell)\right]\left[\prod_{i<\ell\leq n}(\lambda_\ell - \lambda_i)\right]}$$

$$\frac{g'(\rho_j)}{f(\rho_j)} = \sum_{i=1}^{n} (-1)^{2n} \frac{-g(\lambda_i)}{(\lambda_i - \rho_j)^2\left[\prod_{1\leq \ell<i}(\lambda_i - \lambda_\ell)\right]\left[\prod_{i<\ell\leq n}(\lambda_i - \lambda_\ell)\right]}$$

$$= \sum_{i=1}^{n} \frac{-g(\lambda_i)}{(\lambda_i - \rho_j)^2 f'(\lambda_i)}.$$

This proves (3.8), as desired. ∎

We end this section with one final lemma we will use later. This lemma will be proven similarly to (3.8).

LEMMA 3.4. *Given $\lambda_>^{(n)}$ and $\rho^{(n)}$, define polynomials f and g as in (2.2). Then for $j \in [n]$ such that $\rho_j \neq \lambda_\ell$ for all $\ell \in [n]$,*

$$\sum_{i=1}^{n} \frac{g(\lambda_i)}{(\lambda_i - \rho_j) f'(\lambda_i)} = 1.$$

Proof. Define a_i to be the coefficients of the polynomial $g_j(x) := \frac{g(x)}{x-\rho_j} = \prod_{\ell\neq j}(x - \rho_\ell)$, i.e.

$$g_j(x) = \prod_{\ell\neq j}(x - \rho_\ell) = \sum_{i=0}^{n-1} a_i x^i.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

Note that $g_j(\rho_j) = g'(\rho_j)$. Consider the $n \times n$ matrices

$$V = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 1 & & & & a_0 \\ & 1 & & & a_1 \\ & & \ddots & & \vdots \\ & & & 1 & a_{n-2} \\ & & & & a_{n-1} \end{bmatrix}.$$

Noting that $a_{n-1} = 1$, we have

$$\det(VK) = \det(V)\det(K) = \left(\prod_{1\le \ell < k \le n} (\lambda_k - \lambda_\ell)\right) \cdot 1 = \prod_{1\le \ell < k \le n} (\lambda_k - \lambda_\ell).$$

Thus

$$\prod_{1\le \ell < k \le n} (\lambda_k - \lambda_\ell) = \det(VK) = \det\left(\begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-2} & g_j(\lambda_1) \\ 1 & \lambda_2 & \dots & \lambda_2^{n-2} & g_j(\lambda_2) \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_n & \dots & \lambda_n^{n-2} & g_j(\lambda_n) \end{bmatrix}\right) = \sum_{i=1}^{n} (-1)^{n+i}\, g_j(\lambda_i) \left[\prod_{\substack{1\le \ell < k \le n; \\ \ell, k \ne i}} (\lambda_k - \lambda_\ell)\right],$$

by performing cofactor expansion along the last column. Rearranging and simplifying as in Lemma 3.3, we get

$$1 = \sum_{i=1}^{n} (-1)^{n+i} \frac{g_j(\lambda_i)}{\prod_{\substack{1\le \ell < k \le n; \\ \ell = i \text{ or } k = i}} (\lambda_k - \lambda_\ell)} = \sum_{i=1}^{n} \frac{g(\lambda_i)}{(\lambda_i - \rho_j) f'(\lambda_i)}. \qquad \square$$

Equipped now with many useful, albeit technical results, we are ready to start developing an algorithm for the rank 2 Horn problem.

4. Rank 2 Horn—Distinct eigenvalues case. We now assume that eigenvalues $\lambda^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ satisfying the trace equality and Horn inequalities are given such that all but two μ_j's are zero. Our goal is to find $n \times n$ real symmetric matrices A and B such that

$$\sigma(A) = \lambda^{(n)}, \quad \sigma(B) = \mu^{(n)}, \quad \text{and} \quad \sigma(A + B) = \nu^{(n)}.$$

As previously discussed, we can always take $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and since B is rank 2, we have $B = \mu_\gamma vv^T + \mu_\delta ww^T$, where μ_γ and μ_δ are the nonzero eigenvalues of B and v and w are orthonormal vectors. Our goal, therefore, is to find these vectors v and w. Now we claim that it suffices to find the eigenvalues of $A + \mu_\gamma vv^T$, which we denote by $\rho^{(n)}$. Indeed, if we knew $\rho^{(n)}$, we could find v because we know how to solve the rank 1 Horn problem. Then we could take a real orthogonal U such that $U^T(A + \mu_\gamma vv^T)U = \operatorname{diag}(\rho_1, \dots, \rho_n)$ and solve the new rank 1 Horn problem. In particular, we want z such that $\sigma\left(\operatorname{diag}(\rho_1, \dots, \rho_n) + \mu_\delta zz^T\right) = \nu^{(n)}$. Then $w = Uz$. Therefore, the problem boils down to finding the eigenvalues $\rho^{(n)}$ such that these two rank 1 Horn problems are possible and v and w end up orthogonal.

In this section, we will discuss the case when all the λ_i's are distinct from each other and from all ν_k's. Algorithm 1 solves the Horn problem in this case. We will deal with the case of repeated values of λ_i's in Section 6. Finally, in Section 7, we consider the case when $\lambda_i = \nu_k$ for some $i, k \in [n]$. Algorithm 2 solves the rank 2 Horn problem in general.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

4.1. Algorithm. We begin by considering the case when $\lambda_i > \lambda_{i+1}$ for all $i \in [n-1]$ and $\lambda_i \neq \nu_k$ for all $i, k \in [n]$. Because we can simply negate all eigenvalues, we will assume without loss of generality that $\mu_1 > 0$. Thus we either have

$$\mu_1 \geq \mu_2 > 0 = \mu_3 = \cdots = \mu_n \qquad \text{or} \qquad \mu_1 > 0 = \mu_2 = \cdots = \mu_{n-1} > \mu_n.$$

In this case, we find the following conditions on our desired intermediate eigenvalues $\rho^{(n)}$.

THEOREM 4.1. *Given real tuples $\lambda_>^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ satisfying the trace equality, assume $\mu_1 > 0$, only one other μ_j is nonzero, and $\lambda_i \neq \nu_k$ for all $i, k \in [n]$. Then $\lambda^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ satisfy the Horn inequalities if and only if there exists $\rho_>^{(n)}$ satisfying all of the following.*

$$\text{(4.10)} \qquad \textit{(Trace equality)} \sum_{i=1}^{n} \rho_i = \sum_{i=1}^{n} \lambda_i + \mu_1.$$

$$\text{(4.11)} \qquad \textit{(Interlacing Property 1)}\ \lambda_1 + \mu_1 \geq \rho_1 \geq \lambda_1 \geq \rho_2 \geq \lambda_2 \geq \cdots \geq \rho_n \geq \lambda_n.$$

$$\text{(4.12)} \qquad \textit{(Interlacing Property 2)} \begin{cases} \rho_1 + \mu_2 \geq \nu_1 \geq \rho_1 \geq \nu_2 \geq \rho_2 \geq \cdots \geq \nu_n \geq \rho_n, & \textit{if } \mu_2 > 0 \\ \rho_1 \geq \nu_1 \geq \rho_2 \geq \nu_2 \geq \cdots \geq \nu_n \geq \rho_n + \mu_n, & \textit{if } \mu_n < 0 \end{cases}.$$

$$\text{(4.13)} \qquad \textit{(Orthogonality Property)} \sum_{j=1}^{n} r_j \frac{\sqrt{|h(\rho_j) f(\rho_j)|}}{|g'(\rho_j)|} = 0 \textit{ for some } r_1, \ldots, r_n \in \{-1, 1\}.$$

Proof. First assume $\lambda^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ satisfy the trace equality and Horn inequalities. Define

$$\mu_\delta := \begin{cases} \mu_2, & \text{if } \mu_2 > 0 \\ \mu_n, & \text{if } \mu_n < 0 \end{cases}.$$

Then for $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, there exist real orthonormal vectors v and w such that

$$\text{(4.14)} \qquad \sigma\left(\Lambda + \mu_1 vv^T + \mu_\delta ww^T\right) = \nu^{(n)}.$$

Define $\rho^{(n)}$ by

$$\rho^{(n)} := \sigma\left(\Lambda + \mu_1 vv^T\right).$$

We immediately have the trace equality, (4.10), and Theorem 2.2 gives us the first interlacing property, (4.11). Now let U_ρ be a real orthogonal matrix such that

$$U_\rho^T \left(\Lambda + \mu_1 vv^T\right) U_\rho = \operatorname{diag}(\rho_1, \ldots, \rho_n).$$

Then (4.14) gives

$$\text{(4.15)} \qquad \sigma\left(\operatorname{diag}(\rho_1, \ldots, \rho_n) + \mu_\delta U_\rho^T ww^T U_\rho\right) = \nu^{(n)}.$$

Thus applying Theorem 2.2 again, we get the second interlacing property, (4.12). Finally, we know that v and w are orthogonal. We want to know what this tells us about the ρ_ℓ's. Recall that the λ_i's were assumed to be distinct. Further, recall we defined the polynomials f, g, and h in (2.2) as follows.

$$f(x) := \prod_{\ell=1}^{n} (x - \lambda_\ell), \qquad g(x) := \prod_{\ell=1}^{n} (x - \rho_\ell), \qquad h(x) := \prod_{\ell=1}^{n} (x - \nu_\ell).$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

Then by Corollary 3.2, for $i \in [n]$,

$$v_i^2 = \frac{-g(\lambda_i)}{\mu_1 f'(\lambda_i)}. \tag{4.16}$$

Now since we also assumed $\lambda_i \neq \nu_k$ for all $i, k \in [n]$, (4.11) and (4.12) together assure us that the ρ_ℓ's are also distinct. Then using Corollary 3.2 on (4.15), we get

$$\left((U_\rho^T w)_i\right)^2 = \frac{-h(\rho_i)}{\mu_\delta g'(\rho_i)} \implies \left(U_\rho^T w\right)_i = t_i \sqrt{\frac{-h(\rho_i)}{\mu_\delta g'(\rho_i)}} \tag{4.17}$$

for some sign $t_i \in \{-1, 1\}$. Now Lemma 3.3 gave us

$$U_\rho(i,j) = \begin{cases} s_j, & \rho_j = \lambda_i \\ 0, & \rho_j = \lambda_k,\ k \neq i \\ s_j \frac{v_i}{(\lambda_i - \rho_j)} \cdot \sqrt{\frac{\mu_1 f(\rho_j)}{g'(\rho_j)}}, & \rho_j \neq \lambda_\ell\ \forall \ell \in [n] \end{cases}.$$

Fix $i \in [n]$. If there exists $k \in [n]$ such that $\lambda_i = \rho_k$, we have $U_\rho(i,k) = s_k = \pm 1$. Since U_ρ is a real orthogonal matrix, this means all other entries in row i are 0, i.e. $U_\rho(i,j) = 0$ for $j \neq k$. In this case

$$w_i = \sum_{j=1}^n U_\rho(i,j) t_j \sqrt{\frac{-h(\rho_j)}{\mu_\delta g'(\rho_j)}} = s_k t_k \sqrt{\frac{-h(\rho_k)}{\mu_\delta g'(\rho_k)}}. \tag{4.18}$$

Note that in this situation, since $\lambda_i = \rho_k$, $g(\lambda_i) = 0 \implies v_i = 0$. Otherwise, if $\lambda_i \neq \rho_\ell$ for all $\ell \in [n]$, we get

$$w_i = \sum_{j=1}^n U_\rho(i,j) t_j \sqrt{\frac{-h(\rho_j)}{\mu_\delta g'(\rho_j)}} = \sum_{\substack{j:\rho_j \neq \lambda_k \\ \forall k \in [n]}} s_j t_j \frac{v_i}{(\lambda_i - \rho_j)} \cdot \sqrt{\frac{\mu_1 f(\rho_j)}{g'(\rho_j)}} \cdot \sqrt{\frac{-h(\rho_j)}{\mu_\delta g'(\rho_j)}}. \tag{4.19}$$

Then the condition that v and w are orthogonal requires $\sum_{i=1}^n w_i v_i = 0$. For a fixed $i \in [n]$ if $\lambda_i = \rho_\ell$ for some $\ell \in [n]$, we noted previously that $v_i = 0$. Thus,

$$\begin{aligned}
0 = \sum_{i=1}^n w_i v_i &= \sum_{\substack{i:\lambda_i \neq \rho_k \\ \forall k \in [n]}} \left[\left(\sum_{\substack{j:\rho_j \neq \lambda_k \\ \forall k \in [n]}} s_j t_j \frac{v_i}{(\lambda_i - \rho_j)} \cdot \sqrt{\frac{\mu_1 f(\rho_j)}{g'(\rho_j)}} \cdot \sqrt{\frac{-h(\rho_j)}{\mu_\delta g'(\rho_j)}} \right) v_i \right] \\
&= \sum_{\substack{j:\rho_j \neq \lambda_k \\ \forall k \in [n]}} \left(s_j t_j \sqrt{\frac{-h(\rho_j)}{\mu_\delta g'(\rho_j)}} \cdot \sqrt{\frac{\mu_1 f(\rho_j)}{g'(\rho_j)}} \cdot \sum_{\substack{i:\lambda_i \neq \rho_k \\ \forall k \in [n]}} \frac{-g(\lambda_i)}{\mu_1 (\lambda_i - \rho_j) f'(\lambda_i)} \right) \quad \text{by (4.16)} \\
&= \sum_{\substack{j:\rho_j \neq \lambda_k \\ \forall k \in [n]}} \left(s_j t_j \sqrt{\frac{-h(\rho_j)}{\mu_\delta g'(\rho_j)}} \cdot \sqrt{\frac{\mu_1 f(\rho_j)}{g'(\rho_j)}} \cdot \sum_{i=1}^n \frac{-g(\lambda_i)}{\mu_1 (\lambda_i - \rho_j) f'(\lambda_i)} \right) \\
&\qquad \text{since if } \lambda_i = \rho_k \text{ for some } k \in [n],\ g(\lambda_i) = 0 \\
&= \sum_{\substack{j:\rho_j \neq \lambda_k \\ \forall k \in [n]}} \left(s_j t_j \sqrt{\frac{-h(\rho_j)}{\mu_\delta g'(\rho_j)}} \cdot \sqrt{\frac{\mu_1 f(\rho_j)}{g'(\rho_j)}} \cdot \frac{-1}{\mu_1} \right) \quad \text{by Lemma 3.4}
\end{aligned}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

$$= \sum_{j=1}^{n} \left(s_j t_j \sqrt{\frac{-h(\rho_j)}{\mu_\delta g'(\rho_j)}} \cdot \sqrt{\frac{\mu_1 f(\rho_j)}{g'(\rho_j)}} \cdot \frac{-1}{\mu_1} \right) \quad \text{since if } \rho_j = \lambda_k \text{ for some } k \in [n],\ f(\rho_j) = 0.$$

Note $s_j t_j \in \{-1, 1\}$, and the values under the square roots are guaranteed to be nonnegative by the interlacing conditions, so we can take the absolute value. Multiplying both sides by $-\mu_1 \sqrt{\left|\frac{\mu_\delta}{\mu_1}\right|}$ and rearranging, we get (4.13).

Now for the other direction, assume $\rho^{(n)}$ exists satisfying the trace equality, (4.10), the interlacing conditions, (4.11) and (4.12), and the orthogonality condition, (4.13). Then, the interlacing condition (4.11) gives us that $\lambda^{(n)}$, μ_1, and $\rho^{(n)}$ satisfy the Horn inequalities, so combining this with the trace equality, there exists a real unit vector v such that

$$\sigma\left(\Lambda + \mu_1 v v^T\right) = \rho^{(n)}.$$

Analogously, the interlacing condition (4.12) gives us that $\rho^{(n)}$, μ_2, and $\nu^{(n)}$ satisfy the Horn inequalities, so combining this with the trace equality, there exists a real unit vector u such that

$$\sigma\left(\operatorname{diag}(\rho_1, \ldots, \rho_n) + \mu_2 u u^T\right) = \nu^{(n)}.$$

If U_ρ is the real orthogonal matrix that diagonalizes $\Lambda + \mu_1 v v^T$, i.e.

$$U_\rho^T(\Lambda + \mu_1 v v^T)U_\rho = \operatorname{diag}(\rho_1, \ldots, \rho_n) \implies \Lambda + \mu_1 v v^T = U_\rho \operatorname{diag}(\rho_1, \ldots, \rho_n) U_\rho^T,$$

then

$$\sigma\left(U_\rho \operatorname{diag}(\rho_1, \ldots, \rho_n) U_\rho^T + \mu_2 U_\rho u u^T U_\rho^T\right) = \nu^{(n)} \implies \sigma\left(\Lambda + \mu_1 v v^T + \mu_2 U_\rho u u^T U_\rho^T\right) = \nu^{(n)}.$$

Finally, (4.13) guarantees that v and $w := U_\rho u$ are orthogonal. Thus,

$$B := \mu_1 v v^T + \mu_2 U_\rho u u^T U_\rho^T \quad \text{satisfies} \quad \sigma(B) = \{\mu_1, \mu_2, 0, \ldots, 0\},$$

Thus, there is a real Horn solution pair (Λ, B), so that $\lambda^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ satisfy the Horn inequalities. ☐

All in all, if we can find $\rho^{(n)}$ satisfying the four conditions (4.10), (4.11), (4.12), and (4.13), then we can perform two orthogonal rank 1 updates to solve the Horn problem. In particular, we get Algorithm 1.

4.2. Discussion. Note that the heart of Algorithm 1 lies in finding a suitable tuple of values $\rho_>^{(n)}$ in Step 2. The method in which we showed the existence of such values gave no obvious way to actually find them. It is possible, though, to use a solver to numerically find the values $\rho_>^{(n)}$ in Step 2. We have been able to successfully implement this algorithm in MATLAB by using fmincon() to find $\rho_>^{(n)}$. In particular, we set items (i)-(iv) in Step 2 as constraints and optimized over a constant function. Note that the signs $r_j \in \{-1, 1\}$ are unknown to start with, so we ended up running fmincon() for various choices of r_j until a solution was found. To see how fast and accurate the implementation of Algorithm 1 was, we ran 100 random trials in MATLAB for various matrix sizes, computing the time it took and the maximum error. Here the error of a trial was the maximum difference between the inputted eigenvalues and the eigenvalues of the outputted A, B, and $A + B$. The results are summarized in the following tables. Note that the PSD trials took $\mu_1 > \mu_2 > 0$ and the indefinite trials took $\mu_1 > 0 > \mu_n$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

Algorithm 1 Solving the Rank 2 Horn Problem With Distinct Eigenvalues

Given: Real tuples $\lambda_>^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ satisfying the trace equality and Horn inequalities. Assume all but two μ_j's are zero and for all $i,k \in [n]$, $\lambda_i \neq \nu_k$.

Step 1: If $0 > \mu_{n-1} \geq \mu_n$, negate all eigenvalues. Renumber so they are in nonincreasing order.

Step 2: Find $\rho_>^{(n)}$ such that the following hold for

$$f(x) := \prod_{\ell=1}^{n}(x-\lambda_\ell), \quad g(x) := \prod_{\ell=1}^{n}(x-\rho_\ell), \quad h(x) := \prod_{\ell=1}^{n}(x-\nu_\ell).$$

(i) (Trace equality) $\sum_{i=1}^n \rho_i = \sum_{i=1}^n \lambda_i + \mu_1$.

(ii) (Interlacing Property 1) $\lambda_1 + \mu_1 \geq \rho_1 \geq \lambda_1 \geq \rho_2 \geq \lambda_2 \geq \cdots \geq \rho_n \geq \lambda_n$.

(iii) (Interlacing Property 2) $\begin{cases} \rho_1 + \mu_2 \geq \nu_1 \geq \rho_1 \geq \nu_2 \geq \rho_2 \geq \cdots \geq \nu_n \geq \rho_n, & \text{if } \mu_2 > 0 \\ \rho_1 \geq \nu_1 \geq \rho_2 \geq \nu_2 \geq \cdots \geq \nu_n \geq \rho_n + \mu_n, & \text{if } \mu_n < 0 \end{cases}$.

(iv) (Orthogonality Property) $\sum_{j=1}^n r_j \frac{\sqrt{|h(\rho_j)f(\rho_j)|}}{|g'(\rho_j)|} = 0$ for some $r_1, \ldots, r_n \in \{-1, 1\}$.

Step 3: Define the vector $v \in \mathbb{R}^n$ by $v_i = \sqrt{\frac{-g(\lambda_i)}{\mu_1 f'(\lambda_i)}}$ for $i \in [n]$.

Step 4: Take $\mu_\delta = \begin{cases} \mu_2, & \text{if } \mu_2 > 0 \\ \mu_n, & \text{if } \mu_n < 0 \end{cases}$. Define the vector $w \in \mathbb{R}^n$ such that for $i \in [n]$,

$$w_i = \begin{cases} \sqrt{\frac{-h(\rho_\ell)}{\mu_\delta g'(\rho_\ell)}}, & \lambda_i = \rho_\ell \\ \sum_{j=1}^n \frac{r_j}{(\lambda_i - \rho_j)|g'(\rho_j)|} \cdot \sqrt{\frac{h(\rho_j)g(\lambda_i)f(\rho_j)}{\mu_\delta f'(\lambda_i)}}, & \lambda_i \neq \rho_\ell \ \forall \ell \in [n] \end{cases}.$$

Step 5: If we negated all eigenvalues in Step 1, take $A = -\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and $B = -\mu_1 vv^T - \mu_\delta ww^T$. Else, take $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and $B = \mu_1 vv^T + \mu_\delta ww^T$.

Output: $n \times n$ real symmetric matrices A and B with $\sigma(A) = \lambda^{(n)}$, $\sigma(B) = \mu^{(n)}$, and $\sigma(A+B) = \nu^{(n)}$.

100 Random PSD Trials

Matrix Size	Maximum Error	Time (seconds)
3	1.9429×10^{-15}	1.78
10	1.0728×10^{-14}	4.03
20	8.0214×10^{-14}	9.96
50	9.0872×10^{-13}	115.11

100 Random Indefinite Trials

Matrix Size	Maximum Error	Time (seconds)
3	1.5543×10^{-15}	1.51
10	1.6501×10^{-14}	2.16
20	4.6185×10^{-14}	2.68
50	5.5604×10^{-13}	14.41

4.3. Example. To illustrate our algorithm, we provide an example. Suppose we start with

$$\lambda_>^{(3)} = \{8, 4, -8\}, \qquad \mu^{(3)} = \{12, 6, 0\}, \qquad \nu^{(3)} = \left\{16,\, 3+\sqrt{57},\, 3-\sqrt{57}\right\}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

There are many tuples of values $\rho_>^{(3)}$ that satisfy the conditions of Step 2. For example, if we implement Algorithm 1 in MATLAB and use fmincon(), as discussed in Subsection 4.2, we will get

$$\rho_>^{(3)} \approx \{14.522183259496549, 6.805306350191621, -5.327489609688171\}.$$

However, for the sake of this example, we choose the following values that have nice closed forms.

$$\rho_>^{(3)} = \left\{16, 4\sqrt{2}, -4\sqrt{2}\right\}.$$

In particular, we get the polynomials

$$\begin{aligned} f(x) &= (x-8)(x-4)(x+8) \\ g(x) &= (x^2-32)(x-16) \\ h(x) &= ((x-3)^2-57)(x-16). \end{aligned}$$

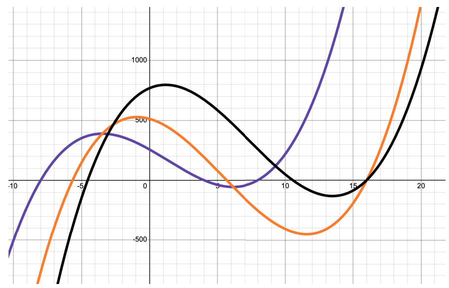

Figure 1: Polynomials f, g, and h – Distinct case.

Note that f and h, shown in Figure 1 in purple and black, respectively, were given. We had to find an interlacing polynomial g, shown in Figure 1 in orange (i.e., the roots of g had to satisfy the interlacing properties). The interlacing of the roots ensured all values under the square roots in Steps 3 and 4 were real. Now g had to be chosen so that the orthogonality property was also satisfied. Here we have

$$\frac{\sqrt{|h(16)f(16)|}}{|g'(16)|} - \frac{\sqrt{|h(4\sqrt{2})f(4\sqrt{2})\,|}}{|g'(4\sqrt{2})|} + \frac{\sqrt{|h(-4\sqrt{2})f(-4\sqrt{2})|}}{|g'(-4\sqrt{2})|} = 0, \tag{4.20}$$

so

$$r_1 = 1, \quad r_2 = -1, \quad \text{and} \quad r_3 = 1.$$

In Step 3, we find

$$\sqrt{\frac{-g(8)}{12f'(8)}} = \frac{1}{\sqrt{3}}, \qquad \sqrt{\frac{-g(4)}{12f'(4)}} = \frac{1}{\sqrt{3}}, \qquad \text{and} \qquad \sqrt{\frac{-g(-8)}{12f'(-8)}} = \frac{1}{\sqrt{3}}.$$

Thus,

$$v = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

In Step 4, $\mu_\delta = \mu_2 = 6$, so

$$w_1 = \frac{1}{(8-16)|g'(16)|}\sqrt{\frac{h(16)g(8)f(16)}{6f'(8)}} - \frac{1}{(8-4\sqrt{2})|g'(4\sqrt{2})|}\sqrt{\frac{h(4\sqrt{2})g(8)f(4\sqrt{2})}{6f'(8)}} + \frac{1}{(8+4\sqrt{2})|g'(-4\sqrt{2})|}\sqrt{\frac{h(-4\sqrt{2})g(8)f(-4\sqrt{2})}{6f'(8)}} = \frac{-1}{\sqrt{6}}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

$$w_2 = \frac{1}{(4-16)|g'(16)|}\sqrt{\frac{h(16)g(4)f(16)}{6f'(4)}} - \frac{1}{(4-4\sqrt{2})|g'(4\sqrt{2})|}\sqrt{\frac{h(4\sqrt{2})g(4)f(4\sqrt{2})}{6f'(4)}} + \frac{1}{(4+4\sqrt{2})|g'(-4\sqrt{2})|}\sqrt{\frac{h(-4\sqrt{2})g(4)f(-4\sqrt{2})}{6f'(4)}} = \frac{2}{\sqrt{6}}$$

$$w_3 = \frac{1}{(-8-16)|g'(16)|}\sqrt{\frac{h(16)g(-8)f(16)}{6f'(-8)}} - \frac{1}{(-8-4\sqrt{2})|g'(4\sqrt{2})|}\sqrt{\frac{h(4\sqrt{2})g(-8)f(4\sqrt{2})}{6f'(-8)}} + \frac{1}{(-8+4\sqrt{2})|g'(-4\sqrt{2})|}\sqrt{\frac{h(-4\sqrt{2})g(-8)f(-4\sqrt{2})}{6f'(-8)}} = \frac{-1}{\sqrt{6}}.$$

Then Step 5 gives

$$A = \begin{bmatrix} 8 & & \\ & 4 & \\ & & -8 \end{bmatrix} \quad \text{and} \quad B = 12vv^T + 6ww^T = \begin{bmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 5 \\ 2 & 8 & 2 \\ 5 & 2 & 5 \end{bmatrix}.$$

It can be verified that $\sigma(A) = \lambda_>^{(3)}$, $\sigma(B) = \mu^{(3)}$, and $\sigma(A+B) = \nu^{(3)}$.

4.4. Uniqueness. In the example given, we noted that there were many choices of $\rho_>^{(3)}$ that would have satisfied the conditions of Step 2. Each different choice of $\rho_>^{(3)}$ would have then led to a different matrix B. In particular, since the λ_i's were unique and A would continue to be diagonal, real Horn solution pairs (A, B) and $(A, \tilde{B})$ corresponding to different choices of $\rho_>^{(3)}$ would not be simultaneously unitarily similar. In general, we have the following proposition when $\mu_1 \neq \mu_2$.

Proposition 4.2. *Given real tuples $\lambda_>^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ satisfying the trace equality and Horn inequalities, assume all but two μ_j's are zero and that the two nonzero μ_j's are different. Further assume for all $i, k \in [n]$, $\lambda_i \neq \nu_k$. Then the real Horn solution pair (A, B) is unique up to simultaneous unitary similarity if and only if there exists a unique tuple $\rho_>^{(n)}$ satisfying the conditions of Step 2 in Algorithm 1.*

Proof. Assume without loss of generality that $\mu_1 > 0$. Take $\mu_\delta = \begin{cases} \mu_2, & \text{if } \mu_2 > 0 \\ \mu_n, & \text{if } \mu_n < 0 \end{cases}$. Assume we have two real symmetric solution pairs,

$$(A, B) := (\Lambda,\ \mu_1 vv^T + \mu_\delta ww^T) \qquad \text{and} \qquad (\tilde{A}, \tilde{B}) := (\Lambda,\ \mu_1 \tilde{v}\tilde{v}^T + \mu_\delta \tilde{w}\tilde{w}^T),$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Suppose these two solution pairs were simultaneously unitarily similar. Then there exists a unitary matrix U such that

$$U\Lambda U^* = \Lambda,$$
$$U(\mu_1 vv^T + \mu_\delta ww^T)U^* = \mu_1 \tilde{v}\tilde{v}^T + \mu_\delta \tilde{w}\tilde{w}^T.$$

Note that since v and w are real vectors, we can say

$$\mu_1 Uvv^*U^* + \mu_\delta Uww^*U^* = \mu_1 \tilde{v}\tilde{v}^* + \mu_\delta \tilde{w}\tilde{w}^*.$$

Since we assumed $\mu_1 \neq \mu_\delta$, there is a unique spectral decomposition, so $Uvv^*U^* = \tilde{v}\tilde{v}^*$ and $Uww^*U^* = \tilde{w}\tilde{w}^*$. Then

$$\sigma(\Lambda + \mu_1 vv^T) = \sigma(U\Lambda U^* + \mu_1 Uvv^*U^*) = \sigma(\Lambda + \mu_1 \tilde{v}\tilde{v}^T).$$

Thus, the two solution pairs produce the same set of tuples $\rho_>^{(n)}$ satisfying the conditions of Step 2. Hence, if the real Horn solution pair (A, B) is unique up to simultaneous unitary similarity, then there exists a unique tuple $\rho_>^{(n)}$ satisfying the conditions of Step 2 in Algorithm 1. Conversely, if there are two different tuples $\rho_>^{(n)}$ and $\tilde{\rho}_>^{(n)}$ satisfying the conditions of Step 2, the corresponding solutions (A, B) and $(\tilde{A}, \tilde{B})$ will not be simultaneously unitarily similar. ∎

Note that a different criterion was given in [13, Corollary in Appendix] for when the solution pair (A, B) is unique up to simultaneous unitary similarity.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

5. Reducing eigenvalue sets. We now wish to generalize Algorithm 1 to solve any rank 2 Horn problem. To this end, we will define when the sets of eigenvalues are reducible. This will allow us to simplify the case when there are repeated eigenvalues. In Section 6, we will use the results obtained in this section to generalize Algorithm 1 to the situation in which the λ_i's are not necessarily distinct. In Section 7, we generalize the algorithm to allow $\lambda_i = \nu_k$ for some $i, k \in [n]$. We then combine the results of these sections to get Algorithm 2, a general algorithm for the rank 2 Horn problem.

5.1. Reducing the eigenvalues. We begin by defining when the sets of eigenvalues are reducible.

DEFINITION 5.1. The sets of eigenvalues $\lambda^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ satisfying the trace equality and Horn inequalities are called reducible if there exist $i, j, k \in [n]$ such that the reduced sets of eigenvalues

$$\begin{aligned} &\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{i-1} \geq \lambda_{i+1} \geq \cdots \geq \lambda_n, \\ &\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{j-1} \geq \mu_{j+1} \geq \cdots \geq \mu_n, \\ \text{and} \quad &\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{k-1} \geq \nu_{k+1} \geq \cdots \geq \nu_n \end{aligned}$$

still satisfy the trace equality and Horn inequalities.

Note this is useful because we could find $\tilde{A}$ and $\tilde{B}$ that solve the Horn problem for the reduced sets of eigenvalues. Then the solution for the original sets of eigenvalues would be

$$A = \left[\begin{array}{c|c} \lambda_i & \\ \hline & \tilde{A} \end{array}\right], \qquad B = \left[\begin{array}{c|c} \mu_j & \\ \hline & \tilde{B} \end{array}\right].$$

Now we will show that in our rank 2 case, there are a couple of situations in which the sets of eigenvalues are guaranteed to be reducible. For this, we first note a corollary to Theorem 2.3 and Corollary 3.1.

COROLLARY 5.2. *Given a tuple of real values $\lambda^{(n)}$, suppose for some $i \in [n]$,*

$$\lambda_{i-1} > \lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+k} > \lambda_{i+k+1},$$

(i.e. there are exactly $k+1$ λ_ℓ's equal to λ_i). For $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and some real unit vector z, suppose $\sigma(\Lambda + \mu z z^T) = \rho^{(n)}$. If there are at least $k+1$ ρ_ℓ's equal to λ_i, then

$$\sum_{\ell=i}^{i+k} |z_\ell|^2 = 0 \implies z_i = z_{i+1} = \cdots = z_{i+k} = 0.$$

Using this we can show the following.

PROPOSITION 5.3. *Suppose $\lambda^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ satisfy the trace equality and Horn inequalities. Assume $\mu_1 \geq \mu_2 > 0 = \mu_3 = \cdots = \mu_n$. If there exists $i \in [n]$ such that $\lambda_i = \nu_i$ or $\lambda_i = \nu_{i+2}$, then the sets of eigenvalues are reducible.*

Proof. Let Λ be a diagonal matrix with diagonal elements equal to the λ_ℓ's (the order will be specified differently for different cases). From the assumptions we know there exist orthonormal vectors v and w such that $\sigma(\Lambda + \mu_1 v v^T + \mu_2 w w^T) = \nu^{(n)}$. Define $\rho^{(n)} := \sigma(\Lambda + \mu_1 v v^T)$. Then by Theorem 2.2 we have the following interlacing conditions.

$$\rho_1 \geq \lambda_1 \geq \rho_2 \geq \lambda_2 \geq \cdots \geq \rho_n \geq \lambda_n \qquad \text{and} \qquad \nu_1 \geq \rho_1 \geq \nu_2 \geq \rho_2 \geq \cdots \geq \nu_n \geq \rho_n.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

These conditions guarantee that if $\lambda_i = \nu_i$, then $\rho_i = \lambda_i$. Similarly, if $\lambda_i = \nu_{i+2}$, then $\rho_{i+1} = \lambda_i$. Thus, there is at least one ρ_k equal to λ_i. Therefore, we can consider the following cases on the number of λ_j's, ρ_k's and ν_ℓ's equal to λ_i.

(I) 1 λ_j, 1 ρ_k, and 1 or more ν_ℓ's.
(II) 1 λ_j, 2 or more ρ_k's, and 1 ν_ℓ.
(III) 1 λ_j, 2 or more ρ_k's, and 2 or more ν_ℓ's.
(IV) 2 or more λ_j's, 1 ρ_k, and 1 or more ν_ℓ's.
(V) 2 or more λ_j's, 2 or more ρ_k's, and 1 or more ν_ℓ's.

We will prove each case separately.

Case (I): Assume

$$\Lambda = \mathrm{diag}(\lambda_i, \lambda_1, \lambda_2, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n).$$

Let α be such that $\rho_\alpha = \lambda_i$ (note α is either i or $i+1$). Then by Corollary 5.2, the first value in v is zero. Then there is a real orthogonal $U = \left[\begin{array}{c|c} 1 & \\ \hline & \hat{U} \end{array}\right]$ such that

$$U^T(\Lambda + \mu_1 vv^T)U = \mathrm{diag}(\rho_\alpha, \rho_1, \rho_2, \dots, \rho_{\alpha-1}, \rho_{\alpha+1}, \dots, \rho_n),$$

and

$$\sigma(\mathrm{diag}(\rho_\alpha, \rho_1, \rho_2, \dots, \rho_{\alpha-1}, \rho_{\alpha+1}, \dots, \rho_n) + \mu_2 U^T ww^T U) = \nu^{(n)}.$$

Using Corollary 5.2 again, we get that the first value in $U^T w$ is zero. By the structure of U, this implies the first value in w is zero. Thus, the solution to the Horn problem has the form

$$A = \left[\begin{array}{c|c} \lambda_i & \\ \hline & \tilde{A} \end{array}\right], \qquad B = \left[\begin{array}{c|c} 0 & \\ \hline & \tilde{B} \end{array}\right],$$

so the sets of eigenvalues are reducible.

Case (II): Since λ_i is unique, by interlacing, we can only have two ρ_k's equal to λ_i. In particular, $\lambda_i = \rho_i = \rho_{i+1}$. Then by the second interlacing condition, $\rho_i = \nu_{i+1} = \rho_{i+1}$. Hence, $\lambda_i = \nu_{i+1}$. By assumption, $\lambda_i = \nu_i$ or $\lambda_i = \nu_{i+2}$, so there are at least two ν_ℓ's equal to λ_i. Thus this case is impossible.

Case (III): Assume

$$\Lambda = \mathrm{diag}(\lambda_i, \lambda_1, \lambda_2, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n).$$

As in Case II, since λ_i is unique, we must have exactly two ρ_k's equal to λ_i. In particular, $\lambda_i = \rho_i = \rho_{i+1}$. Then by Corollary 5.2, the first value in v is zero. Next, there is a real orthogonal $U = \left[\begin{array}{c|c} 1 & \\ \hline & \hat{U} \end{array}\right]$ such that

$$U^T(\Lambda + \mu_1 vv^T)U = \mathrm{diag}(\rho_i, \rho_{i+1}, \rho_1, \rho_2, \dots, \rho_{i-1}, \rho_{i+2}, \dots, \rho_n),$$

and

$$\sigma(\mathrm{diag}(\rho_i, \rho_{i+1}, \rho_1, \rho_2, \dots, \rho_{i-1}, \rho_{i+2}, \dots, \rho_n) + \mu_2 U^T ww^T U) = \nu^{(n)}.$$

Since there are at least two ν_ℓ's equal to $\lambda_i = \rho_i = \rho_{i+1}$, using Corollary 5.2 again, we get that the first two values in $U^T w$ are zero. By the structure of U, this implies the first value in w is zero. As in Case I, this means the sets of eigenvalues are reducible.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

Case (IV): Since ρ_k is unique, we must have exactly two λ_j's equal to $\rho_k = \lambda_i$. In particular, $\lambda_{k-1} = \rho_k = \lambda_k$ (so either $i = k$ or $i = k-1$). Assume

$$\Lambda = \operatorname{diag}(\lambda_{k-1}, \lambda_k, \lambda_1, \lambda_2, \ldots, \lambda_{k-2}, \lambda_{k+1}, \ldots, \lambda_n).$$

Since $\lambda_{k-1} = \lambda_k$, there is a 2×2 real orthogonal W such that for $V = \left[\begin{array}{c|c} W & \\ \hline & I_{n-2} \end{array}\right]$, the first entry of Vv is zero and we still have

$$\sigma(\Lambda + \mu_1 V v v^T V^T + \mu_2 V w w^T V^T) = \nu^{(n)}.$$

Thus, we can take the solution to the Horn problem to be $A = \Lambda$ and

$$B = \mu_1 V v v^T V^T + \mu_2 V w w^T V^T = V(\mu_1 v v^T + \mu_2 w w^T) V^T.$$

Then there is a real orthogonal $U = \left[\begin{array}{c|c} 1 & \\ \hline & \hat{U} \end{array}\right]$ such that

$$U^T(\Lambda + \mu_1 v v^T) U = \operatorname{diag}(\rho_k, \rho_1, \rho_2, \ldots, \rho_{k-1}, \rho_{k+1}, \ldots, \rho_n),$$

and

$$\sigma(\operatorname{diag}(\rho_k, \rho_1, \rho_2, \ldots, \rho_{k-1}, \rho_{k+1}, \ldots, \rho_n) + \mu_2 U^T V w w^T V^T U) = \nu^{(n)}.$$

Since there is at least one ν_ℓ equal to $\lambda_i = \rho_k$, Corollary 5.2 gives us that the first value in $U^T V w$ is zero. By the structure of U, this implies the first value in Vw is zero. Thus a solution to the Horn problem has the form

$$A = \left[\begin{array}{c|c} \lambda_i & \\ \hline & \tilde{A} \end{array}\right], \qquad B = \left[\begin{array}{c|c} 0 & \\ \hline & \tilde{B} \end{array}\right],$$

so the sets of eigenvalues are reducible.

Case (V): Assume

$$\lambda_{\alpha-1} > \lambda_i = \lambda_\alpha = \lambda_{\alpha+1} = \cdots = \lambda_{\alpha+j} > \lambda_{\alpha+j+1},$$

for some $j \geq 1$ (note $i = \alpha + \ell$ for some $\ell \in \{0, 1, \ldots, j\}$). Further, assume

$$\rho_{\beta-1} > \lambda_i = \rho_\beta = \rho_{\beta+1} = \cdots = \rho_{\beta+k} > \rho_{\beta+k+1},$$

for some $k \geq 1$. Assume

$$\Lambda = \operatorname{diag}(\lambda_\alpha, \ldots, \lambda_{\alpha+j}, \lambda_1, \lambda_2, \ldots, \lambda_{\alpha-1}, \lambda_{\alpha+j+1}, \ldots, \lambda_n).$$

If $j = 1$, Corollary 5.2 gives us that the first two values of v are zero. Thus for $V = I_n$, the first two values of Vv are zero. Otherwise, $j \geq 2$, so there is a 3×3 real orthogonal matrix W such that for $V = \left[\begin{array}{c|c} W & \\ \hline & I_{n-3} \end{array}\right]$, the first two entries of Vv is zero and we still have

$$\sigma(\Lambda + \mu_1 V v v^T V^T + \mu_2 V w w^T V^T) = \nu^{(n)}.$$

Next, there is a 2×2 real orthogonal matrix $\tilde{W}$ such that for $\tilde{V} = \left[\begin{array}{c|c} \tilde{W} & \\ \hline & I_{n-2} \end{array}\right]$, the first entry of $\tilde{V} V w$ is zero and we still have

$$\sigma(\Lambda + \mu_1 \tilde{V} V v v^T V^T \tilde{V}^T + \mu_2 \tilde{V} V w w^T V^T \tilde{V}^T) = \nu^{(n)}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

Note that since the first two values of Vv were zeros, the first two values of $\tilde{V}Vv$ are still zeros. Thus, we can take the solution to the Horn problem to be $A = \Lambda$ and

$$B = \mu_1 \tilde{V}Vvv^T V^T \tilde{V}^T + \mu_2 \tilde{V}Vww^T V^T \tilde{V}^T = \tilde{V}V(\mu_1 vv^T + \mu_2 ww^T)V^T \tilde{V}^T.$$

Thus, a solution to the Horn problem has the form

$$A = \left[\begin{array}{c|c} \lambda_i & \\ \hline & \tilde{A} \end{array}\right], \qquad B = \left[\begin{array}{c|c} 0 & \\ \hline & \tilde{B} \end{array}\right],$$

so the sets of eigenvalues are reducible. ∎

We get an analogous result for the indefinite case.

PROPOSITION 5.4. *Suppose $\lambda^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ satisfy the trace equality and Horn inequalities. Assume $\mu_1 > 0 = \mu_2 = \cdots = \mu_{n-1} > \mu_n$. If there exists $i \in [n]$ such that $\lambda_i = \nu_{i-1}$ or $\lambda_i = \nu_{i+1}$, then the sets of eigenvalues are reducible.*

We omit the proof because it follows analogously to that of the previous result. The main difference is that the second interlacing condition becomes

$$\rho_1 \geq \nu_1 \geq \rho_2 \geq \nu_2 \geq \cdots \geq \rho_n \geq \nu_n.$$

Now Proposition 5.3 and Proposition 5.4 get us the following corollary.

COROLLARY 5.5. *Suppose $\lambda^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ satisfy the trace equality and Horn inequalities, and only two μ_j's are nonzero. If the sets of eigenvalues are not reducible, then for any $i \in [n]$ such that $\lambda_i = \lambda_{i+1}$, we know $\lambda_{i-1} \neq \lambda_i \neq \lambda_{i+2}$ and for all $k \in [n]$, $\lambda_i \neq \nu_k$.*

Proof. Let Λ be a diagonal matrix with diagonal elements equal to the λ_ℓ's. From the assumptions, we know there exist orthonormal vectors v and w such that $\sigma(\Lambda + \mu_1 vv^T + \mu_2 ww^T) = \nu^{(n)}$. Define $\rho^{(n)} := \sigma(\Lambda + \mu_1 vv^T)$.

Case 1: First assume $\mu_1 \geq \mu_2 > 0$. Then by Theorem 2.2 we have the following interlacing conditions.

$$\rho_1 \geq \lambda_1 \geq \rho_2 \geq \lambda_2 \geq \cdots \geq \rho_n \geq \lambda_n \qquad \text{and} \qquad \nu_1 \geq \rho_1 \geq \nu_2 \geq \rho_2 \geq \cdots \geq \nu_n \geq \rho_n.$$

By interlacing, if $\lambda_{i-1} = \lambda_i = \lambda_{i+1}$, then $\nu_{i+1} = \lambda_{i-1}$. By Proposition 5.3, this means the eigenvalues are reducible, which is a contradiction. Similarly, if $\lambda_i = \lambda_{i+1} = \lambda_{i+2}$, then $\nu_{i+2} = \lambda_i$. Again Proposition 5.3 gives us that the eigenvalues are reducible, which is a contradiction. Thus, we have $\lambda_{i-1} \neq \lambda_i \neq \lambda_{i+2}$. Interlacing then assures us the only possible ν_k equal to $\lambda_i = \lambda_{i+1}$ are ν_i, ν_{i+1}, and ν_{i+2}. In any case, Proposition 5.3 gives us that the eigenvalues are reducible, which is a contradiction. Hence, for all $k \in [n]$, $\lambda_i \neq \nu_k$.

Case 2: Else, $\mu_1 > 0$ and $\mu_n < 0$. Then by Theorem 2.2 we have the following interlacing conditions.

$$\rho_1 \geq \lambda_1 \geq \rho_2 \geq \lambda_2 \geq \cdots \geq \rho_n \geq \lambda_n \qquad \text{and} \qquad \rho_1 \geq \nu_1 \geq \rho_2 \geq \nu_2 \geq \cdots \geq \rho_n \geq \nu_n.$$

By interlacing, if $\lambda_{i-1} = \lambda_i = \lambda_{i+1}$, then $\nu_i = \lambda_{i-1}$. By Proposition 5.4, this means the eigenvalues are reducible, which is a contradiction. Similarly, if $\lambda_i = \lambda_{i+1} = \lambda_{i+2}$, then $\nu_{i+1} = \lambda_i$. Again Proposition 5.4 gives us that the eigenvalues are reducible, which is a contradiction. Thus, we have $\lambda_{i-1} \neq \lambda_i \neq \lambda_{i+2}$. Interlacing then assures us the only possible ν_k equal to $\lambda_i = \lambda_{i+1}$ are ν_{i-1}, ν_i, ν_{i+1}, and ν_{i+2}. In any case, Proposition 5.4 gives us that the eigenvalues are reducible, which is a contradiction. Hence, for all $k \in [n]$, $\lambda_i \neq \nu_k$. ∎

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

Note that Corollary 5.5 tells us

$$\{\lambda_i : \lambda_i = \lambda_{i+1}\} \cap \{\lambda_i : \lambda_i = \nu_\ell \text{ for some } \ell \in [n]\} = \emptyset.$$

Thus, we will first consider the case when $\lambda_i = \lambda_{i+1}$ for some $i \in [n]$. After, we will separately consider the case when $\lambda_i = \nu_\ell$ for some $i, \ell \in [n]$.

6. Rank 2 Horn – Repeated lambdas case. We return now to our rank 2 Horn problem. In particular, given eigenvalues $\lambda^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ satisfying the trace equality and Horn inequalities, assume all but two μ_ℓ's are zero. This time we further assume that the sets of eigenvalues have already been reduced. In this section, we still keep the condition that the λ_ℓ's are distinct from the ν_k's.

We will now address the case when some λ_ℓ's fail to be distinct from each other, i.e., there exists $i \in [n-1]$ such that $\lambda_i = \lambda_{i+1}$. Note that since the sets of eigenvalues are reduced, Corollary 5.5 guarantees $\lambda_{i-1} \neq \lambda_i \neq \lambda_{i+2}$.

As in Section 4, we assume without loss of generality that $\mu_1 > 0$. Thus, either

$$\mu_1 \geq \mu_2 > 0 = \mu_3 = \cdots = \mu_n \qquad \text{or} \qquad \mu_1 > 0 = \mu_2 = \cdots = \mu_{n-1} > \mu_n.$$

As before, take

$$\mu_\delta := \begin{cases} \mu_2, & \text{if } \mu_2 > 0 \\ \mu_n, & \text{if } \mu_n < 0 \end{cases}.$$

Now define the tuples $i_<^{(\ell)}$ and $j_<^{(n-\ell)}$ by

$$i_<^{(\ell)} := \{i \in [n] : \lambda_i = \lambda_{i-1}\}, \qquad j^{(n-\ell)} = [n] \backslash i_<^{(\ell)}.$$

Take

$$\Lambda = \left[\begin{array}{ccc|ccc} \lambda_{i_1} & & & & & \\ & \ddots & & & & \\ & & \lambda_{i_\ell} & & & \\ \hline & & & \lambda_{j_1} & & \\ & & & & \ddots & \\ & & & & & \lambda_{j_{n-\ell}} \end{array}\right].$$

Then there exist real orthonormal vectors v and w such that

$$\sigma\left(\Lambda + \mu_1 vv^T + \mu_\delta ww^T\right) = \nu^{(n)}.$$

Since the first ℓ elements of Λ are all eigenvalues that occur again later (in particular $\lambda_{i_s} = \lambda_{i_s - 1}$), we can always do a unitary similarity on $\Lambda + \mu_1 vv^T + \mu_\delta ww^T$ to ensure that v starts with k zeros. Thus, we assume without loss of generality that we already have $v_s = 0$ for $s \in [\ell]$.

Taking $\rho^{(n)} := \sigma(\Lambda + \mu_1 vv^T)$ as before, we obtain again the trace equality, (4.10), and interlacing conditions (4.11) and (4.12).

$$\sum_{s=1}^n \rho_s = \mu_1 + \sum_{s=1}^n \lambda_s,$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

$$\lambda_1 + \mu_1 \geq \rho_1 \geq \lambda_1 \geq \rho_2 \geq \lambda_2 \geq \cdots \geq \rho_n \geq \lambda_n,$$
$$\begin{cases} \rho_1 + \mu_\delta \geq \nu_1 \geq \rho_1 \geq \nu_2 \geq \rho_2 \geq \cdots \geq \nu_n \geq \rho_n, & \text{if } \mu_\delta = \mu_2 > 0 \\ \rho_1 \geq \nu_1 \geq \rho_2 \geq \cdots \geq \rho_n \geq \nu_n \geq \rho_n + \mu_\delta, & \text{if } \mu_\delta = \mu_n < 0 \end{cases}.$$

Since $\lambda_{i_s} = \lambda_{i_s - 1}$ for $s \in [k]$, the interlacing conditions ensure

$$\lambda_{i_s} = \rho_{i_s},\ s \in [\ell].$$

Also, Corollary 5.5 ensures $\lambda_{j_1} > \lambda_{j_2} > \cdots > \lambda_{j_{n-\ell}}$. Then, defining the polynomials

$$\hat{f}(x) := \prod_{s=1}^{n-\ell} (x - \lambda_{j_s}) \qquad \text{and} \qquad \hat{g}(x) := \prod_{s=1}^{n-\ell} (x - \rho_{j_s}),$$

we get that the analog to (4.16) is

$$v_{s+\ell}^2 = \frac{-\hat{g}(\lambda_{j_s})}{\mu_1 \hat{f}'(\lambda_{j_s})}, \qquad s \in [n-\ell].$$

Since we are still assuming $\lambda_s \neq \nu_r$ for all $s, r \in [n]$, we still have ρ_s's are distinct. Thus, analogous to (4.17), we have that for polynomials g and h as in (2.2), and unitary U_ρ such that

$$U_\rho^T (\Lambda + \mu_1 v v^T) U_\rho = \left[\begin{array}{ccc|ccc} \rho_{i_1} & & & & & \\ & \ddots & & & & \\ & & \rho_{i_\ell} & & & \\ \hline & & & \rho_{j_1} & & \\ & & & & \ddots & \\ & & & & & \rho_{j_{n-\ell}} \end{array}\right],$$

$$U_\rho^T w = \left[t_{i_1} \sqrt{\frac{-h(\rho_{i_1})}{\mu_\delta g'(\rho_{i_1})}} \quad \cdots \quad t_{i_\ell} \sqrt{\frac{-h(\rho_{i_\ell})}{\mu_\delta g'(\rho_{i_\ell})}} \quad t_{j_1} \sqrt{\frac{-h(\rho_{j_1})}{\mu_\delta g'(\rho_{j_1})}} \cdots \quad t_{j_{n-\ell}} \sqrt{\frac{-h(\rho_{j_{n-\ell}})}{\mu_\delta g'(\rho_{j_{n-\ell}})}} \right]^T.$$

Since $\rho_{i_s} = \lambda_{i_s}$, $U_\rho = I_k \oplus \hat{U}_\rho$ for some $(n-\ell) \times (n-\ell)$ real orthogonal matrix $\hat{U}_\rho$. In particular, since λ_{j_s}'s are unique, and the first ℓ entries of v are zero, we can use Lemma 3.3 to get that for some signs $c_{j_1}, c_{j_2}, \ldots, c_{j_{n-\ell}} \in \{-1, 1\}$, the (a,b)-entry of $\hat{U}_\rho$ is given by

$$\hat{U}_\rho(a,b) = \begin{cases} c_{j_b}, & \rho_{j_b} = \lambda_{j_a} \\ 0, & \rho_{j_b} = \lambda_{j_s},\ s \neq a \\ c_{j_b} \frac{v_{a+\ell}}{(\lambda_{j_a} - \rho_{j_b})} \cdot \sqrt{\frac{\mu_1 \hat{f}(\rho_{j_b})}{\hat{g}'(\rho_{j_b})}}, & \rho_{j_b} \neq \lambda_{j_s}\ \forall s \in [n-\ell] \end{cases}.$$

Then analogous to (4.18) and (4.19), for $s \in [n-\ell]$, the $s+\ell$th element of w is the given by

$$w_{s+\ell} = \begin{cases} c_{j_r} t_{j_r} \sqrt{\frac{-h(\rho_{j_r})}{\mu_\delta g'(\rho_{j_r})}}, & \lambda_{j_s} = \rho_{j_r} \\ \sum_{a=1}^{n-\ell} c_{j_a} t_{j_a} \frac{v_{s+\ell}}{(\lambda_{j_s} - \rho_{j_a})} \cdot \sqrt{\frac{\mu_1 \hat{f}(\rho_{j_a})}{\hat{g}'(\rho_{j_a})}} \cdot \sqrt{\frac{-h(\rho_{j_a})}{\mu_\delta g'(\rho_{j_a})}}, & \lambda_{j_s} \neq \rho_{j_r}\ \forall r \in [n-\ell] \end{cases}.$$

Since $\lambda_{i_s} = \rho_{i_s}$ for $s \in [\ell]$, we have for $a \in [n-\ell]$,

$$\frac{\hat{f}(\rho_{j_a})}{\hat{g}'(\rho_{j_a})} = \frac{f(\rho_{j_a})}{g'(\rho_{j_a})}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

Combining this with the fact that the first ℓ values of v are zeros and proceeding as in the proof of Theorem 4.1, we get that the analog to the orthogonality condition (4.13) is

$$0 = w^T v = \sum_{s=1}^{n-\ell} c_{j_s} t_{j_s} \frac{\sqrt{|h(\rho_{j_s}) f(\rho_{j_s})|}}{|g'(\rho_{j_s})|},$$

where $c_{j_s} t_{j_s} \in \{-1, 1\}$. Now for $s \in [\ell]$, $\rho_{i_s} = \lambda_{i_s}$, so $f(\rho_{i_s}) = 0$. Then since the ρ_s's are distinct, we can add back in the rest of the ρ_s's to get back exactly (4.13),

$$0 = \sum_{s=1}^{n} c_s t_s \frac{\sqrt{|h(\rho_s) f(\rho_s)|}}{|g'(\rho_s)|}.$$

The general algorithm for solving the rank 2 Horn problem is presented in Algorithm 2 in the Appendix. For the case we just discussed, take $k = 0$ and thus $a_{<}^{(k)} = \emptyset$ in the general algorithm.

6.1. Example. To illustrate this case of our algorithm, we provide an example. Suppose we start with

$$\lambda^{(4)} = \{6, 2, 2, -1\}, \qquad \mu^{(4)} = \{5, 2, 0, 0\}, \qquad \nu^{(4)} = \left\{ \frac{9+\sqrt{13}}{2}, 6, \frac{9-\sqrt{13}}{2}, 1 \right\}.$$

We begin by reducing the sets of eigenvalues. In particular, we can remove

$$\lambda_{\alpha_1} = 6, \quad \mu_{\beta_1} = 0, \quad \text{and} \quad \nu_{\gamma_1} = 6.$$

Note that while $\lambda_4 + \mu_2 = -1 + 2 = 1 = \nu_4$, we cannot remove these values because the Horn inequalities would not hold for the reduced sets of eigenvalues. Thus, we get the matrices $\tilde{A} = [6]$ and $\tilde{B} = [0]$. We continue now with the reduced sets

$$\lambda^{(3)} = \{2, 2, -1\}, \qquad \mu^{(3)} = \{5, 2, 0\}, \qquad \nu^{(3)} = \left\{ \frac{9+\sqrt{13}}{2}, \frac{9-\sqrt{13}}{2}, 1 \right\}.$$

There are many tuples of values $\rho_{>}^{(3)}$ that satisfy the conditions of Step 4, but we choose the following values that have nice closed forms.

$$\rho_{>}^{(3)} = \left\{ 3+\sqrt{10}, 2, 3-\sqrt{10} \right\}.$$

In particular, we get

$$\begin{aligned} f(x) &= (x-2)(x-2)(x+1) \\ g(x) &= ((x-3)^2 - 10)(x-2) \\ h(x) &= ((x-9/2)^2 - 13/4)(x-1), \end{aligned}$$

and the orthogonality property is satisfied in the following way.

$$\frac{\sqrt{|h(3+\sqrt{10}) f(3+\sqrt{10})|}}{|g'(3+\sqrt{10})|} + \underbrace{\frac{\sqrt{|h(2) f(2)|}}{|g'(2)|}}_{=0} - \frac{\sqrt{|h(3-\sqrt{10}) f(3-\sqrt{10})|}}{|g'(3-\sqrt{10})|} = 0,$$

so

$$r_1 = 1, \quad r_2 = 1, \quad \text{and} \quad r_3 = -1.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

Note that the sign of r_2 does not matter here. Next, in Step 5, we get $\ell = 1$ and

$$i_1 = 2, \qquad j_1 = 1, \qquad j_2 = 3.$$

Then

$$\hat{f}(x) = (x-2)(x+1), \qquad \hat{g}(x) = ((x-3)^2 - 10).$$

Thus, in Step 6, we get

$$\sqrt{\frac{-\hat{g}(2)}{5\hat{f}'(2)}} = \sqrt{\frac{3}{5}} \qquad \text{and} \qquad \sqrt{\frac{-\hat{g}(-1)}{5\hat{f}'(-1)}} = \sqrt{\frac{2}{5}}, \qquad \text{which gives} \qquad v = \begin{bmatrix} \sqrt{\frac{3}{5}} \\ 0 \\ \sqrt{\frac{2}{5}} \end{bmatrix}.$$

In Step 7, $\mu_\delta = \mu_2 = 2$, so

$$\begin{aligned} w_1 &= w_{j_1} \\ &= \frac{1}{(2-(3+\sqrt{10}))|g'(3+\sqrt{10})|}\sqrt{\frac{h(3+\sqrt{10})\hat{g}(2)f(3+\sqrt{10})}{2\hat{f}'(2)}} - \frac{1}{(2-(3-\sqrt{10}))|g'(3-\sqrt{10})|}\sqrt{\frac{h(3-\sqrt{10})\hat{g}(2)f(3-\sqrt{10})}{2\hat{f}'(2)}} = \frac{-1}{\sqrt{3}}, \end{aligned}$$

$$w_2 = w_{i_1} = \sqrt{\frac{-h(2)}{2g'(2)}} = \frac{1}{\sqrt{6}},$$

$$\begin{aligned} w_3 &= w_{j_2} \\ &= \frac{1}{(-1-(3+\sqrt{10}))|g'(3+\sqrt{10})|}\sqrt{\frac{h(3+\sqrt{10})\hat{g}(-1)f(3+\sqrt{10})}{2\hat{f}'(-1)}} - \frac{1}{(-1-(3-\sqrt{10}))|g'(3-\sqrt{10})|}\sqrt{\frac{h(3-\sqrt{10})\hat{g}(-1)f(3-\sqrt{10})}{2\hat{f}'(-1)}} = \frac{1}{\sqrt{2}}. \end{aligned}$$

In Step 8, we get $\hat{A} = \begin{bmatrix} 2 & & \\ & 2 & \\ & & -1 \end{bmatrix}$ and

$$\hat{B} = 5vv^T + 2ww^T = \begin{bmatrix} 3 & 0 & \sqrt{6} \\ 0 & 0 & 0 \\ \sqrt{6} & 0 & 2 \end{bmatrix} + \begin{bmatrix} \frac{2}{3} & \frac{-\sqrt{2}}{3} & -\sqrt{\frac{2}{3}} \\ \frac{-\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 1 \end{bmatrix} = \begin{bmatrix} \frac{11}{3} & \frac{-\sqrt{2}}{3} & \sqrt{6}-\sqrt{\frac{2}{3}} \\ \frac{-\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{\sqrt{3}} \\ \sqrt{6}-\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 3 \end{bmatrix}.$$

Then for $A = \tilde{A} \oplus \hat{A}$ and $B = \tilde{B} \oplus \hat{B}$, we have $\sigma(A) = \lambda^{(4)}$, $\sigma(B) = \mu^{(4)}$, and $\sigma(A+B) = \nu^{(4)}$.

7. Rank 2 Horn - Lambda equals nu case. There remains one case to be addressed: when there exist $x, y \in [n]$ such that $\lambda_x = \nu_y$. As in previous sections, we can continue to assume without loss of generality that $\mu_1 > 0$ and that the sets of eigenvalues have already been reduced. Now Corollary 5.5 guarantees $\lambda_{x-1} \neq \lambda_x \neq \lambda_{x+1}$. Thus by interlacing, there are at most two ρ_ℓ's equal to λ_x. In particular,

$$\lambda_{x-1} \geq \rho_x \geq \lambda_x \geq \rho_{x+1} \geq \lambda_{x+1}.$$

Thus, it is possible $\rho_x = \lambda_x = \rho_{x+1}$. However, it may not be necessary. Even in this case, one can try implementing Algorithm 1. If $\rho_>^{(n)}$ satisfying the properties in Step 2 exists, then the algorithm will succeed. However, the possibility that $\rho_x = \lambda_x = \rho_{x+1}$ means we are no longer guaranteed to find such $\rho_>^{(n)}$. Thus we must address this case.

As previously noted, Corollary 5.5 gives us that the current case is disjoint from the case of repeated λ_i's. Thus we will assume all our λ_i's are distinct for simplicity of argument. Then we can combine the results of the current case with the results obtained in Section 6 to get the general algorithm.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

7.1. Simplest case. We begin by addressing the simplest instance. In particular, we assume only one λ_x equals to one ν_y. For further simplification of our argument, we assume $x = 1$, i.e. $\lambda_1 = \nu_y$. Since we assumed the eigenvalue sets were already reduced, Proposition 5.3, Proposition 5.4, and the interlacing conditions tell us

$$\nu_y = \begin{cases} \nu_2, & \text{if } \mu_2 > 0 \\ \nu_1 & \text{if } \mu_n < 0 \end{cases}.$$

We will later generalize to the case when multiple λ_i's equal multiple ν_k's.

Now given $\lambda_1 = \nu_y$, we assume that Algorithm 1 failed and thus it is necessary to have $\rho_1 = \lambda_1 = \rho_2$. Note that since we assumed for $i \neq 1$, $\lambda_i \neq \nu_k$ for all $k \in [n]$, $\rho_1 = \rho_2$ are the only repeated ρ_ℓ's. As before, take $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ and

$$\mu_\delta := \begin{cases} \mu_2, & \text{if } \mu_2 > 0 \\ \mu_n, & \text{if } \mu_n < 0 \end{cases}.$$

Then there exist real orthonormal vectors v and w such that

$$\sigma\left(\Lambda + \mu_1 vv^T + \mu_\delta ww^T\right) = \nu^{(n)}. \tag{7.21}$$

Define $\rho^{(n)}$ as before by $\rho^{(n)} := \sigma\left(\Lambda + \mu_1 vv^T\right)$. We continue to have the trace equality (4.10) and interlacing condition (4.11).

$$\sum_{\ell=1}^{n} \rho_\ell = \mu_1 + \sum_{\ell=1}^{n} \lambda_\ell,$$
$$\lambda_1 + \mu_1 \geq \rho_1 \geq \lambda_1 \geq \rho_2 \geq \lambda_2 \geq \cdots \geq \rho_n \geq \lambda_n.$$

All λ_i's are distinct and $\rho_1 = \lambda_1 = \rho_2$. Thus, Corollary 5.2 tells us the first entry of v is zero. Define the vector $\tilde{v} \in \mathbb{R}^{n-1}$ by

$$v = \begin{bmatrix} 0 \\ \tilde{v} \end{bmatrix}.$$

In general, Corollary 3.2 tells us that for f and g as defined in (2.2),

$$v_i^2 = \frac{-g(\lambda_i)}{\mu_1 f'(\lambda_i)}.$$

Then using that $\lambda_1 = \rho_1$, we can define

$$\tilde{g}(x) = \prod_{\ell=2}^{n} (x - \rho_\ell), \qquad \tilde{f}(x) = \prod_{\ell=2}^{n} (x - \lambda_\ell), \tag{7.22}$$

to get

$$\tilde{v}_i^2 = v_{i+1}^2 = \frac{-g(\lambda_{i+1})}{\mu_1 f'(\lambda_{i+1})} = \frac{-\tilde{g}(\lambda_{i+1})}{\mu_1 \tilde{f}'(\lambda_{i+1})}.$$

Now let U_ρ be a real orthogonal matrix such that

$$U_\rho^T \left(\Lambda + \mu_1 vv^T\right) U_\rho = \operatorname{diag}(\rho_1, \dots, \rho_n). \tag{7.23}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

In particular, since $\lambda_1 = \rho_1$ and $v_1 = 0$, we can take $U_\rho = [1] \oplus \tilde{U}_\rho$, where $\tilde{U}_\rho$ is an $(n-1)\times(n-1)$ real orthogonal matrix. Thus, we have

$$\left[\begin{array}{c|c} 1 & \\ \hline & \tilde{U}_\rho^T \end{array}\right]\left(\left[\begin{array}{c|ccc} \lambda_1 & & & \\ \hline & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{array}\right] + \mu_1 \begin{bmatrix} 0 \\ \tilde{v} \end{bmatrix}\begin{bmatrix} 0 & \tilde{v}^T \end{bmatrix}\right)\left[\begin{array}{c|c} 1 & \\ \hline & \tilde{U}_\rho \end{array}\right] = \left[\begin{array}{c|ccc} \rho_1 & & & \\ \hline & \rho_2 & & \\ & & \ddots & \\ & & & \rho_n \end{array}\right].$$

Removing the first row and column, this gives

$$\tilde{U}_\rho^T\left(\operatorname{diag}(\lambda_2,\dots,\lambda_n) + \mu_1\tilde{v}\tilde{v}^T\right)\tilde{U}_\rho = \operatorname{diag}(\rho_2,\dots,\rho_n).$$

Now, as mentioned previously, our assumptions guarantee $\rho_1 = \rho_2$ are the only repeated ρ_ℓ's, i.e., $\rho_2 > \rho_3 > \rho_4 > \cdots > \rho_n$. Thus, we can apply Lemma 3.3 to get that for some signs $s_2, s_3, \dots, s_n \in \{-1, 1\}$, the (i,j)th entry of $\tilde{U}_\rho$ is given by

$$\tilde{U}_\rho(i,j) = \begin{cases} s_{j+1}, & \rho_{j+1} = \lambda_{i+1} \\ 0, & \rho_{j+1} = \lambda_{k+1} \\ s_{j+1}\cdot \frac{v_{i+1}}{\lambda_{i+1}-\rho_{j+1}}\cdot\sqrt{\frac{\mu_1\tilde{f}(\rho_{j+1})}{\tilde{g}'(\rho_{j+1})}}, & \rho_{j+1} \neq \lambda_{\ell+1}\ \forall \ell \in [n-1] \end{cases}, \tag{7.24}$$

since $v_{i+1} = \tilde{v}_i$. Now (7.21) and (7.23) give us

$$\sigma\left(\operatorname{diag}(\rho_1,\dots,\rho_n) + \mu_\delta U_\rho^T w w^T U_\rho\right) = \nu^{(n)}. \tag{7.25}$$

Thus applying Theorem 2.2 again, we get the interlacing condition (4.12) as before.

$$\begin{cases} \rho_1 + \mu_\delta \ge \nu_1 \ge \rho_1 \ge \nu_2 \ge \rho_2 \ge \cdots \ge \nu_n \ge \rho_n, & \text{if } \mu_\delta = \mu_2 > 0 \\ \rho_1 \ge \nu_1 \ge \rho_2 \ge \cdots \ge \rho_n \ge \nu_n \ge \rho_n + \mu_\delta, & \text{if } \mu_\delta = \mu_n < 0 \end{cases}.$$

We would like to use Corollary 3.2 on (7.25), but we cannot yet since $\nu_y = \rho_1 = \rho_2$. To deal with this, define the real orthogonal 2×2 matrix

$$V = \begin{bmatrix} \xi\cos(\theta) & -\sin(\theta) \\ \xi\sin(\theta) & \cos(\theta) \end{bmatrix}, \tag{7.26}$$

for some angle $\theta \in \mathbb{R}$ and sign $\xi \in \{-1, 1\}$ such that

$$V^T \begin{bmatrix} (U_\rho^T w)_1 \\ (U_\rho^T w)_2 \end{bmatrix} = \begin{bmatrix} 0 \\ * \end{bmatrix}.$$

Since $\rho_1 = \rho_2$, we know

$$\left[\begin{array}{c|c} V^T & \\ \hline & I_{n-2} \end{array}\right]\begin{bmatrix} \rho_1 & & \\ & \ddots & \\ & & \rho_n \end{bmatrix}\left[\begin{array}{c|c} V & \\ \hline & I_{n-2} \end{array}\right] = \begin{bmatrix} \rho_1 & & \\ & \ddots & \\ & & \rho_n \end{bmatrix}.$$

Thus, (7.25) gives us

$$\sigma\left(\operatorname{diag}(\rho_1,\dots,\rho_n) + \mu_\delta (V^T \oplus I_{n-2}) U_\rho^T w w^T U_\rho (V \oplus I_{n-2})\right) = \nu^{(n)}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

Note that $\rho_1 = \nu_y$ and by construction, the first entry of $(V^T \oplus I_{n-2})U_\rho^T w$ is zero. Thus, if we define the real vector $z \in \mathbb{R}^{n-1}$ by

$$(V^T \oplus I_{n-2})U_\rho^T w = \begin{bmatrix} 0 \\ z \end{bmatrix}, \tag{7.27}$$

this means

$$\sigma\left(\operatorname{diag}(\rho_2, \ldots, \rho_n) + \mu_\delta z z^T\right) = \nu^{(n)} \backslash \{\nu_y\}.$$

Now we can use Corollary 3.2 to get that for $\tilde{g}$ as defined in (7.22) and $\tilde{h}$ defined as

$$\tilde{h}(x) = \prod_{\ell \neq y} (x - \nu_\ell),$$

$$z_i^2 = \frac{-\tilde{h}(\rho_{i+1})}{\mu_\delta \tilde{g}'(\rho_{i+1})}.$$

Combining this with (7.26) and (7.27), we get that for some signs $t_2, t_3, \ldots, t_n \in \{-1, 1\}$,

$$(U_\rho^T w)_1 = -t_2 \sin(\theta) \sqrt{\frac{-\tilde{h}(\rho_2)}{\mu_\delta \tilde{g}'(\rho_2)}},$$

$$(U_\rho^T w)_2 = t_2 \cos(\theta) \sqrt{\frac{-\tilde{h}(\rho_2)}{\mu_\delta \tilde{g}'(\rho_2)}},$$

$$(U_\rho^T w)_i = t_i \sqrt{\frac{-\tilde{h}(\rho_i)}{\mu_\delta \tilde{g}'(\rho_i)}}, \quad i = 3, 4, \ldots, n.$$

Now, using $U_\rho = [1] \oplus \tilde{U}_\rho$, where $\tilde{U}_\rho$ is as in (7.24), we get the analog to (4.18) is

$$w_1 = -t_2 \sin(\theta) \sqrt{\frac{-\tilde{h}(\rho_2)}{\mu_\delta \tilde{g}'(\rho_2)}},$$

and for $i = 2, 3, \ldots, n$,

$$\begin{aligned} w_i &= U_\rho(i,2) t_2 \cos(\theta) \sqrt{\frac{-\tilde{h}(\rho_2)}{\mu_\delta \tilde{g}'(\rho_2)}} + \sum_{j=3}^n U_\rho(i,j) t_j \sqrt{\frac{\tilde{h}(\rho_j)}{\mu_\delta \tilde{g}'(\rho_j)}} \\ &= \tilde{U}_\rho(i-1,1) t_2 \cos(\theta) \sqrt{\frac{-\tilde{h}(\rho_2)}{\mu_\delta \tilde{g}'(\rho_2)}} + \sum_{j=3}^n \tilde{U}_\rho(i-1,j-1) t_j \sqrt{\frac{\tilde{h}(\rho_j)}{\mu_\delta \tilde{g}'(\rho_j)}} \\ &= \begin{cases} s_2 t_2 \cos(\theta) \frac{v_i}{(\lambda_i - \rho_2)} \sqrt{\frac{\mu_1 \tilde{f}(\rho_2)}{\tilde{g}'(\rho_2)}} \sqrt{\frac{-\tilde{h}(\rho_2)}{\mu_\delta \tilde{g}'(\rho_2)}} + s_k t_k \sqrt{\frac{-\tilde{h}(\rho_k)}{\mu_\delta \tilde{g}'(\rho_k)}}, & \lambda_i = \rho_k \\ s_2 t_2 \cos(\theta) \frac{v_i}{(\lambda_i - \rho_2)} \sqrt{\frac{\mu_1 \tilde{f}(\rho_2)}{\tilde{g}'(\rho_2)}} \sqrt{\frac{-\tilde{h}(\rho_2)}{\mu_\delta \tilde{g}'(\rho_2)}} + \sum_{j=3}^n s_j t_j \frac{v_i}{(\lambda_i - \rho_j)} \sqrt{\frac{\mu_1 \tilde{f}(\rho_j)}{\tilde{g}'(\rho_j)}} \sqrt{\frac{-\tilde{h}(\rho_j)}{\mu_\delta \tilde{g}'(\rho_j)}}, & \lambda_i \neq \rho_\ell \ \forall \ell \in [n] \end{cases}. \end{aligned}$$

Proceeding as in the proof of Theorem 4.1, we get that the analog to the orthogonality condition (4.13) is

$$0 = s_2 t_2 \cos(\theta) \frac{\sqrt{|\tilde{h}(\rho_2) \tilde{f}(\rho_2)|}}{|\tilde{g}'(\rho_2)|} + \sum_{j=3}^n s_j t_j \frac{\sqrt{|\tilde{h}(\rho_j) \tilde{f}(\rho_j)|}}{|\tilde{g}'(\rho_j)|}, \tag{7.28}$$

where $s_j t_j \in \{-1, 1\}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

7.2. General case. Now, so far we were working under the assumption that $\rho_1 = \rho_2$ and all other ρ_ℓ's were distinct. In general, we define sets of indices $a_<^{(k)}$, $b_<^{(n-k)}$, and $c_<^{(n-2k)}$ by

$$\begin{aligned} a_<^{(k)} &:= \{j \in [n-1] : \rho_j = \rho_{j+1}\}, \\ b_<^{(n-k)} &:= [n] \backslash a_<^{(k)}, \\ c_<^{(n-2k)} &:= [n] \backslash \{a_1, a_1+1, a_2, a_2+1, \ldots, a_k, a_k+1\}. \end{aligned}$$

By interlacing we know that for $s \in [m]$,

$$\rho_{a_s} = \lambda_{a_s} = \begin{cases} \nu_{a_s+1}, & \mu_2 > 0 \\ \nu_{a_s}, & \mu_n < 0 \end{cases}. \tag{7.29}$$

Then define the polynomials

$$\tilde{f}(x) := \prod_{s=1}^{n-k} (x - \lambda_{b_s}), \qquad \tilde{g}(x) := \prod_{s=1}^{n-k} (x - \rho_{b_s}),$$

$$\tilde{h}(x) := \begin{cases} \prod_{s=1}^{k} (x - \nu_{a_s}) \prod_{s=1}^{n-2k} (x - \nu_{c_s}), & \mu_2 > 0 \\ \prod_{s=1}^{n-k} (x - \nu_{b_s}), & \mu_n < 0 \end{cases}.$$

Here we are defining $\tilde{h}$ such that we omit the ν_s's that are equal to the repeated ρ_{a_s}'s, as specified in (7.29). Then the orthogonality condition (7.28) becomes that for some real values $\theta_1, \ldots, \theta_k \in \mathbb{R}$ and signs $r_{a_1}, \ldots r_{a_k}, r_{c_1}, \ldots r_{c_{n-2k}} \in \{-1, 1\}$, we need

$$0 = \sum_{s=1}^{k} r_{a_s} \cos(\theta_s) \frac{\sqrt{|\tilde{h}(\rho_{a_s}) \tilde{f}(\rho_{a_s})|}}{|\tilde{g}'(\rho_{a_s})|} + \sum_{s=1}^{n-2k} r_{c_s} \frac{\sqrt{|\tilde{h}(\rho_{c_s}) \tilde{f}(\rho_{c_s})|}}{|\tilde{g}'(\rho_{c_s})|}. \tag{7.30}$$

The general algorithm is presented in Algorithm 2 in the Appendix. This comes from combining the above argument with the arguments of Section 6. When there are no repeated lambda values, as above, we have that $\ell = 0$ and thus $i_<^{(\ell)} = \emptyset$ in the general algorithm.

7.3. Example. To illustrate Algorithm 2 in the case when there is a repeated ρ value, we provide an example. Suppose we start with

$$\lambda^{(3)} = \{4, 3, 1\}, \qquad \mu^{(3)} = \{5, 9/2, 0\}, \qquad \nu^{(3)} = \left\{ \frac{145 + \sqrt{105}}{20}, \frac{145 - \sqrt{105}}{20}, 3 \right\}.$$

This time, we cannot reduce the sets of eigenvalues. Note that in this instance, Algorithm 1 will succeed in finding distinct $\rho_>^{(3)}$ (for example, implementing our algorithm in MATLAB, fmincon() will find $\rho_>^{(3)} = \{7.380652793049324, 3.551959039935155, 2.067388167015521\}$). However, to illustrate the current algorithm, we proceed with finding $\rho^{(3)}$ that has a repeated value. In particular, in Step 4, we can choose

$$\rho^{(3)} = \{7, 3, 3\}.$$

Then,

$$k = 1, \quad a_1 = 2, \quad b_1 = 1, \quad b_2 = 3, \quad c_1 = 1,$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

so since $\mu_2 = 9/2 > 0$,

$$\tilde{f}(x) = (x-4)(x-1)$$
$$\tilde{g}(x) = (x-7)(x-3)$$
$$\tilde{h}(x) = ((x-29/4)^2 - 21/80).$$

This gives

$$\frac{\sqrt{|\tilde{h}(7)\tilde{f}(7)|}}{|\tilde{g}'(7)|} = \frac{3\sqrt{10}}{20} \qquad \text{and} \qquad \frac{\sqrt{|\tilde{h}(3)\tilde{f}(3)|}}{|\tilde{g}'(3)|} = \frac{\sqrt{890}}{20}.$$

Thus, defining θ_1 such that $\cos(\theta_1) = \frac{3}{\sqrt{89}}$ and taking $r_1 = 1$, $r_2 = -1$, the orthogonality property is satisfied in the following way.

$$r_2 \cos(\theta_1) \frac{\sqrt{|\tilde{h}(3)\tilde{f}(3)|}}{|\tilde{g}'(3)|} + r_1 \frac{\sqrt{|\tilde{h}(7)\tilde{f}(7)|}}{|\tilde{g}'(7)|} = 0.$$

In Step 6 we find

$$\sqrt{\frac{-\tilde{g}(4)}{5\tilde{f}'(4)}} = \frac{1}{\sqrt{5}} \qquad \text{and} \qquad \sqrt{\frac{-\tilde{g}(1)}{5\tilde{f}'(1)}} = \frac{2}{\sqrt{5}}, \qquad \text{which gives} \qquad v = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix}.$$

Next, since $\cos(\theta_1) = \frac{3}{\sqrt{89}}$, we have $\sin(\theta_1) = \sqrt{\frac{80}{89}}$. Then in Step 7, we have $\mu_\delta = \mu_2 = 9/2$ and

$$w_1 = w_{b_1} = \frac{-3/\sqrt{89}}{(4-3)|\tilde{g}'(3)|} \sqrt{\frac{\tilde{h}(3)\tilde{g}(4)\tilde{f}(3)}{(9/2)\tilde{f}'(4)}} + \frac{1}{(4-7)|\tilde{g}'(7)|} \sqrt{\frac{\tilde{h}(7)\tilde{g}(4)\tilde{f}(7)}{(9/2)\tilde{f}'(4)}} = \frac{-2}{3\sqrt{5}},$$

$$w_2 = w_{a_1} = -\sqrt{\frac{80}{89}} \sqrt{\frac{-\tilde{h}(3)}{(9/2)\tilde{g}'(3)}} = \frac{-2\sqrt{2}}{3},$$

$$w_3 = w_{b_2} = \frac{-3/\sqrt{89}}{(1-3)|\tilde{g}'(3)|} \sqrt{\frac{\tilde{h}(3)\tilde{g}(1)\tilde{f}(3)}{(9/2)\tilde{f}'(1)}} + \frac{1}{(1-7)|\tilde{g}'(7)|} \sqrt{\frac{\tilde{h}(7)\tilde{g}(1)\tilde{f}(7)}{(9/2)\tilde{f}'(1)}} = \frac{1}{3\sqrt{5}}.$$

Then in Step 8 we get $A = \begin{bmatrix} 4 & & \\ & 3 & \\ & & 1 \end{bmatrix}$ and

$$B = 5vv^T + (9/2)ww^T = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix} + \begin{bmatrix} \frac{2}{5} & \sqrt{\frac{8}{5}} & \frac{-1}{5} \\ \sqrt{\frac{8}{5}} & 4 & -\sqrt{\frac{2}{5}} \\ \frac{-1}{5} & -\sqrt{\frac{2}{5}} & \frac{1}{10} \end{bmatrix} = \begin{bmatrix} \frac{7}{5} & \sqrt{\frac{8}{5}} & \frac{9}{5} \\ \sqrt{\frac{8}{5}} & 4 & -\sqrt{\frac{2}{5}} \\ \frac{9}{5} & -\sqrt{\frac{2}{5}} & \frac{41}{10} \end{bmatrix}.$$

We can verify $\sigma(A) = \lambda^{(3)}$, $\sigma(B) = \mu^{(3)}$, and $\sigma(A+B) = \nu^{(3)}$.

8. Connection to Li and Poon's problem. The Horn problem has a connection to a related problem, found in Li and Poon's 2003 paper [14]. The problem, which we will call the Li Poon problem, was, given eigenvalues $\lambda^{(n)}$, $\mu^{(m)}$, and $\nu^{(n+m)}$, when does there exist an $(n+m) \times (n+m)$ real symmetric matrix D

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

such that the eigenvalues of D are $\nu^{(n+m)}$, the eigenvalues of the top left $n \times n$ block of D are $\lambda^{(n)}$, and the eigenvalues of the bottom right $m \times m$ block of D are $\mu^{(m)}$? To illustrate the connection between the two problems, we state Li and Poon's result, with proof, from their 2003 paper [14, Theorem 2.2].

THEOREM 8.1. *Given tuples of real numbers $\lambda^{(n)}$, $\mu^{(m)}$, and $\nu^{(n+m)}$, set $a \geq |\min\{0, \lambda_n, \mu_m, \nu_{n+m}\}|$. The $(n+m)$-tuples*

$$\begin{aligned} &(\lambda_1 + a, \lambda_2 + a, \ldots, \lambda_n + a, \underbrace{0, \ldots, 0}_{m \text{ copies}}), \\ &(\mu_1 + a, \mu_2 + a, \ldots, \mu_m + a, \underbrace{0, \ldots, 0}_{n \text{ copies}}), \\ \text{and} \quad &(\nu_1 + a, \nu_2 + a, \ldots, \nu_{n+m} + a), \end{aligned} \tag{8.31}$$

satisfy the trace equality and Horn inequalities if and only if there exists an $(n+m) \times (n+m)$ real symmetric matrix

$$D = \left[\begin{array}{cccc|cccc} \lambda_1 & & & & & & & \\ & \lambda_2 & & & & \bigstar & & \\ & & \ddots & & & & & \\ & & & \lambda_n & & & & \\ \hline & & & & \mu_1 & & & \\ & \bigstar & & & & \mu_2 & & \\ & & & & & & \ddots & \\ & & & & & & & \mu_m \end{array}\right] \quad \textit{such that} \quad \sigma(D) = \nu^{(n+m)}. \tag{8.32}$$

Proof. First assume the $(n+m)$-tuples given in (8.31) satisfy the trace equality and Horn inequalities. Then by Theorem 2.1, we know there are real symmetric $(n+m) \times (n+m)$ matrices X and Y such that

$$\begin{aligned} \sigma(X) &= \{\lambda_1 + a, \ldots, \lambda_n + a, 0, \ldots, 0\}, \\ \sigma(Y) &= \{\mu_1 + a, \ldots, \mu_m + a, 0, \ldots, 0\}, \\ \sigma(X+Y) &= \{\nu_1 + a, \ldots, \nu_{n+m} + a\}. \end{aligned}$$

By definition of a, X and Y are both positive semidefinite. Further, X is rank at most n and Y is rank at most m. Thus, there is an $(n+m) \times n$ real matrix R and an $(n+m) \times m$ real matrix S such that

$$X = RR^T \qquad \text{and} \qquad Y = SS^T.$$

Then

$$X + Y = RR^T + SS^T = \left[\begin{array}{c|c} R & S \end{array}\right] \left[\begin{array}{c} R^T \\ \hline S^T \end{array}\right],$$

so the $(n+m) \times (n+m)$ matrix

$$\left[\begin{array}{c} R^T \\ \hline S^T \end{array}\right] \left[\begin{array}{c|c} R & S \end{array}\right] = \left[\begin{array}{c|c} R^T R & R^T S \\ \hline S^T R & S^T S \end{array}\right],$$

has the same eigenvalues as $X+Y$, namely $\nu_1 + a, \ldots, \nu_{n+m} + a$. Further, $R^T R$ has the same nonzero eigenvalues as RR^T, namely $\lambda_1 + a, \ldots, \lambda_n + a$. Analogously, we get that $S^T S$ has the eigenvalues $\mu_1 +$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

$a, \ldots, \mu_m + a$. Taking the $n \times n$ real orthogonal matrix U_1 and $m \times m$ real orthogonal matrix U_2 such that $U_1 R^T R U_1^T$ and $U_2 S^T S U_2^T$ are diagonal, we get that

$$\left[\begin{array}{c|c} U_1 & \\ \hline & U_2 \end{array}\right] \left[\begin{array}{c|c} R^T R & R^T S \\ \hline S^T R & S^T S \end{array}\right] \left[\begin{array}{c|c} U_1^T & \\ \hline & U_2^T \end{array}\right] - aI_{n+m},$$

satisfies (8.32).

For the other direction, assume there exists an $(n+m) \times (n+m)$ real symmetric matrix D satisfying (8.32). Then by definition of a, $D + aI_{n+m}$ is positive semidefinite, so there exists an $(n+m) \times (n+m)$ real matrix P such that $D + aI_{n+m} = P^T P$. Define the $(n+m) \times n$ matrix P_1 and $(n+m) \times m$ matrix P_2 by

$$P = \left[\begin{array}{c|c} P_1 & P_2 \end{array}\right].$$

Then

$$D + aI_{n+m} = P^T P = \left[\begin{array}{c} P_1^T \\ \hline P_2^T \end{array}\right] \left[\begin{array}{c|c} P_1 & P_2 \end{array}\right] = \left[\begin{array}{c|c} P_1^T P_1 & P_1^T P_2 \\ \hline P_2^T P_1 & P_2^T P_2 \end{array}\right]. \tag{8.33}$$

Thus by definition of D, we know $P_1^T P_1 = \operatorname{diag}(\lambda_1 + a, \ldots, \lambda_n + a)$ and $P_2^T P_2 = \operatorname{diag}(\mu_1 + a, \ldots, \mu_m + a)$. Take

$$A = P_1 P_1^T \qquad \text{and} \qquad B = P_2 P_2^T.$$

Then A and B are both $(n+m) \times (n+m)$ real symmetric matrices. Further,

$$\sigma(A) = \{\lambda_1 + a, \ldots, \lambda_n + a, 0, \ldots, 0\} \quad \text{and} \quad \sigma(B) = \{\mu_1 + a, \ldots, \mu_m + a, 0, \ldots, 0\}.$$

Also,

$$A + B = P_1 P_1^T + P_2 P_2^T = \left[\begin{array}{c|c} P_1 & P_2 \end{array}\right] \left[\begin{array}{c} P_1^T \\ \hline P_2^T \end{array}\right],$$

so by (8.33) and the fact that $\sigma(D) = \nu^{(n+m)}$,

$$\sigma(A + B) = \{\nu_1 + a, \ldots, \nu_{n+m} + a\}.$$

Thus, the $(n+m)$-tuples specified in (8.31) must satisfy the trace equality and Horn inequalities. ∎

This result and its proof allow us to use our solution to the rank 2 Horn problem to solve the Li Poon problem when $m = 2$. In particular, if we are given $\lambda^{(n)}$, $\mu^{(2)}$, and $\nu^{(n+2)}$, we can solve the Horn problem for the $(n+2)$-tuples as specified in (8.31). Then, as in the presented proof of Theorem 8.1, we can use this solution to solve the Li Poon problem.

9. The next step. In this paper, we developed a general algorithm, Algorithm 2, for finding a real symmetric solution to the rank 2 Horn problem. Note that, in particular, this algorithm gives us a solution to the general 3×3 Horn problem. This is because we can always shift the eigenvalues so that one of the μ_j's is zero. Then we are in a rank 2 situation. While the 3×3 Horn problem was previously solved in [3], the algorithm there only gave Hermitian solutions. Our algorithm here gives real symmetric solutions.

Suppose we wanted to generalize the ideas used here to solve higher rank Horn problems. For a rank 3 matrix B, we would want to perform 3 rank 1 updates. Then we would be looking for 3 orthonormal vectors v, w, and u such that

$$\sigma\left(\operatorname{diag}(\lambda_1, \ldots, \lambda_n) + \mu_1 v v^T + \mu_2 w w^T + \mu_3 u u^T\right) = \nu^{(n)}.$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

We would need two sets of intermediate eigenvalues

$$\rho^{(n)} := \sigma\left(\operatorname{diag}(\lambda_1, \ldots, \lambda_n) + \mu_1 vv^T\right),$$
$$\tau^{(n)} := \sigma\left(\operatorname{diag}(\lambda_1, \ldots, \lambda_n) + \mu_1 vv^T + \mu_2 ww^T\right).$$

This would give rise to two trace equality conditions and three interlacing conditions. Then we would also need three orthogonality conditions since we require $v \perp w$, $w \perp u$, and $v \perp u$. Increasing the rank of B, of course yields more requirements. We believe that a key ingredient for solving the higher rank case is to have a good understanding of the orthogonality condition (4.13). Indeed, as illustrated in Subsection 4.3, the key is to find a polynomial $g(x)$ that "interlaces" the given polynomials $f(x)$ and $h(x)$, while at the same time satisfying an equation like (4.20). The fact that we seek $r_i \in \{-1, 1\}$, means in fact that the quantities $\frac{\sqrt{|h(\rho_i)f(\rho_i)|}}{|g'(\rho_i)|}$ need to be "balanced," in the sense that a subset of them needs to yield the same sum as the complementary set. A deeper understanding of this requirement will be helpful in the higher rank cases.

Acknowledgment. We thank the reviewer for their thorough reading of our paper and for their extensive and helpful comments.

REFERENCES

[1] J.R. Bunch, C.P. Nielsen, and D.C. Sorensen. Rank-one modification of the symmetric eigenproblem. *Numer. Math.*, 31(1):31–48, 1978/79.
[2] L. Cao and H.J. Woerdeman. A normal variation of the Horn problem: The rank 1 case. *Ann. Funct. Anal.*, 5(2):138–146, 2014.
[3] L. Cao and H.J. Woerdeman. Real zero polynomials and A. Horn's problem. *Linear Algebra Appl.*, 552:147–158, 2018.
[4] K. Fan and G. Pall. Imbedding conditions for Hermitian and normal matrices. *Canadian J. Math.*, 9:298–304, 1957.
[5] C. Franks. Operator scaling with specified marginals. In: *STOC'18—Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing.* ACM, New York, 190–203, 2018.
[6] W.C. Franks. *A Simple Algorithm for Horn's Problem and Two Results on Discrepancy.* PhD thesis, Rutgers The State University of New Jersey, 2019.
[7] G.H. Golub. Some modified matrix eigenvalue problems. *SIAM Rev.*, 15:318–334, 1973.
[8] A. Grinshpan, D.S. Kaliuzhnyi-Verbovetskyi, V. Vinnikov, and H.J. Woerdeman. Stable and real-zero polynomials in two variables. *Multidimens. Syst. Signal Process.*, 27(1):1–26, 2016.
[9] A. Horn. Eigenvalues of sums of Hermitian matrices. *Pacific J. Math.*, 12:225–241, 1962.
[10] R.A. Horn and C.R. Johnson. *Matrix Analysis*, 2nd edition. Cambridge University Press, Cambridge, 2013.
[11] A.A. Klyachko. Stable bundles, representation theory and Hermitian operators. *Selecta Math. (N.S.)*, 4(3):419–445, 1998.
[12] A. Knutson and T. Tao. The honeycomb model of $\mathrm{GL}_n(\mathbf{C})$ tensor products. I. Proof of the saturation conjecture. *J. Amer. Math. Soc.*, 12(4):1055–1090, 1999.
[13] A. Knutson, T. Tao, and C. Woodward. The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone. *J. Amer. Math. Soc.*, 17(1):19–48, 2004.
[14] C.-K. Li and Y.-T. Poon. Principal submatrices of a Hermitian matrix. *Linear Multilinear Algebra*, 51(2):199–208, 2003.
[15] L. Mirsky. Matrices with prescribed characteristic roots and diagonal elements. *J. London Math. Soc.*, 33:14–21, 1958.
[16] J.F. Queiró and A.P. Santana. The inverse Horn problem. *Electron. J. Linear Algebra*, 39:90–93, 2023.
[17] R.C. Thompson. The behavior of eigenvalues and singular values under perturbations of restricted rank. *Linear Algebra Appl.*, 13(1–2):69–78, 1976. Collection of articles dedicated to Olga Taussky Todd.
[18] H. Wielandt. *Topics in the Analytic Theory of Matrices.* Department of Mathematics, University of Wisconsin, Madison, Wisconsin, 1967.
[19] J.H. Wilkinson. *The Algebraic Eigenvalue Problem.* Clarendon Press, Oxford, 1965.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

Appendix. Here we present the general algorithm for finding real symmetric solutions to the rank 2 Horn problem.

Algorithm 2 Solving the Rank 2 Horn Problem

Given: Real tuples $\lambda^{(n)}$, $\mu^{(n)}$, and $\nu^{(n)}$ satisfying the trace equality and Horn inequalities.

Step 1: Reduce the sets of eigenvalues. In particular, if we remove $\lambda_{\alpha_1}, \dots, \lambda_{\alpha_m}$, $\mu_{\beta_1}, \dots, \mu_{\beta_m}$, and $\nu_{\gamma_1}, \dots, \nu_{\gamma_m}$, ordered such that for all $\ell \in [m]$, $\lambda_{\alpha_\ell} + \mu_{\beta_\ell} = \nu_{\gamma_\ell}$. Then set

$$\tilde{A} = \operatorname{diag}(\lambda_{\alpha_1}, \lambda_{\alpha_2}, \dots, \lambda_{\alpha_m}), \quad \text{and} \quad \tilde{B} = \operatorname{diag}(\mu_{\beta_1}, \mu_{\beta_2}, \dots, \mu_{\beta_m}).$$

Step 2: Now consider only the remaining eigenvalues, renumbered 1 to $n-m$, still in nonincreasing order. Assume at this point that exactly two remaining μ_ℓ's are nonzero.

Step 3: If $0 > \mu_{n-m-1} \geq \mu_{n-m}$, negate all eigenvalues. Renumber so they are in nonincreasing order.

Step 4: Find $\rho^{(n-m)}$ such that for sets

$$a_<^{(k)} := \{j \in [n-m-1] : \rho_j = \rho_{j+1}\}, \qquad b_<^{(n-m-k)} := [n-m] \backslash a_<^{(k)},$$
$$c_<^{(n-m-2k)} := [n-m] \backslash \{a_1, a_1+1, a_2, a_2+1, \dots, a_k, a_k+1\},$$

and polynomials

$$\tilde{f}(x) := \prod_{s=1}^{n-m-k} (x - \lambda_{b_s}), \quad \tilde{g}(x) := \prod_{s=1}^{n-m-k} (x - \rho_{b_s}), \quad \tilde{h}(x) := \begin{cases} \prod_{s=1}^{k} (x - \nu_{a_s}) \prod_{s=1}^{n-m-2k} (x - \nu_{c_s}), & \mu_2 > 0 \\ \prod_{s=1}^{n-m-k} (x - \nu_{b_s}), & \mu_n < 0 \end{cases},$$

the following hold.

(i) (Trace equality) $\sum_{s=1}^{n-m} \rho_s = \sum_{s=1}^{n-m} \lambda_s + \mu_1$.

(ii) (Interlacing Property 1) $\lambda_1 + \mu_1 \geq \rho_1 \geq \lambda_1 \geq \rho_2 \geq \lambda_2 \geq \cdots \geq \rho_{n-m} \geq \lambda_{n-m}$.

(iii) (Interlacing Property 2) $\begin{cases} \rho_1 + \mu_2 \geq \nu_1 \geq \rho_1 \geq \nu_2 \geq \rho_2 \geq \cdots \geq \nu_{n-m} \geq \rho_{n-m}, & \text{if } \mu_2 > 0 \\ \rho_1 \geq \nu_1 \geq \rho_2 \geq \nu_2 \geq \cdots \geq \nu_{n-m} \geq \rho_{n-m} + \mu_{n-m}, & \text{if } \mu_{n-m} < 0 \end{cases}.$

(iv) (Orthogonality Property) $\sum_{s=1}^{k} r_{a_s} \cos(\theta_s) \frac{\sqrt{|\tilde{h}(\rho_{a_s}) \tilde{f}(\rho_{a_s})|}}{|\tilde{g}'(\rho_{a_s})|} + \sum_{s=1}^{n-m-2k} r_{c_s} \frac{\sqrt{|\tilde{h}(\rho_{c_s}) \tilde{f}(\rho_{c_s})|}}{|\tilde{g}'(\rho_{c_s})|} = 0$ for some real values $\theta_1, \dots, \theta_k \in \mathbb{R}$ and signs $r_{a_1}, \dots, r_{a_k}, r_{c_1}, \dots r_{c_{n-m-2k}} \in \{-1, 1\}$.

Step 5: Define

$$i_<^{(\ell)} := \{i \in [n-m] : \lambda_i = \lambda_{i-1}\}, \quad j_<^{(n-m-k-\ell)} := b^{(n-m-k)} \backslash i_<^{(\ell)}, \quad d_<^{(n-m-2k-\ell)} := c_<^{(n-m-2k)} \backslash i_<^{(\ell)},$$

$$\hat{f}(x) = \prod_{s=1}^{n-m-k-\ell} (x - \lambda_{j_s}), \qquad \hat{g}(x) = \prod_{s=1}^{n-m-k-\ell} (x - \rho_{j_s}).$$

Step 6: Define the vector v such that $v_{a_s} = 0$, $v_{i_s} = 0$, and $v_{j_s} = \sqrt{\frac{-\hat{g}(\lambda_{j_s})}{\mu_1 \hat{f}'(\lambda_{j_s})}}$.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 41, pp. 304-337, May 2025.

Step 7: For $\mu_\delta = \begin{cases}\mu_2, & \text{if } \mu_2 > 0\\ \mu_n, & \text{if } \mu_n < 0\end{cases}$, define the vector w by $w_{i_s} = \sqrt{\frac{-\tilde{h}(\rho_{i_s})}{\mu_\delta \tilde{g}'(\rho_{i_s})}}$, $w_{a_s} = -\sin(\theta_s)\sqrt{\frac{-\tilde{h}(\rho_{a_s})}{\mu_\delta \tilde{g}'(\rho_{a_s})}}$, and

$$w_{j_s} = \begin{cases} \left(\sum_{t=1}^{k} \frac{r_{a_t}\cos(\theta_t)}{(\lambda_{j_s}-\rho_{a_t})|\tilde{g}'(\rho_{a_t})|}\sqrt{\frac{\tilde{f}(\rho_{a_t})\hat{g}(\lambda_{j_s})\tilde{h}(\rho_{a_t})}{\mu_\delta \hat{f}'(\lambda_{j_s})}}\right) + r_{j_y}\sqrt{\frac{-\tilde{h}(\rho_{j_y})}{\mu_\delta \tilde{g}'(\rho_{j_y})}}, & \lambda_{j_s} = \rho_{j_y} \\ \left(\sum_{t=1}^{k} \frac{r_{a_t}\cos(\theta_t)}{(\lambda_{j_s}-\rho_{a_t})|\tilde{g}'(\rho_{a_t})|}\sqrt{\frac{\tilde{f}(\rho_{a_t})\hat{g}(\lambda_{j_s})\tilde{h}(\rho_{a_t})}{\mu_\delta \hat{f}'(\lambda_{j_s})}}\right) & \\ \quad + \left(\sum_{t=1}^{n-m-2k-\ell} \frac{r_{d_t}}{(\lambda_{j_s}-\rho_{d_t})|\tilde{g}'(\rho_{d_t})|}\sqrt{\frac{\tilde{h}(\rho_{d_t})\hat{g}(\lambda_{j_s})\tilde{f}(\rho_{d_t})}{\mu_\delta \hat{f}'(\lambda_{j_s})}}\right), & \lambda_{j_s} \neq \rho_{j_y} \ \forall y \in [n-m-k-\ell]. \end{cases}$$

Step 8: If we negated all eigenvalues in Step 3, take $\hat{A} = -\operatorname{diag}(\lambda_1, \ldots, \lambda_{n-m})$ and $\hat{B} = -\mu_1 vv^T - \mu_\delta ww^T$. Else, take $\hat{A} = \operatorname{diag}(\lambda_1, \ldots, \lambda_{n-m})$ and $\hat{B} = \mu_1 vv^T + \mu_\delta ww^T$.

Output: $n \times n$ real symmetric matrices $A = \tilde{A} \oplus \hat{A}$ and $B = \tilde{B} \oplus \hat{B}$ satisfying $\sigma(A) = \lambda^{(n)}$, $\sigma(B) = \mu^{(n)}$, and $\sigma(A+B) = \nu^{(n)}$.