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EDGE-DISJOINT SPANNING TREES AND BALLOONS IN (MULTI-)GRAPHS FROM SIZE OR SPECTRAL RADIUS*

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Abstract. A multigraph is a graph that may have multiple edges, but has no loops. The multiplicity of a multigraph is the maximum number of edges between any pair of vertices. The spanning tree packing number of a graph G, denoted by $\tau(G)$, is the maximum number of edge-disjoint spanning trees contained in G. A balloon of a graph G is a maximal 2-edge-connected subgraph that is joined to the rest of G by exactly one cut edge. By $b(G)$, $e(G)$, and $\kappa(G)$, we denote the number of balloons, the size, and the vertex-connectivity of G, respectively. In this paper, we show that for a positive integer k and any multigraph G of order $n \geq 2r$ with multiplicity $m \leq k$ and minimum degree $\delta \geq 2k$, if $e(G) \geq m[\binom{r}{2} + \binom{n-r}{2}] + k$, then $\tau(G) \geq k$, where $r = \lceil (\delta+1)/m \rceil$. This extends the result of Fan, Gu and Lin (J. Graph Theory, 2023). Analogous results involving the size to characterize $\kappa(G) \geq k$ or $b(G) \leq k-1$ of a multigraph G are also presented. In addition, we prove a tight sufficient condition to guarantee $b(G) \leq k-1$ in terms of the spectral radius of a simple graph G, with extremal graphs characterized.

Key words. Spanning tree packing, Vertex-connectivity, Balloon, Size, Spectral radius.

AMS subject classifications. 05C40, 05C50, 05C70.

1. Introduction.

A *multigraph* is a graph with possible multiple edges, but no loops. The *multiplicity* of a multigraph is the maximum number of edges between any pair of vertices. Clearly, a simple graph can be seen as a multigraph whose multiplicity is 1. In this paper, we consider finite undirected (multi-)graphs and k always denotes a positive integer. For graph theoretic notation and terminology not defined here, we refer to [1, 2].

Let $G = (V(G), E(G))$ be a (multi-)graph with vertex set $V(G)$ and edge (multi-)set $E(G)$. The *order* of G is its number of vertices, and the *size* is its number of edges, denoted by $|G|$ and $e(G)$, respectively. As usual, let K_n and $K_{t,n-t}$ be a complete graph and a complete bipartite graph of order n, respectively. In particular, $K_{1,n-1}$ is called a *star*, which is also written as S_n. For a vertex $v \in V(G)$, let $N_G(v)$ be the set of all neighbors of v in G and $d_G(v) = |N_G(v)|$ be the degree of v in G. In particular, $\delta(G) = \min_{v\in V(G)} d_G(v)$ is the *minimum degree* of G. The vertex with degree $n-1$ in a star S_n is called the *center* of S_n. For simplicity, we may omit the subscripts G for our notations when there is no danger of confusion. The *vertex-connectivity* of a connected graph G, denoted by $\kappa(G)$, is the minimum cardinality of a vertex-cut of G.

The study of edge-disjoint spanning trees has been shown to be very important to graphs and has many applications in fault-tolerance networks as well as network reliability [5, 14]. So it is quite interesting to explore how many edge-disjoint spanning trees there are in a given graph. For a connected graph G, the *spanning tree packing number*, denoted by $\tau(G)$, is the maximum number of edge-disjoint spanning trees in G. A survey on $\tau(G)$ can be found in [26]. In 1961, Nash-Williams and Tutte, independently, gave a sufficient and necessary condition to guarantee that a graph contains at least k edge-disjoint spanning trees (see Theorem 2.2 in Section 2).

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Inspired by Kirchhoff's MatrixTree Theorem and a problem posed by Seymour, Cioabă and Wong [4] started to study the spanning tree packing number via the second largest eigenvalue of the adjacency matrix of a simple graph. From then on, more and more researchers focused their efforts on the relationship between $\tau(G)$ and the eigenvalues of a graph G. For more results on this topic, we refer to [3, 6, 9, 11, 15–19, 21, 28]. Recently, Fan, Gu and Lin [7] studied the spanning tree packing number in terms of size. They proved that for a connected simple graph G with minimum degree $\delta \geq 2k$ and order $n \geq 2(\delta+1)$, if $e(G) \geq \binom{\delta+1}{2}+\binom{n-\delta-1}{2}+k$, then $\tau(G) \geq k$. Our first main result extends their conclusion from simple graphs to multigraphs.

Theorem 1.1. *Let G be a connected multigraph with multiplicity $m \leq k$, minimum degree $\delta \geq 2k$ and order $n \geq 2r$. If $e(G) \geq m\left[\binom{r}{2}+\binom{n-r}{2}\right]+k$, then $\tau(G) \geq k$, where $r = \lceil(\delta+1)/m\rceil$.*

Remark 1. *By constructing a class of graphs, Fan, Gu and Lin [7] presented that the condition in Theorem 1.1 is tight when $m = 1$.*

Our second main result establishes a sufficient condition by the size of a multigraph G with given multiplicity to ensure $\kappa(G) \geq k$.

Theorem 1.2. *Let G be a multigraph with multiplicity m, minimum degree $\delta \geq mk$, and order n. If $e(G) \geq m[\binom{k-1}{2}+\binom{p}{2}+\binom{n-p-k+1}{2}+(k-1)(n-k+1)]+1$, then $\kappa(G) \geq k$, where $p = \lceil\delta/m\rceil - k + 2$.*

Remark 2. *The condition in Theorem 1.2 is tight when $m = 1$. Let K_n^m be the graph obtained from K_n by replacing each edge with m multiple edges. Obviously, $K_n^1 = K_n$. For $n \geq 2n_1 + s$, let $H_{n,n_1}^{m,s}$ be the graph obtained from $K_s^m \cup K_{n_1}^m \cup K_{n-s-n_1}^m$ by adding all possible multiple edges between $V(K_s^m)$ and $V(K_{n_1}^m) \cup V(K_{n-s-n_1}^m)$. Taking $m = 1$, we have $p = \delta - k + 2$. It is easy to check that the graph $H_{n,p}^{1,k-1}$ has multiplicity 1, minimum degree δ and $e(H_{n,p}^{m,k-1}) = \binom{k-1}{2}+\binom{p}{2}+\binom{n-p-k+1}{2}+(k-1)(n-k+1)$, but $\kappa(H_{n,p}^{m,k-1}) < k$.*

Given a simple graph G, its *adjacency matrix* $A(G)$ is an $n \times n$ 0-1 square matrix whose (u, v)-entry is 1 if and only if the vertices u and v are adjacent in G. As $A(G)$ is real symmetric, all of its eigenvalues are real, and so we can always arrange them as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. In particular, $\lambda_1(G)$ is referred to as the *spectral radius* of G. Let $D(G)$ be *degree diagonal matrix* of G. Then we call $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ the *Laplacian matrix* and the *signless Laplacian matrix* of G, respectively. We may arrange the eigenvalues of $L(G)$ and $Q(G)$ by $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$ and $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$, respectively.

For a (multi-)graph G, a *cut edge* of G is an edge whose deletion increases the number of components. A maximal 2-edge-connected subgraph of G incident to exactly one cut edge of G is called a *balloon* in G. Denote by $b(G)$ the number of balloons in G. The problem of counting the number of balloons in a graph was originally raised by O and West [25]. A cut edge that joins a balloon to the rest of the graph is called a *string*, the vertex incident to the cut edge is called the *neck* of the balloon. O and West [25] showed that $b(G)$ has a close relationship with the matching number, the number of cut edges, and the total domination number.

Very recently, the relationship between $b(G)$ and the eigenvalues of a (multi-)graph G has been studied in the literature. O and Cioabă [24] showed that, for $k \geq 3$ and any connected d-regular simple graph G, if $\lambda_k(G) \leq \theta(d)$, then $b(G) \leq k - 1$, where $d \geq 3$ is an odd integer and $\theta(d)$ is the largest root of $x^3 - (d-2)x^2 - 2dx + d - 1 = 0$. Gu [10] showed that for $k \geq 3$ and any connected multigraph G with multiplicity m and minimum degree δ, $b(G) \leq k - 1$ if $\mu_{n-k+1}(G) \geq \frac{1}{l}$ or $\lambda_k(G) \leq \delta - \frac{1}{l}$ or $q_k(G) \leq 2\delta - \frac{1}{l}$, where $l = \max\{\lceil(\delta+1)/m\rceil, 2\}$. Gu and Liu [12] studied the relationship between the ratio $\frac{\mu_{n-1}(G)}{\mu_1(G)}$ and $b(G)$ of a simple graph G.

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Motivated by the above results, we aim to find sufficient conditions on the size or the spectral radius which assure that $b(G) \leq k-1$. To formulate our counting results for balloons, we need to introduce some graphs.

Recall that K_n^m is the graph obtained from K_n by replacing each edge with m multiple edges. Let $\mathcal{G}^{m,k}_{n_0,n_1,\dots,n_k}$ be the set of multigraphs obtained from $K^m_{n_0}, K^m_{n_1}, \dots, K^m_{n_k}$ by adding an edge to connect one vertex in $K^m_{n_0}$ and one vertex in $K^m_{n_i}$ for each $i \in \{1,\dots,k\}$. For convenience, we use $\mathcal{G}^{k}_{n_0,n_1,\dots,n_k}$ for $\mathcal{G}^{1,k}_{n_0,n_1,\dots,n_k}$ (see Figure 1, both graphs are in $\mathcal{G}^3_{6,5,5,5}$). For any positive integer q, we denote

$$\mathcal{N}^{m,k}_{n,q} = \left\{ G \in \mathcal{G}^{m,k}_{n_0,n_1,\dots,n_k} : \min\{n_0,n_1,\dots,n_k\} \geq q, \sum_{i=0}^{k} n_i = n \right\},$$

For convenience, we use $\mathcal{N}^k_{n,q}$ for $\mathcal{N}^{1,k}_{n,q}$.

Let S_{k+1} be a star with vertex set $\{v_0, v_1, \dots, v_k\}$, where v_0 is the center of S_{k+1}. If $n \geq \sum_{j=1}^k n_j + 1$, then denote by $U^k_{n,(n_1,\dots,n_k)}$ the n-vertex simple graph obtained from S_{k+1} by identifying vertex v_i of S_{k+1} with a vertex of K_{n_i}, where $0 \leq i \leq k$ and $n_0 = n - \sum_{j=1}^k n_j$. If $n_i = q$ for each $i \in \{1,\dots,k\}$, then we set $U^k_{n,q} = U^k_{n,(q,\dots,q)}$ for short (see Fig. 1).

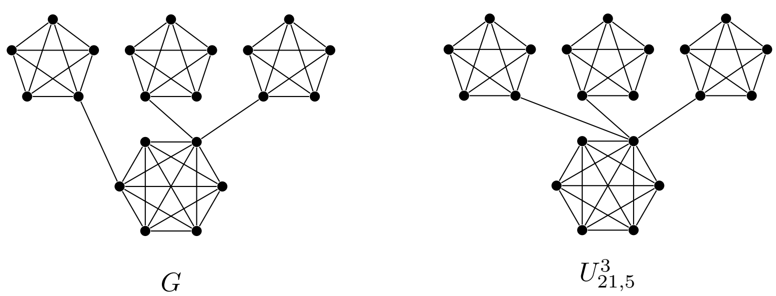

FIGURE 1. *The number of balloons of G (resp. $U^3_{21,5}$) is 3.*

Our third main result establishes a sufficient condition by the size of a multigraph G with given multiplicity to ensure $b(G) \leq k-1$.

THEOREM 1.3. *Let G be a connected multigraph with multiplicity m, minimum degree $\delta > k \geq 3$ and order n, and let $l = \max\{\lceil(\delta+1)/m\rceil, 2\}$. If $e(G) \geq m[k\binom{l}{2} + \binom{n-kl}{2}] + k + 1$, then $b(G) \leq k-1$.*

REMARK 3. *Taking $m = 1$ gives $l = \max\{\delta+1, 2\} = \delta+1$. Then for $n \geq (k+1)(\delta+1)$, any graph G in $\mathcal{G}^k_{n-k\delta-k,\delta+1,\dots,\delta+1}$ has minimum degree δ, and size $e(G) = m[k\binom{l}{2} + \binom{n-kl}{2}] + k$, but $b(G) = k$. This implies that the sufficient condition in Theorem 1.3 is tight when $m = 1$.*

We then focus on a spectral analog for a simple graph G. We succeed in discovering a sufficient condition for $b(G) \leq k-1$ via the spectral radius and characterize the unique spectral extremal graph $U^k_{n,\delta+1}$ among the structural extremal graph family $\mathcal{N}^k_{n,\delta+1}$.

THEOREM 1.4. *Let G be an n-vertex connected simple graph with minimum degree $\delta > k \geq 3$, where $n \geq k(\delta+1) + k + 1 + \lceil\frac{\delta}{2}\rceil$. If $\lambda_1(G) \geq \rho(n,\delta,k)$, then $b(G) \leq k-1$ unless $G \cong U^k_{n,\delta+1}$, where $\rho(n,\delta,k)$ is the largest zero of $P(x)$, where*

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$$P(x) = x^4 + (3 + k\delta + k - n - \delta)x^3 + [3 + (\delta - 2)(n - k\delta - k) - 3\delta - k]x^2 + [(2\delta + k - 1)(n - k\delta - k) + \delta k + 1 - 3\delta - 3k]x + (\delta + k - \delta k)(n - k\delta - k) + 2\delta k - 2k - \delta. \tag{1.1}$$

The remainder of this paper is organized as follows: In Section 2, we give some essential definitions and some necessary preliminaries, including spanning tree packing theorem (Nash-Williams 1961; Tutte 1961) and quotient matrices. Section 3 is entirely devoted to the proofs for our main results. Some concluding remarks are given in the last section.

2. Preliminaries. In this section, we present some of the preliminaries and former results to be used in our arguments.

For any subset $U \subseteq V(G)$, $d(U)$ denotes the number of edges each of which has exactly one endpoint in U, that is $d(U) = e(U, V(G) \setminus U)$.

Lemma 2.1 (Gu [10]). *Let G be a multigraph with multiplicity m and minimum degree δ, and U be a nonempty proper subset of $V(G)$. If $d(U) < \delta$, then $|U| \geq l$, where $l = \max\{\lceil(\delta+1)/m\rceil, 2\}$.*

Assume that U and W are two disjoint vertex subsets of $V(G)$. Then let $e(U, W)$ denote the number of edges between U and W in G. For any partition $\pi = (V_1, \ldots, V_t)$ of $V(G)$, let $e_G(\pi)$ be the number of edges in G whose endpoints lie in different parts of π, that is, $e_G(\pi) = \sum_{1 \leq i < j \leq t} e(V_i, V_j)$.

Theorem 2.2 (Nash-Williams [22] and Tutte [27]). *Let G be a connected graph. Then $\tau(G) \geq k$ if and only if for any partition π of $V(G)$, $e_G(\pi) \geq k(t-1)$, where t is the number of parts in the partition π.*

By the Perron–Frobenius Theorem, if G is connected, then $\lambda_1(G)$ is simple, and there exists a positive eigenvector, say $\mathbf{x}$, corresponding to $\lambda_1(G)$, which is called the *Perron vector* of G. It will be convenient for us to normalize so that the usual Euclidean norm is 1. Hereafter, we write x_v for the entry of $\mathbf{x}$ corresponding to the vertex $v \in V(G)$.

Lemma 2.3 (Nikiforov [23]). *Let G be a connected graph with vertex set $\{v_1, v_2, \ldots, v_n\}$, and let $\mathbf{x} = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^T$ be the Perron vector of $A(G)$ corresponding to $\lambda_1(G)$. If there exist v_i, v_j in $V(G)$ such that $N_G(v_i) \setminus \{v_j\} = N_G(v_j) \setminus \{v_i\}$, then $x_{v_i} = x_{v_j}$.*

Lemma 2.4 (Liu, Lu and Tian [20], Wu, Xiao and Hong [29]). *Let G be a connected graph and let $\mathbf{x}$ be the Perron vector of $A(G)$. Assume $u, v \in V(G)$ and $w_1, w_2, \ldots, w_s \in N_G(v) \setminus N_G(u)$ with $s \geq 1$. Let G^* be the graph obtained from G by deleting the edges vw_i and adding the edges uw_i for $1 \leq i \leq s$. If $x_u \geq x_v$, then $\lambda_1(G) < \lambda_1(G^*)$.*

Let M be an $n \times n$ real matrix, whose rows and columns are indexed by $V = \{1, 2, \ldots, n\}$. Assume that $\pi = (V_1, V_2, \ldots, V_t)$ is a partition of V. Then M can be partitioned based on π as

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1t} \\ \vdots & \ddots & \vdots \\ M_{t1} & \cdots & M_{tt} \end{bmatrix},$$

where M_{ij} denotes the submatrix of M, indexed by the rows and columns of V_i and V_j, respectively. Let b_{ij} be the average row sum of M_{ij} for $1 \leq i, j \leq t$. Usually, the $t \times t$ matrix $M_\pi = (b_{ij})$ is called the *quotient matrix* of M. The previous partition is *equitable* if for each $1 \leq i, j \leq t$, any row sum of M_{ij} equals to b_{ij}.

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Lemma 2.5 (Brouwer and Haemers [2], Godsil and Royle [8], Haemers [13]). *Let M be a real symmetric matrix, and let M_π be its equitable quotient matrix. Then the eigenvalues of the quotient matrix M_π are eigenvalues of M. Furthermore, if M is nonnegative and irreducible, the spectral radius of M_π equals the spectral radius of M.*

Lemma 2.6. *If $n \geq k(\delta+1)+3$, then $\lambda_1(U^k_{n,\delta+1}) = \rho(n,\delta,k)$, where $\rho(n,\delta,k)$ is the largest zero of $P(x)$ defined in (1.1).*

Proof. Recall that $U^k_{n,\delta+1}$ is an n-vertex simple graph obtained from S_{k+1} by identifying each leaf of S_{k+1} with a vertex of $K_{\delta+1}$ and identifying the center of S_{k+1} with a vertex of $K_{n-k(\delta+1)}$. Let $n_0 = n - k(\delta+1)$, then $U^k_{n,\delta+1}$ contains K_{n_0} as a proper subgraph. We partition the vertex set $V(U^k_{n,\delta+1})$ into the following four parts: $n_0 - 1$ vertices of $V(K_{n_0})$ with degree $n_0 - 1$, the vertex of $V(K_{n_0})$ with degree $n_0 + k - 1$, k vertices of degree $\delta+1$, and the remaining $k\delta$ vertices of degree δ. Clearly, this partition, say π, is equitable, and the corresponding quotient matrix is

$$M_\pi = \begin{bmatrix} n_0-2 & 1 & 0 & 0 \\ n_0-1 & 0 & k & 0 \\ 0 & 1 & 0 & \delta \\ 0 & 0 & 1 & \delta-1 \end{bmatrix}.$$

By an elementary computation, we get the characteristic polynomial of M_π as

$$\begin{aligned} P_0(x) =& x^4 + (3-n_0-\delta)x^3 + (\delta n_0 + 3 - 3\delta - k - 2n_0)x^2 + (1+\delta k + 2\delta n_0 + kn_0 - 3\delta - 3k - n_0)x + \\ & \delta n_0 + kn_0 + 2\delta k - 2k - \delta - \delta k n_0. \end{aligned}$$

Substituting $n_0 = n - k(\delta+1)$ into $P_0(x)$ gives (1.1). Recall that the partition π is equitable and $\rho(n,\delta,k)$ is the largest root of $P(x) = 0$. By Lemma 2.5, $\lambda_1(U^k_{n,\delta+1}) = \rho(n,\delta,k)$, as desired. ◻

3. Proofs of our main results.

In this section, we present the proofs of Theorems 1.1–1.4.

3.1. The proof of Theorem 1.1.

In order to give the proof of Theorem 1.1, we need the following lemmas.

Lemma 3.1 (Fan, Gu and Lin [7]). *Let a and b be two positive integers. If $a \geq b$, then*

$$\binom{a}{2} + \binom{b}{2} < \binom{a+1}{2} + \binom{b-1}{2}.$$

Lemma 3.2. *Let a and b be two integers. If $b \geq a \geq 0$ and $b \geq 3$, then*

$$a + \binom{b-a}{2} \leq \binom{b}{2}.$$

Proof. Note that $b \geq a \geq 0$ and $b \geq 3$. Then

$$\binom{b}{2} - a - \binom{b-a}{2} = \frac{a(2b-a-3)}{2} \geq 0,$$

as required. ◻

Lemma 3.3. *Let a and b be two integers. If $b \geq a \geq 1$, then*

$$\binom{a}{2} + \binom{b-a+1}{2} \leq \binom{b}{2}.$$

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Proof. Note that $b \geq a \geq 1$. Then

$$\binom{b}{2} - \binom{a}{2} - \binom{b-a+1}{2} = (b-a)(a-1) \geq 0,$$

as desired. □

For any partition π of $V(G)$, we call a part *trivial* if it contains only one single vertex, otherwise we call it *nontrivial*. For any subset $U \subseteq V(G)$, let $G[U]$ be the subgraph of G *induced* by U, and let $e_G(U)$ be the size of $G[U]$.

Proof of Theorem 1.1. We show our result by contradiction. Suppose that $\tau(G) \leq k-1$. By Theorem 2.2, there exists a partition of $V(G)$, say $\pi = (v_1, \ldots, v_{t_1}, V_1, \ldots, V_{t_2})$, with t_1 trivial parts $v_1, \ldots, v_{t_1}$ and t_2 nontrivial parts $V_1, \ldots, V_{t_2}$, such that

$$e_G(\pi) \leq k(t-1) - 1, \tag{3.2}$$

where $t = t_1 + t_2$. If $t = 1$, then $e_G(\pi) \leq k(t-1) - 1 = -1$, which is impossible, so $t \geq 2$. We first claim that $t_2 \geq 2$. Suppose to the contrary that $t_2 \leq 1$. Then $t_1 = t - t_2 \geq t - 1$. Note that $d_G(v_i) \geq \delta$ for $1 \leq i \leq t_1$. Combining this with $\delta \geq 2k$ gives

$$e_G(\pi) = \frac{1}{2}\left(\sum_{1\leq i\leq t_1} d_G(v_i) + \sum_{1\leq j\leq t_2} d(V_j)\right) \geq \frac{1}{2}\sum_{1\leq i\leq t_1} d_G(v_i) \geq \frac{1}{2}\delta t_1 \geq k(t-1),$$

a contradiction to (3.2).

Claim 1. *π contains at least two nontrivial parts (say V_1, V_2), such that $d(V_i) \leq \delta - 1$ for $i = 1, 2$.*

Proof of Claim 1. Suppose that the partition π contains at most one nontrivial part, say V_a $(1 \leq a \leq t_2)$, such that $d(V_a) \leq \delta - 1$. Then $d(V_i) \geq \delta$ for all $i \in \{1, \ldots, t_2\} \setminus \{a\}$. Since G is connected, we have $d(V_a) \geq 1$. Note that $\delta \geqslant 2k$ and $t_1 + t_2 = t$. Hence,

$$\begin{aligned} e_G(\pi) &= \frac{1}{2}\left[\sum_{1\leq j\leq t_1} d_G(v_j) + \sum_{1\leq i\leq t_2} d(V_i)\right] \\ &\geq \frac{1}{2}\left[\delta t_1 + (t_2 - 1)\delta + 1\right] \\ &\geq \frac{1}{2}\left[2k(t-1) + 1\right] \\ &> k(t-1), \end{aligned}$$

this contradicts (3.2). □

By Claim 1 and Lemma 2.1, we have $|V_i| \geq l$ for $i = 1, 2$, where $l = \max\{\lceil(\delta+1)/m\rceil, 2\}$. Since $\delta \geq 2k$ and $m \leq k$. We have $\lceil(\delta+1)/m\rceil \geq 3$, and thus

$$l = \max\{\lceil(\delta+1)/m\rceil, 2\} \geq 3. \tag{3.3}$$

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If $|V_1| = \max\{|V_1|, |V_2|, \ldots, |V_{t_2}|\}$ or $|V_2| = \max\{|V_1|, |V_2|, \ldots, |V_{t_2}|\}$, then without loss of generality, we assume that $|V_1| = \max\{|V_1|, |V_2|, \ldots, |V_{t_2}|\}$. Since $|V_i| \geq l$ and $|V_j| \geq 2$ for $i = 1, 2$, and $3 \leq j \leq t_2$, respectively, we have

$$l \leq |V_1| = n - t_1 - \sum_{j=3}^{t_2} |V_j| - |V_2| \leq n - t_1 - 2(t_2 - 2) - l. \tag{3.4}$$

Combining (3.3) with (3.4) gives us

$$n - (t + l - 2) = n - t_1 - (t_2 - 2) - l \geq l + (t_2 - 2) \geq \max\{3, t_2 - 2\}. \tag{3.5}$$

Hence,

$$\begin{aligned}
\sum_{1\leq i\leq t_2} e_G(V_i) &= e_G(V_1) + e_G(V_2) + \sum_{3\leq i\leq t_2} e_G(V_i) \\
&\leq m\left[\binom{|V_1|}{2} + \binom{|V_2|}{2} + \sum_{3\leq i\leq t_2}\binom{|V_i|}{2}\right] \\
\text{(by Lemma 3.1)} \quad &\leq m\left[\binom{n - t_1 - 2(t_2 - 2) - l}{2} + \binom{l}{2} + (t_2 - 2)\binom{2}{2}\right] \\
&= m\left[\binom{n - (t + l - 2) - (t_2 - 2)}{2} + \binom{l}{2} + (t_2 - 2)\right] \\
\text{(by (3.5) and Lemma 3.2)} \quad &\leq m\left[\binom{l}{2} + \binom{n - (t + l - 2)}{2}\right].
\end{aligned}$$

If there exists a nontrivial part, say V_j, such that $|V_j| = \max\{|V_1|, |V_2|, \ldots, |V_{t_2}|\}$ for some $j \in \{3, \ldots, t_2\}$, then

$$l \leq \max\{|V_1|, |V_2|\} \leq |V_j| = n - t_1 - |V_1| - |V_2| - \sum_{3\leq i\leq t_2, i\neq j} |V_i| \leq n - t_1 - 2l - 2(t_2 - 3). \tag{3.6}$$

Together with (3.3) and (3.6), we get

$$n - (t + 2l - 3) = n - t_1 - 2l - (t_2 - 3) \geq l + (t_2 - 3) \geq \max\{3, t_2 - 3\}. \tag{3.7}$$

Thus,

$$\begin{aligned}
\sum_{1\leq i\leq t_2} e_G(V_i) &= e_G(V_1) + e_G(V_2) + \sum_{3\leq i\leq t_2} e_G(V_i) \\
&\leq m\left[\binom{|V_1|}{2} + \binom{|V_2|}{2} + \sum_{3\leq i\leq t_2}\binom{|V_i|}{2}\right] \\
\text{(by Lemma 3.1)} \quad &\leq m\left[2\binom{l}{2} + (t_2 - 3)\binom{2}{2} + \binom{n - t_1 - 2l - 2(t_2 - 3)}{2}\right] \\
&= m\left[2\binom{l}{2} + (t_2 - 3) + \binom{n - (t + 2l - 3) - (t_2 - 3)}{2}\right] \\
\text{(by (3.7) and Lemma 3.2)} \quad &\leq m\left[2\binom{l}{2} + \binom{n - (t + 2l - 3)}{2}\right].
\end{aligned}$$

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By Lemma 3.3, we have

$$\binom{l}{2}+\binom{n-(t+2l-3)}{2}\le\binom{n-(t+l-2)}{2}. \tag{3.8}$$

Now, combining (3.2), (3.7), (3.8) with $\sum_{1\le i\le t_1} e_G(v_i)=0$, we have

$$\begin{aligned}
e(G) &= \sum_{1\le i\le t_2} e_G(V_i)+\sum_{1\le i\le t_1} e_G(v_i)+e_G(\pi)\\
&\le m\cdot\max\left\{\binom{l}{2}+\binom{n-(t+l-2)}{2},\,2\binom{l}{2}+\binom{n-(t+2l-3)}{2}\right\}+k(t-1)-1\\
&= m\left[\binom{l}{2}+\binom{n-(t+l-2)}{2}\right]+k(t-1)-1\\
&= \frac{mt^2}{2}+\left(-mn+lm-\frac{3m}{2}+k\right)t+m-k-2lm+\frac{3mn}{2}+l^2m+\frac{mn^2}{2}-lmn-1.
\end{aligned}$$

Let $g(t)=\frac{mt^2}{2}+\left(-mn+lm-\frac{3m}{2}+k\right)t+m-k-2lm+\frac{3mn}{2}+l^2m+\frac{mn^2}{2}-lmn-1$ be a real function in t for $t\in[2,n-2l+2]$. Taking the derivative of $g(t)$ gives us $g'(t)=mt+(-mn+lm-\frac{3m}{2}+k)$. Hence,

$$\begin{aligned}
&(\text{since } t\le n-2l+2) & g'(t) &\le m\left(-l+\frac{1}{2}\right)+k\\
&(\text{by (3.3)}) & &= m\left(-\left\lceil\frac{\delta+1}{m}\right\rceil+\frac{1}{2}\right)+k\\
& & &\le -\delta-1+\frac{m}{2}+k\\
&(\text{since } m\le k) & &\le -\delta-1+\frac{3k}{2}\\
&(\text{since } \delta\ge 2k) & &\le -\frac{k}{2}-1\\
& & &< 0.
\end{aligned}$$

Then $g(t)$ is decreasing with respect to $2\le t\le n-2l+2$. Therefore, $g(t)\le g(2)$.

By (3.3),

$$e(G)\le g(t)\le g(2)=m\left[\binom{l}{2}+\binom{n-l}{2}\right]+k-1=m\left[\binom{r}{2}+\binom{n-r}{2}\right]+k-1,$$

where $r=\lceil(\delta+1)/m\rceil$, which contradicts the hypothesis. This completes the proof. □

3.2. Proof of Theorem 1.2. In this subsection, we present the proof of Theorems 1.2. In order to show Theorem 1.2, we need the following lemma.

LEMMA 3.4. *Let G be a multigraph of order n with minimum degree δ and multiplicity m. Let S be an arbitrary vertex-cut with κ vertices and A be the minimum component of $G-S$. Then $|A|\ge\max\{\lceil\delta/m\rceil-\kappa+1,1\}$.*

Proof. For each vertex $v\in V(A)$, one sees that v sends at most m multiple edges to each of the vertices in $(V(A)\setminus\{v\})\cup S$. Thus, we have

$$\delta|A|\le\sum_{v\in V(A)} d_G(v)\le m|A|(|A|+\kappa-1),$$

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which implies $|A| \geq \lceil \delta/m \rceil - \kappa + 1$. Note that $V(A)$ is nonempty. Therefore, $|A| \geq \max\{\lceil \delta/m \rceil - \kappa + 1, 1\}$, as required. ∎

For two graphs H and G, $H \subseteq G$ means that H is a subgraph of G. Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let S be an arbitrary minimum vertex-cut and A be a minimum component of $G - S$. Clearly, G is a spanning subgraph of $H^{m,|S|}_{n,|A|}$.

Suppose to the contrary that $\kappa = \kappa(G) \leq k - 1$. Combining $\delta \geq mk$ with Lemma 3.4 gives us

$$|A| + \kappa \geq \max\{\lceil \delta/m \rceil + 1, \kappa + 1\} = \lceil \delta/m \rceil + 1. \tag{3.9}$$

Due to the structure of $H^{m,s}_{n,n_1}$, it is easy to see that $G \subseteq H^{m,|S|}_{n,|A|} \subseteq H^{m,k-1}_{n,c}$, where $c = |A| - (k - 1 - \kappa)$. According to (3.9), we get

$$c = |A| - (k - 1 - \kappa) \geq \lceil \delta/m \rceil - k + 2 = p.$$

Since A is the minimum component of $G - S$, we know that $n \geq 2|A| + \kappa \geq 2c + k - 1$. Then

$$e(G) \leq e(H^{m,k-1}_{n,c}),$$

with equality if and only if $G \cong H^{m,k-1}_{n,c}$. Combining this with Lemma 3.1 gives

$$\begin{aligned} e(G) \leq e(H^{m,k-1}_{n,c}) &= m\left[\binom{k-1}{2} + \binom{c}{2} + \binom{n-c-k+1}{2} + (k-1)(n-k+1)\right] \\ &\leq m\left[\binom{k-1}{2} + \binom{p}{2} + \binom{n-p-k+1}{2} + (k-1)(n-k+1)\right], \end{aligned}$$

which contradicts the hypothesis. This completes the proof. ∎

3.3. Proof of Theorem 1.3. In order to prove Theorem 1.3, we need the following key lemma.

LEMMA 3.5. *Let l be a positive integer, $k \geq 3$. If $G \in \mathcal{N}^{m,k}_{n,l}$, then*

$$e(G) \leq m\left[k\binom{l}{2} + \binom{n-kl}{2}\right] + k. \tag{3.10}$$

Proof. By the definition of $\mathcal{N}^{m,k}_{n,l}$, we assume that $G \in \mathcal{G}^{m,k}_{n_0,n_1,\ldots,n_k}$ with $\min\{n_0, \ldots, n_k\} \geq l$ and $\sum_{i=0}^{k} n_i = n$. Combining with Lemma 3.1 gives

$$\begin{aligned} e(G) &= m\left[\binom{n_0}{2} + \binom{n_1}{2} + \cdots + \binom{n_k}{2}\right] + k \\ &\leq m\left[k\binom{l}{2} + \binom{n-kl}{2}\right] + k, \end{aligned}$$

as desired. ∎

Now we are ready to present the proof of Theorem 1.3.

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Proof of Theorem 1.3. We prove our result by contradiction. Suppose that $b(G) \geq k$. By the definition of $b(G)$, there exist at least k pairwise disjoint 2-edge-connected subgraphs $B_1, B_2, \ldots, B_k$ such that $d_G(V(B_i)) = 1$ for $1 \leq i \leq k$. Note that $b(G) \geq k \geq 3$, one has $e(V(B_i), V(B_j)) = 0$ for $1 \leq i \neq j \leq k$. Let $S = V(G) \setminus \bigcup_{1 \leq i \leq k} V(B_i)$, then $d_G(S) = k < \delta$, and so $|S| \geq l$ by Lemma 2.1. For convenience, let $n_i = |V(B_i)|$ for $1 \leq i \leq k$. Recall that $d_G(V(B_i)) = 1 < \delta$. We have $n_i \geq l$ by Lemma 2.1. Then, $n = n_1 + \cdots + n_k + |S|$ and $\min\{|S|, n_1, \ldots, n_k\} \geq l$. Clearly, G is a spanning subgraph of some graph H in $\mathcal{N}_{n,l}^{m,k}$. Then

$$e(G) \leq e(H),$$

with equality if and only if $G \cong H$. Combining this with Lemma 3.5, we conclude that

$$e(G) \leq e(H) \leq m\left[k\binom{l}{2} + \binom{n-kl}{2}\right] + k,$$

which is a contradiction to the hypothesis. Hence, $b(G) \leq k - 1$. This completes the proof. ◻

3.4. The proof of Theorem 1.4. In this section, we present the proof of Theorem 1.4. Before doing that, we prove the following theorem, which characterizes the unique spectral extremal graph $U_{n,\delta+1}^k$ among the structural extremal graph family $\mathcal{N}_{n,\delta+1}^k$.

THEOREM 3.6. *Let $G \in \mathcal{N}_{n,\delta+1}^k$, where $\delta > k \geq 3$ and $n \geq k(\delta+1) + k + 1 + \lceil \frac{\delta}{2} \rceil$. Then $\lambda_1(G) \leq \lambda_1(U_{n,\delta+1}^k)$, with equality if and only if $G \cong U_{n,\delta+1}^k$.*

Proof. Choose G' among $\mathcal{N}_{n,\delta+1}^k$ such that its spectral radius is as large as possible. Then for every $G \in \mathcal{N}_{n,\delta+1}^k$, we have

$$\lambda_1(G) \leq \lambda_1(G'). \tag{3.11}$$

Without loss of generality, we assume $G' \in \mathcal{G}_{a_0,a_1,\ldots,a_k}^k$, where $a_1 \geq a_2 \geq \cdots \geq a_k$, $\min\{a_0, a_k\} \geq \delta + 1$ and $\sum_{i=0}^k a_i = n$. By the definition of $\mathcal{G}_{a_0,a_1,\ldots,a_k}^k$, there exist k balloons in G', i.e., $K_{a_1}, K_{a_2}, \ldots, K_{a_k}$. Then we partition $V(G')$ into $V_0 \cup V_1 \cup \cdots \cup V_k$ with $V_i = V(K_{a_i})$, for $0 \leq i \leq k$. Let $\mathbf{x}$ be the Perron vector of $A(G')$, and let $u_1 \in V_0$ with $x_{u_1} = \max_{u \in V_0} x_u$. We are to describe the structure of G' through the following four claims.

CLAIM 2. *Every neck of balloon K_{a_i} $(1 \leq i \leq k)$ in G' is adjacent to u_1.*

Proof of Claim 2. Suppose to the contrary that there exists a balloon K_{a_i} such that its neck (say w) is not adjacent to u_1. Note that $G' \in \mathcal{G}_{a_0,a_1,\ldots,a_k}^k$. Hence, there exists $u \in V_0 \setminus \{u_1\}$ such that $w \sim u$. Now let $G_1 = G' - wu + wu_1$. Clearly, $G_1 \in \mathcal{N}_{n,\delta+1}^k$. Combining $x_{u_1} \geq x_u$ with Lemma 2.4 gives us $\lambda_1(G_1) > \lambda_1(G')$, which contradicts the choice of G'. Thus, Claim 2 holds. ◻

By Claim 2, one sees that $G' \cong U_{n,(a_1,a_2,\ldots,a_k)}^k$. For convenience, let w_i be the neck of balloon K_{a_i} of G' for $i \in \{1, \ldots, k\}$.

CLAIM 3. *$x_{w_i} = \max_{v \in V(K_{a_i})} x_v$, for $i \in \{1, 2, \ldots, k\}$. Moreover, for any $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$, if $a_i \geq a_j$, then $x_{w_i} \geq x_{w_j}$ with equality if and only if $a_i = a_j$.*

Proof of Claim 3. By Claim 2, we have $d_{G'}(u_1) = a_0 + k - 1$. For $i \in \{1, 2, \ldots, k\}$, let

$$V(K_{a_i}) \setminus \{w_i\} = \{v_1^i, v_2^i, \ldots, v_{a_i-1}^i\}.$$

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By Lemma 2.3, one has $x_{v_1^i} = x_{v_t^i}$ for $t \in \{2, \ldots, a_i - 1\}$. Then by $A(G')\mathbf{x} = \lambda_1(G')\mathbf{x}$, we have

$$\lambda_1(G')x_{v_1^i} = x_{w_i} + (a_i - 2)x_{v_1^i} \quad \text{and} \quad \lambda_1(G')x_{w_i} = x_{u_1} + (a_i - 1)x_{v_1^i}. \tag{3.12}$$

Thus,

$$(\lambda_1(G') - a_i + 2)x_{v_1^i} = x_{w_i} \quad \text{and} \quad (\lambda_1(G') + 1)x_{v_1^i} = (\lambda_1(G') + 1)x_{w_i} - x_{u_1}. \tag{3.13}$$

Note that K_{a_i} is a proper subgraph of G' for $i \in \{1, 2, \ldots, k\}$. Then

$$\lambda_1(G') > \max_{1 \le i \le k}\{\lambda_1(K_{a_i})\} = \max_{1 \le i \le k}\{a_i - 1\}.$$

Together with the first part of (3.13), we have $x_{v_1^i} < x_{w_i}$ for $i \in \{1, 2, \ldots, k\}$, which implies $x_{w_i} = \max_{v \in V(K_{a_i})} x_v$. This completes the first part of our result.

Now, we prove the additional statement. Choose $j \in \{1, 2, \ldots, k\} \setminus \{i\}$. Then by the second part of (3.12) and the second part of (3.13), we have

$$\lambda_1(G')(x_{w_i} - x_{w_j}) = (a_i - 1)x_{v_1^i} - (a_j - 1)x_{v_1^j} \quad \text{and} \quad x_{v_1^i} - x_{v_1^j} = x_{w_i} - x_{w_j}. \tag{3.14}$$

By the latter of (3.14), we may deduce $\lambda_1(G')(x_{v_1^i} - x_{v_1^j}) = \lambda_1(G')(x_{w_i} - x_{w_j})$. Together with the first part of (3.14) one has $(\lambda_1(G') - a_i + 1)x_{v_1^i} = (\lambda_1(G') - a_j + 1)x_{v_1^j}$. If $a_i \ge a_j$, then $x_{v_1^i} \ge x_{v_1^j}$, with equality if and only if $a_i = a_j$. Regarding in the second part of (3.14), one has $x_{w_i} \ge x_{w_j}$. This completes the proof. □

Claim 4. $a_2 = \delta + 1$.

Proof of Claim 4. Suppose to the contrary that $a_2 \ge \delta + 2$. By Claim 2, we have $G' \cong U^k_{n,(a_1,a_2,\ldots,a_k)}$. Recall that $V_i = V(K_{a_i})$ and w_i is the neck of balloon K_{a_i} in G', for $i \in \{1, 2, \ldots, k\}$. By $A(G')\mathbf{x} = \lambda_1(G')\mathbf{x}$, we have

$$\lambda_1(G')x_{w_1} = x_{u_1} + \sum_{v \in V_1 \setminus \{w_1\}} x_v \quad \text{and} \quad \lambda_1(G')x_{w_2} = x_{u_1} + \sum_{v \in V_2 \setminus \{w_2\}} x_v. \tag{3.15}$$

Note that $a_1 \ge a_2$. Hence, by Claim 3, we have $x_{w_1} \ge x_{w_2}$. Combining this with (3.15) gives us

$$\sum_{v \in V_1} x_v \ge \sum_{v \in V_2} x_v. \tag{3.16}$$

We partition V_2 into $A_1 \cup A_2$ such that $|A_1| = \delta + 1$ and $e(A_2, V_0) = 0$. Then, $e(A_1, V_0) = 1$ and $|A_2| = a_2 - (\delta + 1) \ge 1$. Therefore,

$$\sum_{v \in V_2} x_v > \sum_{v \in A_1} x_v. \tag{3.17}$$

Let G_2 be a simple graph obtained from G' by deleting all edges between A_1 and A_2 and adding all possible edges between A_2 and V_1. Clearly, $G_2 \cong U^k_{n,(a_1+|A_2|,\delta+1,a_3,\ldots,a_k)}$. Thus, $G_2 \in \mathcal{N}^k_{n,\delta+1}$ and

$$\begin{aligned}
\lambda_1(G_2) - \lambda_1(G') &\ge \mathbf{x}^T(A(G_2) - A(G'))\mathbf{x} \\
&= 2\sum_{v \in A_2} x_v \left(\sum_{v \in V_1} x_v - \sum_{v \in A_1} x_v\right) \\
\text{(by (3.16) and (3.17))} \qquad &> 0,
\end{aligned}$$

i.e., $\lambda_1(G') < \lambda_1(G_2)$, a contradiction to the choice of G'. Thus, Claim 4 holds. □

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Due to Claim 4, we have $G' \cong U^k_{n,(a_1,\delta+1,\dots,\delta+1)}$.

Claim 5. $a_1 = \delta + 1$.

Proof of Claim 5. Note that $G' \cong U^k_{n,(a_1,\delta+1,\dots,\delta+1)}$. Suppose to the contrary that $a_1 \geq \delta + 2$. We label $V_1 = V(K_{a_1}) = \{v_1, v_2, \dots, v_{a_1}\}$ and $V_0 = V(K_{a_0}) = \{u_1, u_2, \dots, u_{a_0}\}$. Recall that $x_{u_1} = \max_{u\in V_0} x_u$. Let $x_{v_1} = \max_{v\in V_1} x_v$. Then by Claim 3, we have $w_1 = v_1$. Now, we partition V_1 into $F_1 \cup F_2$ satisfying $|F_1| = \delta + 1$ and $e(F_2, V_0) = 0$. Thus, $e(F_1, V_0) = 1$ and $|F_2| = a_1 - (\delta + 1) \geq 1$. We proceed by considering the following two possible cases.

Case 1. $\sum_{u\in V_0} x_u > \sum_{v\in F_1} x_v$.

Let G_3 be a simple graph obtained from G' by deleting all edges between F_1 and F_2 and adding all possible edges between F_2 and V_0. Clearly, $G_3 \cong U^k_{n,\delta+1}$. Thus, $G_3 \in \mathcal{N}^k_{n,\delta+1}$ and

$$\begin{aligned}\lambda_1(G_3) - \lambda_1(G') &\geq \mathbf{x}^T(A(G_3) - A(G'))\mathbf{x} \\ &= 2\sum_{v\in F_2} x_v \left(\sum_{u\in V_0} x_u - \sum_{v\in F_1} x_v\right) \\ &> 0,\end{aligned}$$

i.e., $\lambda_1(G') < \lambda_1(G_3)$, a contradiction to the choice of G'.

Case 2. $\sum_{u\in V_0} x_u \leq \sum_{v\in F_1} x_v$.

Note that K_{a_0} and K_{a_1} are proper subgraphs of G'. Then,

$$\lambda_1(G') > \max\{\lambda_1(K_{a_0}), \lambda_1(K_{a_1})\} = \max\{a_0 - 1, a_1 - 1\}. \tag{3.18}$$

Together with Lemma 2.3 and the definition of $U^k_{n,(a_1,\delta+1,\dots,\delta+1)}$, one has $x_{u_2} = x_{u_p}$ for $p \in \{3, \dots, a_0\}$ and $x_{v_2} = x_{v_q}$ for $q \in \{3, \dots, a_1\}$.

Now, by $A(G')\mathbf{x} = \lambda_1(G')\mathbf{x}$, we have

$$\lambda_1(G')x_{u_2} = x_{u_1} + (a_0 - 2)x_{u_2} \quad \text{and} \quad \lambda_1(G')x_{v_2} = x_{v_1} + (a_1 - 2)x_{v_2}.$$

Combining these with (3.18) gives us

$$x_{u_2} = \frac{x_{u_1}}{\lambda_1(G') - a_0 + 2} \quad \text{and} \quad x_{v_2} = \frac{x_{v_1}}{\lambda_1(G') - a_1 + 2}. \tag{3.19}$$

If $a_0 \geq a_1$, then by $\sum_{u\in V_0} x_u \leq \sum_{v\in F_1} x_v$, we have

$$x_{u_1} + (a_0 - 1)x_{u_2} = \sum_{u\in V_0} x_u \leq \sum_{v\in F_1} x_v = x_{v_1} + \delta x_{v_2}. \tag{3.20}$$

Substituting (3.19) in (3.20) gives us

$$\left(1 + \frac{a_0 - 1}{\lambda_1(G') - a_0 + 2}\right) x_{u_1} \leq \left(1 + \frac{\delta}{\lambda_1(G') - a_1 + 2}\right) x_{v_1}. \tag{3.21}$$

Together with (3.18), $a_0 \geq a_1$ and $a_1 \geq \delta + 2$, we obtain

$$\frac{a_0 - 1}{\lambda_1(G') - a_0 + 2} > \frac{\delta}{\lambda_1(G') - a_1 + 2}.$$

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Combining this with (3.21) gives

$$x_{u_1} < x_{v_1}.$$

Let G_4 be the simple graph obtained from G' by deleting edges between u_1 and w_i and adding edges between v_1 (i.e., w_1) and w_i, where $i \in \{2, \dots, k\}$. Clearly, $G_4 \in \mathcal{N}^k_{n,\delta+1}$. In view of (14) and Lemma 2.4, we have $\lambda_1(G_4) > \lambda_1(G')$, a contradiction to the choice of G'.

If $a_0 < a_1$, by $a_0 \geq \delta + 1$, then we are to show $a_0 = \delta + 1$. Suppose that $a_0 \geq \delta + 2$. Now, we partition V_0 into $D_1 \cup D_2$ such that $|D_1| = \delta + 1$ and $e(D_2, V(G') \setminus V_0) = 0$. Thus, $e(D_1, V(G') \setminus V_0) = k$ and $|D_2| = a_0 - (\delta + 1) \geq 1$. Recall that $\sum_{u \in V_0} x_u \leq \sum_{v \in F_1} x_v$ and $|F_2| \geq 1$. Then

$$\sum_{u \in D_1} x_u < \sum_{u \in V_0} x_u \leq \sum_{v \in F_1} x_v < \sum_{v \in V_1} x_v. \tag{3.22}$$

Let G_5 be the simple graph obtained from G' by deleting all edges between D_1 and D_2, and adding all possible edges between D_2 and V_1. Clearly, $G_5 \cong U^k_{n,(a_1+|D_2|,\delta+1,\dots,\delta+1)}$. Thus, $G_5 \in \mathcal{N}^k_{n,\delta+1}$, and

$$\begin{aligned} \lambda_1(G_5) - \lambda_1(G') &\geq \mathbf{x}^T (A(G_5) - A(G'))\mathbf{x} \\ &= 2 \sum_{u \in D_2} x_u \left(\sum_{v \in V_1} x_v - \sum_{u \in D_1} x_u \right) \\ \text{(by (3.22))} \qquad &> 0, \end{aligned}$$

i.e., $\lambda_1(G') < \lambda_1(G_5)$, a contradiction to the choice of G'. So we do indeed have $a_0 = \delta + 1$. Note that $a_1 > a_0, n \geq k(\delta + 1) + k + 1 + \lceil \frac{\delta}{2} \rceil$ and $a_0 + a_1 + (k - 1)(\delta + 1) = n$. Hence,

$$a_1 \geq \max \left\{ \delta + 2, k + 1 + \left\lceil \frac{\delta}{2} \right\rceil \right\}. \tag{3.23}$$

Note that $a_1 \geq \delta + 2$ and $a_2 = a_i = \delta + 1$ for $i \in \{3, \dots, k\}$. Hence, by Claim 3, we have $x_{w_2} = x_{w_i}$ for $i \in \{3, \dots, k\}$ and $x_{w_2} < x_{v_1}$. By $A(G')\mathbf{x} = \lambda_1(G')\mathbf{x}$, one sees

$$\lambda_1(G')x_{u_1} = \delta x_{u_2} + x_{v_1} + (k - 1)x_{w_2} < \delta x_{u_2} + kx_{v_1}.$$

Combining this with the first part of (3.19) and $a_0 = \delta + 1$ gives

$$\left(\lambda_1(G') - \frac{\delta}{\lambda_1(G') - \delta + 1} \right) x_{u_1} < kx_{v_1}. \tag{3.24}$$

By (3.18) and (3.23), we have

$$\begin{aligned} \lambda_1(G') - \frac{\delta}{\lambda_1(G') - \delta + 1} &> a_1 - 1 - \frac{\delta}{a_1 - \delta} \\ &\geq a_1 - 1 - \frac{\delta}{2} \\ &\geq k. \end{aligned}$$

Together with (3.24), we obtain $x_{u_1} < x_{v_1}$.

Let G_6 be the simple graph obtained from G' by deleting edges between u_1 and w_i and adding edges between v_1 (i.e., w_1) and w_i, $i = 2, \dots, k$. Clearly, $G_6 \cong U^k_{n,\delta+1}$. Note that $x_{u_1} < x_{v_1}$. Hence, by Lemma 2.4, we have $\lambda(G_6) > \lambda(G')$, which contradicts the choice of G'. Thus, Claim 5 holds. ∎

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By Claim 5, $G' \cong U^k_{n,\delta+1}$. Together with (3.11), we can deduce that $\lambda_1(G) \leq \lambda_1(U^k_{n,\delta+1})$, with equality if and only if $G \cong U^k_{n,\delta+1}$. This completes the proof of Theorem 3.6. ◻

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. We prove our result by contradiction. Suppose that $b(G) \geq k \geq 3$. By the definition of $b(G)$, there exist at least k pairwise disjoint 2-edge-connected subgraphs $B_1, B_2, \ldots, B_k$ such that $d_G(V(B_i)) = 1$ for $1 \leq i \leq k$. Note that $k \geq 3$. Then, $e\left(V(B_i), V(B_j)\right) = 0$ for $1 \leq i \neq j \leq k$.

Let $S = V(G) \setminus \bigcup_{1 \leq i \leq k} V(B_i)$. Then $d_G(S) = k < \delta$, and so $|S| \geq \delta + 1$ by Lemma 2.1. Let $n_i = |V(B_i)|$ for $1 \leq i \leq k$. Notice that $d_G(V(B_i)) = 1 < \delta$. Hence, by Lemma 2.1, we have $n_i \geq \delta + 1$. Then, $n = n_1 + \cdots + n_k + |S|$ and $\min\{|S|, n_1, \ldots, n_k\} \geq \delta + 1$. Clearly, G is a spanning subgraph of some graph H in $\mathcal{N}^k_{n,\delta+1}$. Then

$$\lambda_1(G) \leq \lambda_1(H),$$

with equality if and only if $G \cong H$. Combining this with $n \geq k(\delta + 1) + k + 1 + \lceil \frac{\delta}{2} \rceil$, Lemma 2.6 and Theorem 3.6 give us

$$\lambda_1(G) \leq \lambda_1(H) \leq \lambda_1(U^k_{n,\delta+1}) = \rho(n, \delta, k),$$

with equality if and only if $G \cong U^k_{n,\delta+1}$, a contradiction to the assumption $\lambda_1(G) \geq \rho(n, \delta, k)$ and $G \ncong U^k_{n,\delta+1}$. Hence, $b(G) \leq k - 1$. This completes the proof. ◻

4. Concluding remarks. In this paper, we mainly head in determining some relationships between the size or the spectral radius of a (muti-)graph G with $\tau(G), \kappa(G)$, or $b(G)$. Theorem 1.1 (resp. Theorem 1.2) establishes a sufficient condition by the size of a multigraph G with given multiplicity to ensure $\tau(G) \geq k$ (resp. $\kappa(G) \geq k$). Theorem 1.3 ensures $b(G) \leq k - 1$ in condition similar to the previous results, whereas Theorem 1.4 gives a sufficient condition by the spectral radius of a simple graph G to guarantee $b(G) \leq k-1$.

For any Hermitian matrix M, we use $\theta_i(M)$ to denote the i-th largest eigenvalue of M. Thus, $\lambda_i(G) = \theta_i(A(G))$, $q_i(G) = \theta_i(Q(G))$, and $\mu_i(G) = \theta_i(L(G))$. Inspired by Theorem 1.4, it is natural and interesting to count the number of balloons in a simple graph G via $\mu_1(G)$ or $q_1(G)$.

The following result is commonly referred to as the Weyl inequalities (see Page 29 of [2]).

THEOREM 4.1. *Let M and N be Hermitian matrices of order n. Then, for $1 \leq i, j \leq n$,*

(i) $\theta_i(M) + \theta_j(N) \leq \theta_{i+j-n}(M + N)$ *if* $i + j \geq n + 1$.
(ii) $\theta_i(M) + \theta_j(N) \geq \theta_{i+j-1}(M + N)$ *if* $i + j \leq n + 1$.

COROLLARY 4.2. *Let Δ be the maximum degree of a graph G. Then,*

(i) $\Delta + \lambda_1(G) \geq q_1(G)$.
(ii) $\mu_1(G) + \lambda_1(G) \geq \Delta$.

Proof. (i) By Theorem 4.1(ii), $\theta_1(D(G)) + \theta_1(A(G)) \geq \theta_1(D(G) + A(G)) = \theta_1(Q(G))$, i.e., $\Delta + \lambda_1(G) \geq q_1(G)$.

(ii) By Theorem 4.1(ii), $\theta_1(L(G)) + \theta_1(A(G)) \geq \theta_1(L(G) + A(G)) = \theta_1(D(G))$, hence, $\mu_1(G) + \lambda_1(G) \geq \Delta$. ◻

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Recall that $\rho(n, \delta, k)$ is the largest zero of $P(x)$ defined in (1.1). The subsequent result follows from Theorem 1.4 and Corollary 4.2.

THEOREM 4.3. *Let G be a connected simple graph with maximum degree Δ, minimum degree $\delta > k \geq 3$, and order $n \geq k(\delta+1)+k+1+\lceil \frac{\delta}{2} \rceil$.*

(i) *If $q_1(G) \geq \rho(n,\delta,k)+\Delta$, then $b(G) \leq k-1$ unless $G \cong U^k_{n,\delta+1}$.*

(ii) *If $\mu_1(G) \leq \Delta - \rho(n,\delta,k)$, then $b(G) \leq k-1$ unless $G \cong U^k_{n,\delta+1}$.*

Proof. (i) Combining the assumption with Corollary 4.2(i), we get

$$\Delta + \lambda_1(G) \geq q_1(G) \geq \rho(n,\delta,k) + \Delta.$$

Thus, $\lambda_1(G) \geq \rho(n,\delta,k)$. Now, the result follows from Theorem 1.4.

(ii) Combining the assumption with Corollary 4.2(ii), we get

$$\Delta - \lambda_1(G) \leq \mu_1(G) \leq \Delta - \rho(n,\delta,k).$$

Thus, $\lambda_1(G) \geq \rho(n,\delta,k)$. Now, the result follows from Theorem 1.4. ◻

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