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CHARACTERIZATIONS OF COMPLEX P-MATRICES*

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Abstract. A P-matrix is a square matrix all of whose principal minors are positive. The characterization of real P-matrices as matrices that do not reverse the sign of any nonzero real vector is generalized to complex P-matrices by associating them with the reflection of complex vectors. This prompts the extension of other P-matrix properties and related real matrix classes to the complex field. In particular, semipositivity of real P-matrices is generalized to complex P-matrices. Principal pivot transforms and Cayley transforms of complex P-matrices are also considered.

Key words. P-matrix, Principal Minors, Reflection, Semipositive matrix, Cone, Principal Pivot Transform, Cayley Transform.

AMS subject classifications. 15A48, 15A15, 15A23, 15A09, 15A18.

1. Introduction.

P-matrices, that is, matrices all of whose principal minors are positive, are ubiquitous in theory and applications. In most settings, P-matrices are real square matrices, because they historically arose as a generalization and unification of real matrix classes (e.g., M-matrices and symmetric positive definite matrices) and because of their prominent role in the study of systems of inequalities (e.g., the linear complementarity problem). Naturally, the defining property of P-matrices points to real matrices since the positivity of principal minors is easier to bring to bear in the real field. Nevertheless, there is theoretical and applied interest (e.g., in signal processing and quantum mechanics) in paying more attention to complex P-matrices, which are not rare; for example, hermitian positive definite matrices are P-matrices. We note that the class of complex P-matrices is nontrivial in the sense that they contain nonreal matrices that are neither hermitian positive definite, nor diagonally similar to real P-matrices. By definition, complex P-matrices have characteristic polynomials with real coefficients and have positive diagonal entries, although all non-diagonal entries of a complex P-matrix may be nonreal. In fact, all complex P-matrices can be constructed recursively by the method developed in [10].

Example 1.1. The complex P-matrix

$$A = \begin{bmatrix} 3.5 & -i & 1+i & \frac{1}{3} \\ -2i & 3 & -1-2i & 3-i \\ -1+i & -\frac{1}{2}+i & 1 & 2-i \\ \frac{2}{3} & -3-i & -4-2i & 2 \end{bmatrix},$$

is constructed in [10, Example 4.5]. Note that A is neither positive definite, nor diagonally similar to a real P-matrix. To see the latter, consider the cycle $1 \to 2 \to 3 \to 1$ in the directed graph of A and note that the product of the weights on that cycle is nonreal, and so it would be nonreal for any diagonal similarity of A.

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A well-known characterization of real $n \times n$ P-matrices (see Theorem 3.1) states that $A \in \mathbf{M}_n(\mathbb{R})$ is a P-matrix if and only if for every nonzero $x = [x_j] \in \mathbb{R}^n$, there exists at least one j such that $x_j\,(Ax)_j > 0$; that is, the jth entries of x and Ax are both nonzero and have the same sign. In a figure of speech, the real P-matrices are precisely the matrices that do not 'reverse the sign' of any nonzero real vector. Complex P-matrices also do not reverse the sign of any nonzero real vector; however, this is not a sufficient property to characterize complex P-matrices; see Remark 3.2. Other properties of real P-matrices are also implausible for complex P-matrices; for example, it is known that every real P-matrix A is semipositive (i.e., there exists an entrywise nonnegative $x \in \mathbb{R}^n$ such that Ax is entrywise positive), which is not necessarily the case for a complex P-matrix; see Example 4.5.

In this article, we define and extend the characterizations, properties, and matrix relations from real to complex P-matrices. To accomplish our goals, we generalize non-reversal of the sign of a real vector to the notion of non-reflexetivity. We also use a notion of cone semipositivity to extend the class of semipositive matrices into a class that contains the complex P-matrices. The Schur complements, principal pivot transforms, and Cayley transforms of complex P-matrices admit analogous treatment and extensions.

2. Preliminaries.

The set of complex (resp., real) $n\times n$ matrices is denoted by $\mathbf{M}_n(\mathbb{C})$ (resp. $\mathbf{M}_n(\mathbb{R})$). The following notation and concepts are used for vectors $x = [x_j] \in \mathbb{C}^n$ and matrices $A = [a_{ij}] \in \mathbf{M}_n(\mathbb{C})$:

- For a positive integer n, we denote $\langle n\rangle = \{1, 2, \ldots, n\}$.
- We refer to $x = [x_j] \in \mathbb{C}^n$ as *strictly nonzero* if $x_j \neq 0$ for all $j \in \langle n\rangle$.
- $A^T = [a_{ji}]$ and $A^* = [\overline{a_{ji}}]$ denote the *transpose* and *conjugate transpose* of A, respectively.
- We write $A \geq 0$ and $x \geq 0$ if $a_{ij} \geq 0$ and $x_j \geq 0$ for all $i, j \in \langle n\rangle$. We refer to such an A and x as *nonnegative*. Analogously, we consider *positive* arrays denoted by $A > 0$ and $x > 0$.
- We refer to $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$ as the *nonnegative orthant* of $\mathbb{R}^n$.
- $\sigma(A) = \{\lambda \in \mathbb{C} : \det(A - \lambda I) = 0\}$ denotes the *spectrum* of A.
- $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ is the *spectral radius* of A.
- Every nonempty subset $\alpha \subseteq \langle n\rangle$ under consideration is always arranged so that its elements are in the ascending order.
- We let α^c denote the complement of $\alpha \subseteq \langle n\rangle$ in $\langle n\rangle$ and $|\alpha|$ denote the cardinality of α.
- For $\alpha, \beta \subseteq \langle n\rangle$, $A[\alpha, \beta]$ denotes the submatrix of A that lies in the rows and columns indexed by α and β, respectively. When α or β is empty, the corresponding submatrix is considered vacuous.
- We abbreviate $A[\alpha, \alpha]$ by $A[\alpha]$ and refer to it as a *principal submatrix* of A.
- We refer to $\det A[\alpha]$ as a *principal minor* of A. By convention, if $\alpha = \emptyset$, then $\det A[\alpha] = 1$.
- We denote by $\mathrm{diag}(d_1, d_2, \ldots, d_n) \in \mathbf{M}_n(\mathbb{C})$ the diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$.
- It is convenient to denote the set of diagonal matrices $\mathrm{diag}(t_1, t_2, \ldots, t_n)$, where $t_j \in [0, 1]$ ($j = 1, 2, \ldots, n$), by the matrix interval $[0, I]$.

Next, we formally recall some concepts and matrix classes for which a good comprehensive reference is [7].

Definition 2.1. Given $A \in M_n(\mathbb{C})$, and $\alpha \subseteq \langle n\rangle$ such that $A[\alpha]$ is invertible, $A/A[\alpha]$ denotes the *Schur complement* of $A[\alpha]$, that is,

$$A/A[\alpha] = A[\alpha^c] - A[\alpha^c, \alpha]A[\alpha]^{-1}A[\alpha, \alpha^c].$$

Note that $\det(A/A[\alpha]) = \det A/\det(A[\alpha])$.

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Definition 2.2. Given a nonempty $\alpha \subseteq \langle n \rangle$ and provided that $A[\alpha]$ is invertible, we define the *principal pivot transform* of $A \in M_n(\mathbb{C})$ *relative* to α as the matrix $\mathrm{ppt}\,(A, \alpha)$ obtained from A by replacing

$$\begin{array}{llllll} A[\alpha] & \text{by} & A[\alpha]^{-1}, & A[\alpha,\alpha^c] & \text{by} & -A[\alpha]^{-1}A[\alpha,\alpha^c], \\ A[\alpha^c,\alpha] & \text{by} & A[\alpha^c,\alpha]A[\alpha]^{-1}, & \text{and } A[\alpha^c] & \text{by} & A/A[\alpha]. \end{array}$$

By convention, $\mathrm{ppt}\,(A, \emptyset) = A$.

To illustrate this definition, when $\alpha = \{1, \ldots, k\}$ $(0 < k < n)$, then

$$B = \mathrm{ppt}(A,\alpha) = \begin{bmatrix} A[\alpha]^{-1} & -A[\alpha]^{-1}A[\alpha,\alpha^c] \\ A[\alpha^c,\alpha]\,A[\alpha]^{-1} & A/A[\alpha] \end{bmatrix}.$$

The fundamental (defining) property of the principal pivot transform B above is that for any $x \in \mathbb{C}^n$,

$$A \begin{bmatrix} x[\alpha] \\ x[\alpha^c] \end{bmatrix} = \begin{bmatrix} y[\alpha] \\ y[\alpha^c] \end{bmatrix} \quad \text{if and only if} \quad B \begin{bmatrix} y[\alpha] \\ x[\alpha^c] \end{bmatrix} = \begin{bmatrix} x[\alpha] \\ y[\alpha^c] \end{bmatrix}.$$

For the history and theory of principal pivot transforms, see [9].

Definition 2.3. For $A \in M_n(\mathbb{C})$ with $-1 \notin \sigma(A)$, consider the fractional linear map $F_A = (I+A)^{-1}(I-A)$. This map is an involution, namely, $A = (I+F_A)^{-1}(I-F_A)$. The matrix F_A is refered to as the *Cayley transform* of A.

Definition 2.4. $A \in \mathbf{M}_n(\mathbb{C})$ is a *P-matrix* if for all $\alpha \subseteq \langle n \rangle$, $\det A[\alpha] > 0$. We then write $A \in \mathbb{P}_n(\mathbb{C})$ or $A \in \mathbb{P}_n(\mathbb{R})$ to distinguish its order and the field of entries.

Definition 2.5. A *permutation matrix* $Q \in \mathbf{M}_n(\mathbb{R})$ is a matrix all of whose entries belong to $\{0, 1\}$ and each row (and thus each column) of Q contains exactly one entry equal to 1. A *signature matrix* $S \in \mathbf{M}_n(\mathbb{R})$ is a diagonal matrix all of whose diagonal entries belong to $\{-1, 1\}$.

In the theorem below, we include some basic facts about P-matrices and outline their proofs, because they are essential to the generalizations pursued in this paper.

Theorem 2.6. *Assume that $A \in M_n(\mathbb{C})$ is a P-matrix. Then the following hold.*

(1) $A^T \in \mathbb{P}_n(\mathbb{C})$ *and* $A^* \in \mathbb{P}_n(\mathbb{C})$.
(2) $QAQ^T \in \mathbb{P}_n(\mathbb{C})$ *for every permutation matrix* $Q \in \mathbf{M}_n(\mathbb{R})$.
(3) $SAS \in \mathbb{P}_n(\mathbb{C})$ *for every signature matrix* $S \in \mathbf{M}_n(\mathbb{R})$.
(4) $DAE \in \mathbb{P}_n(\mathbb{C})$ *for all diagonal matrices* $D \in \mathbf{M}_n(\mathbb{C})$ *and* $E \in \mathbf{M}_n(\mathbb{C})$ *such that the diagonal entries of* DE *are positive.*
(5) $A[\alpha] \in \mathbb{P}_k(\mathbb{C})$ *for all nonempty* $\alpha \subseteq \langle n \rangle$ *with* $|\alpha| = k$.
(6) $A + D \in \mathbb{P}_n(\mathbb{C})$ *for all diagonal matrices* D *with nonnegative diagonal entries.*
(7) $A/A[\alpha] \in \mathbb{P}_{n-k}(\mathbb{C})$ *for all* $\alpha \subseteq \langle n \rangle$ *with* $|\alpha| = k$.
(8) $\mathrm{ppt}\,(A, \alpha) \in \mathbb{P}_n(\mathbb{C})$ *for all* $\alpha \subseteq \langle n \rangle$. *In particular, when* $\alpha = \langle n \rangle$, *we have* $\mathrm{ppt}\,(A, \langle n \rangle) = A^{-1} \in \mathbb{P}_n(\mathbb{C})$.
(9) $TI + (I - T)A \in \mathbb{P}_n(\mathbb{C})$ *for all diagonal matrices* T *whose diagonal entries belong to* $[0, 1]$.
(10) $I + F_A$, $I - F_A \in \mathbb{P}_n(\mathbb{C})$, *where* F_A *is the Cayley transform of* A.

Proof. Notice that statements (1)–(5) are consequences of determinantal properties and the definition of P-matrix.

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(6) Notice that if $A = [a_{ij}] \in \mathbb{P}_n(\mathbb{C})$, then we have $\dfrac{\partial \det A}{\partial a_{ii}} = \det A[\{i\}^c] > 0$; that is, $\det A$ is an increasing function of the diagonal entries. Thus, as the diagonal entries of D are added in succession to the diagonal of A, the determinant of A, and similarly every principal minor of A, remain positive.

(7) Since A is a P-matrix, A and $A[\alpha]$ are invertible and A^{-1}, up to a permutation similarity, has the block representation (see [6, Section 0.7.3])

$$A^{-1} = \begin{bmatrix} (A/A[\alpha^c])^{-1} & -A[\alpha]^{-1}A[\alpha,\alpha^c](A/A[\alpha])^{-1} \\ -(A/A[\alpha])^{-1}A[\alpha^c,\alpha]A[\alpha]^{-1} & (A/A[\alpha])^{-1} \end{bmatrix}.$$

Therefore, every principal submatrix of A^{-1} is of the form $(A/A[\alpha])^{-1}$ for some $\alpha \subseteq \langle n\rangle$ and its determinant is $\det A[\alpha]/\det A > 0$. This means that A^{-1} is a P-matrix, and so $(A/A[\alpha])^{-1}$ and $A/A[\alpha]$ are P-matrices for every $\alpha \subseteq \langle n\rangle$.

(8) Let A be a P-matrix and without loss of generality let us consider the case where α is the first singleton, that is, $\alpha = \{1\}$. Let $B = \mathrm{ppt}(A,\alpha)$. By definition, the principal submatrices of B that do not include entries from the first row coincide with principal submatrices of $A/A[\alpha]$ and thus have positive determinants (by clause (7)). The principal submatrices of B that include entries from the first row of B are equal to the corresponding principal submatrices of the matrix B' obtained from B using $b_{11} = (A[\alpha])^{-1} > 0$ as the pivot and eliminating the nonzero entries below it. Now, notice that

$$B' = \begin{bmatrix} 1 & 0 \\ -A[\alpha^c,\alpha] & I \end{bmatrix} \begin{bmatrix} b_{11} & -b_{11}A[\alpha,\alpha^c] \\ A[\alpha^c,\alpha]b_{11} & A/A[\alpha] \end{bmatrix} = \begin{bmatrix} b_{11} & -b_{11}A[\alpha,\alpha^c] \\ 0 & A[\alpha^c] \end{bmatrix},$$

that is, B' is itself a P-matrix for it is block upper triangular and the diagonal blocks are P-matrices. It follows that all the principal minors of B are positive and thus B itself is a P-matrix. Next, consider the case $\alpha = \{i_1,\ldots,i_k\} \subseteq \langle n\rangle$ with $k \geq 1$. By the argument so far, the sequence of matrices obtained from A by successive principal pivot transformations relative to the elements of α, that is,

$$A_0 = A, \quad A_j = \mathrm{ppt}\,(A_{j-1},\{i_j\}), \quad j = 1,\ldots,k,$$

is well defined and comprises P-matrices. Moreover, from uniqueness of $B = \mathrm{ppt}(A,\alpha)$ shown in [9, Theorem 3.1], it follows that $A_k = \mathrm{ppt}\,(A,\alpha) = B$. Thus, B is a P-matrix, completing the proof of clause (8).

(9) Let $T = \mathrm{diag}(t_1,\ldots,t_n)$, where each $t_i \in [0,1]$. Since T and $I - T$ are diagonal, we have (see e.g., [2])

$$\det(TI + (I-T)A) = \sum_{\alpha\subseteq\langle n\rangle} \prod_{i\notin\alpha} t_i \, \det(((I-T)A)[\alpha]).$$

As $t_i \in [0,1]$ and $A \in \mathbb{P}_n(\mathbb{C})$, all summands in this determinantal expansion are nonnegative. Unless $T = 0$, in which case $TI + (I-T)A = A$, one of these summands is positive. Hence, $\det(TI + (I-T)A) > 0$. The same argument can be applied to any principal submatrix of $TI+(I-T)A$, proving that $TI+(I-T)A \in \mathbb{P}_n$.(10) First, since A is a P-matrix, $I + A$ is invertible by clause (6) and thus F_A is well defined. It can also

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be verified (see [3]) that

$$I + F_A = 2\,(I + A)^{-1} \text{ and } I - F_A = 2\,(I + A^{-1})^{-1}.$$

As addition of positive diagonal matrices and inversion are operations that preserve P-matrices (see clauses (6), (8)), we have $I + F_A = 2(I + A)^{-1}$ and $I - F_A = 2(I + A^{-1})^{-1}$ are P-matrices. □

REMARK 2.7. It is noted that each clause (1)(9) of Theorem 2.6 represents a necessary and sufficient condition that $A \in \mathbf{M}_n(\mathbb{C})$ be a P-matrix. Clause (10) does not imply that A is a P-matrix; see [3, Example 3.2] combined with Theorem 2.6 (8).

Two important observations about P-matrices follow.

THEOREM 2.8. *Let $A \in \mathbf{M}_n(\mathbb{C})$. Then the following hold.*

(a) *If $A \in \mathbb{P}_n(\mathbb{C})$, then any real eigenvalue of A is positive.*
(b) *If $A \in \mathbb{P}_n(\mathbb{C})$, then the characteristic polynomial, $\det(tI - A)$, of A is a real monic polynomial whose coefficients have alternating signs.*
(c) *$A \in \mathbb{P}_n(\mathbb{C})$ if and only if for every nonempty $\alpha \subseteq \langle n\rangle$, the characteristic polynomial of $A[\alpha]$ is a real polynomial and the real eigenvalues of $A[\alpha]$ are positive.*

Proof.
(a) If an eigenvalue λ of A were negative, then Theorem 2.6 (6) would be violated for $D = -\lambda I$.
(b) The coefficients r_k $(k = 0, 1, \ldots, n)$ of $\det(tI - A)$ corresponding to t^k are given by:

$$r_k = (-1)^{n-k} \sum_{\alpha \subseteq \langle n\rangle, |\alpha| = n-k} \det A[\alpha];$$

that is, r_k $(k = 0, 1, \ldots, n)$ are real and have alternating signs.
(c) Let $A \in \mathbb{P}_n(\mathbb{C})$ so that for every nonempty $\alpha \in \langle n\rangle$, $A[\alpha]$ is a P-matrix. By part (a), all real eigenvalues of $A[\alpha]$ are positive, and by part (b) the characteristic polynomial of $A[\alpha]$ is a real polynomial. Conversely, suppose that for every nonempty $\alpha \subseteq \langle n\rangle$, the characteristic polynomial of $A[\alpha]$ is a real polynomial and the real eigenvalues of $A[\alpha]$ are positive. As a consequence, the nonreal eigenvalues of $A[\alpha]$ occur in complex conjugate pairs and the real eigenvalues are positive, that is, $\det(A[\alpha]) > 0$. □

3. From real to complex P-matrices. We begin with the formal statement (and proof) of the theorem to be generalized from real to complex P-matrices; it appeared first in [4, Theorem (3,3)].

THEOREM 3.1 (Real Case). *Let $A \in M_n(\mathbb{R})$. Then A is a P-matrix if and only if for each nonzero vector $x \in \mathbb{R}^n$, there exists $j \in \langle n\rangle$ such that $x_j(Ax)_j > 0$.*

Proof.

(⇒) Suppose that $A \in M_n(\mathbb{R})$ and that there exists $x \in \mathbb{R}^n$ such that for all $j \in \langle n\rangle$, $x_j(Ax)_j \leq 0$. Then for the nonnegative diagonal matrix D defined by

$$D = \operatorname{diag}(d_1, d_2, \ldots, d_n), \quad \text{where} \quad d_j = \begin{cases} -(Ax)_j/x_j & \text{if } x_j \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

we have that $Ax = -Dx$, that is, $(A + D)x = 0$. By Theorem 2.6 (6), it follows that A is not a P-matrix. This proves that when $A \in \mathbb{P}_n(\mathbb{R})$, then for every nonzero $x \in \mathbb{R}^n$, there exists $j \in \langle n\rangle$ such that $x_j(Ax)_j > 0$.

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($\Leftarrow$) Suppose now that $A \in M_n(\mathbb{R})$ and that for each nonzero $x \in \mathbb{R}^n$, there exists $j \in \langle n \rangle$ such that $x_j(Ax)_j > 0$. Notice that the same holds for every principal submatrix $A[\alpha]$ of A, by simply considering a nonzero vector x such that $x[\alpha^c] = 0$. Thus, all the real eigenvalues of $A[\alpha]$ are positive, for all non-empty $\alpha \subseteq \langle n \rangle$. As complex eigenvalues of a real matrix come in complex conjugate pairs, it follows that all the principal minors of A are positive. Hence, A is a P-matrix. ∎

Remark 3.2.

- Theorem 3.1 can informally be interpreted as follows: Multiplication of a real nonzero vector by a real P-matrix cannot reverse the sign of the vector; conversely, a real matrix that does not reverse the sign of any nonzero real vector is a P-matrix.
- Note that the forward proof of Theorem 3.1 applies to complex P-matrices, as well; that is, if $A \in \mathbf{M}_n(\mathbb{C})$ is a P-matrix, then for each nonzero vector $x \in \mathbb{R}^n$, there exists $j \in \langle n \rangle$ such that $x_j(Ax)_j > 0$. However, the converse is not true. To see this let $A = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$ and notice that there is no nonzero vector $x = [x_1 \;\; x_2]^T \in \mathbb{R}^2$ such that $x_j(Ax)_j \leq 0$, simply because

$$Ax = [ix_1 + x_2 \;\; x_1 + ix_2]^T,$$

and so either $x_1(ix_1 + x_2) \notin \mathbb{R}$ or $x_2(x_1 + ix_2) \notin \mathbb{R}$. However, A is not a P-matrix.

To generalize Theorem 3.1 to complex matrices, we consider the following concept.

Definition 3.3. A vector $y = [y_j] \in \mathbb{C}^n$ is called an (entrywise) *reflection* of $x = [x_j] \in \mathbb{C}^n$ if for each $j = 1, 2, \ldots, n$, there exists $d_j \geq 0$ such that

$$y_j = -d_j x_j \quad (j = 1, 2, \ldots, n).$$

By definition, the zero vector is a reflection of every vector. When the vectors x and y are real, being a reflection simply means that $x_j\, y_j \leq 0$ for all $j = 1, 2, \ldots, n$.

Lemma 3.4. *Let $A \in M_n(\mathbb{C})$. Then Ax is a reflection of a nonzero $x \in \mathbb{C}^n$ if and only if there exists a nonnegative diagonal matrix D such that* $\det(A + D) = 0$.

Proof.
($\Rightarrow$) Assume that Ax is a reflection of $x \in \mathbb{C}^n \setminus \{0\}$. This means that, there exist $d_j \geq 0$ $(j = 1, 2, \ldots, n)$ such that

$$Ax = -D\,x, \;\text{ where }\; D = \operatorname{diag}(d_1, d_2, \ldots, d_n) \geq 0.$$

Then, $(A + D)x = 0$, $x \neq 0$, which means that $\det(A + D) = 0$.

($\Leftarrow$) Suppose that $\det(A + D) = 0$ for some nonnegative diagonal matrix D. Then, $(A + D)x = 0$ for some $x \neq 0$; that is, $Ax = -Dx$ is a reflection of x. ∎

Remark 3.5. If $A \in \mathbb{P}_n(\mathbb{C})$, then by Theorem 2.6 (6), $\det(A + D) \neq 0$, for all nonnegative diagonal matrices D. The converse does not hold in general; see for example, the matrix $A \in M_2(\mathbb{C})$ in Remark 3.2. As shown next, in order for the converse to hold, we must further assume that the characteristic polynomials of A and of all of its principal submatrices are real polynomials.

Proposition 3.6. *Let $A \in M_n(\mathbb{C})$ such that the characteristic polynomials of all principal submatrices of A are real. If $\det(A + D) \neq 0$ for every nonnegative diagonal matrix D, then A is a P-matrix.*

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Proof. Suppose that A is as prescribed and that $\det(A+D) \neq 0$ for every nonnegative diagonal matrix D. Taking $D = -\lambda I$, where $\lambda \in \sigma(A) \cap \mathbb{R}$, we have that $\det(A+D) = 0$. It follows that every real eigenvalue λ of A must be positive. Recall that the nonreal eigenvalues of A come in complex conjugate pairs. Thus, by Theorem 2.8 (c), $\det A > 0$. The same is clearly true for all principal submatrices of A. Thus, all principal minors of A are positive and so A is a P-matrix. ∎

We now can state our characterization of complex P-matrices.

THEOREM 3.7 (Complex Case). *Let $A \in M_n(\mathbb{C})$ be such that all principal submatrices of A have real characteristic polynomials. Then A is a P-matrix if and only if for every nonzero $x \in \mathbb{C}^n$, Ax is not a reflection of x.*

Proof.
($\Rightarrow$) Let $A \in M_n(\mathbb{C})$ and assume that A is a P-matrix. By Theorem 2.6 (6), we have that $\det(A+D) \neq 0$, for all nonnegative diagonal matrices D. By Lemma 3.4, Ax is not a reflection of the vector x for any nonzero $x \in \mathbb{C}^n$.

($\Leftarrow$) Suppose that Ax is not a reflection of x for any nonzero $x \in \mathbb{C}^n$. By Lemma 3.4, $A+D$ is invertible for all nonnegative diagonal matrices D. Since all principal submatrices of A have real characteristic polynomials, by Proposition 3.6, it must be that A is a P-matrix. ∎

We note that spectra of complex P-matrices (diagonal stabilization) are discussed in [1] and [5]. The next theorem follows from [5, Theorem 4.6], which generalizes [1, Theorem 2].

THEOREM 3.8. *Let $A \in \mathbf{M}_n(\mathbb{C})$ be a P-matrix. Then there is a nonnegative invertible diagonal matrix $D \in \mathbf{M}_n(\mathbb{R})$ such that all the eigenvalues of DA are positive and simple.*

4. Complex P-matrices and x-Semipositivity. Let us recall the notion of semipositivity of a real matrix:

DEFINITION 4.1. $A \in \mathbf{M}_n(\mathbb{R})$ is *semipositive* if there exists $x \geq 0$ such that $Ax > 0$.

The following are two well-known theorems relating real P-matrices to semipositivity; see [7, Chapter 6] for the proofs. In this section, we aim to generalize these two theorems to complex P-matrices.

THEOREM 4.2. *Let $A \in \mathbf{M}_n(\mathbb{R})$ be a P-matrix. Then A is a semipositive matrix.*

THEOREM 4.3. *$A \in \mathbf{M}_n(\mathbb{R})$ is a P-matrix if and only if for every signature matrix $S \in \mathbf{M}_n(\mathbb{R})$, SAS is semipositive.*

EXAMPLE 4.4. We note that not every semipositive matrix is a P-matrix, as shown by $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$. Indeed, A maps the all-ones vector in $\mathbb{R}^2$ to a positive vector, but A is not a P-matrix.

EXAMPLE 4.5. Complex P-matrices are not necessarily semipositive; for example, $A = \begin{bmatrix} 1 & i \\ i & 2 \end{bmatrix}$ is a P-matrix but clearly not semipositive.

To generalize Theorems 4.2 and 4.3 from real to complex P-matrices, we need to introduce a meaningful generalization of semipositivity, applicable to complex matrices. For this purpose, we first recall the concept of a cone in $\mathbb{C}^n$.

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Definition 4.6. We call a convex set $K \subseteq \mathbb{C}^n$ a *cone* if $r\,K \subset K$ for all $r \geq 0$. Note that a cone is by definition a convex set; as such a cone K is a closed set (in the topology induced by the standard inner product in $\mathbb{C}^n$).

Definition 4.7. Consider a nonzero $x \in \mathbb{C}^n$. We refer to the set

$$W(x) = \{Dx \,:\, D \in \mathbf{M}_n\,(\mathbb{R}) \text{ is a nonnegative diagonal matrix}\},$$

which is clearly a cone in $\mathbb{C}^n$, as the *refraction cone* of x. Note that $W(x)$ is a cone whose topological interior in $\mathbb{C}^n$ is empty if x is nonzero with at least one zero entry; in that case its relative interior is

$$\operatorname{ri} W(x) = \{Dx \,:\, D \in \mathbf{M}_n\,(\mathbb{R}) \text{ is a nonnegative invertible diagonal matrix}\}.$$

If x is strictly nonzero, then the interior, $\operatorname{int} W(x)$, and relative interior of $W(x)$ coincide.

The following concept generalizes semipositivity of real matrices.

Definition 4.8. Let $x \in \mathbb{C}^n$ be a nonzero vector. We call $A \in \mathbf{M}_n\,(\mathbb{C})$ an *x-semipositive* matrix if there exists $y \in W(x)$ such that Ay is in the relative interior of $W(x)$. Note that x-semipositivity coincides with semipositivity if $x > 0$.

Theorem 4.9. *If $A \in \mathbf{M}_n\,(\mathbb{C})$ is a P-matrix, then there exists a nonzero $x \in \mathbb{C}^n$ such that A is x-semipositive.*

Proof. By Theorem 3.8, there exists an invertible nonnegative diagonal matrix $E \in \mathbf{M}_n\,(\mathbb{R})$ such that EA has distinct positive eigenvalues. As a consequence, there exist a nonzero $x \in \mathbb{C}^n$ and $\lambda > 0$ such that $EAx = \lambda x$. Thus, $Ax = Dx$, where $D = \lambda E^{-1} \in \mathbf{M}_n\,(\mathbb{R})$ is an invertible nonnegative diagonal matrix. Since $x \neq 0$, it follows that $x = Ix \in W(x)$ and also $Ax = Dx \in \operatorname{ri} W(x)$; thus, A is x-semipositive. ∎

Remark 4.10.

1. Theorem 4.9 says that every complex P-matrix is x-semipositive for some nonzero $x \in \mathbb{C}^n$. This generalizes semipositivity of real P-matrices.
2. If $x > 0$ in Theorem 4.9, then we recover Theorem 4.2.
3. It is interesting to note that the proof of Theorem 4.9 is purely linear-algebraic. This is important because, to our knowledge, the proofs of Theorem 4.2 accessible in the literature always rely on a 'Theorem of the Alternative' for systems of inequalities; see [7, Chapter 6].
4. As is evident from the proof of Theorem 4.9, determining a nonzero $x \in \mathbb{C}^n$ such that a P-matrix $A \in \mathbf{M}_n\,(\mathbb{C})$ is x-semipositive amounts to finding a nonnegative invertible diagonal matrix D such that $A - D$ has a nonzero nullvector x. This is illustrated in the next two examples.

Example 4.11. Consider the P-matrix $A = \begin{bmatrix} 3 & i \\ -i & 2 \end{bmatrix}$. Note that for $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $A - D$ is singular with nonzero nullvector $x = \begin{bmatrix} i \\ -1 \end{bmatrix}$. Thus, $Ax = Dx \in \operatorname{int} W(x)$; that is, A is x-semipositive.

Example 4.12. Let $\alpha = \{1, 2, 3\}$ and $B = A[\alpha]$, where A is the P-matrix in Example 1.1. Then $Bx = \lambda x$, where $\lambda = 0.965$ and

$$x = [-0.4061 - 0.1044i \quad 0.4564 + 0.3093i \quad 0.7212]^T,$$

approximately. This means that $Bx = Dx \in \operatorname{int} W(x)$, where $D = \lambda I$; that is, the P-matrix B is x-semipositive.

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We state the following conjecture that, if true, would generalize Theorem 4.3 to complex P-matrices.

Conjecture 4.13. *Let $A \in \mathbf{M}_n(\mathbb{C})$ such that the characteristic polynomials of all principal submatrices of A are real. Then $A \in \mathbf{M}_n(\mathbb{C})$ is a P-matrix if and only if there exists a strictly nonzero $x \in \mathbb{C}^n$ such that for every unitary diagonal matrix $D \in \mathbf{M}_n(\mathbb{C})$, D^*AD is x-semipositive.*

5. Factorizations of complex P-matrices. We conclude with the generalization of characterizations of real P-matrices in factored form found in [8, Theorems 3.3 and 3.4]. Again, we need to impose the condition on the characteristic polynomial of A and of all its principal submatrices.

Theorem 5.1. (Complex case) *Let $A = BC^{-1} \in \mathbf{M}_n(\mathbb{C})$, where $B, C \in \mathbf{M}_n(\mathbb{C})$ and such that all principal submatrices of A have real characteristic polynomials. Then A is a P-matrix if and only if the matrix $TC + (I - T)B$ is invertible for every $T \in [0, I]$.*

Proof. If $A = BC^{-1}$ is a P-matrix, by Theorem 2.6 (9), $TI + (I-T)BC^{-1} \in \mathbb{P}_n(\mathbb{C})$ for every $T \in [0, I]$. Thus, $TC + (I - T)B$ is invertible for every $T \in [0, I]$.

For the converse, suppose $TI+(I-T)BC^{-1}$ is invertible for every $T \in [0, I]$ and, by way of contradiction, suppose $BC^{-1} \notin \mathbb{P}_n(\mathbb{C})$. By Theorem 3.7, there exists a nonzero $x \in \mathbb{C}^n$ such that $y = BC^{-1}x$ is a reflection of x. Then for each $j \in \langle n \rangle$, $t_j \in [0, 1]$ can be selected so that

$$t_j x_j + (1 - t_j) y_j = 0.$$

Then for $T = \text{diag}(t_1, t_2, \dots, t_n)$, we have

$$Tx + (I - T)BC^{-1}x = 0,$$

a contradiction to the invertibility of $TI + (I - T)BC^{-1}$ that completes the proof. ∎

The following result is analogous to Theorem 5.1.

Theorem 5.2. *Let $A = B^{-1}C \in \mathbf{M}_n(\mathbb{C})$, where $B, C \in \mathbf{M}_n(\mathbb{C})$ and such that all principal submatrices of A have real characteristic polynomials. Then A is a P-matrix if and only if the matrix $CT + B(I - T)$ is invertible for every $T \in [0, I]$.*

Proof. Note that by Theorem 2.6 (1), $A = B^{-1}C$ is a P-matrix if and only if $A^T = C^T(B^T)^{-1}$ is a P-matrix. By Theorem 5.1, the latter is a P-matrix if and only if $\hat{T}B^T + (I - \hat{T})C^T$ is invertible for every $\hat{T} \in [0, I]$. In turn, $\hat{T}B^T + (I - \hat{T})C^T$ is invertible if and only if $B\hat{T} + C(I - \hat{T})$ is invertible for every $\hat{T} \in [0, I]$. The stated result follows by taking $T = I - \hat{T} \in [0, I]$. ∎

Remark 5.3. Matrices $A \in \mathbf{M}_n(\mathbb{C})$ all whose principal minors are nonnegative are referred to as P_0-matrices. It is well known that $A \in \mathbf{M}_n(\mathbb{C})$ is a P_0-matrix if and only $A + \epsilon I$ is a P-matrix for every $\epsilon > 0$. As a consequence, the results herein can be adapted to P_0-matrices.

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