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ORDER RELATIONS OF THE WASSERSTEIN MEAN AND THE SPECTRAL GEOMETRIC MEAN*

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Abstract. On the space of positive definite matrices, several operator means are popular and have been studied extensively. In this paper, we investigate the near order and the Löwner order relations on the curves defined by the Wasserstein mean and the spectral geometric mean. We show that the near order $\preceq$ is stronger than the eigenvalue entrywise order and that $A\natural_t B \preceq A \diamond_t B$ for $t \in [0,1]$. We prove the monotonicity properties of the curves originated from the Wasserstein mean and the spectral geometric mean in terms of the near order. The Löwner order properties of the Wasserstein mean and the spectral geometric mean are also explored.

Key words. Wasserstein mean, Spectral geometric mean, Near order, Löwner order, Eigenvalue entrywise order.

AMS subject classifications. 15A42, 15A45, 15B48.

1. Introduction. The primary objective of this manuscript is to investigate the near order and the Löwner order relations concerning the Wasserstein mean and the spectral geometric mean, as well as the curves induced by these two means.

Let $\mathbb{C}_{n\times n}$ be the space of all $n \times n$ complex matrices. In $\mathbb{C}_{n\times n}$, let $\mathbb{H}_n$ (resp. $\mathbb{P}_n$, $\overline{\mathbb{P}}_n$) be the set of $n \times n$ Hermitian (resp. positive definite, positive semidefinite) matrices and $\mathrm{U}(n)$ the group of $n \times n$ unitary matrices. Given $A \in \mathbb{C}_{n\times n}$, we denote $|A| = (A^*A)^{1/2}$. For $A \in \mathbb{P}_n$, let $\lambda(A)$ denote the n-tuple of eigenvalues of A with nonincreasing order, that is, $\lambda(A) = (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$ and $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$.

On the space of positive definite matrices, several operator means are popular and have been studied extensively. Given $A, B \in \mathbb{P}_n$, *the arithmetic mean*, *the Wasserstein mean*, *the metric geometric mean*, and *the spectral geometric mean* are defined for $t \in [0,1]$:

$$A\nabla_t B = (1-t)A + tB, \tag{1.1}$$

$$A \diamond_t B = (1-t)^2 A + t^2 B + t(1-t)[(AB)^{1/2} + (BA)^{1/2}], \tag{1.2}$$

$$A\sharp_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}, \tag{1.3}$$

$$A\natural_t B = (A^{-1}\sharp B)^t A (A^{-1}\sharp B)^t, \tag{1.4}$$

where $A\sharp B = A\sharp_{1/2} B$. Many discovered relations between these means are related to their spectra.

The metric geometric mean was first introduced by Pusz and Woronowicz [24] for $t = 1/2$ and extended to $t \in [0,1]$ by Kubo and Ando [20]. The mean was extended to multiple variables by Ando, Li, and Mathias [5]. The properties of this mean were studied extensively by Lim [23]. The spectral geometric mean was introduced by Fiedler and Pták [10] for $t = 1/2$ and extended to $t \in [0,1]$ by Lee and Lim [22]. Properties of spectral geometric mean can be found in [2, 11, 13, 17, 18] and the references therein. The

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Wasserstein mean is linked to the barycenter in the Wasserstein space of Gaussian distributions, which is one of the popular topics in matrix analysis and probability theory [1, 3]. Many interesting inequalities and properties of the Wasserstein mean has been given in [8, 11, 14, 15]. Hwang and Kim in [16, Lemma 2.4] gave another form of the Wasserstein mean (1.2) in terms of the metric geometric mean and arithmetic mean as:

$$A \diamond_t B = [I\nabla_t(A^{-1}\sharp B)]A[I\nabla_t(A^{-1}\sharp B)]. \tag{1.5}$$

This form is analogous to the form of the spectral geometric mean (1.4). In consequence, many properties of these two means are similar.

The following interesting relations between positive definite matrices will be discussed:

1. We use $A \geq 0$ to denote that A is positive semidefinite. Two matrices $A, B \in \mathbb{P}_n$ satisfy *the Löwner order* $A \leq B$ if $B - A \geq 0$.
2. Define *the near order* on $\mathbb{P}_n$: $A \preceq B$ if $A^{-1}\sharp B \geq I$. The relation is introduced by Dumitru and Franco [9].
3. Given $A, B \in \mathbb{P}_n$, we define the *eigenvalue entrywise order* relation: $A \leq_\lambda B$ if $\lambda_i(A) \leq \lambda_i(B)$ for $1 \leq i \leq n$. We say that $A =_\lambda B$ if $\lambda(A) = \lambda(B)$.
4. Given $A, B \in \mathbb{P}_n$, we write $A \prec_{w\log} B$ if $\lambda(A)$ is *weakly log-majorized* by $\lambda(B)$, that is,

$$\prod_{i=1}^{k} \lambda_i(A) \leq \prod_{i=1}^{k} \lambda_i(B), \quad k = 1, 2, \ldots, n. \tag{1.6}$$

   In particular, we say that $\lambda(A)$ is *log-majorized* by $\lambda(B)$, denoted by $A \prec_{\log} B$, if (1.6) is true for $k = 1, 2, \ldots, n-1$ and equality holds for $k = n$.

The above relations satisfy the transitive property except for the near order. The Löwner order is the strongest condition among these relations. The near order is weaker than the Löwner order but stronger than the eigenvalue entrywise order, which we will prove in Theorem 2.4. It is straightforward that $A \leq_\lambda B$ implies $A \prec_{w\log} B$. So for $A, B \in \mathbb{P}_n$, we have the following relationships:

$$A \leq B \Longrightarrow A \preceq B \Longrightarrow A \leq_\lambda B \Longrightarrow A \prec_{w\log} B. \tag{1.7}$$

In Theorem 3.1, we show that $A\natural_t B \preceq A \diamond_t B$ for $t \in (0, 1)$, which completes a chain of order relations among means in (2.8). Consequently, there is the eigenvalue entrywise relation $A\natural_t B \leq_\lambda A \diamond_t B$ for $t \in (0, 1)$.

The metric geometric mean satisfies that if $A \leq B$, then the induced geodesic curve $\{A\sharp_t B \mid t \geq 0\}$ is monotonically increasing in terms of the Löwner order $\leq$ with respect to t. Similar properties hold for the curves induced by the Euclidean mean $A\nabla_t B$ and the log-Euclidean mean $\exp((1-t)\log A + t\log B)$, but not for the curves induced by the Wasserstein mean and the spectral geometric mean. We show in Theorems 3.6 that if $A \preceq B$, then $\{A \diamond_t B \mid t \geq 0\}$ and $\{A\natural_t B \mid t \in \mathbb{R}\}$ are monotonically increasing in terms of the near order $\preceq$ with respect to t. In Theorem 3.7, the near order relations between $A \diamond_t B$ and $A \diamond_s B$, between $A\natural_t B$ and $A\natural_s B$, and between $A\natural_t B$ and $A \diamond_s B$ are compared, respectively. For certain real powers p, the near order on the curves $\{A^p \diamond_t B^p \mid t \geq 0\}$ and $\{B^{-p} \diamond_t A^{-p} \mid t \geq 0\}$ are also discussed. The results disclose the existence of abundant near order relations and the corresponding eigenvalue entrywise order relations in the Wasserstein mean, the spectral geometric mean, and the curves induced by two means.

In Section 4, we study the Löwner order properties of Wasserstein mean and spectral geometric mean, both of which are connected to the metric geometric mean. In particular, Theorem 4.2 shows that if $A \diamond_t B \leq A \diamond_t C$ or $A\natural_t B \leq A\natural_t C$ for one $t \in (0, 1]$, then $A^{-1}\sharp_s B \leq A^{-1}\sharp_s C$ for all $s \in [0, 1/2]$.

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Some preliminary results of the metric geometric mean, the Wasserstein mean, the spectral geometric mean, and their relations are explored in Section 2.

2. Preliminaries. Extensive investigations have been done on the properties of matrix means. Here, we list some basic properties of the metric geometric mean [7, 20, 22], the spectral geometric mean [22, 23], and the Wasserstein mean [14, 16, 19]. They display different characteristics of these three means.

THEOREM 2.1. *Let $A, B, C, D \in \mathbb{P}_n$ and let $s, u, t \in [0, 1]$. The following are satisfied:*

1. $A\sharp_t B = B\sharp_{1-t}A$.
2. $(A\sharp_t B)^{-1} = A^{-1}\sharp_t B^{-1}$.
3. $(aA)\sharp_t(bB) = a^{1-t}b^t(A\sharp_t B)$ *for any* $a, b > 0$.
4. $\det(A\sharp_t B) = (\det A)^{1-t}(\det B)^t$.
5. $M(A\sharp_t B)M^* = (MAM^*)\sharp_t(MBM^*)$ *for non-singular* M.
6. $(A^{-1}\nabla_t B^{-1})^{-1} \le A\sharp_t B \le A\nabla_t B$, *where* $A\nabla_t B = (1-t)A + tB$.
7. $(A\sharp_s B)\sharp_t(A\sharp_u B) = A\sharp_{(1-t)s+tu}B$.
8. *If* $A \le C$ *and* $B \le D$, *then* $A\sharp_t B \le C\sharp_t D$.
9. $A^{-1}\sharp(BAB) = B$.

THEOREM 2.2. *Let $A, B \in \mathbb{P}_n$ and let $s, u, t \in [0, 1]$. The following are satisfied:*

1. $A\natural_t B = B\natural_{1-t}A$.
2. $(A\natural_t B)^{-1} = A^{-1}\natural_t B^{-1}$.
3. $(aA)\natural_t(bB) = a^{1-t}b^t(A\natural_t B)$ *for any* $a, b > 0$.
4. $(A\natural_s B)\natural_t(A\natural_u B) = A\natural_{(1-t)s+tu}B$.
5. $\det(A\natural_t B) = (\det A)^{1-t}(\det B)^t$.

THEOREM 2.3. *Let $A, B \in \mathbb{P}_n$ and let $s, t, u \in [0, 1]$. The following are satisfied:*

1. $A \diamond_t B = B \diamond_{1-t} A$.
2. $(A \diamond_t B)^{-1} = A^{-1} \diamond_t B^{-1}$ *if and only if* $A = B$.
3. $(aA) \diamond_t (aB) = a(A \diamond_t B)$ *for any* $a > 0$.
4. $(A \diamond_s B) \diamond_t (A \diamond_u B) = A \diamond_{(1-t)s+tu} B$.
5. $\det(A \diamond_t B) \ge (\det A)^{1-t}(\det B)^t$.
6. $A \diamond_t B \le A\nabla_t B$.

Besides the individual properties of matrix means, many relations among these means have been discovered. The near order relation $\preceq$ on $\mathbb{P}_n$ is recently introduced by Dumitru and Franco [9]:

$$A \preceq B \text{ if and only if } A^{-1}\sharp B \ge I.$$

It is natural that $A \preceq B$ if and only if $B^{-1} \preceq A^{-1}$. However, the relation "$\preceq$" does not satisfy transitive property [9], that is, $A \preceq B$ and $B \preceq C$ do not necessarily imply that $A \preceq C$. So the relation "$\preceq$" is called the near order.

On the one hand, the near order relation is weaker than the Löwner order, since $A \le B$ implies that $A^{-1}\sharp B \ge A^{-1}\sharp A = I$ so that $A \preceq B$. On the other hand, we show below that the near order is stronger than the eigenvalue entrywise relation.

THEOREM 2.4. *Let $A, B \in \mathbb{P}_n$. If $A \preceq B$, then $A \le_\lambda B$.*

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Proof. Assume $A \preceq B$, we have $B^{-1}\sharp A \leq I$. Then it is straightforward to have

$$\begin{aligned} A &=_\lambda B^{1/2}B^{-1/2}(B^{1/2}AB^{1/2})^{1/2}B^{-1}(B^{1/2}AB^{1/2})^{1/2}B^{-1/2}B^{1/2} \\ &= B^{1/2}(B^{-1}\sharp A)^2B^{1/2} \\ &\leq B^{1/2}IB^{1/2} = B. \end{aligned}$$

Thus, $A \leq_\lambda B$. ∎

From [4, 6, 12], we have the log-majorization relations between the metric geometric mean, the log-Euclidean mean, the fidelity, and the spectral geometric mean as:

$$A\sharp_t B \prec_{\log} \exp((1-t)\log A + t\log B) \prec_{\log} B^{t/2}A^{1-t}B^{t/2} \prec_{\log} A\natural_t B.$$

It is well known (e.g., [8, 21]) that there are Löwner orders:

$$(A^{-1}\nabla_t B^{-1})^{-1} \leq A\sharp_t B \leq A\nabla_t B \quad \text{and} \quad A \diamond_t B \leq A\nabla_t B.$$

Our new result in Theorem 3.1 shows that $A\natural_t B \preceq A \diamond_t B$ for all $t \in (0,1)$. It completes a chain of order relations among means as:

$$\begin{aligned} (A^{-1}\nabla_t B^{-1})^{-1} \leq A\sharp_t B &\prec_{\log} \exp((1-t)\log A + t\log B) \\ &\prec_{\log} B^{t/2}A^{1-t}B^{t/2} \prec_{\log} A\natural_t B \preceq A \diamond_t B \leq A\nabla_t B. \end{aligned} \tag{2.8}$$

The relation between any two of the above means can be derived from (2.8) and the transitive properties. Here are some relations:

1. (2.8) and [9, Theorem 2] imply that

$$A\natural_t B \preceq A\nabla_t B.$$

2. From (2.8), we get

$$A\sharp_t B \prec_{w\log} A \diamond_t B.$$

It is the known strongest relation. The following counterexample shows that $A\sharp_t B$ and $A \diamond_t B$ do not have eigenvalue entrywise relation, near order relation, or Löwner order relation. Let

$$A = \begin{bmatrix} 39.1195 & 42.1116 \\ 42.1116 & 61.1568 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 26.3279 & 13.3485 \\ 13.3485 & 12.2727 \end{bmatrix}.$$

Then we have

$$A\sharp_t B = \begin{bmatrix} 32.2446 & 29.2497 \\ 29.2497 & 39.8872 \end{bmatrix} \quad \text{and} \quad A \diamond_t B = \begin{bmatrix} 35.6339 & 33.9111 \\ 33.9111 & 45.3815 \end{bmatrix}.$$

The spectra of $A\sharp_t B$ is $\{65.5641, 6.5677\}$ and the spectra of $A \diamond_t B$ is $\{74.7672, 6.2481\}$. Thus, $A\sharp_t B$ and $A \diamond_t B$ do not have eigenvalue entrywise relation. (1.7) implies that $A\sharp_t B$ and $A \diamond_t B$ do not have either near order relation or Löwner order relation.

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3. (2.8) may not always provide the strongest relation between means. For example, (2.8) implies that $(A^{-1}\nabla_t B^{-1})^{-1} \prec_{w\log} A \diamond_t B$. However, Theorem 2.3(6) and Theorem 3.9 show that

$$(A^{-1}\nabla_t B^{-1})^{-1} \leq (A^{-1} \diamond_t B^{-1})^{-1} \preceq A \diamond_t B \leq A\nabla_t B,$$

hence by [9, Theorem 2], a stronger relation exists:

$$(A^{-1}\nabla_t B^{-1})^{-1} \preceq A \diamond_t B.$$

A relation between the near order and the Löwner order is given as follows, which strengthen a result in [16, Hwang and Kim, 2022].

THEOREM 2.5. *Let $A \in \mathbb{P}_n$. Let $P, Q \in \mathbb{H}_n$ be nonsingular such that $P^{-1}Q \in \mathbb{P}_n$. Then $I \leq P^{-1}Q$ if and only if $PAP \preceq QAQ$.*

Proof. Since $P, Q \in \mathbb{H}_n$ and $P^{-1}Q \in \mathbb{P}_n$, we have $P^{-1}Q = (P^{-1}Q)^* = QP^{-1}$ so that P and Q commute. By Theorem 2.1 (9), $I \leq P^{-1}Q$ if and only if

$$I \leq P^{-1}Q = (PAP)^{-1}\sharp[(P^{-1}Q)(PAP)(P^{-1}Q)] = (PAP)^{-1}\sharp(QAQ),$$

if and only if $PAP \preceq QAQ$. ∎

COROLLARY 2.6. *Let $A, P, Q \in \mathbb{P}_n$ and $PQ = QP$, then $P \leq Q$ if and only if $PAP \preceq QAQ$.*

By [26, Theorem II.1], on the manifold $\overline{P_n}$ of $n \times n$ positive semidefinite matrices, the formula (1.2) of $A \diamond_t B$ can be extended to all $t \in \mathbb{R}$, and

$$A \diamond_t B = B \diamond_{1-t} A = |(1-t)A^{1/2} + tU^* B^{1/2}|^2 = |A^{1/2}\nabla_t(U^* B^{1/2})|^2, \tag{2.9}$$

in which U is a unitary matrix in the polar decomposition $B^{1/2}A^{1/2} = U|B^{1/2}A^{1/2}|$, and ∇_t in (1.1) is extended to all $t \in \mathbb{R}$. When $A \in \mathbb{P}_n$, (2.9) is equivalent to that for $t \in \mathbb{R}$:

$$A \diamond_t B = [I\nabla_t(A^{-1}\sharp B)]A[I\nabla_t(A^{-1}\sharp B)]. \tag{2.10}$$

Likewise, the formula (1.4) of $A\natural_t B$ can be extended to all $t \in \mathbb{R}$ such that

$$A\natural_t B = (A^{-1}\sharp B)^t A (A^{-1}\sharp B)^t. \tag{2.11}$$

(2.10) and (2.11) immediately imply the following result.

PROPOSITION 2.7. *Let $A, B \in \mathbb{P}_n$. Let $\mu_1, \ldots, \mu_n$ be the eigenvalues of $A^{-1}\sharp B$. Then for $t \in \mathbb{R}$,*

$$\det(A \diamond_t B) = \det(A) \prod_{i=1}^n (1 - t + t\mu_i)^2,$$

$$\det(A\natural_t B) = \det(A)^{1-t} \det(B)^t = \det(A) \prod_{i=1}^n \mu_i^{2t}.$$

All properties of the spectral geometric mean in Theorem 2.2 still hold for $s, u, t \in \mathbb{R}$, and they can be proved by (2.11). The extension of Theorem 2.2(4) to $s, u, t \in \mathbb{R}$ is proved below.

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PROPOSITION 2.8. *Let $A, B \in \mathbb{P}_n$. Then for $s, u, t \in \mathbb{R}$,*

$$(A\natural_s B)\natural_t(A\natural_u B) = A\natural_{(1-t)s+tu}B.$$

Proof. (2.11) and Theorem 2.1(9) imply that for $A, B \in \mathbb{P}_n$ and $s \in \mathbb{R}$,

$$A^{-1}\sharp(A\natural_s B) = A^{-1}\sharp[(A^{-1}\sharp B)^s A(A^{-1}\sharp B)^s] = (A^{-1}\sharp B)^s,$$

so that for $t, s \in \mathbb{R}$,

$$A\natural_t(A\natural_s B) = [(A^{-1}\sharp B)^s]^t A[(A^{-1}\sharp B)^s]^t = A\natural_{ts}B.$$

Therefore,

$$(A\natural_s B)\natural_t B = B\natural_{1-t}(B\natural_{1-s}A) = B\natural_{(1-t)(1-s)}A = A\natural_{(1-t)s+t}B,$$

so that

$$(A\natural_s B)\natural_t(A\natural_u B) = [A\natural_{\frac{s}{u}}(A\natural_u B)]\natural_t(A\natural_u B) = A\natural_{(1-t)\frac{s}{u}+t}(A\natural_u B) = A\natural_{(1-t)s+tu}B.$$

The proof is completed. ∎

However, the counterpart of Proposition 2.8 for $A \diamond_t B$ is not true, since the Wasserstein curve $\{A \diamond_t B \mid t \in \mathbb{R}\}$ given by (1.2), (2.9), or (2.10) consists of several geodesic curves joint at some boundary points of $\overline{P_n}$. A difference between $\natural_t$ and $\diamond_t$ lies in the fact that: in (2.11), $(A^{-1}\sharp B)^t$ for $t \in \mathbb{R}$ is always positive definite, but in (2.10), $I\nabla_t(A^{-1}\sharp B)$ is not.

EXAMPLE 2.9. *If $A, B \in \mathbb{P}_n$ are commuting, then (2.9) shows that*

$$A \diamond_t B = (A^{1/2}\nabla_t B^{1/2})^2 = |A^{1/2}\nabla_t B^{1/2}|^2.$$

Let $A = \operatorname{diag}(4, 1)$ and $B = \operatorname{diag}(1, 4)$. Then for $t, s \in \mathbb{R}$,

$$\begin{aligned} A \diamond_{ts} B &= \operatorname{diag}((2-ts)^2, (1+ts)^2) \\ A \diamond_t (A \diamond_s B) &= \operatorname{diag}([2(1-t) + |2-s|t]^2, (1-t+|1+s|t)^2). \end{aligned}$$

When $t \neq 0, 1$ and $s > 2$, the $(1, 1)$ entries of $A \diamond_{ts} B$ and $A \diamond_t (A \diamond_s B)$ are different.

When A and B has near order relation, the following properties hold for Wasserstein curve.

PROPOSITION 2.10. *Suppose that $A, B \in \mathbb{P}_n$.*

1. *If $A \preceq B$. Then for $t, s \in [0, \infty)$,*

$$A \diamond_t (A \diamond_s B) = A \diamond_{ts} B.$$

2. *If $A \succeq B$. Then for $t, s \in (-\infty, 1]$,*

$$(A \diamond_s B) \diamond_t B = A \diamond_{(1-t)s+t} B.$$

Proof. 1. $A \preceq B$ implies that $A^{-1}\sharp B \geq I$. For $s \geq 0$, we have $I\nabla_s(A^{-1}\sharp B) \geq I$. By Theorem 2.1(9), $A^{-1}\sharp(A \diamond_s B) = I\nabla_s(A^{-1}\sharp B)$, and for $t \geq 0$,

$$\begin{aligned} A^{-1}\sharp[A \diamond_t (A \diamond_s B)] &= I\nabla_t[A^{-1}\sharp(A \diamond_s B)] = I\nabla_t[I\nabla_s(A^{-1}\sharp B)] \\ &= I\nabla_{ts}(A^{-1}\sharp B) = A^{-1}\sharp(A \diamond_{ts} B). \end{aligned}$$

Therefore, when $t, s \geq 0$, we have $A \diamond_t (A \diamond_s B) = A \diamond_{ts} B$.

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2. It can be proved by using $A \diamond_t B = B \diamond_{1-t} A$ and the preceding result. ∎

In the coming sections, we will focus on exploring the near order relation and the Löwner order relations between the spectral geometric mean and the Wasserstein mean, and these relations on the curves defined by two means.

3. The near orders on Wasserstein and spectral geometric curves. There are abundant near order relations on the curves originated from the Wasserstein mean and the spectral geometric mean. We will explore these relations here.

$A \diamond_t B$ and $A\natural_t B$ have the following near order relations.

THEOREM 3.1. *Suppose $A, B \in \mathbb{P}_n$.*

1. *For $t \in (0,1)$, the spectral geometric mean and the Wasserstein mean satisfy that*

$$A\natural_t B \preceq A \diamond_t B. \tag{3.12}$$

2. *If $A \preceq B$, then for $t \in (1,\infty)$,*

$$A\natural_t B \succeq A \diamond_t B. \tag{3.13}$$

3. *If $A \succeq B$, then for $t \in (-\infty,0)$,*

$$A\natural_t B \succeq A \diamond_t B. \tag{3.14}$$

Moreover, any equality in (3.12), (3.13), or (3.14) holds if and only if $A = B$.

Proof. Given $a > 0$, define the function $f_a(t) = a^t - 1 + t - at$. If $a = 1$, then $f_1(t) = 0$ for all $t \in \mathbb{R}$. Otherwise, we have $f_a(0) = f_a(1) = 0$ and $f_a''(t) = (\ln a)^2 a^t > 0$ for all $t \in \mathbb{R}$. Hence, $f_a(t) < 0$ for $t \in (0,1)$ and $f_a(t) > 0$ for $t \in (-\infty,0) \cup (1,\infty)$.

Let $X = A^{-1}\sharp B$. Let $\mu_1 \geq \cdots \geq \mu_n$ be the eigenvalues of X. Then $X > 0$ and X^t commutes with $I\nabla_t X$ for $t \in \mathbb{R}$. Therefore, X^t and $I\nabla_t X$ are simultaneously unitarily diagonlizable and the eigenvalues of $X^t - I\nabla_t X$ are $f_{\mu_i}(t)$ for $i = 1, 2, \ldots, n$. The preceding argument shows that:

1. For $t \in (0,1)$, we have the Löwner order $0 < X^t \leq I\nabla_t X$. By Corollary 2.6,
$$A\natural_t B = X^t A X^t \preceq (I\nabla_t X)A(I\nabla_t X) = A \diamond_t B.$$
2. Suppose that $A \preceq B$. Then all $\mu_i \geq 1$. So for $t \in (1,\infty)$, we have $0 < I\nabla_t X \leq X^t$. By Corollary 2.6, we get $A\natural_t B \succeq A \diamond_t B$.
3. Suppose that $A \succeq B$. Then all $\mu_i \in (0,1]$. For $t \in (-\infty,0)$, we have $0 < I\nabla_t X \leq X^t$. Corollary 2.6 implies that $A\natural_t B \succeq A \diamond_t B$.

The values of $f_{\mu_i}(t)$ show that the equality in (3.12), (3.13), or (3.14) holds if and only if all $\mu_i = 1$, that is, $A = B$. ∎

EXAMPLE 3.2. *For $t \in (0,1)$, we have the near order $A\natural_t B \preceq A \diamond_t B$. However, they don't satisfy the stronger Löwner order. Here is a counterexample:*

$$A = \begin{bmatrix} 50 & 0 \\ 0 & 10 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 57.8906 & 19.8885 \\ 19.8885 & 62.1094 \end{bmatrix}$$

when $t = 1/2$, the spectrum of $A \diamond_t B - A\natural_t B$ is $\{-0.21, 6.3338\}$.

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Theorem 2.4 shows that the near order is stronger than the eigenvalue entrywise relation. So, (3.12) implies the following result.

COROLLARY 3.3. *If $A, B \in \mathbb{P}_n$, then for $t \in (0,1)$,*

$$A\natural_t B \leq_\lambda A \diamond_t B,$$

where the equality holds if and only if $A = B$.

Corollary 3.3 strengthens the weak log-majorization result in [11]: for $t \in (0,1)$, $A\natural_t B \prec_{w\log} A \diamond_t B$.

COROLLARY 3.4. *For $t \in (0,1)$, we have $A\natural_t B \preceq A\nabla_t B$.*

Proof. It is known [21, 8] that $A \diamond_t B \leq A\nabla_t B$ for $t \in (0,1)$, and (3.12) shows that $A\natural_t B \preceq A \diamond_t B$. By [9], we get $A\natural_t B \preceq A\nabla_t B$. ∎

Theorem 2.3(5) shows that for $t \in (0,1)$, we have

$$\det(A \diamond_t B) \geq (\det A)^{1-t}(\det B)^t = \det(A\natural_t B).$$

The following is a direct consequence of Theorem 3.1.

COROLLARY 3.5. *If $A \preceq B$ and $t \in (1,\infty)$, or $A \succeq B$ and $t \in (-\infty, 0)$, then*

$$\det(A \diamond_t B) \leq (\det A)^{1-t}(\det B)^t.$$

Next, we study the monotonicity of near order on the curves defined by $\diamond_t$ and $\natural_t$.

THEOREM 3.6. *The following are equivalent for $A, B \in \mathbb{P}_n$ and $t, s \in \mathbb{R}$:*

(1) $A \preceq B$;
(2) $A \diamond_t B \preceq A \diamond_s B$ for certain $0 \leq t < s \leq 1$;
(3) $A\natural_t B \preceq A\natural_s B$ for certain $t < s$.

Moreover, when $A \preceq B$, the parametric curves $\{A \diamond_t B \mid t \geq 0\}$ and $\{A\natural_t B \mid t \in \mathbb{R}\}$ are monotonically increasing with respect to the near order, that is,

$$0 \leq t < s \Longrightarrow A \diamond_t B \preceq A \diamond_s B,$$
$$t < s \Longrightarrow A\natural_t B \preceq A\natural_s B.$$

Proof. Obviously, (1) implies (2) and (3).

Suppose that (2) holds, namely, there are $t, s \in [0,1]$ with $t < s$ such that $A \diamond_t B \preceq A \diamond_s B$. By (1.5),

$$[I\nabla_t(A^{-1}\sharp B)]A[I\nabla_t(A^{-1}\sharp B)] \preceq [I\nabla_s(A^{-1}\sharp B)]A[I\nabla_s(A^{-1}\sharp B)].$$

Note that $I\nabla_t(A^{-1}\sharp B)$ and $I\nabla_s(A^{-1}\sharp B)$ are in $\mathbb{P}_n$ and they commute. By Corollary 2.6,

$$I\nabla_t(A^{-1}\sharp B) \leq I\nabla_s(A^{-1}\sharp B).$$

So $I \leq A^{-1}\sharp B$, and thus $A \preceq B$. We get (1).

A similar argument shows that (3) implies (1).

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Now suppose that $A \preceq B$. Then $A^{-1}\sharp B \geq I$. For any $0 \leq t < s$, we have

$$0 < I \leq I\nabla_t(A^{-1}\sharp B) \leq I\nabla_s(A^{-1}\sharp B),$$

so that by (2.10) and Corollary 2.6,

$$A \diamond_t B = [I\nabla_t(A^{-1}\sharp B)]A[I\nabla_t(A^{-1}\sharp B)] \preceq [I\nabla_s(A^{-1}\sharp B)]A[I\nabla_s(A^{-1}\sharp B)] = A \diamond_s B.$$

Similarly, for any $t < s$, we have $0 \leq (A^{-1}\sharp B)^t \leq (A^{-1}\sharp B)^s$, so that by (2.11) and Corollary 2.6,

$$A\natural_t B = (A^{-1}\sharp B)^t A(A^{-1}\sharp B)^t \preceq (A^{-1}\sharp B)^s A(A^{-1}\sharp B)^s = A\natural_s B.$$

This completes the proof. ∎

Likewise, Theorem 3.6 has a counterpart theorem for $A \succeq B$ and the proof can be obtained by using $A \diamond_t B = B \diamond_{1-t} A$ and $A\natural_t B = B\natural_{1-t} A$. Indeed, more precise comparisons can be done by Theorem 2.5 together with (2.10) and (2.11) as follows.

THEOREM 3.7. *Suppose that $A, B \in \mathbb{P}_n$ are distinct. Let $\mu_1 \geq \cdots \geq \mu_n$ be the eigenvalues of $A^{-1}\sharp B$.*

1. *For real numbers $t < s$,*

$$\mu_n^{2s-2t}(A\natural_t B) \preceq A\natural_s B \preceq \mu_1^{2s-2t}(A\natural_t B). \tag{3.15}$$

2. *For real numbers $t < s$, if one of the following cases occurs:*
   (a) $t, s \in [0,1]$, or
   (b) $A \preceq B$ or $A \succeq B$, $\mu_1 \neq 1$, and $t, s \in (\frac{1}{1-\mu_1}, \infty)$, or
   (c) $A \preceq B$ or $A \succeq B$, $\mu_n \neq 1$, and $t, s \in (-\infty, \frac{1}{1-\mu_n})$,
   then

$$\left(\frac{1-s+s\mu_n}{1-t+t\mu_n}\right)^2 (A \diamond_t B) \preceq A \diamond_s B \preceq \left(\frac{1-s+s\mu_1}{1-t+t\mu_1}\right)^2 (A \diamond_t B). \tag{3.16}$$

3. *For $t, s \in \mathbb{R}$ such that $1-s+s\mu_1 > 0$ and $1-s+s\mu_n > 0$, let*

$$m_{s,t} = \min\left\{\frac{1-s+s\mu_i}{\mu_i^t} \mid i \in [n]\right\}, \quad M_{s,t} = \max\left\{\frac{1-s+s\mu_i}{\mu_i^t} \mid i \in [n]\right\},$$

   then

$$m_{s,t}^2(A\natural_t B) \preceq A \diamond_s B \preceq M_{s,t}^2(A\natural_t B). \tag{3.17}$$

Proof. 1. For $t < s$,

$$A\natural_s B = (A^{-1}\sharp B)^s A(A^{-1}\sharp B)^s = (A^{-1}\sharp B)^{s-t}(A\natural_t B)(A^{-1}\sharp B)^{s-t}.$$

Since $\mu_n^{s-t} I \leq (A^{-1}\sharp B)^{s-t} \leq \mu_1^{s-t} I$, Corollary 2.6 implies (3.15).

2. Fixing $t < s$, we define the function of μ:

$$f_{s,t}(\mu) = \frac{1-s+s\mu}{1-t+t\mu}. \tag{3.18}$$

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Since $f'_{s,t}(\mu) > 0$, $f_{s,t}(\mu)$ is monotonically increasing in any interval not containing $1 - \frac{1}{t}$. Denote

$$l_{s,t} = \min\{f_{s,t}(\mu_i) \mid i \in [n]\}, \qquad L_{s,t} = \max\{f_{s,t}(\mu_i) \mid i \in [n]\}.$$

When $l_{s,t} > 0$, we get

$$0 < l_{s,t}I \leq [I\nabla_s(A^{-1}\sharp B)][I\nabla_t(A^{-1}\sharp B)]^{-1} \leq L_{s,t}I.$$

Note that $I\nabla_s(A^{-1}\sharp B)$ and $[I\nabla_t(A^{-1}\sharp B)]^{-1}$ are commuting, and (2.10) implies that

$$A \diamond_s B = [I\nabla_s(A^{-1}\sharp B)][I\nabla_t(A^{-1}\sharp B)]^{-1}(A \diamond_t B)[I\nabla_s(A^{-1}\sharp B)][I\nabla_t(A^{-1}\sharp B)]^{-1}.$$

Applying Theorem 2.5, we get

$$l_{s,t}^2(A \diamond_t B) \preceq A \diamond_s B \preceq L_{s,t}^2(A \diamond_t B).$$

To prove (3.16) for cases 2(a), 2(b), and 2(c) in the theorem, it suffices to prove that $0 < f_{s,t}(\mu_n) \leq \cdots \leq f_{s,t}(\mu_1)$ for these cases.

(a) $t, s \in [0,1]$. Then $\mu_n \leq \cdots \leq \mu_1$ are in the interval $(0,\infty)$ not containing $1 - \frac{1}{t}$. We have $0 < \frac{(1-s)+s\mu_n}{(1-t)+t\mu_n} = f_{s,t}(\mu_n) \leq \cdots \leq f_{s,t}(\mu_1)$.

(b) $A \preceq B$ or $A \succeq B$, $\mu_1 \neq 1$, and $t, s \in (\frac{1}{1-\mu_1}, \infty)$. We have

$$1 + (\mu - 1)t = \frac{\mu - 1}{\mu_1 - 1}[1 + (\mu_1 - 1)t] + \frac{\mu_1 - \mu}{\mu_1 - 1}. \tag{3.19}$$

If $A \preceq B$, then $1 \leq \mu_n \leq \cdots \leq \mu_1$ and $1 < \mu_1$; by $\frac{1}{1-\mu_1} < t$ and (3.19), we have $1+(\mu-1)t > 0$ for $\mu \in [\mu_n, \mu_1]$. If $A \succeq B$, then $0 < \mu_n \leq \cdots \leq \mu_1 < 1$; by $\frac{1}{1-\mu_1} < t$ and (3.19), we have $1 + (\mu - 1)t < 0$ for $\mu \in [\mu_n, \mu_1]$. In either case, $f_{s,t}(\mu) = 1 + \frac{(\mu-1)(s-t)}{1+(\mu-1)t}$ is positive and monotonically increasing for $\mu \in [\mu_n, \mu_1]$.

(c) $A \preceq B$ or $A \succeq B$, $\mu_n \neq 1$, and $t, s \in (-\infty, \frac{1}{1-\mu_n})$. Replace μ_1 by μ_n in (3.19). Analogous arguments shows that $0 < f_{s,t}(\mu_n) \leq \cdots \leq f_{s,t}(\mu_1)$.

3. For $t, s \in \mathbb{R}$, $(A^{-1}\sharp B)^t$ and $[I\nabla_s(A^{-1}\sharp B)]$ commute. Moreover, when $\mu > 1$ and $s \in (\frac{1}{1-\mu}, \infty)$, or $0 < \mu < 1$ and $s \in (-\infty, \frac{1}{1-\mu})$, we have $\frac{1-s+s\mu}{\mu^t} > 0$. Now

$$A \diamond_s B = [I\nabla_s(A^{-1}\sharp B)](A^{-1}\sharp B)^{-t}(A\natural_t B)[I\nabla_s(A^{-1}\sharp B)](A^{-1}\sharp B)^{-t}$$

in which $m_{s,t}I \leq [I\nabla_s(A^{-1}\sharp B)](A^{-1}\sharp B)^{-t} \leq M_{s,t}I$. Applying Theorem 2.5, we get (3.17). ◻

Corollary 3.8. *Let $A, B \in \mathbb{P}_n$. Let $\mu_1 \geq \cdots \geq \mu_n$ be the eigenvalues of $A^{-1}\sharp B$. The following statements hold:*

1. *For $\mu \in (0, \mu_n]$, the parametric curves $\{(\mu^2 A)\natural_t B \mid t \in \mathbb{R}\}$ and $\{(\mu^2 A)\diamond_t B \mid t \geq 0\}$ are monotonically increasing with respect to the near order.*
2. *For $\mu \in [\mu_1, \infty)$, the parametric curves $\{(\mu^2 A)\natural_t B \mid t \in \mathbb{R}\}$ and $\{(\mu^2 A) \diamond_t B \mid t \leq 1\}$ are monotonically decreasing with respect to the near order.*

Proof. Let $t < s$. (3.15) shows that for $\mu \in (0, \mu_n]$,

$$\mu^{2s-2t}(A\natural_t B) \leq \mu_n^{2s-2t}(A\natural_t B) \preceq A\natural_s B,$$

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so that

$$(\mu^2 A)\natural_t B = \mu^{2-2t}(A\natural_t B) \preceq \mu^{2-2s}(A\natural_s B) = (\mu^2 A)\natural_s B.$$

Hence, $\{(\mu^2 A)\natural_t B \mid t \in \mathbb{R}\}$ is monotonically increasing with respect to the near order. By Theorem 3.6, $\{(\mu^2 A)\diamond_t B \mid t \geq 0\}$ is monotonically increasing with respect to the near order. The proof of second statement is analogous. ∎

By Theorem 2.3 (2), when $t \in (0,1)$, $(A \diamond_t B)^{-1} = A^{-1} \diamond_t B^{-1}$ if and only if $A = B$. In general, they satisfy the following near order relation.

Theorem 3.9. *Let $A, B \in \mathbb{P}_n$. Then for $t \in (0,1)$,*

$$(A^{-1} \diamond_t B^{-1})^{-1} \preceq A\natural_t B,$$
$$(A^{-1} \diamond_t B^{-1})^{-1} \preceq A \diamond_t B,$$

and either of the equalities holds if and only if $A = B$.

Proof. According to (2.10),

$$\begin{aligned}(A^{-1} \diamond_t B^{-1})^{-1} &= [I\nabla_t(A\sharp B^{-1})]^{-1} A [I\nabla_t(A\sharp B^{-1})]^{-1} \\ &= [I\nabla_t(A^{-1}\sharp B)^{-1}]^{-1} A [I\nabla_t(A^{-1}\sharp B)^{-1}]^{-1}.\end{aligned}$$

The following Löwner order relations exist for $t \in (0,1)$:

$$[I\nabla_t(A^{-1}\sharp B)^{-1}]^{-1} \leq (A^{-1}\sharp B)^t \leq I\nabla_t(A^{-1}\sharp B).$$

By Corollary 2.6, we get $(A^{-1} \diamond_t B^{-1})^{-1} \preceq A\natural_t B$ and $(A^{-1} \diamond_t B^{-1})^{-1} \preceq A \diamond_t B$. Moreover, either of the equalities holds if and only if $A^{-1}\sharp B = I$, that is, $A = B$. ∎

Remark 3.10. *Theorem 2.3 (6) shows that for $A, B \in \mathbb{P}_n$ and $t \in [0,1]$,*

$$(A^{-1} \diamond_t B^{-1})^{-1} \geq (A^{-1}\nabla_t B^{-1})^{-1}.$$

By Theorems 3.1 and 3.9, the Wasserstein mean, the spectral geometric mean, the harmonic mean, and the arithmetic mean have the following relations:

$$(A^{-1}\nabla_t B^{-1})^{-1} \leq (A^{-1} \diamond_t B^{-1})^{-1} \preceq (A^{-1}\natural_t B^{-1})^{-1} = A\natural_t B \preceq A \diamond_t B \leq A\nabla_t B. \tag{3.20}$$

Equivalently,

$$(A\nabla_t B)^{-1} \leq (A \diamond_t B)^{-1} \preceq (A\natural_t B)^{-1} = A^{-1}\natural_t B^{-1} \preceq A^{-1} \diamond_t B^{-1} \leq A^{-1}\nabla_t B^{-1}. \tag{3.21}$$

By [9, Theorem 2], we conclude that any two means in the sequence (3.20) or (3.21) have at least the near order relation.

By [9, Proposition 4, Remark 5], or by an analogous result of [16, Theorem 3.6], if $A, B \in \mathbb{P}_n$ and $A \preceq B$, then $A^p \preceq B^p$ for $p \geq 1$; if $A \leq B$ or $\log A \leq \log B$, then $A^p \preceq B^p$ for $p \geq 0$. Note that $A \preceq B$ if and only if $B^{-1} \preceq A^{-1}$. We summarize the results as follows and skip their proofs.

Theorem 3.11. *Let $A, B \in \mathbb{P}_n$. Suppose that one of the following holds:*

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1. *$A \preceq B$ and $p \geq 1$, or*
2. *$A \leq B$ and $p \geq 0$, or*
3. *$\log A \leq \log B$ and $p \geq 0$.*

Then $A^p \preceq B^p$ and $B^{-p} \preceq A^{-p}$. Moreover,

(i) the parametric curves $\{A^p \diamond_t B^p \mid t \geq 0\}$ and $\{B^{-p} \diamond_t A^{-p} \mid t \geq 0\}$ are geodesics monotonically increasing with respect to the near order;
(ii) the parametric curve $\{A^p \natural_t B^p \mid t \in \mathbb{R}\}$ is monotonically increasing with respect to the near order.

Remark 3.12. *The statement "$A \preceq B$ and $p \geq 1$ imply that $A^p \preceq B^p$" can be viewed as a special case of [9, Theorem 6], which can be applied to obtain other monotonic curves with respect to $\preceq$.*

Because the Löwner order is stronger than the near order and the near order is stronger than the eigenvalue entrywise order, it is straightforward to have the following corollary.

Corollary 3.13. *Let $A, B \in \mathbb{P}_n$ and $p \geq 1$. If $A \diamond_t B \preceq A \diamond_s B$ for some $0 \leq t < s \leq 1$, or $A \natural_t B \preceq A \natural_s B$ for some $t < s$, then the matrices on the curves $\{A^p \diamond_t B^p \mid t \in [0, \infty)\}$ and $\{A^p \natural_t B^p \mid t \in \mathbb{R}\}$ have entrywise monotonically increasing eigenvalues. In particular,*

1. *$A^p \leq_\lambda A^p \diamond_t B^p \leq_\lambda B^p$ for all $t \in (0, 1)$,*
2. *$A^p \leq_\lambda A^p \natural_t B^p \leq_\lambda B^p$ for all $t \in (0, 1)$.*

Corollary 3.13 can be applied to the following two cases.

Corollary 3.14. *Let $A, B \in \mathbb{P}_n$. If $A \geq I$ and $A \diamond_t B \leq I$ for some $t \in (0, 1)$, then*

$$A^p \diamond_s B^p \leq \left(\frac{1}{1 - t + t\lambda_n(A^{-1} \sharp B)} \right)^{2p} I$$

for all $p \geq 1$ and all $s \in [0, 1]$.

Proof. Since $A \diamond_t B \leq I$, according to (1.5),

$$A \leq [I \nabla_t (A^{-1} \sharp B)]^{-2} \leq (1 - t + t\lambda_n(A^{-1} \sharp B))^{-2} I.$$

Since $A \diamond_t B \leq I \leq A$, by Corollary 3.13, for all $p \geq 1$ and all $s \in [0, 1]$,

$$\lambda_1(A^p \diamond_s B^p) \leq \lambda_1(A^p) \leq \left(\frac{1}{1 - t + t\lambda_n(A^{-1} \sharp B)} \right)^{2p},$$

which means that $A^p \diamond_s B^p \leq \left(\frac{1}{1-t+t\lambda_n(A^{-1} \sharp B)} \right)^{2p} I$. □

A similar argument on $A^p \natural_s B^p$ leads to the following result. We skip its proof here.

Corollary 3.15. *Let $A, B \in \mathbb{P}_n$. If $A \geq I$ and $A \natural_t B \leq I$ for some $t \in (0, 1)$, then*

$$A^p \natural_s B^p \leq \left(\frac{1}{\lambda_n(A^{-1} \sharp B)} \right)^{2pt} I$$

for all $p \geq 1$ and all $s \in [0, 1]$.

Using $A \diamond_t B = B \diamond_{1-t} A$ and $A \natural_t B = B \natural_{1-t} A$, Theorem 3.11 can be changed to an equivalent theorem for curves monotonically decreasing with respect to the near order.

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THEOREM 3.16. *Let $A, B \in \mathbb{P}_n$. Suppose that one of the following holds:*

1. $A \succeq B$ *and* $p \geq 1$, *or*
2. $A \geq B$ *and* $p \geq 0$, *or*
3. $\log A \geq \log B$ *and* $p \geq 0$.

Then $A^p \succeq B^p$ *and* $B^{-p} \succeq A^{-p}$. *Moreover,*

(i) the parametric curves $\{A^p \diamond_t B^p \mid t \leq 1\}$ *and* $\{B^{-p} \diamond_t A^{-p} \mid t \leq 1\}$ *are geodesics monotonically decreasing with respect to the near order;*
(ii) the parametric curve $\{A^p \natural_t B^p \mid t \in \mathbb{R}\}$ *is monotonically decreasing with respect to the near order.*

4. The Löwner orders on two means. We explore the Löwner order properties of Wasserstein mean and spectral geometric mean in this section. These properties further display the interesting similarities between the two means. Some of them can be extended to the curves induced by the two means.

(2.10), (2.11), and Theorem 2.1(9) imply that for $A, B \in \mathbb{P}_n$ and $t \in [0, 1]$:

$$A^{-1} \sharp (A \diamond_t B) = B \sharp (B^{-1} \diamond_t A^{-1}) = I \nabla_t (A^{-1} \sharp B), \tag{4.22}$$

$$A^{-1} \sharp (A \natural_t B) = B \sharp (B^{-1} \natural_t A^{-1}) = (A^{-1} \sharp B)^t. \tag{4.23}$$

They can be used to derive identities like:

$$\begin{aligned} [A \sharp (A \diamond_t B)^{-1}] \nabla_t [B \sharp (A \diamond_t B)^{-1}] = I, \\ [A \sharp (A \natural_t B)^{-1}]^{1-t} [B \sharp (A \natural_t B)^{-1}]^t = I. \end{aligned}$$

(4.22) and (4.23) also imply the following Löwner order relations.

THEOREM 4.1. *For $A, B \in \mathbb{P}_n$ and $t \in [0, 1]$,*

$$A^{-1} \sharp (A \natural_t B) \leq A^{-1} \sharp (A \diamond_t B), \tag{4.24}$$

$$B^{-1} \sharp (A \natural_t B) \leq B^{-1} \sharp (A \diamond_t B), \tag{4.25}$$

(4.24) and (4.25) can be rephrased as follows:

$$|(A \natural_t B)^{1/2} A^{1/2}| \leq |(A \diamond_t B)^{1/2} A^{1/2}|, \qquad |(A \natural_t B)^{1/2} B^{1/2}| \leq |(A \diamond_t B)^{1/2} B^{1/2}|.$$

The above inequlities lead to the following results.

THEOREM 4.2. *Let $A, B, C \in \mathbb{P}_n$.*

1. *If* $A \diamond_t B \leq A \diamond_t C$ *or* $A \natural_t B \leq A \natural_t C$ *for one* $t \in (0, 1]$, *then* $A^{-1} \sharp_s B \leq A^{-1} \sharp_s C$ *for all* $s \in [0, 1/2]$.
2. *If* $A \diamond_t B = A \diamond_t C$ *or* $A \natural_t B = A \natural_t C$ *for one* $t \in (0, 1]$, *then* $B = C$.

Proof. We prove the statements related to $\diamond_t$, and the proofs for statements related to $\natural_t$ are analogous.

1. By (4.22) and the assumption $A \diamond_t B \leq A \diamond_t C$,

$$\begin{aligned} A^{1/2} [I \nabla_t (A^{-1} \sharp B)] A^{1/2} &= (A^{1/2} (A \diamond_t B) A^{1/2})^{1/2} \\ &\leq (A^{1/2} (A \diamond_t C) A^{1/2})^{1/2} \\ &= A^{1/2} [I \nabla_t (A^{-1} \sharp C)] A^{1/2}, \end{aligned}$$

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which gives $A^{-1}\sharp B \leq A^{-1}\sharp C$. In other words,

$$(A^{1/2}BA^{1/2})^{1/2} \leq (A^{1/2}CA^{1/2})^{1/2}.$$

Therefore, for any $s \in [0,1/2]$, we have $2s \in [0,1]$ so that by Löwner–Heinz inequality (see [25])

$$(A^{1/2}BA^{1/2})^{s} \leq (A^{1/2}CA^{1/2})^{s}.$$

In other words, $A^{-1}\sharp_s B \leq A^{-1}\sharp_s C$.

2. Similar to the above analysis, $A \diamond_t B = A \diamond_t C$ leads to $A^{-1}\sharp B = A^{-1}\sharp C$, which gives $B = C$. ◻

Theorem 4.2 implies the following result, in which $A^{-1} \diamond_t A = (A^{-1/2}\nabla_t A^{1/2})^2$ for $A \in \mathbb{P}_n$.

THEOREM 4.3. *Let $A, B \in \mathbb{P}_n$. Then $A \leq B$ if one of the following holds for any $t \in (0,1)$:*

1. $A^{-1} \diamond_t A \leq A^{-1} \diamond_t B$,
2. $B^{-1} \diamond_t B \leq A^{-1} \diamond_t B$,
3. $B^{-1} \diamond_t A \leq A^{-1} \diamond_t A$,
4. $B^{-1} \diamond_t A \leq B^{-1} \diamond_t B$.

Proof. We prove the first statement and the others can be done similarly.

If $A^{-1} \diamond_t A \leq A^{-1} \diamond_t B$ for any $t \in (0,1)$, then by Theorem 4.2, $A = A\sharp A \leq A\sharp B$, so that $I \leq |B^{1/2}A^{-1/2}|$ and thus $A \leq B$. ◻

The analogous results for the spectral geometric mean are given as follows, and we omit the proofs here. Note that $A^{-1}\natural_t A = A^{2t-1}$ for $A \in \mathbb{P}_n$.

THEOREM 4.4. *Let $A, B \in \mathbb{P}_n$. Then $A \leq B$ if one of the following holds for $t \in (0,1)$:*

1. $A^{-1}\natural_t A \leq A^{-1}\natural_t B$,
2. $B^{-1}\natural_t B \leq A^{-1}\natural_t B$,
3. $B^{-1}\natural_t A \leq A^{-1}\natural_t A$,
4. $B^{-1}\natural_t A \leq B^{-1}\natural_t B$.

The identity matrix I commutes with any matrix $X \in \mathbb{P}_n$ so that for $t \in (0,1)$:

$$I\nabla_t X \geq I \diamond_t X = (I\nabla_t X^{1/2})^2 \geq X^t = I\natural_t X,$$

and either equality holds if and only if $X = I$.

The following result is only for the Wasserstein mean.

THEOREM 4.5. *If $A, B \in \overline{\mathbb{P}_n}$, then for $t \in (0,1)$,*

$$|(A \diamond_t B)^{1/2}A^{1/2}| \geq A \diamond_t |B^{1/2}A^{1/2}|.$$

Proof. By (2.9), we have

$$|(A \diamond_t B)^{1/2}A^{1/2}| = (1-t)A + t|B^{1/2}A^{1/2}| = A\nabla_t|B^{1/2}A^{1/2}| \geq A \diamond_t |B^{1/2}A^{1/2}|.$$

The inequality is proved. ◻

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