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CHARACTERIZATION OF INVARIANT SUBSPACES FOR A NILPOTENT LINEAR OPERATOR THAT ADMIT COMPLEMENTARY INVARIANT SUBSPACES*

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Abstract. The aim of this work is to solve the problem of determining the necessary and sufficient conditions for a vector subspace invariant by a nilpotent endomorphism to admit a complementary invariant subspace for the same linear operator. As applications, we offer results about Jordan bases associated with nilpotent linear maps and reflexive generalized inverses of finite potent endomorphisms and square matrices.

Key words. Nilpotent endomorphism, Invariant subspace, Jordan bases, Finite potent endomorphism, Reflexive generalized inverse.

AMS subject classifications. 15A09, 15A04, 15A20.

1. Introduction.

Given an arbitrary k-vector space V, it is well-known that for every vector subspace $H \subset V$, there exists a complementary subspace to H: that is, there exists a vector subspace $S \subset V$ such that $H \oplus S = V$.

If $f \in \operatorname{End}_k(V)$ is a linear operator on V and H is f-invariant, in general, the existence of a complementary subspace that is invariant under f is not guaranteed.

This work aims to provide a solution to the problem of the existence of vector subspaces invariant by a nilpotent endomorphism that admit complementary vector subspaces invariant by the same linear operator.

The solution of this problem is given, using Zorn's Lemma, in Theorem 3.4 from the following statement: "given an arbitrary k-vector space V and a nilpotent endomorphism $g \in \operatorname{End}_k(V)$, a non-trivial g-invariant subspace $H \subset V$ admits a g-invariant complementary subspace in V if and only if H satisfies the following property: $u_1 \in H \cap [\operatorname{Im} g]$ if and only if there exists $u_0 \in H$ with $g(u_0) = u_1$."

As an application of this characterization, we study different properties of the Jordan bases for a nilpotent endomorphism in an arbitrary vector space, whose existence was proved by M. López-Pellicer and R. Bru in [3] and, using a different technique, by the author of this work in [4]. Moreover, we offer some new results about reflexive generalized inverses of finite potent endomorphisms and square matrices. In particular, Proposition 4.9 shows that "if $A \in \operatorname{Mat}_{n\times n}(k)$ is a square matrix with entries in an arbitrary field k, $R(A)$ denotes the range of a A, $N(A)$ is the Nullspace of A and $v \in k^n$ is a vector with $v \notin R(A)$, then for every vector $\tilde{v} \in k^n$ with $\tilde{v} \notin N(A)$ there exists a reflexive generalized inverse $A^+_{v,\tilde{v}} \in A(1,2)$ such that $A^+_{v,\tilde{v}} \cdot v^t = \tilde{v}^t$."

The paper is organized as follows. In Section 2, we briefly recall some statements of the papers [1], [4], [5], and [7]. Section 3 deals with the main results of this work: to determine the necessary and sufficient

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conditions for a vector subspace invariant by a nilpotent endomorphism to admit a complementary subspace invariant for the same linear operator. Finally, Section 4 is devoted to offer new results about Jordan bases associated with nilpotent linear operators and reflexive generalized inverses of finite potent endomorphisms and square matrices.

2. Preliminaries.

2.1. Finite potent endomorphisms. Let k be an arbitrary field, let V be a k-vector space, and let us consider an endomorphism φ of V. We say that φ is "finite potent" if $\varphi^n V$ is finite dimensional for some n, where the power means the composition $\varphi \circ \overset{n)}{\ldots\ldots} \circ \varphi$. This definition was introduced by J. Tate in [8] as a basic tool for his elegant definition of Abstract Residues.

In 2007, M. Argerami, F. Szechtman, and R. Tifenbach showed in [2] that an endomorphism φ is finite potent if and only if V admits a φ-invariant decomposition $V = U_\varphi \oplus W_\varphi$ such that $\varphi_{|_{U_\varphi}}$ is nilpotent, W_φ is finite dimensional, and $\varphi_{|_{W_\varphi}} : W_\varphi \xrightarrow{\sim} W_\varphi$ is an isomorphism.

Indeed, if $k[x]$ is the algebra of polynomials in the variable x with coefficients in k, we may view V as an $k[x]$-module via φ, and the explicit definition of the above φ-invariant subspaces of V is:

- $U_\varphi = \{v \in V \text{ such that } \varphi^m(v) = 0 \text{ for some } m\}$;
- $W_\varphi = \left\{\begin{array}{l} v \in V \text{ such that } p(\varphi)(v) = 0 \text{ for some } p(x) \in k[x] \\ \text{relatively prime to } x \end{array}\right\}$.

Note that if the annihilator polynomial of φ is $x^m \cdot p(x)$ with $(x, p(x)) = 1$, then $U_\varphi = \operatorname{Ker} \varphi^m$ and $W_\varphi = \operatorname{Ker} p(\varphi)$.

Hence, this decomposition is unique. We shall call this decomposition the φ-invariant AST-decomposition of V.

Moreover, we shall call "index of φ", $i(\varphi)$, to the nilpotent order of $\varphi_{|_{U_\varphi}}$. One has that $i(\varphi) = 0$ if and only if V is a finite-dimensional vector space and φ is an automorphism.

Basic examples of finite potent endomorphisms are all endomorphisms of a finite-dimensional vector spaces and finite rank or nilpotent endomorphisms of infinite-dimensional vector spaces.

2.2. CN decomposition of a finite potent endomorphism. Let V be again an arbitrary k-vector space. Given a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$, there exists a unique decomposition $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in \operatorname{End}_k(V)$ are finite potent endomorphisms satisfying that:

- $i(\varphi_1) \leq 1$;
- φ_2 is nilpotent;
- $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1 = 0$.

According to [5, Theorem 3.2], if φ^D is the Drazin inverse of φ, one has that $\varphi_1 = \varphi \circ \varphi^D \circ \varphi$ is the core part of φ. Also, φ_2 is named the nilpotent part of φ and one has that

$$\varphi = \varphi_1 \Longleftrightarrow U_\varphi = \operatorname{Ker} \varphi \Longleftrightarrow W_\varphi = \operatorname{Im} \varphi \Longleftrightarrow (\varphi^D)^D = \varphi \Longleftrightarrow i(\varphi) \leq 1\,. \tag{2.1}$$

Moreover, if $V = W_{\varphi} \oplus U_{\varphi}$ is the AST-decomposition of V induced by φ, then φ_1 and φ_2 are the unique linear maps such that:

$$\varphi_1(v) = \begin{cases} \varphi(v) & \text{if } v \in W_{\varphi} \\ 0 & \text{if } v \in U_{\varphi} \end{cases} \quad \text{and} \quad \varphi_2(v) = \begin{cases} 0 & \text{if } v \in W_{\varphi} \\ \varphi(v) & \text{if } v \in U_{\varphi} \end{cases} . \tag{2.2}$$

2.3. Jordan bases of a nilpotent endomorphism. Let V be a vector space over an arbitrary field k and let $g \in \operatorname{End}_k(V)$ be a nilpotent endomorphism. If n is the nilpotency index of g, according to the statements of [4], setting $U_i^g = \operatorname{Ker} g^i/[\operatorname{Ker} g^{i-1} + g(\operatorname{Ker} g^{i+1})]$ with $i \in \{1, 2, \dots, n\}$, $\mu_i(V, g) = \dim_k U_i^g$ and $S_{\mu_i(V,g)}$ a set such that $\# S_{\mu_i(V,g)} = \mu_i(V, g)$ with $S_{\mu_i(V,g)} \cap S_{\mu_j(V,g)} = \emptyset$ for all $i \neq j$, one has that there exists a family of vectors $\{v_{s_i}\}$ that determines a Jordan basis of V for g:

$$B_V = \bigcup_{\substack{s_i \in S_{\mu_i(V,g)} \\ 1 \leq i \leq n}} \{v_{s_i}, g(v_{s_i}), \dots, g^{i-1}(v_{s_i})\} . \tag{2.3}$$

Moreover, if we write $H^g_{s_i} = \langle v_{s_i}, g(v_{s_i}), \dots, g^{i-1}(v_{s_i})\rangle$, the basis B induces a decomposition

$$V = \bigoplus_{\substack{s_i \in S_{\mu_i(V,g)} \\ 1 \leq i \leq n}} H^g_{s_i} . \tag{2.4}$$

2.4. Bases of a finite potent endomorphism. Let us now consider a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$ with CN-decomposition $\varphi = \varphi_1 + \varphi_2$ and that induces the AST-decomposition $V = U_\varphi \oplus W_\varphi$. Keeping the above notation, if n is the nilpotency order of φ_2, we can construct a basis $B_V = B_{W_\varphi} \cup B_{U_\varphi}$ of V where

$$B_{W_\varphi} = \{w_1, \dots, w_r\},$$

is a basis of W_φ ($r = \dim_k W_\varphi$) and

$$B_{U_\varphi} = \bigcup_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \leq i \leq n}} \{v_{s_i}, \varphi(v_{s_i}), \dots, \varphi^{i-1}(v_{s_i})\},$$

is a Jordan basis of U_φ determined by $\varphi_{|U_\varphi}$.

If $\varphi = \varphi_1 + \varphi_2$ is the CN decomposition of φ, it is clear that

$$B_{U_\varphi} = \bigcup_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \leq i \leq n}} \{v_{s_i}, \varphi_2(v_{s_i}), \dots, \varphi_2^{i-1}(v_{s_i})\},$$

and

$$\operatorname{Ker} \varphi = \bigoplus_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \leq i \leq n}} \langle \varphi^{i-1}(v_{s_i})\rangle = \bigoplus_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \leq i \leq n}} \langle \varphi_2^{i-1}(v_{s_i})\rangle .$$

2.5. {1}-Inverses of a finite potent endomorphism. Let V be a k-vector space and let $f \in \operatorname{End}_k(V)$ be an arbitrary endomorphism. Recall that $f^- \in \operatorname{End}_k(V)$ is a $\{1\}$-inverse of $f \in \operatorname{End}_k(V)$ when

$$f \circ f^- \circ f = f\,.$$

The set of $\{1\}$-inverses of f will be denoted by $X_f(1)$.

It is known from [7, Lemma 3.2] that $f^- \in \operatorname{End}_k(V)$ is a $\{1\}$-inverse of f if and only if for every $v \in V$ we have that

$$f^-(f(v)) = v + u,$$

with $u \in \operatorname{Ker} f$.

With the above notation, according to [7, Proposition 3.3], one has that:

Proposition 2.1. *If $\varphi \in \operatorname{End}_k(V)$ is a finite potent endomorphism, then an endomorphism $\hat{\varphi} \in \operatorname{End}_k(V)$ is a $\{1\}$-inverse of φ if an only if $\hat{\varphi}$ satisfies that*

- $\hat{\varphi}(w_h) = (\varphi_{|W_\varphi})^{-1}(w_h) + u_h$ *for each* $h \in \{1, \dots, r\}$;
- $\hat{\varphi}(\varphi^j(v_{s_i})) = \varphi^{j-1}(v_{s_i}) + u^j_{s_i}$ *for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ *and* $j \in \{1, \dots, i-1\}$;
- $\hat{\varphi}(v_{s_i}) = \tilde{v}_{s_i}$ *for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$;

where $\tilde{v}_{s_i} \in V$ *and* $u_h, u^j_{s_i} \in \operatorname{Ker}\varphi$ *for each* $h \in \{1, \dots, r\}$ *and for every* $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ *and* $j \in \{1, \dots, i-1\}$.

2.6. Reflexive generalized inverses of finite potent endomorphisms. Let V be again an arbitrary k-vector space. Given an endomorphism $f \in \operatorname{End}_k(V)$, we say that $f^+ \in \operatorname{End}_k(V)$ is a "reflexive generalized inverse" of f when it satisfies that:

- $f^+ \circ f \circ f^+ = f^+$;
- $f \circ f^+ \circ f = f$.

It is clear that f^+ is a reflexive generalized inverse of f when f^+ is a $\{1\}$-inverse of f and f is a $\{1\}$-inverse of f^+. The set of reflexive generalized inverses of f will be denoted by $X_f(1,2)$.

Let $\varphi \in \operatorname{End}_k(V)$ be a finite potent endomorphism. With the above notation, if $C = (c_{ij})$ is the matrix associated to $\varphi_{|W_\varphi}$ in the basis B_{W_φ}, we have that

$$\varphi(w_j) = \sum_{i=1}^{r} c_{ij} w_i,$$

for every $j \in \{1, \dots, r\}$.

According to [1, Proposition 3.2], the explicit characterization of the reflexive generalized inverses of a finite potent endomorphism is the following:

Proposition 2.2. *Let V be an arbitrary k-vector space and let us consider a finite potent endomorphism $\varphi \in \operatorname{End}_k(V)$. With the previous notation, an endomorphism $\varphi^+ \in \operatorname{End}_k(V)$ is a reflexive generalized inverse if and only if it satisfies the following conditions:*

- $\varphi^+(w_h) = (\varphi_{|W_\varphi})^{-1}(w_h) + \sum_{\substack{s_{i'} \in S_{\mu_{i'}(U_\varphi,\varphi)} \\ 1 \leq i' \leq n}} \lambda^h_{s_{i'}} \cdot \varphi^{i'-1}(v_{s_{i'}})$, *with* $\lambda^h_{s_{i'}} \in k$ *for each* $s_{i'} \in S_{\mu_{i'}(U_\varphi,\varphi)}$ *and each* $h \in \{1, \dots, r\}$ *and where only a finite number of the scalars* $\{\lambda^h_{s_{i'}}\}$ *are different from zero.*

- $\varphi^+(\varphi^j(v_{s_i})) = \varphi^{j-1}(v_{s_i}) + \sum_{\substack{s_{i'} \in S_{\mu_{i'}(U_\varphi,\varphi)} \\ 1 \leq i' \leq n}} \beta^{s_i,j}_{s_{i'}} \cdot \varphi^{i'-1}(v_{s_{i'}})$, *with* $\beta^{s_i,j}_{s_{i'}} = 0$ *for almost all* $s_{i'} \in S_{\mu_i(U_\varphi,\varphi)}$ *and* $j \in \{1,\dots,i-1\}$.
- $\varphi^+(v_{s_i}) = \sum_{j=1}^r \gamma_j^{s_i} \cdot w_j + \sum_{\substack{s_{i'} \in S_{\mu_{i'}(U_\varphi,\varphi)} \\ 1 \leq i' \leq n}} [\sum_{l=0}^{i'-1} \xi^{s_i,l}_{s_{i'}} \varphi^l(v_{s_{i'}})]$ *with*

$$\xi^{s_i,i'-1}_{s_{i'}} = \sum_{j,h=1}^{r} (\lambda^h_{s_{i'}} \cdot c_{hj} \cdot \gamma_j^{s_i}) + \sum_{\substack{s_{i''} \in S_{\mu_{i''}(U_\varphi,\varphi)} \\ 1 \leq i'' \leq n}} (\sum_{l=0}^{i''-2} [\xi^{s_i,l}_{s_{i''}} \cdot \beta^{s_{i''},l+1}_{s_{i'}}]),$$

and $\xi^{s_i,l}_{s_{i'}} = 0$ *for almost all* $s_{i'} \in S_{\mu_i(U_\varphi,\varphi)}$ *and* $l \in \{1,\dots,i'-1\}$.

It is important to remark that the first and the second conditions of Proposition 2.2 coincide with the first and the second conditions of Proposition 2.1.

Also, we wish to point out that, in general, a reflexive generalized inverse of a finite potent endomorphism is not a finite potent endomorphism.

3. Invariant subspaces of nilpotent endomorphisms.

This section contains the main result of this work: the explicit characterization of the vector subspaces of an arbitrary vector space V that are invariant by a nilpotent endomorphism $g \in \operatorname{End}_k(V)$ and admitting complementary vector subspaces that are also invariant by g.

Henceforth, we shall say that a vector subspace $H \subset V$ is non-trivial when $H \neq \{0\}$ and $H \neq V$.

LEMMA 3.1. *Let* V *be an arbitrary* k*-vector space, let* $g \in \operatorname{End}_k(V)$ *be a nilpotent endomorphism, and let* $H \subset V$ *be a subspace satisfying the following condition:*

$$\text{“}u_1 \in H \cap [Im\ g] \textit{ if and only if there exists } u_0 \in H \textit{ with } g(u_0) = u_1 \text{.''} \tag{3.5}$$

Then $H = V$ *if and only if* $\operatorname{Ker} g \subseteq H$.

Proof. Let us assume that H satisfies condition (3.5).

If $H = V$, it is clear that $\operatorname{Ker} g \subseteq H$.

Conversely, let us assume that $\operatorname{Ker} g \subseteq H$ and let us consider an arbitrary vector $v \in V$.

If $v \in \operatorname{Ker} g$, then $v \in H$, and when $v \notin \operatorname{Ker} g$, there exists a positive integer $r \geq 2$ such that $v \in \operatorname{Ker} g^r$ and $v \notin \operatorname{Ker} g^{r-1}$.

In the second case, one has that $g^{r-1}(v) \in \operatorname{Ker} g$ and, therefore, $g^{r-1}(v) \in H \cap [\operatorname{Im} g]$. Then, it follows from condition (3.5) that there exists $v_0 \in H$ such that $g(u_0) = g^{r-1}(v)$. Bearing in mind that $g(u_0 - g^{r-2}(v)) = 0$, we deduce that $g^{r-2}(v) = u_0 + h$ for a certain $h \in \operatorname{Ker} g$. Accordingly, $g^{r-2}(v) \in H$ and, arguing in a similar way, recurrently we get that $v \in H$.

Accordingly, if $\operatorname{Ker} g \subseteq H$, then $H = V$ and the claim is proved. □

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PROPOSITION 3.2. *Let V be an arbitrary k-vector space, let $g \in \operatorname{End}_k(V)$ be a nilpotent endomorphism, and let H be a non-trivial g-invariant subspace of V satisfying condition (3.5). Then, there exists a g-invariant subspace $\widetilde{H} \subset V$ such that $V = H \oplus \widetilde{H}$.*

Proof. Let us consider a subspace $H \subset V$ satisfying the hypothesis of this proposition and let X_H be the set consisting of all g-invariant subspaces $H' \subset V$ such that $H \cap H' = \{0\}$.

Since $H \neq V$, it is known from Lemma 3.1 that there exists a vector $\widetilde{u} \in \operatorname{Ker} g$ with $\widetilde{u} \notin H$. Then, we have that $\langle \widetilde{u} \rangle \in X_H$ and, therefore, X_H is not the empty set.

It is clear that X_H is a partially ordered set with the inclusion of vector subspaces of V.

Moreover, if $\{H'_i\}_{i \in I}$ is a totally ordered subset of X_H, then $\sum_{i \in I} H'_i \in X_H$ is an upper bound of $\{H'_i\}_{i \in I}$.

Hence, applying Zorn's Lemma we obtain that X_H contains at least one maximal element $\widetilde{H}$. In particular, we have that $\widetilde{H} \cap H = \{0\}$.

With this notation, the k-subspace $\widetilde{H} \subset V$ satisfies that if $\widetilde{u} \in \widetilde{H} \cap [\operatorname{Im} g]$, then $v \in H + \widetilde{H}$ for every $v \in V$ with $g(v) = \widetilde{u}$, because if $g(v) = \widetilde{u}$ is such that $v \notin H + \widetilde{H}$, $g^{r-1}(v) \neq 0$ and $g^r(v) = 0$, then

$$\widetilde{H} \subsetneqq \widetilde{H} + \langle v, g(v), \ldots, g^{r-1}(v) \rangle \in X_H \,,$$

which is impossible. Accordingly, $H + \widetilde{H}$ satisfies condition (3.5).

To conclude the proof, if the nilpotency index of g is n, we shall check that $\operatorname{Ker} g^i \subseteq H + \widetilde{H}$ for each $i \in \{1, \ldots, n\}$.

We will proceed by induction on i.

For $i = 1$, if there exists $u' \in \operatorname{Ker} g$ with $u' \notin H + \widetilde{H}$, then $\widetilde{H} \subsetneqq \widetilde{H} + \langle u' \rangle \in X_H$, because $\widetilde{H} + \langle u' \rangle$ is g-invariant, and we come to a contradiction with the maximality of $\widetilde{H}$ in X_H. Thus, we have that $\operatorname{Ker} g \subseteq H + \widetilde{H}$.

Suppose that $\operatorname{Ker} g^r \subseteq H + \widetilde{H}$ for certain $r \leq n-1$ and we shall prove that $\operatorname{Ker} g^{r+1} \subseteq H + \widetilde{H}$.

Let us consider a vector $v' \in \operatorname{Ker} g^{r+1}$. Since $g(v') \in \operatorname{Ker} g^r \subseteq H + \widetilde{H}$, by condition (3.5) we know that there exists $v \in H + \widetilde{H}$ with $g(v) = g(v')$. Hence, since $v' = v + u$ with $u \in \operatorname{Ker} g \subseteq H + \widetilde{H}$, then $v' \in H + \widetilde{H}$ and we have that $\operatorname{Ker} g^{r+1} \subseteq H + \widetilde{H}$, from where we obtain that $V = \operatorname{Ker} g^n \subseteq H + \widetilde{H}$ and the claim is deduced. □

PROPOSITION 3.3. *Let V be an arbitrary k-vector space and let $g \in \operatorname{End}_k(V)$ be a nilpotent endomorphism. If $H_1, H_2 \subseteq V$ are non-trivial g-invariant subspaces of V such that $V = H_1 \oplus H_2$, then H_i satisfies (3.5) for $i \in \{1, 2\}$.*

Proof. Let us consider a vector $v_1 \in H_1 \cap \operatorname{Im} g$. If $v \in V$ is such that $g(v) = v_1$ and $v = u_1 + u_2$ with $u_1 \in H_1$ and $u_2 \in H_2$, then one has that $v_1 = g(u_1) + g(u_2)$ and

$$v_1 - g(u_1) = g(u_2) \in H_1 \cap H_2 = \{0\} \,,$$

from where we obtain that $g(u_1) = v_1$ and we deduce that H_1 satisfies (3.5).

Analogously, we can check that H_2 satisfies condition (3.5) and the assertion is proved. □

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Theorem 3.4. *Given an arbitrary k-vector space V and a nilpotent endomorphism $g \in \operatorname{End}_k(V)$, a non-trivial g-invariant subspace $H \subset V$ admits a g-invariant complementary subspace in V if and only if H satisfies the following property:*

$$\text{“}u_1 \in H \cap [Im\ g] \text{ if and only if there exists } u_0 \in H \text{ with } g(u_0) = u_1 \text{.”}$$

Proof. The statement is immediately deduced from Propositions 3.2 and 3.3. ∎

Remark 3.5. *Let $g \in \operatorname{End}_k(V)$ be a nilpotent endomorphism of an arbitrary k-vector space V with nilpotency index of g equal to n. It is clear that* $\operatorname{Ker} g$ *is a g-invariant subspace of V. Then, if $n = 1$, one has that* $\operatorname{Ker} g = V$ *and, when $n \geq 2$, we know that* $\operatorname{Ker} g \neq V$. *Hence, since Lemma 3.1 shows that* $\operatorname{Ker} g$ *does not satisfy condition (3.5) for $n \geq 2$, it follows from Theorem 3.4 that* $\operatorname{Ker} g$ *does not admit a complementary g-invariant subspace in V in this case.*

4. Applications.

The final section of this work is devoted to apply the previous results to study Jordan bases associated with nilpotent linear operators and reflexive generalized inverses of finite potent endomorphisms and square matrices.

4.1. Jordan bases associated with nilpotent endomorphisms.

Given again an arbitrary k-vector space V and a nilpotent linear map $g \in \operatorname{End}_k(V)$, and fixing a vector $u \in V$ different from zero, we shall now show that there exists a Jordan basis B_V of V induced for g and such that $u \in B_V$.

Lemma 4.1. *Let V be an arbitrary k-vector space, let $g \in \operatorname{End}_k(V)$ be a nilpotent endomorphism, and let $v \in V$ be a vector such that $v \neq 0$. Then, there exists a vector $u \in V$ with $u \notin Im\ g$, $g^{n-1}(u) \neq 0$ and $g^n(u) = 0$ for a certain $n \in \mathbb{N}$, such that $v \in \{u, g(u), \dots, g^{n-1}(u)\}$.*

Proof. If $v \notin \operatorname{Im} g$, $g^{n-1}(v) \neq 0$ and $g^n(v) = 0$, then one has that

$$v \in \{v, g(v), \dots, g^{n-1}(v)\}\,.$$

On the other hand, when $v \in \operatorname{Im} g^r$ and $v \notin \operatorname{Im} g^{r+1}$ with $r \geq 1$, let us consider a vector $u \in V$ such that $g^r(u) = v$. Since $u \notin \operatorname{Im} g$, one has that $v \in \{u, g(u), \dots, g^{n-1}(u)\}$ for a certain $n \geq r+1$ and the statement is proved. ∎

Proposition 4.2. *Given an arbitrary k-vector space V, a nilpotent endomorphism $g \in \operatorname{End}_k(V)$ and a vector $v \in V$ with $v \neq 0$, then there exists a Jordan basis B_V of V for g such that $v \in B_V$.*

Proof. It follows from Lemma 4.1 that there exists a vector $u \in V$ with $u \notin \operatorname{Im} g$, $g^{n-1}(u) \neq 0$, $g^n(u) = 0$ and such that $v \in \{u, g(u), \dots, g^{n-1}(u)\}$.

If $\langle u, g(u), \dots, g^{n-1}(u)\rangle = V$, then $B_V = \{u, g(u), \dots, g^{n-1}(u)\}$ is a Jordan basis of V for g such that $v \in B_V$.

Let us now suppose that $\langle u, g(u), \dots, g^{n-1}(u)\rangle \neq V$.

In this case, $\langle u, g(u), \dots, g^{n-1}(u)\rangle$ is a non-trivial subspace of V satisfying condition (3.5), and from Theorem 3.4, we can consider a non-trivial g-invariant vector subspace $\widetilde{H} \subset V$ satisfying that

$$V = \langle u, g(u), \dots, g^{n-1}(u)\rangle \oplus \widetilde{H}\,.$$

Then, if the nilpotency index of $g_{|_{\widetilde{H}}}$ is m and

$$B_{\widetilde{H}} = \bigcup_{\substack{s_i \in S_{\mu_i(\widetilde{H}, g_{|_{\widetilde{H}}})} \\ 1 \leq i \leq m}} \{v_{s_i}, g(v_{s_i}), \dots, g^{i-1}(v_{s_i})\},$$

is a Jordan basis of $\widetilde{H}$ for $g_{|_{\widetilde{H}}}$, one has that

$$B_V = \{u, g(u), \dots, g^{n-1}(u)\} \cup B_{\widetilde{H}},$$

is a Jordan basis of V for g such that $v \in B_V$. □

Lemma 4.3. *Given an arbitrary k-vector space V, a nilpotent linear map $g \in \operatorname{End}_k(V)$ and a vector $v \in V$ with $v \neq 0$ and $v \notin \operatorname{Ker} g$, there exists a Jordan basis B_V of V for g such that $\{v, g(v)\} \subseteq B_V$.*

Proof. If B_V is the Jordan basis of V for g constructed in Proposition 4.2 verifying that $v \in B_V$, when $v \notin \operatorname{Ker} g$ it is easy to check that $\{v, g(v)\} \subseteq B_V$. □

Let V be again an arbitrary k-vector space and recall that a nilpotent linear map is a finite potent endomorphism. To finish this paragraph, we shall use Proposition 2.2 to study reflexive generalized inverses of a nilpotent endomorphism $g \in \operatorname{End}_k(V)$ associated with a fixed $v \in V$ such that $v \notin \operatorname{Ker} g$.

Lemma 4.4. *If V is an arbitrary k-vector space, $g \in \operatorname{End}_k(V)$ is a nilpotent endomorphism, and $v \in V$ is a vector such that $v \notin \operatorname{Ker} g$, there exist nilpotent reflexive generalized inverses $g^+ \in X_g(1,2)$ satisfying that $g^+(g(v)) = v$.*

Proof. Let B_V be a basis of V such that $\{v, g(v)\} \subseteq B_V$, constructed as in the proof of Proposition 4.2. If the structure of B_V is

$$B_V = \bigcup_{\substack{s_i \in S_{\mu_i(V,g)} \\ 1 \leq i \leq n,}} \{v_{s_i}, g(v_{s_i}), \dots, g^{i-1}(v_{s_i})\},$$

then we can define the endomorphism $g^+_{B_V} \in \operatorname{End}_k(V)$ as the unique linear map satisfying that:

- $g^+_{B_V}(g^j(v_{s_i})) = g^{j-1}(v_{s_i})$ for every $s_i \in S_{\mu_i(V,g)}$ and $j \in \{1, \dots, i-1\}$;
- $g^+_{B_V}(v_{s_i}) = 0$ for each $s_i \in S_{\mu_i(V,g)}$.

It follows from Proposition 2.2 that $g^+_{B_V} \in X_g(1,2)$, and it is clear that $g^+_{B_V}$ is nilpotent. Hence, bearing in mind that $g^+_{B_V}(g(v)) = v$, the claim is proved. □

Remark 4.5. *We wish to point out that for an arbitrary Jordan basis B_V of V for g we can define a nilpotent reflexive generalized inverse $g^+_{B_V} \in X_g(1,2)$ as the one defined in Lemma 4.4.*

Moreover, we wish to remark that the reflexive generalized inverses $g^+_{B_V} \in X_g(1,2)$ satisfying that $g^+_{B_V}(g(v)) = v$ depend on the Jordan bases B_V chosen (in fact, bases with $\{v, g(v)\} \subseteq B_V$).

4.2. Reflexive generalized inverses of finite potent endomorphisms. To finish this work, we shall generalize Lemma 4.4 to study reflexive generalized inverses of finite potent endomorphisms.

If V is an arbitrary k-vector space and $\varphi \in \operatorname{End}_k(V)$ is a finite potent endomorphism, let $\varphi = \varphi_1 + \varphi_2$ be the CN decomposition of φ as it was described in Section 2.2 and let $V = W_\varphi \oplus U_\varphi$ be the AST-decomposition of V induced by φ.

We can define the reflexive generalized inverse $\varphi_1^+ \in X_{\varphi_1}(1,2)$ as the unique linear map satisfying that:

$$\varphi_1^+(v) = \varphi^D(v) = \begin{cases} (\varphi_{|W_\varphi})^{-1}(v) & \text{if } v \in W_\varphi \\ 0 & \text{if } v \in U_\varphi \end{cases},$$

where φ^D is the Drazin inverse of φ studied in [6].

Moreover, if B_{U_φ} is a Jordan basis of U_φ for $\varphi_{|U_\varphi}$ and $\varphi^+_{B_{U_\varphi}}$ is the reflexive generalized inverse of $\varphi_{|U_\varphi}$ constructed in Lemma 4.4, we can define a reflexive generalized inverse $(\varphi_2^+)_{B_{U_\varphi}} \in X_{\varphi_2}(1,2)$ as the unique linear operator such that:

$$(\varphi_2^+)_{B_{U_\varphi}}(v) = \begin{cases} 0 & \text{if } v \in W_\varphi \\ \varphi^+_{B_{U_\varphi}}(v) & \text{if } v \in U_\varphi \end{cases}.$$

Lemma 4.6. *If V is k-vector space, $\varphi \in \operatorname{End}_k(V)$ is a finite potent endomorphism, $u' \in U_\varphi$ is a vector such that $u' \notin \operatorname{Ker}\varphi$ and B_{U_φ} is a Jordan basis of U_φ for $\varphi_{B_{U_\varphi}}$ such that $u' \in B_{U_\varphi}$, then $\varphi^+_{B_{U_\varphi}} = \varphi_1^+ + (\varphi_2^+)_{B_{U_\varphi}} \in X_\varphi(1,2)$ and $\varphi^+_{B_{U_\varphi}}(\varphi(w+u')) = w + u'$ for all $w \in W_\varphi$. Moreover, $\varphi^+_{B_{U_\varphi}}$ is a finite potent endomorphism.*

Proof. All the assertions are immediately deduced from the definitions of φ_1^+ and $(\varphi_2^+)_{B_{U_\varphi}}$. □

With the previous notation, it is clear that $(\varphi^+_{B_{U_\varphi}})_1 = \varphi_1^+$ and $(\varphi^+_{B_{U_\varphi}})_2 = (\varphi_2^+)_{B_{U_\varphi}}$.

Proposition 4.7. *Let V be an arbitrary k-vector space and let $\varphi \in \operatorname{End}_k(V)$ be a finite potent endomorphism. If $v \in V$ is a vector with $v \notin \operatorname{Ker}\varphi$, then there exists a finite potent reflexive generalized inverse $\varphi^+ \in X_\varphi(1,2)$ such that $\varphi^+(\varphi(v)) = v$.*

Proof. If $V = W_\varphi \oplus U_\varphi$ is again the AST-decomposition of V determined by φ, we can write $v = w + u'$ with $w \in W_\varphi$ and $u' \in U_\varphi$.

If $u' \notin \operatorname{Ker}\varphi$ and B_{U_φ} is a Jordan basis of U_φ for $\varphi_{B_{U_\varphi}}$ such that $u' \in B_{U_\varphi}$, one has that the finite potent reflexive generalized inverse $\varphi^+_{B_{U_\varphi}}$ defined in Lemma 4.6 satisfies that $\varphi^+_{B_{U_\varphi}}(\varphi(v)) = v$.

Moreover, when $u' \in \operatorname{Ker}\varphi$, one has that $w \neq 0$. In this case, let us consider a basis $B_{W_\varphi} = \{\varphi(w), w_2, \dots, w_r\}$ of W_φ and a Jordan basis B_{U_φ} of U_φ for $\varphi_{|U_\varphi}$ such that $u' \in B_{U_\varphi}$.

Accordingly, if

$$B_{U_\varphi} = \bigcup_{\substack{s_i \in S_{\mu_i(U_\varphi,\varphi)} \\ 1 \leq i \leq n}} \{v_{s_i}, \varphi(v_{s_i}), \dots, \varphi^{i-1}(v_{s_i})\},$$

then we can consider the unique linear map $\varphi^+ \in \operatorname{End}_k(V)$ satisfying the conditions:

- $\varphi^+(\varphi(w)) = w + u' = v$;
- $\varphi^+(w_i) = (\varphi_{|W_\varphi})^{-1}(w_i)$ for all $i \in \{2, \dots, r\}$;
- $\varphi^+(\varphi^j(v_{s_i})) = \varphi^{j-1}(v_{s_i})$ for every $s_i \in S_{\mu_i(U_\varphi,\varphi)}$ and $j \in \{1, \dots, i-1\}$;
- $\varphi^+(v_{s_i}) = 0$ for every $s_i \in S_{\mu_i(U_\varphi,\varphi)}$.

By definition of φ^+, we have that $\varphi^+(\varphi(v)) = v$, and it follows from Proposition 2.2 that $\varphi^+ \in X_\varphi(1,2)$.

Furthermore, if $u' \in \{v_{s_{i_0}}, \varphi(v_{s_{i_0}}), \dots, \varphi^{i_0-1}(v_{s_{i_0}})\}$ (in fact, $u' = \varphi^{i_0-1}(v_{s_{i_0}})$) and the index of φ is n, then $\operatorname{Im}(\varphi^+)^n \subseteq W_\varphi + \langle v_{s_{i_0}}, \varphi(v_{s_{i_0}}), \dots, \varphi^{i_0-1}(v_{s_{i_0}})\rangle$, from where we deduce that φ^+ is finite potent and the statement is proved. ∎

Proposition 4.8. *Let V be an arbitrary k-vector space, let $\varphi \in \operatorname{End}_k(V)$ be a finite potent endomorphism, and let $v \in V$ be a vector such that $v \notin$ Im φ. Then, for every $\tilde v \in V$ with $\tilde v \notin \operatorname{Ker}\varphi$, there exists a finite potent reflexive generalized inverse $\widetilde{\varphi}^+_{v,\tilde v} \in X_\varphi(1,2)$ with $\widetilde{\varphi}^+_{v,\tilde v}(v) = \tilde v$.*

Proof. If $V = W_\varphi \oplus U_\varphi$ is the AST-decomposition of V induced by φ, we have that $v = w + \bar u$ with $w \in W_\varphi$, $\bar u \in U_\varphi$, and $\bar u \notin \operatorname{Im}\varphi_{|_{U_\varphi}}$. If $\bar u \in \operatorname{Ker}\varphi^s$ and $\bar u \notin \operatorname{Ker}\varphi^{s-1}$, it follows from Lemma 3.1 and Theorem 3.4 that there exists a $\varphi_{|_{U_\varphi}}$-invariant vector subspace $\tilde H \subset U_\varphi$ such that

$$U_\varphi = \langle \bar u, \varphi(\bar u), \dots, \varphi^{s-1}(\bar u)\rangle \oplus \tilde H\,.$$

Thus, we can consider a Jordan basis B_{U_φ} of U_φ for $\varphi_{|_{U_\varphi}}$ with the following structure:

$$\tilde B_{U_\varphi} = \{\bar u, \varphi(\bar u), \dots, \varphi^{s-1}(\bar u)\} \cup \Big[\bigcup_{\substack{s_i \in S_{\mu_i(\tilde H,\varphi)} \\ 1 \le i \le n}} \{v_{s_i}, \varphi(v_{s_i}), \dots, \varphi^{i-1}(v_{s_i})\}\Big]\,.$$

If, similar to above, we set $\tilde B_V = \{w_1, \dots, w_r\} \cup \tilde B_{U_\varphi}$ with $\{w_1, \dots, w_r\}$ a basis of W_φ, we can define the endomorphism $\varphi^+_{v,\tilde v} \in \operatorname{End}_k(V)$ as the unique linear map that satisfies the following conditions:

- $\varphi^+_{v,\tilde v}(w_h) = (\varphi_{|_{W_\varphi}})^{-1}(w_h) + \lambda^h_{\bar u}\varphi^{s-1}(\bar u) + \sum_{\substack{s_{i'} \in S_{\mu_{i'}(U_\varphi,\varphi)} \\ 1 \le i' \le n}} \lambda^h_{s_{i'}} \cdot \varphi^{i'-1}(v_{s_{i'}})$, with $\lambda^h_{\bar u} \in k$, $\lambda^h_{s_{i'}} \in k$ for each $s_{i'} \in S_{\mu_{i'}(U_\varphi,\varphi)}$ and each $h \in \{1, \dots, r\}$ and where only a finite number of the scalars $\{\lambda^h_{s_{i'}}\}$ are different from zero;
- $\varphi^+_{v,\tilde v}(\varphi^j(v_{s_i})) = \varphi^{j-1}(v_{s_i}) + \beta^{s_i,j}_{\bar u}\varphi^{s-1}(\bar u) + \sum_{\substack{s_{i'} \in S_{\mu_{i'}(\tilde H,\varphi)} \\ 1 \le i' \le n}} \beta^{s_i,j}_{s_{i'}} \cdot \varphi^{i'-1}(v_{s_{i'}})$, with $\beta^{s_i,j}_{s_{i'}} = 0$ for almost all $s_{i'} \in S_{\mu_i(\tilde H,\varphi)}$ and $j \in \{1, \dots, i-1\}$;
- $\varphi^+_{v,\tilde v}(v_{s_i}) = 0$ for every $s_i \in S_{\mu_i(\tilde H,\varphi)}$;
- $\varphi^+_{v,\tilde v}(\varphi^j(\bar u)) = \varphi^{j-1}(\bar u) + \sum_{\substack{s_{i'} \in S_{\mu_{i'}(\tilde H,\varphi)} \\ 1 \le i' \le n}} \beta^{\bar u,j}_{s_{i'}} \cdot \varphi^{i'-1}(v_{s_{i'}}) + \beta^{\bar u,j}_{\bar u}\varphi^{s-1}(\bar u)$, with $\beta^{\bar u,j}_{s_{i'}} = 0$ for almost all $s_{i'} \in S_{\mu_i(\tilde H,\varphi)}$ and $j \in \{1, \dots, s-1\}$;
- $\varphi^+_{v,\tilde v}(\bar u) = \tilde v - \varphi^+_{v,\tilde v}(w) = \sum_{j=1}^r \gamma^{\bar u}_j \cdot w_j + \sum_{t=0}^{s-1} \xi^{\bar u,t}_{\bar u}\varphi^t(\bar u)$

$$+ \sum_{\substack{s_{i'} \in S_{\mu_{i'}(\tilde H,\varphi)} \\ 1 \le i' \le n}} \Big[\sum_{l=0}^{i'-1} \xi^{\bar u,l}_{s_{i'}}\varphi^l(v_{s_{i'}})\Big]\,,$$

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with $\xi^{\bar{u},l}_{s_{i'}} = 0$ for almost all $s_{i'} \in S_{\mu_i(\bar{H},\varphi)}$ and $l \in \{1,\dots,i'-1\}$,

$$\xi^{\bar{u},i'-1}_{s_{i'}} = \sum_{j,h=1}^{r} (\lambda^h_{s_{i'}} \cdot c_{hj} \cdot \gamma^{\bar{u}}_j) + \sum_{\substack{s_{i''} \in S_{\mu_{i''}(\tilde{H},\varphi)} \\ 1 \le i'' \le n}} \left(\sum_{l=0}^{i''-2} [\xi^{\bar{u},l}_{s_{i''}} \cdot \beta^{s_{i''},l+1}_{s_{i'}}] \right) + \sum_{t=0}^{s-2} [\xi^{\bar{u},t}_{\bar{u}} \cdot \beta^{\bar{u},t+1}_{s_{i'}}]$$

and

$$\xi^{\bar{u},s-1}_{\bar{u}} = \sum_{j,h=1}^{r} (\lambda^h_{\bar{u}} \cdot c_{hj} \cdot \gamma^{\bar{u}}_j) + \sum_{\substack{s_{i''} \in S_{\mu_{i''}(\tilde{H},\varphi)} \\ 1 \le i'' \le n}} \left(\sum_{l=0}^{i''-2} [\xi^{\bar{u},l}_{s_{i''}} \cdot \beta^{s_{i''},l+1}_{\bar{u}}] \right) + \sum_{t=0}^{s-2} [\xi^{\bar{u},t}_{\bar{u}} \cdot \beta^{\bar{u},t+1}_{\bar{u}}] \, ,$$

which is well-defined because, since $\varphi^+_{v,\tilde{v}}(\bar{u}) \notin \operatorname{Ker} \varphi$, we have that $\xi^{\bar{u},t}_{\bar{u}} \neq 0$ for a certain $t \in \{0,\dots,s-2\}$ or $\xi^{\bar{u},l}_{s_{i'}} \neq 0$ for some $s_{i'} \in S_{\mu_{i'}(\tilde{H},\varphi)}$ and $l \in \{0,\dots,i'-2\}$, and therefore, we can obtain all the coefficients $\{\xi^{\bar{u},i'-1}_{s_{i'}}\}_{s_{i'} \in S_{\mu_{i'}(\tilde{H},\varphi)}} \cup \{\xi^{\bar{u},s-1}_{\bar{u}}\}$ from the elements of the set

$$\{\xi^{\bar{u},t}_{\bar{u}}\}_{t \in \{0,\dots,s-2\}} \cup \{\xi^{\bar{u},l}_{s_{i'}}\}_{s_{i'} \in S_{\mu_{i'}(\tilde{H},\varphi)}; l \in \{0,\dots,i'-2\}},$$

and a finite number of non-zero coefficients $\{\lambda^h_{\bar{u}}, \lambda^h_{s_{i'}}, \beta^{s_i,j}_{s_{i'}}, \beta^{s_i,j}_{\bar{u}}, \beta^{\bar{u},j}_{s_{i'}}, \beta^{\bar{u},j}_{\bar{u}}\}$.

By construction, we have that $\varphi^+_{v,\tilde{v}} \in X_\varphi(1,2)$ and $\widetilde{\varphi}^+_{v,\tilde{v}}(v) = \tilde{v}$. Moreover, there exists a finite number of indices $\{s_{i_1},\dots,s_{i_h}\}$ such that

$$\operatorname{Im} \varphi^+_{v,\tilde{v}} \subseteq W_\varphi + \langle \bar{u}, \varphi(\bar{u}), \dots, \varphi^{s-1}(\bar{u}) \rangle + \sum_{l=1}^{h} \langle v_{s_{i_l}}, \varphi(v_{s_{i_l}}), \dots, \varphi^{i_l-1}(v_{s_{i_l}}) \rangle \, ,$$

from where we obtain that $\varphi^+_{v,\tilde{v}}$ is finite potent and the assertion is deduced. □

Note that, in general, the reflexive generalized inverse $\widetilde{\varphi}^+_{v,\tilde{v}}$ such that $\widetilde{\varphi}^+_{v,\tilde{v}}(v) = \tilde{v}$, that has been constructed in the previous proposition, is not unique.

Moreover, in Matrix Theory, Proposition 4.8 shows that:

PROPOSITION 4.9. *If $A \in Mat_{n\times n}(k)$ is a square matrix with entries in an arbitrary field k, $R(A)$ denotes the range of a A, $N(A)$ is the Nullspace of A and $v \in k^n$ is a vector with $v \notin R(A)$, then for every vector $\tilde{v} \in k^n$ with $\tilde{v} \notin N(A)$ there exists a reflexive generalized inverse $A^+_{v,\tilde{v}} \in A(1,2)$ such that $A^+_{v,\tilde{v}} \cdot v^t = \tilde{v}^t$.*

EXAMPLE 4.10. *To finish this work, we shall offer an illustrative example of Proposition 4.9. Let $A \in Mat_{7\times 7}(\mathbb{R})$ be the matrix:*

$$A = \begin{pmatrix} -9 & -15 & -4 & 2 & -6 & -3 & 16 \\ -25 & -37 & -6 & 6 & -14 & -12 & 43 \\ -17 & -26 & -5 & 4 & -10 & -7 & 29 \\ -34 & -50 & -8 & 8 & -19 & -16 & 58 \\ -4 & -5 & 0 & 1 & -2 & -2 & 6 \\ -2 & -4 & -1 & 1 & -2 & -1 & 4 \\ -29 & -44 & -8 & 7 & -17 & -13 & 50 \end{pmatrix} .$$

Readers can check that:

- *the index of* A *is* 3;
- $N(A) = \langle (0,1,0,1,0,1,1),(1,0,1,0,0,1,1)\rangle$;
- $R(A) = \langle (1,0,1,1,1,0,1),(-1,2,0,2,0,0,1),(0,1,-1,0,0,1,1),$
  $(2,0,1,0,-1,0,1),(1,-1,1,1,0,1,0)\rangle$;
- $W_f = R(A^3) = \langle (1,0,1,1,1,0,1),(-1,2,0,2,0,0,1)\rangle$;
- $U_f = N(A^3) = \langle (0,1,-1,0,0,1,1),(2,0,1,0,-1,0,1),(1,-1,1,1,0,1,0),$
  $(0,1,0,1,0,1,1),(1,0,1,0,0,1,1)\rangle$;

where $f \in \operatorname{End}_{\mathbb{R}}(\mathbb{R}^7)$ *is an endomorphism such that* A *is the matrix associated with* f *in the standard basis of* $\mathbb{R}^7$, *and* $\mathbb{R}^7 = W_f \oplus U_f$ *is the AST-decomposition of* $\mathbb{R}^7$ *determined by* f.

Moreover, one has that

$$f_{|W_f} \equiv \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix},$$

in the basis $\{(1,0,1,1,1,0,1),(-1,2,0,2,0,0,1)\}$ *of* W_f.

If we consider $v = (3,0,0,1,1,0,2), \tilde{v} = (1,1,1,1,1,1,1) \in \mathbb{R}^7$, *we have that* $v \notin R(A)$ *and* $\tilde{v} \notin N(A)$ *and, with the notation of Proposition 4.9, our purpose is to calculate a reflexive generalized inverse* $A^+_{v,\tilde{v}} \in A(1,2)$.

Since $v = w + u$, *with* $w = (3,-2,2,0,2,0,1) \in W_f$ *and* $u = (0,2,-2,1,-1,0,1) \in U_f$, *then*

$$\begin{aligned} B_{U_f} = \{&(0,2,-2,1,-1,0,1),(2,1,1,1,-1,1,2),(0,1,0,1,0,1,0), \\ &(1,-1,1,1,0,1,0),(1,0,1,0,0,1,1)\} \end{aligned}$$

is a Jordan basis of U_f *for* $f_{|U_f}$ *that allows us to compute* $A^+_{v,\tilde{v}}$.

Bearing in mind that

$$A = P \cdot \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \cdot P^{-1},$$

with

$$P = \begin{pmatrix} 1 & -1 & 0 & 2 & 0 & 1 & 1 \\ 0 & 2 & 2 & 1 & 1 & -1 & 0 \\ 1 & 0 & -2 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 0 & 1 \end{pmatrix},$$

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and

$$P^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & -1 \\ -5 & -7 & -1 & 1 & -3 & -2 & 8 \\ -5 & -8 & -2 & 1 & -3 & -2 & 9 \\ 6 & 9 & 2 & -1 & 3 & 2 & -10 \\ 11 & 17 & 3 & -2 & 7 & 5 & -19 \\ -3 & -5 & -1 & 1 & -2 & -1 & 5 \\ -14 & -21 & -4 & 2 & -8 & -5 & 24 \end{pmatrix},$$

then, if $v' \notin N(A)$, every reflexive generalized inverse $A^+_{v,v'}$ constructed as in the proof of Proposition 4.8 has the structure

$$A^+_{v,v'} = P \cdot \begin{pmatrix} -3 & 2 & \gamma_{13} & 0 & 0 & 0 & 0 \\ 2 & -1 & \gamma_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_{33} & 1 & 0 & 0 & 0 \\ 0 & 0 & \xi_{43} & 0 & 1 & 0 & 0 \\ \lambda_{51} & \lambda_{52} & \xi_{53} & \beta_{54} & \beta_{55} & 0 & \beta_{57} \\ 0 & 0 & \xi_{63} & 0 & 0 & 0 & 1 \\ \lambda_{71} & \lambda_{72} & \xi_{73} & \beta_{74} & \beta_{75} & 0 & \beta_{77} \end{pmatrix} \cdot P^{-1},$$

with

$$\xi_{53} = \gamma_{13}(\lambda_{51} + 2\lambda_{52}) + \gamma_{23}(2\lambda_{51} + 3\lambda_{52}) + \xi_{33}\beta_{54} + \xi_{43}\beta_{55} + \xi_{63}\beta_{57},$$

and

$$\xi_{73} = \gamma_{13}(\lambda_{71} + 2\lambda_{72}) + \gamma_{23}(2\lambda_{71} + 3\lambda_{72}) + \xi_{33}\beta_{74} + \xi_{43}\beta_{75} + \xi_{63}\beta_{77}.$$

Now, since

$$P^{-1}\begin{pmatrix} 3 \\ -2 \\ 2 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad P^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -9 \\ -10 \\ 11 \\ 22 \\ -6 \\ -26 \end{pmatrix};$$

$$\begin{pmatrix} -3 & 2 & \gamma_{13} & 0 & 0 & 0 & 0 \\ 2 & -1 & \gamma_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_{33} & 1 & 0 & 0 & 0 \\ 0 & 0 & \xi_{43} & 0 & 1 & 0 & 0 \\ \lambda_{51} & \lambda_{52} & \xi_{53} & \beta_{54} & \beta_{55} & 0 & \beta_{57} \\ 0 & 0 & \xi_{63} & 0 & 0 & 0 & 1 \\ \lambda_{71} & \lambda_{72} & \xi_{73} & \beta_{74} & \beta_{75} & 0 & \beta_{77} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -8 \\ 5 \\ 0 \\ 0 \\ 2\lambda_{51} - \lambda_{52} \\ 0 \\ 2\lambda_{71} - \lambda_{72} \end{pmatrix},$$

and

$$\begin{aligned} &(2, -9, -10, 11, 22, -6, -26) - (-8, 5, 0, 0, 2\lambda_{51} - \lambda_{52}, 0, 2\lambda_{71} - \lambda_{72}) = \\ &= (10, -14, -10, 11, 22 - 2\lambda_{51} + \lambda_{52}, -6, -26 - 2\lambda_{71} + \lambda_{72}), \end{aligned}$$

we need to solve the equations

$$22 - 2\lambda_{51} + \lambda_{52} = 10(\lambda_{51} + 2\lambda_{52}) - 14(2\lambda_{51} + 3\lambda_{52}) - 10\beta_{54} + 11\beta_{55} - 6\beta_{57},$$

and

$$-26 - 2\lambda_{71} + \lambda_{72} = 10(\lambda_{71} + 2\lambda_{72}) - 14(2\lambda_{71} + 3\lambda_{72}) - 10\beta_{74} + 11\beta_{55} - 6\beta_{77} \,.$$

Bearing in mind that a solution of these equations is:

- $\beta_{55} = 2$; $\beta_{74} = 2$ *and* $\beta_{77} = 1$;
- $\lambda_{51} = \lambda_{52} = \lambda_{71} = \lambda_{72} = 0$;
- $\beta_{54} = \beta_{57} = \beta_{75} = 0$;

we obtain that

$$A^{+}_{v,\tilde{v}} = P \cdot \begin{pmatrix} -3 & 2 & 10 & 0 & 0 & 0 & 0 \\ 2 & -1 & -14 & 0 & 0 & 0 & 0 \\ 0 & 0 & -10 & 1 & 0 & 0 & 0 \\ 0 & 0 & 11 & 0 & 1 & 0 & 0 \\ 0 & 0 & 22 & 0 & 2 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 & 0 & 1 \\ 0 & 0 & -26 & 2 & 0 & 0 & 1 \end{pmatrix} \cdot P^{-1} =$$

$$= \begin{pmatrix} -84 & -128 & -29 & 15 & -52 & -30 & 145 \\ 118 & 180 & 37 & -21 & 72 & 46 & -202 \\ -75 & -114 & -25 & 13 & -45 & -27 & 129 \\ 31 & 48 & 9 & -6 & 20 & 14 & -53 \\ -75 & -115 & -25 & 14 & -46 & -29 & 129 \\ 12 & 19 & 3 & -3 & 8 & 7 & -20 \\ 22 & 34 & 5 & -4 & 13 & 11 & -37 \end{pmatrix} .$$

Readers can easily check that $A^{+}_{v,\tilde{v}} \in A(1,2)$ *and*

$$A^{+}_{v,\tilde{v}} \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} .$$

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