



## THE NUMBER OF SPANNING TREES OF THE BIPARTITE COMPLEMENT OF A SEMIREGULAR BIPARTITE GRAPH\*

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**Abstract.** For a bipartite graph  $G$  on parts of cardinality  $m$  and  $n$ , the bipartite complement of  $G$  is defined as the graph obtained from the complete bipartite graph  $K_{m,n}$  by removing the edges of an isomorph of  $G$ . In this paper, based on the matrix-tree theorem and the Schur complement technique, we give a determinant expression for the number of spanning trees of the bipartite complement of a semiregular bipartite graph. As by-products, we show that the corresponding result can generalize some previous results on the problem of enumerating spanning trees and obtain an explicit formula for the number of spanning trees of the bipartite complement of the subdivision of a regular circulant graph.

**Key words.** Spanning tree, Semiregular, Bipartite graph, Schur complement.

**AMS subject classifications.** 05C30, 05C05.

**1. Introduction.** In this paper, we only consider simple graphs. The problem of enumerating spanning trees of various kinds of graphs is a long-studied subject of study in combinatorics and has attracted extensive attention in statistical physics and theoretical computer science [3, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 20, 22, 24].

We denote by  $\tau(G)$  the number of spanning trees of a graph  $G$ . Among various techniques for counting spanning trees of different classes of graphs, algebraic methods have proven to be one of the most efficient approaches. The most-commonly used algebraic method is Kirchoff's matrix-tree theorem [13] which interprets the spanning tree number as the co-factor of any element in the Laplacian matrix of the considered graph, or equivalently

$$\tau(G) = (-1)^{i+j} \det(L_{ij}(G)),$$

holds for an arbitrary graph  $G$ , where  $L_{ij}(G)$  be the minor of the  $(i, j)$  element in the Laplacian matrix  $L(G)$  which is exactly the difference of the degree matrix and the adjacency matrix of  $G$ . As analogous to the matrix-tree theorem, another expression for the spanning tree number of a graph  $G$  is

$$\tau(G) = \frac{1}{n^2} \det(nI_n - D(\bar{G}) + A(\bar{G})),$$

where  $\bar{G}$  is the usual graph complement of  $G$  and the matrix  $nI_n - D(\bar{G}) + A(\bar{G})$  is called the complement spanning tree matrix of  $G$ . For a detailed description of this method, readers may refer to [11]. Moreover, Kirby et al. [14] developed an interesting algebraic method called the cycle theorem to formulate the number of spanning trees of general chemical graphs.

A graph  $G = (V, E)$  is called bipartite if its vertex set  $V$  can be partitioned as  $V = V_1 \cup V_2$  such that no two vertices in  $V_1$ , or in  $V_2$ , are adjacent. We often refer the bipartition as  $(V_1, V_2)$  and denote by

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$G = (V_1, V_2; E)$  a bipartite graph with the bipartition  $(V_1, V_2)$  and  $E$  the set of edges between them. A bipartite graph  $G = (V_1, V_2; E)$  is called  $(n_1, n_2, d_1, d_2)$ -semiregular if  $|V_1| = n_1$ ,  $|V_2| = n_2$  and  $d_G(v) = d_i$  for each  $v \in V_i$  ( $i = 1, 2$ ). Moreover, the bipartite complement of  $G = (V_1, V_2; E)$  is defined as the bipartite graph with the same bipartition  $(V_1, V_2)$  and the edge set  $\{(u, v) | u \in V_1, v \in V_2, uv \notin E\}$  and denoted by  $\overline{G}_{m,n}$ . Suppose  $m \geq n_1$ ,  $n \geq n_2$ , let  $K_{m,n} - G$  denote the graph obtained from  $K_{m,n}$  by removing the edges of an isomorph of  $G$  in  $K_{m,n}$ . In a sence,  $\overline{G}_{m,n} = K_{m,n} - G$  in the case of  $m = |V_1|$  and  $n = |V_2|$ .

In 2009, Zhang and Yan [22] proved

$$\tau(K_{n,n} - pK_2) = (n^2 - 2n + p)(n - 2)^{p-1}n^{(2n-p-3)}, (p \leq n).$$

Particularly,  $\tau(K_{n,n} - nK_2) = n^{(n-2)}(n - 2)^{(n-1)}(n - 1)$ . It should be pointed out that Spiro [19] proved the number of spanning trees of an  $(n_1, n_2; x, y)$ -semiregular graph  $G$  has the following form:

$$\tau(G) = \frac{(d_1 + d_2)d_1^{n_1-n_2} \prod_{i=2}^{n_2} (d_1d_2 - \lambda_i^2(G))}{n_1 + n_2},$$

where  $\lambda_1(G), \lambda_2(G), \dots, \lambda_{n_2}(G)$  are the first  $n_2$  largest eigenvalues of the adjacency matrix of  $G$  in a decending order. In addition, Sato [17] found an expression for the zeta function of the middle graph of a semiregular bipartite graph  $G$  and further obtained the number of spanning trees of the middle graph of the bipartite graph  $G$ . In this paper, we are concerned with the problem of enumerating spanning trees of the bipartite complement of a semiregular graph. By using the matrix-tree theorem and the Schur complement technique, we obtain a determinant expression for the number of spanning trees of the bipartite complement of a semiregular graph and discuss some other related applications.

**2. Notations and terminology.** In this section, we will review some basic terminologies and give several results of linear algebra that will be used in the subsequent sections. For the sake of convenience, the complete graph and cycle of order  $n$  are denoted by  $K_n$  and  $C_n$  respectively; the complete bipartite graph on sets with  $m$  and  $n$  vertices is denoted by  $K_{m,n}$ . Let  $I_n$  denote the identity matrix of size  $n$  and  $J_{s \times t}$  (resp.  $J_n$ ) denote the  $s \times t$  (resp.  $n \times n$ ) matrix with all entries equal to 1. Let  $G = (V_1, V_2; E)$  be a bipartite graph with two parts  $V_1 = \{v_1^1, v_2^1, \dots, v_m^1\}$  and  $V_2 = \{v_1^2, v_2^2, \dots, v_n^2\}$ . The biadjacency matrix  $B(G)$  of  $G$  whose rows and columns are indexed by  $V_1$  and  $V_2$ , respectively, is the  $m \times n$   $(0, 1)$ -matrix  $(b_{ij})$  such that  $b_{ij} = 1$  if and only if the vertex  $v_i^1$  and the vertex  $v_j^2$  are adjacent in  $G$ . In other words, the biadjacency matrix  $B(G)$  of  $G$  is the upper-right submatrix in the partitioned adjacency matrix  $A(G)$  of  $G$ , i.e.  $A(G) = \begin{pmatrix} 0 & B(G) \\ B(G)^T & 0 \end{pmatrix}$ , where  $B(G)^T$  is the transpose of  $B(G)$ .

LEMMA 2.1 (Klee and Stamps [15]). *Let  $G$  be a graph of order  $n$ . Then*

$$\det(L(G) + \mathbf{x}\mathbf{y}^T) = \left( \sum_{i=1}^n x_i \right) \left( \sum_{i=1}^n y_i \right) \tau(G),$$

where  $\mathbf{x} = (x_i)$  and  $\mathbf{y} = (y_i)$  ( $i \in [n]$ ) are column vectors in  $\mathbb{R}^n$ .

LEMMA 2.2 (Zhang [21]). *Let  $A \in \mathbb{F}^{p \times p}, B \in \mathbb{F}^{p \times q}, C \in \mathbb{F}^{q \times p}, D \in \mathbb{F}^{q \times q}$ . Suppose  $A$  and  $D$  are nonsingular matrices. Then*

$$\begin{aligned} \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det(A) \det(D - CA^{-1}B) \\ &= \det(D) \det(A - BD^{-1}C), \end{aligned}$$

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where  $D - CA^{-1}B$  and  $A - BD^{-1}C$  are called the Schur complements of  $A$  and  $D$ , respectively.

LEMMA 2.3. Let  $k \in \mathbb{Z}^+$  and  $M = aI_k + bJ_k$  such that  $a \neq 0, a + bk \neq 0$ . Then  $M$  is a nonsingular matrix and  $M^{-1} = \frac{1}{a}I_k + \frac{-b}{(a + bk)a}J_k$ .

*Proof.* Since  $\det(M) = a^{(k-1)}(a + bk) \neq 0$  by assumption, it follows that  $M$  is a nonsingular matrix. Clearly,

$$(aI_k + bJ_k) \left( \frac{1}{a}I_k + \frac{-b}{(a + bk)a}J_k \right) = I_k.$$

This implies

$$M^{-1} = \frac{1}{a}I_k + \frac{-b}{(a + bk)a}J_k. \quad \square$$

LEMMA 2.4. Let  $G = (V_1, V_2; E)$  ( $|V_1| = m, |V_2| = n$ ) be an  $(m, n; d_1, d_2)$ -semiregular bipartite graph. Then

$$(2.1) \quad B(G)B(G)^T = (N_{ij}^1)_{m \times m}, \quad B(G)J_nB(G)^T = d_1^2J_m;$$

$$(2.2) \quad B(G)^TB(G) = (N_{ij}^2)_{n \times n}, \quad B(G)^TJ_mB(G) = d_2^2J_n,$$

where, for every pair of vertices  $v_i^k, v_j^k \in V_k$  ( $k = 1, 2$ ), the number of vertices adjacent to both  $v_i^k$  and  $v_j^k$  is defined as:

$$N_{ij}^k = \begin{cases} |N_G(v_i^k) \cap N_G(v_j^k)| & i \neq j, \\ d_G(v_i^k) & i = j. \end{cases}$$

*Proof.* Note that the biadjacency matrix of  $G$  whose rows and columns are indexed by  $V_1$  and  $V_2$ , respectively, has the form:

$$B(G) = (b_{ij})_{m \times n} = \begin{matrix} & v_1^2 & v_2^2 & \cdots & v_n^2 \\ \begin{matrix} v_1^1 \\ v_2^1 \\ \vdots \\ v_m^1 \end{matrix} & \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \end{matrix},$$

where  $b_{ij} = 1$  if and only if  $v_i^1$  and  $v_j^2$  are adjacent in  $G$ , and  $b_{ij} = 0$  otherwise. By definition, it is not difficult to verify that (1)  $B(G)B(G)^T = (N_{ij}^1)_{m \times m}$  and  $B(G)J_nB(G)^T = d_1^2J_m$ ; (2)  $B(G)^TB(G) = (N_{ij}^2)_{n \times n}$  and  $B(G)^TJ_mB(G) = d_2^2J_n$ .  $\square$

### 3. Main results.

THEOREM 3.1. Let  $G$  be an  $(m, n; x, y)$ -semiregular bipartite graph. Then

$$(3.1) \quad \tau(\overline{G}_{m,n}) = \frac{(n-x)^{(m-1)}(m+n-x)}{(m+n)^2} \cdot \det \left( (m-y)I_n + \frac{(n-x)(m+n-x) + y^2}{(n-x)(m+n-x)}J_n - \frac{1}{n-x}B(G)^TB(G) \right)$$

$$(3.2) \quad = \frac{(m-y)^{(n-1)}(m+n-y)}{(m+n)^2} \cdot \det \left( (n-x)I_m + \frac{(m-y)(m+n-y) + x^2}{(m-y)(m+n-y)}J_m - \frac{1}{m-y}B(G)B(G)^T \right).$$

*Proof.* Without loss of generality, let  $(U, V)$  ( $|U| = m, |V| = n$ ) be the bipartition of the vertex set of  $G$  and  $B(G)$  be the biadjacency matrix of  $G$  whose rows and columns are indexed by  $U$  and  $V$ , respectively. By Lemma 2.1, we have

$$\det \left( L(\overline{G}_{m,n}) + J_{m+n} \right) = (m+n)^2 \tau(\overline{G}_{m,n}).$$

Observe that

$$L(K_{m,n}) = \begin{pmatrix} nI_m & -J_{m \times n} \\ -J_{n \times m} & mI_n \end{pmatrix} \text{ and } L(G) = \begin{pmatrix} xI_m & -B(G) \\ -B(G)^T & yI_n \end{pmatrix}.$$

Thus,

$$\begin{aligned} & L(\overline{G}_{m,n}) + J_{m+n} \\ &= L(K_{m,n}) - L(G) + J_{m+n} \\ &= \begin{pmatrix} A & B(G) \\ B(G)^T & C \end{pmatrix}. \end{aligned}$$

where  $A$  denotes  $(n-x)I_m + J_m$  and  $C$  denotes  $(m-y)I_n + J_n$ . Because  $\det A = (n-x)^{(m-1)}(m+n-x) \neq 0$ ,  $\det C = (m-y)^{(n-1)}(m+n-y) \neq 0$ , so  $A$  and  $C$  are nonsingular matrices. By Lemma 2.2, we get

$$\begin{aligned} \det \left( \begin{pmatrix} A & B(G) \\ B(G)^T & C \end{pmatrix} \right) &= \det(A) \det \left( C - B(G)^T A^{-1} B(G) \right) \\ &= \det(C) \det \left( A - B(G) C^{-1} B(G)^T \right). \end{aligned}$$

Recall that  $A = (n-x)I_m + J_m$ , and the matrix  $A$  is invertible. According to Lemma 2.3,  $A^{-1} = pI_m + qJ_m$  with  $p = \frac{1}{n-x}$  and  $q = \frac{-1}{(n-x)(m+n-x)}$ . It follows that

$$\det A = (n-x)^{(m-1)}(m+n-x),$$

and

$$C - B(G)^T A^{-1} B(G) = C - pB(G)^T B(G) - qB(G)^T J_m B(G).$$

In other words,

$$\begin{aligned} & (m+n)^2 \tau(\overline{G}_{m,n}) \\ &= \det \left( L(\overline{G}_{m,n}) + J_{m+n} \right) \\ &= \det A \cdot \det \left( C - pB(G)^T B(G) - qB(G)^T J_m B(G) \right) \\ &= (n-x)^{(m-1)}(m+n-x) \det \left( C - pB(G)^T B(G) - qB(G)^T J_m B(G) \right). \end{aligned}$$

Therefore, by Lemma 2.4,

$$\begin{aligned} \tau(\overline{G}_{m,n}) &= \frac{(n-x)^{(m-1)}(m+n-x)}{(m+n)^2} \\ &\quad \cdot \det \left( (m-y)I_n + \frac{(n-x)(m+n-x) + y^2}{(n-x)(m+n-x)} J_n - \frac{1}{n-x} B(G)^T B(G) \right). \end{aligned}$$

By using the similar method in above discussion, we obtain equation (3.2) easily.  $\square$

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**COROLLARY 3.2.** *Let  $G$  be an  $(m, n; x, y)$ -semiregular bipartite graph with parts  $U$  ( $|U| = m$ ) and  $V$  ( $|V| = n$ ). If every pair of vertices in  $U$  (resp.  $V$ ) have the same number  $\delta$  of neighbors in  $V$  (resp.  $U$ ), then*

$$\tau(\overline{G}_{m,n}) = \frac{1}{(m+n)^2} \cdot (n-x)^{(m-n-1)} \cdot (mn - xm - yn + xy - y + \delta)^{n-1} \cdot \left[ (m+n-x) \left( (n(n-x-\delta) + (mn - xm - yn + xy - y + \delta)) + y^2n \right) \right].$$

*Proof.* By note of the fact  $B(G)^T B(G) = (y-\delta)I_n + \delta J_n$ , the corollary is the straightforward application of Eq. (3.1) in Theorem 3.1.  $\square$

**COROLLARY 3.3.** *Let  $G$  be an  $(m, n; x, y)$ -semiregular bipartite graph with parts  $U$  ( $|U| = m$ ) and  $V$  ( $|V| = n$ ). If any two vertices in  $V$  have the same number  $\delta$  of neighbors in  $U$ , then*

$$\tau(\overline{G}_{m,n}) = \frac{1}{(m+n)^2} \cdot (m-y)^{(n-m-1)} \cdot (mn - xm - yn + xy - x + \delta)^{m-1} \cdot \left[ (m+n-y) \left( (m(m-y-\delta) + (mn - xm - yn + xy - x + \delta)) + x^2n \right) \right].$$

*Proof.* By note of the fact  $B(G)B(G)^T = (x-\delta)I_m + \delta J_m$ , the corollary is the straightforward application of Eq. (3.2) in Theorem 3.1.  $\square$

**4. Counting spanning trees for several kinds of bipartite graphs.** In this section, some applications of the results of work conducted under Section 3 will be provided in this section.

**4.1. The bipartite complements of some bipartite graphs.** (1) For  $n \geq 3$ , the  $n$ -crown graph is the bipartite graph with vertex set  $\{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_n\}$  and edge set  $\{(u_i, v_j) : 1 \leq i, j \leq n, i \neq j\}$ . Let us reflect in another way, the  $n$ -crown graph is the bipartite complement of the disjoint union of  $n$  copies of  $K_2$ . Since  $G = nK_2$  is an  $(n, n; 1, 1)$ -semiregular bipartite graph and  $B(G)^T B(G) = I_n$ . Thus, by Theorem 3.1, the number of spanning trees of the  $n$ -crown graph is given by:

$$\begin{aligned} & \tau(\overline{G}_{n,n}) \\ &= \frac{(n-1)^{(n-1)}(2n-1)}{(2n)^2} \det \left( (n-1 - \frac{1}{n-1})I_n + \frac{(n-1)(n-1+n)+1}{(n-1)(n-1+n)} J_n \right) \\ &= \frac{(n-1)^{(n-1)}(2n-1)}{(2n)^2} \left( \frac{n^2-2n}{n-1} \right)^{(n-1)} \left( \frac{n^2-2n}{n-1} + \frac{(n-1)(2n-1)+1}{(n-1)(2n-1)} n \right) \\ &= \frac{(n-1)^{(n-1)}(2n-1)}{4n^2} \frac{(n^2-2n)^{(n-1)}}{(n-1)^{(n-1)}} \frac{4n(n^2-2n+1)}{(n-1)(2n-1)} \\ &= n^{(n-2)}(n-2)^{(n-1)}(n-1) \text{ (firstly appeared in [22]).} \end{aligned}$$

(2) Let  $G$  be an  $(m, n; 1, y)$ -semiregular bipartite graph such that  $m = ny$ . Note that  $G = \cup_{i=1}^n K_{1,y}$  and  $B(G)^T B(G) = yI_n$ . Then by Theorem 3.1, we have

$$\tau(\overline{G}_{m,n}) = \frac{(n-1)^{(m-1)}(n-1+m)}{(n+m)^2} \det \left( (m-y - \frac{y}{n-1})I_n + \frac{(n-1)(n-1+m)+y^2}{(n-1)(n-1+m)} J_n \right)$$

$$\begin{aligned}
 &= \frac{(n-1)^{(m-1)}(n-1+m)}{(n+m)^2} \left(m-y-\frac{y}{n-1}\right)^{(n-1)} \\
 &\quad \cdot \left(m-y-\frac{y}{n-1} + \frac{(n-1)(n-1+m)+y^2}{(n-1)(n-1+m)}n\right) \\
 &= \frac{(n-1)^{(m-n-1)}m^{(n-1)}(n-2)^{(n-1)}}{(m+n)^2} \left[(m+n-1)(mn-2m+n^2-n)+my\right].
 \end{aligned}$$

REMARK 4.1. For a positive integer  $n$ , let  $[n] = \{1, 2, \dots, n\}$ . Suppose  $U$  and  $V$  are the set of all  $k$ -subsets,  $(n-k)$ -subsets of  $[n]$ , respectively. The bipartite Kneser graph  $H(n, k)$  has  $U \cup V$  as its two parts of the vertex set, and two vertices  $u \in U$ ,  $v \in V$  are adjacent if and only if  $u \subset v$  or  $v \subset u$ . Clearly,  $H(n, 1)$  is an  $(n, n; n-1, n-1)$ -semiregular bipartite graph,  $B^T(H(n, 1))B(H(n, 1)) = I_n + (n-2)J_n$ . By Theorem 3.1, it is not difficult to obtain the number of spanning trees of the bipartite complement of the bipartite Kneser graph  $H(n, 1)$ ; here, we omit the detail result.

**4.2. The bipartite complements of the subdivision of a circulant graph.** Let  $n$  and  $m$  be positive integers. The circulant graph  $G[n, \{s_1, s_2, \dots, s_m\}]$  with distinct jumps  $s_1, s_2, \dots, s_m$  (positive integers) is defined as the undirected graph with vertex set  $V = \{0, 1, \dots, n-1\}$  and edge set  $E = \{\{i, i \pm s_k \pmod n\} : 0 \leq i \leq n-1, 1 \leq k \leq m\}$ . If  $s_k \not\equiv \pm s_l \pmod n$  for all  $1 \leq k, l \leq m$ , then  $G[n, \{s_1, s_2, \dots, s_m\}]$  is  $d$ -regular, where

$$d = \begin{cases} 2m & \text{if } s_k \not\equiv n/2 \pmod n \text{ for all } 1 \leq k \leq m, \\ 2m-1 & \text{otherwise.} \end{cases}$$

If  $\gcd(s_1, s_2, \dots, s_m, n) = 1$ , then  $G[n, \{s_1, s_2, \dots, s_m\}]$  is connected [2]. It is worth mentioning that the adjacency matrix  $A(G)$  of  $G$  is a circulant matrix (a square matrix with the property that each row is a cyclic permutation of the first row) and has the following form:

$$\begin{pmatrix} c_0 & c_1 & \dots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & & c_{n-2} \\ \vdots & c_{n-1} & c_0 & \ddots & \vdots \\ c_2 & & \ddots & \ddots & c_1 \\ c_1 & c_2 & \dots & c_{n-1} & c_0 \end{pmatrix} \stackrel{Def.}{=} C(c_0, c_1, \dots, c_{n-1}),$$

where  $c_0 = 0$  and  $c_i = c_{n-i} = 0$  if vertices  $i$  and  $n-i$  are not adjacent, and  $c_i = c_{n-i} = 1$  otherwise. Observe that  $A(G)$  can be uniquely represented as polynomial of the elementary circulant matrix  $\Pi = C(0, 1, 0, \dots, 0)$ :

$$A(G) = c_1\Pi^1 + c_2\Pi^2 + \dots + c_{n-1}\Pi^{n-1} = \sum_{k=1}^{n-1} c_k\Pi^k.$$

Hence, the circulant matrix  $A(G)$  have the complex eigenvectors  $(1, \omega, \omega^2, \dots, \omega^{n-1})$  corresponding to the eigenvalues  $F(\omega)$ , respectively, as  $\omega$  ranges over the  $n$ -th roots of unity, where

$$F(x) = c_1x^1 + c_2x^2 + \dots + c_{n-1}x^{n-1} = \sum_{k=1}^{n-1} c_kx^k.$$

As we already know, circulant graphs form a large family of graphs including many common graphs, such as the cycles, complete graphs, complete bipartite graphs  $K_{m,n}$ ,  $n$ -crown graphs, prism or antiprism graphs,

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cocktail party graphs  $K_{n \times 2}$ , Möbius ladders, torus grid graphs  $C_m \square C_n$  (here  $m$  and  $n$  relatively prime,  $\square$  denotes a Cartesian product), and so on. The explicit formulas of the number of spanning trees are known for some circulant graphs [4, 1, 23]. The subdivision graph  $S(G)$  of a graph  $G$  is the graph obtained by inserting a new vertex into every edge of  $G$ . The following formula is an indirect application of Theorem 3.1.

**THEOREM 4.2.** *Let  $G = G[n, \{s_1, s_2, \dots, s_m\}]$  be a  $d$ -regular circulant graph with the adjacency matrix  $A(G) = C(c_0, c_1, \dots, c_{n-1})$ . Then*

$$(4.1) \quad \tau(\overline{S(G)}_{n, nd/2}) = \frac{(nd/2 - d)^{(n-1)}(n + nd/2 - d)}{(n + nd/2)^2} \prod_{k=0}^{n-1} F(\omega_k).$$

where,  $R_1 = \frac{nd}{2} - d - \frac{d}{n-2}$ ,  $R_2 = \frac{(n-2)(n+nd/2-2)+d^2}{(n-2)(n+nd/2-2)}$ ,  $R_3 = \frac{-1}{n-2}$ ,

$$F(x) = (R_1 + R_2) + (R_2 + R_3c_1)x + (R_2 + R_3c_2)x^2 + \dots + (R_2 + R_3c_{n-1})x^{n-1},$$

$$\omega_k = e^{\frac{2k\pi i}{n}} = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \quad \text{for } 0 \leq k \leq n-1;$$

and  $\omega_k$  ranges over the  $n$ -th roots of unity.

*Proof.* We denote  $I(G)$  for the set of those new vertices by the subdivision of edges in  $G$ . Obviously,  $(V(G), I(G))$  is the bipartition of the graph  $S(G)$ . Let  $B(S(G))$  be the biadjacency matrix of  $S(G)$  whose rows and columns indexed by  $V(G)$  and  $I(G)$ , respectively. Then it is easy to verify that  $B(S(G))B(S(G))^T = dI_n + A(G)$ , namely a circulant matrix with the first row  $(d, c_1, \dots, c_{n-1})$ . Note that  $S(G)$  is an  $(n, nd/2; d, 2)$ -semiregular bipartite graph. According to equation (3.2) in Theorem 3.1,

$$\left(\frac{nd}{2} - d\right)I_n + \frac{(n-2)(n + nd/2 - 2) + d^2}{(n-2)(n + nd/2 - 2)}J_n - \frac{1}{n-2}B(S(G))B(S(G))^T,$$

is exactly a circulant matrix with the first row

$$(R_1 + R_2, R_2 + R_3c_1, \dots, R_2 + R_3c_{n-1}),$$

where  $R_1 = \frac{nd}{2} - d - \frac{d}{n-2}$ ,  $R_2 = \frac{(n-2)(n+nd/2-2)+d^2}{(n-2)(n+nd/2-2)}$ ,  $R_3 = \frac{-1}{n-2}$ . Consequently,

$$\begin{aligned} & \det \left( \left(\frac{nd}{2} - d\right)I_n + \frac{(n-2)(n + nd/2 - 2) + d^2}{(n-2)(n + nd/2 - 2)}J_n - \frac{1}{n-2}B(S(G))B(S(G))^T \right) \\ &= \prod_{k=0}^{n-1} F(\omega_k). \end{aligned}$$

where,

$$F(x) = (R_1 + R_2) + (R_2 + R_3c_1)x + (R_2 + R_3c_2)x^2 + \dots + (R_2 + R_3c_{n-1})x^{n-1},$$

and

$$\omega_k = e^{\frac{2k\pi i}{n}} = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \quad (0 \leq k \leq n-1),$$

ranges over the  $n$ -th roots of unity. □

**5. Conclusions.** In this paper, we studied the problem of enumerating the spanning trees of the bipartite complement of a semiregular bipartite graph and obtained a determinant expression for the number of spanning trees of the bipartite complement of a semiregular bipartite graph. As a by-product, we obtained an explicit formula for the number of spanning trees of the bipartite complement of the subdivision of a regular circulant graph. In a sense, the new results show a unified enumerating approach which relies on the use of fundamental linear algebra about the biadjacency matrix of the bipartite graph.

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