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CONNECTIONS BETWEEN POINTS OF CONVEXITY OF FUNCTIONS AND CENTRALITY OF ELEMENTS IN C^*-ALGEBRAS*

GERGŐ NAGY†

Abstract. In this paper, we give a characterization of central elements in a C^*-algebra $\mathcal{A}$ in terms of points of convexity of scalar functions. We prove that if D is an open interval and $f \in \mathcal{C}^2(D)$ is a convex function satisfying a certain inequality, then a self-adjoint element $a \in \mathcal{A}$ with spectrum in D is central if and only if it is a point of convexity of f. The class of functions with these properties contains the nontrivial real exponential ones and the power ones with exponent outside $[-1, 2]$.

Key words. C^*-algebras, Centrality, Continuous functional calculus, Convexity, Divided differences.

AMS subject classifications. Primary: 46L05. Secondary: 15A16.

1. Introduction and statement of the main results. This paper forms a continuation of our investigations concerning local properties of functions on C^*-algebras presented in [8, 9]. Here, we focus on convexity properties. Relations between the commutativity of a C^*-algebra (or locally, the centrality of an element) and the global (or local) convexity of some functions have been studied in the literature. As for related results, we mention [11, Thm. 4.] stating that a C^*-algebra $\mathcal{A}$ is commutative exactly when one has a convex function $f\colon]0,\infty[\to \mathbb{R}$ such that the map $A \mapsto f(A)$ is not convex on the cone of complex positive definite 2×2 matrices, but it is so on the set of positive invertible elements of $\mathcal{A}$.

In this result, the convexity of the functions in question yields consequences on the underlying C^*-algebra. It also happens that the latter property of a function f on sets of self-adjoint elements has implications on that map. An important example of this case is the one where $f\colon D \to \mathbb{R}$ is operator convex, meaning that for all real numbers $\lambda \in [0, 1]$, complex Hilbert spaces $\mathcal{K}$ and self-adjoint operators A, B acting on $\mathcal{K}$ with spectra in the real interval D, one has

$$f(\lambda A + (1-\lambda)B) \le \lambda f(A) + (1-\lambda) f(B).$$

In other words, the map $A \mapsto f(A)$ is convex on the set of such operators. Functions of this kind appear, for example, in the theory of operator means and quantum divergences. They have very strong regularity properties, for example, each of them has an analytic continuation to the upper half-plane. This shows that in the case of such functions f, the convexity of the map $A \mapsto f(A)$ on sets of self-adjoint operators has considerable consequences on f.

Recently, the relations between "localized" versions of the commutativity of a C^*-algebra and those of the convexity of some functions have been investigated. For example, Virosztek [13] has shown that a large class of functions possesses the property that each map in that set is locally convex exactly at central elements, that is, those members of a C^*-algebra $\mathcal{A}$ which commute with all $a \in \mathcal{A}$ (cf. [13, Theorem 3]).

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†Institute of Mathematics, University of Debrecen, P. O. Box 400, H-4002 Debrecen, Hungary and HUN-REN-UD Equations, Functions, Curves and their Applications Research Group (nagyg@science.unideb.hu).

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Consequently, any of these functions f has the property that the map $a \mapsto f(a)$ is convex on the set of self-adjoint elements in $\mathcal{A}$ with spectrum in dom f.

In this paper, we prove a similar theorem for a certain class of functions. The research presented here is a contribution to the general topic of properties of scalar functions on C^*-algebras. This area is investigated, for example, in the case of monotonicity properties. Several papers are devoted to the study of connections between commutativity in C^*-algebras and global and local monotonicity of scalar functions on them, see, for example, [5, 7, 9, 10, 12, 14].

Now we shall introduce some definitions and notation that will be used in the paper. Let $\mathcal{A}$ be a C^*-algebra, denote by $\mathcal{A}_{sa}$ the space of all self-adjoint elements of $\mathcal{A}$ and let $\sigma(a)$ stand for the spectrum of an element $a \in \mathcal{A}$. We say that a is positive if it is self-adjoint and $\sigma(a) \subset [0, \infty)$. The set of all positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}_+$. The usual order $\leq$ on $\mathcal{A}_{sa}$ is defined as follows: $a \leq b$ if $b - a \in \mathcal{A}_+$ $(a, b \in \mathcal{A}_{sa})$.

In the rest of the paper, $D \subset \mathbb{R}$ denotes an interval and $f\colon D \to \mathbb{R}$ stands for a continuous function.

Let $a \in \mathcal{A}_{sa}$ be an element for which $\sigma(a) \subset D$. We say that a is a point of convexity of f, or, in other words, f is locally convex at a if for any $b \in \mathcal{A}_{sa}$ satisfying $\sigma(a \pm b) \subset D$, one has

$$f(a) \leq \frac{1}{2}(f(a+b) + f(a-b)).$$

For such elements a, b, we define the Jensen difference $J_f(a,b)$ of f by:

$$J_f(a,b) \doteq \frac{1}{2}(f(a+b) + f(a-b)) - f(a).$$

We remark that a is a point of convexity of f exactly when $J_f(a,b) \geq 0$ for all $b \in \mathcal{A}_{sa}$ with $\sigma(a \pm b) \subset D$. Local convexity is a kind of local mid-point convexity. As for global mid-point convexity, that is, the property that all elements $a \in \mathcal{A}_{sa}$ with $\sigma(a) \subset D$ are points of convexity of f, since f is continuous, a standard argument shows that it is equivalent to the convexity of the map $a \mapsto f(a)$ on the set of such elements.

In the main result of the paper, an inequality containing divided differences appears. These quantities are defined as follows. Let $m \in \mathbb{N}$, further $x_0, \ldots, x_m \in D$ be $m+1$ distinct numbers. Then,

$$f^{[0]}[x_0] = f(x_0),$$
$$f^{[m]}[x_0, \ldots, x_m] = \frac{f^{[m-1]}[x_1, \ldots, x_m] - f^{[m-1]}[x_0, \ldots, x_{m-1}]}{x_m - x_0}.$$

Of course, these definitions are valid for arbitrary real functions. The latter quantity is called an mth order divided difference of f. It is well known that if $f \in \mathcal{C}^m(D)$, then the divided differences of f have a unique continuous extension to the case of repeated arguments, and

$$f^{[m]}[x_0, \ldots, x_0] = \frac{f^{(m)}(x_0)}{m!}. \tag{1}$$

Moreover, they are symmetric in their variables.

In the rest of the paper, we will use the following notation. The symbol $\mathcal{C}^2(D)$ stands for the set of all twice continuously differentiable functions from D to $\mathbb{R}$ (this concerns the whole manuscript). If $f \in \mathcal{C}^2(D),\ x, y \in D$, then

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$$d_f(x,y) = \frac{f''(x)}{2} f^{[2]}[x,y,y] - f^{[2]}[x,x,y]^2.$$

The main result of the manuscript reads as follows.

Theorem 1.1. *Let $a \in \mathcal{A}_{sa}$ be an element such that $\sigma(A) \subset D$. Moreover, assume that D is open, $f \in \mathcal{C}^2(D)$, it is convex and satisfies*

$$d_f(x,y) < 0 \qquad \forall x, y \in D;\ x \neq y.$$

Then we have that a is a point of convexity of f if and only if it is central.

This result is strongly related to the following characterization of matrix convex functions of order $n \in \mathbb{N}$, that is, real-valued functions on D that are convex on the set of all $n \times n$ Hermitian matrices with spectrum in D (see [1, Theorem 2.4.3.]).

Theorem 1.2 (Hiai). *Let $a, b \in \mathbb{R},\ a < b$ and $g\colon (a,b) \to \mathbb{R}$ be a function. Then g is matrix convex of order n if and only if $g \in \mathcal{C}^2(a,b)$ and for each tuple $(\lambda_1, \dots \lambda_n) \in D^n$, one has $(g^{[2]}[\lambda_1, \lambda_i, \lambda_j])_{i,j=1}^n \geq 0$.*

To see the connection between the last two results, first assume that $f \in \mathcal{C}^2(D)$ and $\lambda_1, \lambda_2 \in D$. Observe that $d_f(\lambda_1, \lambda_2) = \det(f^{[2]}[\lambda_1, \lambda_i, \lambda_j])_{i,j=1}^2$. Since the first-order divided difference of a convex function is increasing in its variables, the second-order divided differences of such a map are nonnegative. Now suppose that f is also convex. It follows that in this case, the elements of the 2×2 real symmetric matrix $(f^{[2]}[\lambda_1, \lambda_i, \lambda_j])_{i,j=1}^2$ are nonnegative, therefore so is its trace, hence it is straightforward to see that this matrix is positive semidefinite exactly when its determinant is nonnegative. By the previous discussion, the remark after the definition of points of convexity, referring to the last two theorems, we conclude that the following alternative like statement holds.

Corollary 1.3. *Assume that D is open, $f \in \mathcal{C}^2(D)$ and it is convex. Then we have the below statements whose conditions represent "extreme cases."*

(i) If $\det(f^{[2]}[\lambda_1, \lambda_i, \lambda_j])_{i,j=1}^2 \geq 0$ for all pairs $(\lambda_1, \lambda_2) \in D^2$, then each 2×2 Hermitian matrix with spectrum contained in D is a point of convexity of f.

(ii) If $\det(f^{[2]}[\lambda_1, \lambda_i, \lambda_j])_{i,j=1}^2 < 0$ for all pairs $(\lambda_1, \lambda_2) \in D^2$ with $\lambda_1 \neq \lambda_2$, then, among the latter matrices, only the central (i.e., scalar) ones are points of convexity of f.

In what follows, we establish that the members of certain classes of functions satisfy the conditions of Theorem 1.1.

Lemma 1.4. *Suppose that $p \in \mathbb{R} \setminus [-1, 2]$, furthermore let $D = (0, \infty)$ and $f(z) = z^p$ $(z \in D)$. Then,*

$$d_f(x,y) < 0 \qquad \forall x, y \in D;\ x \neq y.$$

As for the exponents $p \in [-1, 2]$, one can see immediately that the power functions with exponent in $(0, 1)$ are strictly concave; hence, no elements are points of convexity of them. Moreover, it is known that such functions with exponent in $[-1, 0] \cup [1, 2]$ are operator convex; thus, all admissible elements are points of convexity of them.

The next two assertions tell us that the real exponential function and the members of a relatively large family of functions fulfill the conditions of Theorem 1.1. This entails that it is a (far reaching) generalization of [8, Theorem 2] which states that for an arbitrary element $a \in \mathcal{A}_{sa}$, one has that $\exp|_{\mathbb{R}}$ is locally convex at a if and only if a is central.

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Lemma 1.5.

$$d_{\exp|_{\mathbb{R}}}(x,y) < 0 \qquad \forall x,y \in (0,\infty);\ x \neq y.$$

Lemma 1.6. *Assume that D is open and unbounded from above, further $f \in \mathcal{C}^2(D)$, it is convex and f'' is strictly concave. Then*

$$d_f(x,y) < 0 \qquad \forall x,y \in (0,\infty);\ x \neq y.$$

[13, Theorem 3] states that Theorem 1.1 holds for all functions defined on a real open interval that satisfy the conditions of this lemma. Unfortunately, we were unable to decide if every such map fulfills the last displayed inequality. However, the proof of Lemma 1.6 shows that they satisfy it with nonstrict inequality. Therefore, we strongly conjecture that they also fulfill it with strict relation.

As for points of convexity of functions, we are aware only of such results which tell us that one has just two extreme cases: all admissible elements are points of convexity of them or only the central ones. To see an other case, we can consider the power function with exponent 4. It is known that the first case is not valid for that map and the next assertion shows that neither the second.

Lemma 1.7. *Assume that $D = \mathbb{R}$ and f is the function $f(z) = z^4$ $(z \in \mathbb{R})$. Moreover, let $a \in \mathcal{A}_{sa}$ be an element such that a^2 is central. Then a is a point of convexity of f.*

To prove this, first observe that for all $r, s \in \mathcal{A}_{sa}$ we have $\frac{1}{2}(r^2 + s^2) \geq \big(\frac{1}{2}(r+s)\big)^2$. Let $b \in \mathcal{A}_{sa}$ be an element. By the previous observation,

$$J_f(a,b) \geq \left(\frac{1}{2}((a+b)^2 + (a-b)^2)\right)^2 - a^4 = b^4 + 2a^2b^2.$$

Trivially, $a^2b^2 \in \mathcal{A}_+$ and now the statement of Lemma 1.7 follows.

Theorem 1.1 has the following global version which establishes a characterization of commutativity of C^*-algebras in terms of the order $\leq$ and the linear space operations on $\mathcal{A}$.

Corollary 1.8. *Assume that D is open, moreover $f \in \mathcal{C}^2(D)$, it is convex and satisfies*

$$d_f(x,y) < 0 \qquad \forall x,y \in D;\ x \neq y.$$

Then $\mathcal{A}$ is commutative exactly when the map $a \mapsto f(a)$ $(a \in \mathcal{A}_{sa},\ \sigma(a) \subset D)$ is convex.

In the particularly important special case where $\mathcal{A}$ is the operator algebra $\mathcal{B}(\mathcal{H})$ on a complex Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} \geq 2$, this statement tells us that for any open interval $D \subset \mathbb{R}$ and convex function $f \in \mathcal{C}^2(D)$ satisfying the last displayed inequality, the transformation $A \mapsto f(A)$ $(A \in \mathcal{B}(\mathcal{H})_{sa},\ \sigma(A) \subset D)$ is not convex.

2. Proofs. This section is devoted to the verification of our results formulated in the previous one. We will use the following notation in it. $\mathcal{H}$ denotes a complex Hilbert space with $\dim \mathcal{H} = 2$ and $\mathcal{B}(\mathcal{H})$ stands for the operator algebra on $\mathcal{H}$.

In parts of the proof of the next assertion and Theorem 1.1, we will apply arguments similar to those in [13]. The statement below has a crucial role in verifying the main result of the paper.

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LEMMA 2.1. *Pick an operator $A \in \mathcal{B}(\mathcal{H})_{sa}$, moreover, assume that D is open and $f \in \mathcal{C}^2(D)$, it is convex and satisfies*

$$d_f(x,y) < 0 \qquad \forall x, y \in D;\ x \neq y.$$

Then we have that the hypothesis $A \notin \mathbb{R}I$ implies the existence of elements $t_0 > 0,\ w \in \mathcal{H}$ for which the quantity

$$\Delta(\lambda_1, \lambda_2) = \langle (J_f(A, t_0 B)w, w\rangle, \tag{2}$$

is negative and $t_0,\ \Delta(\lambda_1, \lambda_2)$ depend only on $\sigma(A)$.

Proof. We start with the main step in which it will be shown that there is a number $t_0 > 0$ for which $J_f(A, t_0 B)$ is not positive semidefinite. In the rest of this argument, we denote the eigenvalues of A by λ_1 and λ_2; furthermore, we identify operators on $\mathcal{H}$ and elements of $\mathcal{H}$ with their matrices and coordinate row vectors, respectively, with respect to an eigenbasis for A. We are going to show that there is a number $t_0 > 0$ such that $\det(J_f(A, t_0 B)) < 0$.

In order to do so, first we observe that the Jensen difference of f is intimately connected to the second Frèchet derivative $\partial^2 f$ of the function:

$$X \mapsto f(X) \qquad (X \in \mathcal{B}(\mathcal{H})_{sa},\ \sigma(X) \subset D).$$

Namely, it is well known that, for any given $B \in \mathcal{B}(\mathcal{H})_{sa}$,

$$\begin{aligned} \partial^2 f(A)(B,B) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} f(A + tB)_{|t=0} \\ &= \lim_{t\to 0} \frac{1}{t^2}(f(A+tB) - 2f(A) + f(A - tB)) = 2 \lim_{t \to 0} \frac{1}{t^2} J_f(A, tB). \end{aligned} \tag{3}$$

By [2, Example 3.34], if $B = (B_{ij})_{i,j=1}^2$, then

$$\partial^2 f(A)(B,B) = 2\left(\sum_{k=1}^2 f^{[2]}[\lambda_i, \lambda_k, \lambda_j] B_{ik} B_{kj}\right)_{i,j=1}^2.$$

Now set $B_{12} = 1,\ B_{22} = 0$. It follows that

$$\partial^2 f(A)(B,B) = 2\begin{pmatrix} \frac{f''(\lambda_1)}{2} B_{11}^2 & f^{[2]}[\lambda_1, \lambda_1, \lambda_2] B_{11} \\ f^{[2]}[\lambda_1, \lambda_1, \lambda_2] B_{11} & f^{[2]}[\lambda_1, \lambda_2, \lambda_2] \end{pmatrix},$$

therefore,

$$\det \partial^2 f(A)(B,B) = 4(d_f(\lambda_1, \lambda_2) B_{11}^2 + f^{[2]}[\lambda_1, \lambda_1, \lambda_2] f^{[2]}[\lambda_1, \lambda_2, \lambda_2]).$$

This equality, the equation (3) and the conditions of Lemma 2.1 imply

$$\lim_{t \to 0} \frac{1}{t^4} \det J_f(A, tB) \to -\infty \qquad (B_{11} \to +\infty).$$

Hence, $\lim_{t\to 0} t^{-4} \det J_f(A, tB) < 0$ for all large enough $B_{11} \in \mathbb{R}$, thus $t^{-4} \det J_f(A, tB) < 0$ for all large enough $B_{11} \in \mathbb{R}$ and each $0 \neq t \in \mathbb{R}$ with small enough magnitude. We conclude that there is a number $t_0 > 0$ such that $\det(J_f(A, t_0 B)) < 0$ yielding that $J_f(A, t_0 B)$ is not positive semidefinite.

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To sum up, we have verified that, identifying A with the diagonal matrix whose diagonal entries are λ_1, λ_2, for the latter matrix, one has a scalar $t_0 > 0$ and a vector $w \in \mathbb{C}^2$ such that the quantity in (2) is negative. Moreover, clearly, that quantity depends on B only through λ_1, λ_2 which depend just on $\sigma(A)$. This holds also for t_0. We deduce that $\Delta(l_1, \lambda_2)$ depends on B and t_0 only through $\sigma(A)$. Now the statement of Lemma 2.1 follows very easily. ∎

Next, we proceed with the verification of the main result of the paper.

Proof of Theorem 1.1. Now we use the argument in [13, Section 3.2.]. One can check that the reasoning in its steps can be applied also here, without modification, except the proof of the most crucial one. In that step, the statement below was obtained.

(**) (6) on [13, p.6] holds for some vector w_n ($n \in \mathbb{N}$) and number $t_0 > 0$, and the left-hand side of the inequality in (6) does not depend on n.

In [13], (**) is derived from an assertion that is a counterpart of Lemma 2.1 for other functions f. It is almost trivial to see that Lemma 2.1 implies (**) in the case where f has the properties postulated in the conditions of Theorem 1.1. Now its verification is complete. ∎

In the next two proofs, we will use the identities:

$$f^{[2]}[x,x,y] = \frac{\frac{f(y)-f(x)}{y-x} - f'(x)}{y-x},$$
$$f^{[2]}[x,y,y] = \frac{f'(y) - \frac{f(y)-f(x)}{y-x}}{y-x},$$

which is valid, by (1), for all functions $f \in \mathcal{C}^2(D)$ and each pair of numbers $x, y \in D;\ x \neq y$. Now we proceed with the verification of Lemmas 1.4, 1.5.

Proof of Lemma 1.4. Let $x, y \in (0, \infty);\ x \neq y$. We compute

$$\begin{aligned} d_f(x,y) = {} & \frac{1}{2(x-y)^4}(p(p-1)x^{p-2}(py^{p-1}(y-x)^3 + (x^p - y^p)(y-x)^2) \\ & - (px^{p-1}(y-x) + x^p - y^p)^2). \end{aligned}$$

Let $z = \frac{y}{x}$. Then it follows

$$\begin{aligned} d_f(x,y) = {} & \frac{x^{2p-4}}{2(1-z)^4}(p(p-1)(pz^{p-1}(z-1)^3 + (1-z^p)(z-1)^2) \\ & -(p(z-1) + 1 - z^p)^2), \end{aligned}$$

implying, by Maple, that

$$\frac{2(1-z)^4}{x^{2p-4}} d_f(x,y) = P(z), \tag{4}$$

where

$$\begin{aligned} P(z) \doteq {} & -2z^{2p} + (p^3 - 2p^2 + p)z^{p+2} + (-3p^3 + 5p^2 + 2p)z^{p+1} \\ & + (3p^3 - 4p^2 - 3p + 4)z^p + (-p^3 + p^2)z^{p-1} + (-p^2 - p)z^2 \\ & + (2p^2 - 2p)z + (-p^2 + 3p - 2) = -2z^{2p} + p(p-1)^2 z^{p+2} \\ & + p(2-p)(3p+1)z^{p+1} + (p^2-1)(3p-4)z^p + p^2(1-p)z^{p-1} \\ & - p(1+p)z^2 + 2p(p-1)z + (1-p)(p-2). \end{aligned}$$

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In what follows, we investigate the zeroes of P as a function on $(0,\infty)$. Using Maple, we obtain that $P^{(k)}(1)=0$ $(k=0,\dots 5)$; therefore, 1 is a zero of P with multiplicity at least 6. In what follows, we count the zeroes of P according to their multiplicities. By [4, Theorem 3.1.], the number of sign changes in the sequence of the coefficients of P (ordered decreasingly w.r.t. the exponent in the corresponding term) is at least the number of zeroes of P. Obviously, the former quantity must be at most 7. Moreover, using the last displayed chain of equalities and separating the corresponding cases according to the value of p, it can be seen very easily that in the previous sequence, there are at least two adjacent terms with the same sign. By the latter discussion, we deduce that P has exactly six zeroes and their common value is 1.

A tiny computation using the last displayed form of P shows that $\lim_{z\to 0} P(z), \lim_{z\to\infty} P(z) < 0$; hence, $P(z) < 0$ if z is small or large enough. Since 1 is the only zero of P, this implies, by Bolzano's theorem, $P(z) < 0$ for all $0 < z \neq 1$, which, together with (4), yields the statement of Lemma 1.4. ∎

Proof of Lemma 1.5. Let $f = \exp|_{\mathbb{R}};\ x, y \in \mathbb{R},\ x \neq y$ and define $z = y - x$. We compute

$$\begin{aligned} 2z^4 e^{-2x} d_f(x,y) &= z^3 e^z - z^2 e^z + z^2 - 2(z - e^z + 1)^2 \\ &= -2e^{2z} + (z^3 - z^2 + 4z + 4)e^z - z^2 - 4z - 2 \doteq \tilde{f}(z). \end{aligned} \tag{5}$$

Then for the function $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$, we have

$$\begin{aligned} \tilde{f}'(z) &= (z^3 + 2z^2 + 2z + 8)e^z - 4e^{2z} - 2z - 4 \\ \tilde{f}''(z) &= (z^3 + 5z^2 + 6z + 10)e^z - 8e^{2z} - 2 \\ \tilde{f}'''(z) &= (z^3 + 8z^2 + 16z + 16)e^z - 16e^{2z} \\ &= e^z \left(z^3 + 8z^2 + 16z + 16 - 16 - 16z - 8z^2 - \frac{16}{6} z^3 - \sum_{n=4}^{\infty} \frac{z^n}{n!} \right). \end{aligned}$$

By Taylor's theorem, one has

$$\sum_{n=4}^{\infty} \frac{z^n}{n!} = e^z - \sum_{n=0}^{3} \frac{z^n}{n!} = \frac{\exp^{(4)}(z_0)}{4!} z^4 = \frac{e^{z_0}}{24} z^4,$$

with some number $z_0 \in \mathbb{R}$ strictly between 0 and z. We deduce that

$$\tilde{f}'''(z) = \frac{e^z}{24} z^3 (e^{z_0} z - 40) = -e^z \left(\frac{5}{3} z^3 + \sum_{n=4}^{\infty} \frac{z^n}{n!} \right),$$

which gives the inequality $\tilde{f}'''(z)z < 0$. We infer that $\tilde{f}''$ is strictly increasing on $(-\infty, 0)$ and strictly decreasing on $(0,\infty)$. Since $\tilde{f}''(0) < 0$, it follows that $\tilde{f}'' < 0$ and therefore $\tilde{f}$ is strictly concave. The chain of equalities $\tilde{f}(0) = \tilde{f}'(0) = 0$ implies that the tangent line to the graph of $\tilde{f}$ at $(0,0)$ is $y = 0$; hence, we obtain that $\tilde{f}(z) < 0$ $(z \neq 0)$. This together with (5) yields the statement of Lemma 1.5. ∎

We finish this section with the proof of Lemma 1.6. The assertion below will be used for the verification of the latter statement. We guess that it is a folk result, but its proof is included here for the sake of completeness.

LEMMA 2.2. *If $E \subset \mathbb{R}$ is an interval which is unbounded from above and $f\colon E \to \mathbb{R}$ is a concave function that is bounded from below, then f is increasing.*

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Proof. Assume on the contrary that for some numbers $x, y \in E$ with $x < y$ one has $f(x) > f(y)$. Let $z > y$ be an arbitrary point in E. Then, we have

$$\frac{f(y) - f(x)}{y - x} \geq \frac{f(z) - f(y)}{z - y},$$

yielding

$$f(z) \leq \frac{z - y}{y - x}(f(y) - f(x)) + f(y),$$

which entails

$$\limsup_{z \to \infty} f(z) \leq -\infty,$$

contradicting the boundedness property of f. □

Finally, we present the verification of Lemma 1.6.

Proof of Lemma 1.6. Since f is convex, f'' is nonnegative, therefore bounded from below. It is also concave and D is unbounded from above, hence, by the previous lemma, we infer that f'' is increasing. Since it is also strictly concave, it follows that f'' is strictly increasing.

Let $x_0, x_1, x_2 \in D$. By [3, Chapter 6, Section 1, Corollary 1],

$$f^{[2]}[x_0, x_1, x_2] = \int_0^1 \int_0^{t_1} f''(t_2(x_2 - x_1) + t_1(x_1 - x_0) + x_0) dt_1 dt_2.$$

Choose numbers $x_2' \in D,\ t_1 \in [0, 1],\ t_2 \in (0, 1]$ such that $x_2' > x_2,\ t_1 \geq t_2$. Then referring to the end of the previous paragraph,

$$f''(t_2(x_2' - x_1) + t_1(x_1 - x_0) + x_0) > f''(t_2(x_2 - x_1) + t_1(x_1 - x_0) + x_0),$$

thus, this holds for all pairs (t_1, t_2) in a set of positive Lebesgue measure. By the latter discussion, we deduce that if $x_0, x_1, x_2, x_2' \in D$ are such that $x_2 < x_2'$, then $f^{[2]}[x_0, x_1, x_2] < f^{[2]}[x_0, x_1, x_2']$ which gives us that the second-order divided difference of f is strictly increasing in its variables.

Now let $x, y \in D$ with $x \neq y$ and define

$$a = f^{[2]}[x, x, x],\ b = f^{[2]}[x, x, y],\ c = f^{[2]}[x, y, y].$$

Clearly, $d_f(x, y) = ac - b^2$ and the end of the last paragraph yields that $a \neq c$. Applying the arithmetic-geometric mean inequality, it follows that

$$4d_f(x, y) < (a + c)^2 - 4b^2 = (a + 2b + c)(a - 2b + c).$$

On the one hand, since f is convex, its second-order divided differences are nonnegative, so $a + 2b + c \geq 0$. On the other hand,

$$\begin{gathered} a - 2b + c = (a - b) - (b - c) \\ = (x - y)(f^{[3]}[x, x, x, y] - f^{[3]}[x, x, y, y]) = (x - y)^2 f^{[4]}[x, x, x, y, y]. \end{gathered}$$

We have that $-f \in \mathcal{C}^2(D)$ and $(-f)''$ is convex, therefore, applying [6, Theorem 6], we get that $-f$ is 4-convex which means that its fourth-order divided differences (with distinct arguments) are nonnegative. Referring to the continuity of divided differences, now it follows that $f^{[4]}[x, x, x, y, y] \leq 0$, which, together with the last two displayed chains of relations and the nonnegativity of $a + 2b + c$, entails the statement of Lemma 1.6. □

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