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MULTIPLICATIVITY OF PERMANENTS OVER MATRIX SEMIRINGS*

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Abstract. In this paper, we investigate the conditions for the multiplicativity of the permanent over a matrix semiring. We prove that if S is either a commutative antiring or a commutative semiring where the set $V(S)$ of all additively invertible elements coincides with the set of all nilpotents, then the permanent is multiplicative on the group of invertible matrices over S if and only if $1+2V(S)^2=1$. We then use this result to investigate the number of invertible matrices over S with a specified permanent.

Key words. Semiring, Permanent, Invertible matrix.

AMS subject classifications. 16Y60, 15A15.

1. Introduction. A *semiring* is a set S equipped with binary operations $+$ and $\cdot$ such that $(S,+)$ is a commutative monoid with identity element 0 and $(S,\cdot)$ is a monoid with identity element 1. In addition, operations $+$ and $\cdot$ are connected by distributivity and 0 annihilates S. A semiring is *commutative* if $ab=ba$ for all $a,b\in S$. The theory of semirings has many applications in optimization theory, automatic control, models of discrete event networks, and graph theory (see e.g. [1, 6, 12, 16]). For an extensive theory of semirings, we refer the reader to [11]. There are many natural examples of commutative semirings, for example, the set of nonnegative integers (or reals) with the usual operations of addition and multiplication. Other examples include distributive lattices, tropical semirings, dioïds, fuzzy algebras, inclines, and bottleneck algebras. A semiring S is called *entire* or *zero-divisor-free* if $ab=0$ for some $a,b\in S$ implies that $a=0$ or $b=0$, and it is called *antinegative* or *zero-sum-free* if $a+b=0$ for some $a,b\in S$ implies that $a=b=0$. Antinegative semirings are also called *antirings*. The simplest example of an antinegative semiring is the binary Boolean semiring, the set $\{0,1\}$ in which addition and multiplication are the same as in $\mathbb{Z}$ except that $1+1=1$.

For a (semi)ring S, the permanent of an n-by-n matrix A with entries in S is defined by $\text{per}(A)=\sum_{\pi\in S_n}\prod_{i=1}^n A_{i\pi(i)}$, where S_n denotes the symmetric group on n elements, i.e., the group of all permutations of the numbers $1,2,\ldots,n$, and A_{ij} denotes the entry of matrix A at position (i,j). Studies on permanents, since their introduction in 1812 by Binet [4] and Cauchy [5] have mainly focused on matrices over fields and commutative rings. The permanent of a square matrix has significant graph theoretic interpretations. For example, the number of vertex-disjoint cycle covers in a directed graph is equal to the permanent of its adjacency matrix. Also, the number of perfect matchings in a bipartite graph is equal to the permanent of its bipartite adjacency matrix. Theory of permanents also provides us with an effective tool in dealing with order statistics corresponding to random variables which are independent but possibly nonidentically distributed [2]. While the determinant of a matrix can be computed in polynomial time, computing the permanent of a matrix is a "#P-complete problem," which cannot be done in polynomial time unless $P=NP$ [14, 15].

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In [7], the author studied the multiplicativity of a generalized permanent d_H corresponding to a subgroup H of S_n with matrices having entries in a semiring. Over antinegative cancellation semirings, the author proved some results for the permanent that are similar to the results over fields. For chain semirings, the semigroup of all matrices A such that $d_H(A) \neq 0$ is maximal with respect to the equation of multiplicativity of the permanent over the 2-element Boolean semiring. In [3], the author proved that the set of all nonsingular matrices of the form PD, where P is a permutation matrix and D is a diagonal matrix with complex entries is a maximal group on which the permanent function is multiplicative.

Let us give some basic definitions we shall be using throughout the paper. We shall denote the set of all multiplicatively invertible elements in S by S^* and the set of all additively invertible elements in S by $V(S)$. Furthermore, let $\mathcal{N}(S)$ denote the set of all nilpotent elements in S. For a semiring S and a subset $T \subseteq S$, we shall denote the semiring of n by n matrices with entries in T by $M_n(T)$. We shall denote the diagonal matrix with elements $d_1, d_2, \ldots, d_n \in S$ along the diagonal with $\text{Diag}(d_1, d_2, \ldots, d_n)$. A matrix with 1 at (i, j)-th entry and zeroes elsewhere will be denoted by E_{ij}. For a permutation $\sigma \in S_n$, the permutation matrix P_σ will denote the $0/1$ matrix where the (i, j)-th entry of P_σ is equal to 1 if and only if $i = \sigma(j)$. We shall often denote the (i, j)-th entry of matrix A simply by A_{ij}. An element $e \in S$ is called *idempotent* if $e^2 = e$ and e is called *nilidempotent* if $e^2 = e + x$ for some $x \in \mathcal{N}(S)$. A set $\{a_1, a_2, \ldots, a_r\} \subseteq S$ of nonzero elements is called an *orthogonal decomposition of* 1 *of length* r in S if $a_1 + a_2 + \ldots + a_r = 1$ and $a_i a_j = 0$ for all $i \neq j$. A matrix $A \in M_n(S)$ is *an orthogonal combination* of matrices $A_1, A_2, \ldots, A_r$ if there exists an orthogonal decomposition $\{a_1, a_2, \ldots, a_r\}$ of 1, such that $A = \sum_{i=1}^{r} a_i A_i$. The group of all invertible matrices in $M_n(S)$ will be denoted by $GL_n(S)$.

In this paper, we shall investigate the properties of permanents in two different settings: commutative antirings and commutative semirings with $V(S) = \mathcal{N}(S)$. The latter family obviously includes all antirings without (nonzero) nilpotent elements, as well as all semirings of the form $S + R$, where R is a nilpotent ring which is also an S-semimodule. Some further examples of such semirings are also given in [10]. The paper is organized as follows. In the next section, we examine the conditions for the permanent to be multiplicative on the group of all invertible matrices over S. The main result of the section is Theorem 2.5, where we prove that in each of the two above settings, the permanent is multiplicative on the group of invertible matrices over S if and only if $1 + 2V(S)^2 = 1$. In the final section, we use these results to investigate the number of (invertible) matrices with a specified permanent (see Theorem 3.2 and Proposition 3.5).

2. Multiplicativity of permanents. In this section, we investigate the conditions on the semiring S under which the permanent is multiplicative on $GL_n(S)$.

We start with the following lemma, which shows that the condition of the multiplicativity of the permanent on the set of all matrices is too restrictive to be of any real interest.

LEMMA 2.1. *Let S be a commutative semiring such and $n \geq 2$. Then* $\text{per}(AB) = \text{per}(A)\text{per}(B)$ *for every $A, B \in M_n(S)$ if and only if S is a ring of characteristic* 2.

Proof. Denote $A = E_{11} + E_{12} + E_{33} + \ldots + E_{nn}$ and $B = A + E_{21} + E_{22} \in M_n(S)$ and observe that we have $E_{ij}E_{kl} = E_{il}$ for $j = k$, and 0 otherwise. This implies that $BA = B$. Since $\text{per}(A) = 0$ and $\text{per}(B) = 2$, we get $2 = 0$. This implies that $V(S) = S$, so S is a ring of characteristic 2. On the contrary, if S is a ring of characteristic 2, the permanent is equal to the determinant, which is of course multiplicative. ∎

Therefore, we loosen the multiplicativity condition somewhat and observe the commutative semirings such that the multiplicativity of the permanent holds only for all the invertible matrices.

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Lemma 2.2. *Let S be a commutative semiring and $n \geq 2$. If* $\operatorname{per}(AB) = \operatorname{per}(A)\operatorname{per}(B)$ *for every* $A, B \in GL_n(S)$ *then* $1 + 2V(S)^2 = 1$.

Proof. Choose $x, y \in V(S)$ and denote $A(x) = I + xE_{12}$ and $B(y) = I + yE_{21}$, where I is the identity matrix. Observe that $A(x)A(-x) = B(y)B(-y) = I$, so $A(x), B(y) \in GL_n(S)$. However, $1 + 2xy = \operatorname{per}(A(x)B(y)) = \operatorname{per}(A(x))\operatorname{per}(B(y)) = 1$. ∎

We shall also need the following lemma, which appears in [10] but we include it here for the sake of completeness.

Lemma 2.3. *Let S be a commutative semiring. If $1 = f_1 + f_2 + \ldots + f_k$ is an orthogonal decomposition of 1 of maximal length and $1 = e_1 + e_2 + \ldots + e_l$ is an orthogonal decomposition of 1, then for every $1 \leq i \leq l$ there exists a set $\emptyset \neq I_i \subseteq \{1, 2, \ldots, k\}$ such that $e_i = \sum_{j \in I_i} f_j$.*

Proof. By multiplying the two equations, we get $\sum_{j=1}^{k} \sum_{i=1}^{l} f_j e_i = 1$. Suppose there exist $1 \leq j_1 \leq k$ and $1 \leq i_1, i_2 \leq l$ such that $f_{j_1} e_{i_1}, f_{j_1} e_{i_2} \neq 0$. Since the maximal length of an orthogonal decomposition of 1 is k, there exists $1 \leq j_2 \leq k$ such that $f_{j_2} e_i = 0$ for every $1 \leq i \leq l$, so $f_{j_2} = 0$, a contradiction. This implies that for every $1 \leq j \leq k$ there exists a unique $1 \leq i_j \leq l$ such that $f_j e_{i_j} \neq 0$ and observe that $f_j e_{i_j} = f_j$. For every $1 \leq i \leq l$ now define $I_i = \{1 \leq j \leq k; e_i f_j = f_j\}$ and observe that $I_i \neq \emptyset$ and $e_i = e_i \left(\sum_{j \in I_i} f_j\right) = \sum_{j \in I_i} f_j$. ∎

Before proving our main result of this section, we also need the following lemma.

Lemma 2.4. *Let S be a commutative semiring and $n \geq 2$. Suppose that*

$$A = D\left(\sum_{\sigma \in X} f_\sigma P_\sigma\right)B,$$

for some $X \subseteq S_n$ and an invertible diagonal matrix D, where P_σ is a permutation matrix for every $\sigma \in X$, $\sum_{\sigma \in X} f_\sigma = 1$ is an orthogonal decomposition of 1, and $B \in M_n(S)$. Then $\operatorname{per}(A) = \operatorname{per}(D)\operatorname{per}(B)$.

Proof. Observe first that $A = DA'$ for $A' = (\sum_{\sigma \in X} f_\sigma P_\sigma)B$ and that for every $1 \leq i \leq n$, each entry in the i-th row of A' is multiplied with the i-th diagonal element from D, thus implying that $\operatorname{per}(A) = \operatorname{per}(D)\operatorname{per}(A')$. We now have to prove that $\operatorname{per}(A') = \operatorname{per}(B)$. Note that for every $1 \leq i, j \leq n$, the (i,j)-th entry of A' equals $\sum_{\sigma \in X} f_\sigma B_{\sigma^{-1}(i)j}$. Since for every $\sigma \neq \sigma' \in X$, idempotents f_σ and f'_σ are orthogonal, we conclude that $\operatorname{per}(A') = \sum_{\tau \in S_n} \left(\prod_{i=1}^{n} \left(\sum_{\sigma \in X} f_\sigma B_{\sigma^{-1}(i)\tau(i)}\right)\right) = \sum_{\tau \in S_n} \left(\sum_{\sigma \in X} f_\sigma \left(\prod_{i=1}^{n} B_{\sigma^{-1}(i)\tau(i)}\right)\right) = \sum_{\sigma \in X} f_\sigma \left(\sum_{\tau \in S_n} \left(\prod_{i=1}^{n} B_{\sigma^{-1}(i)\tau(i)}\right)\right)$. Since P_σ is a permutation matrix, we have $\operatorname{per}(P_\sigma B) = \operatorname{per}(B)$ for every $\sigma \in X$, which finally yields $\operatorname{per}(A') = \sum_{\sigma \in X} f_\sigma \operatorname{per}(B) = \operatorname{per}(B)$. ∎

We now have the following theorem, which is the main theorem of this section.

Theorem 2.5. *Let $n \geq 2$ and S be a commutative antiring or a commutative semiring with $V(S) = \mathcal{N}(S)$ a finitely generated S-semimodule. Then* $\operatorname{per}(AB) = \operatorname{per}(A)\operatorname{per}(B)$ *for every $A, B \in GL_n(S)$ if and only if* $1 + 2V(S)^2 = 1$.

Proof. One side of the implication follows directly from Lemma 2.2.

Suppose that S is a commutative antiring. Then by [8, Theorem 1], every $A \in GL_n(S)$ is of the form $A = D\sum_{\sigma \in S_n} a_\sigma P_\sigma$, where D is an invertible diagonal matrix, P_σ is a permutation matrix, and $\sum_{\sigma \in S_n} a_\sigma$ is an orthogonal decomposition of 1. So, choose two invertible matrices $A = D\sum_{\sigma \in S_n} a_\sigma P_\sigma$ and $A' = D'\sum_{\sigma' \in S_n} a'_{\sigma'} P'_{\sigma'}$. Observe first that $\operatorname{per}(A) = \Pi_{i=1}^{n} d_{ii}$ and $\operatorname{per}(A') = \Pi_{i=1}^{n} d'_{ii}$. By Lemma

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2.3, we can assume that $a_\sigma a'_{\sigma'}$ is an idempotent and $\sum_{\sigma,\sigma'\in S_n} a_\sigma a'_{\sigma'}$ is an orthogonal decomposition of 1. Therefore, we have $AA' = D\sum_{\sigma,\sigma'\in S_n} a_\sigma a'_{\sigma'} P_\sigma D' P'_{\sigma'}$. Now, [9, Lemma 5] implies that for every $\sigma \in S_n$ we have $P_\sigma D' = D'' P_\sigma$, where D'' is an invertible diagonal matrix with $\mathrm{per}(D'') = \mathrm{per}(D')$. So, $\mathrm{per}(DD'') = \mathrm{per}(DD')$ and $AA' = \sum_{\sigma,\sigma'\in S_n} DD'' a_\sigma a'_{\sigma'} P_\sigma P'_{\sigma'}$, which implies that $\mathrm{per}(AA') = \mathrm{per}(DD') = \mathrm{per}(D)\mathrm{per}(D') = \mathrm{per}(A)\mathrm{per}(A')$.

Next, assume that S is a commutative semiring with $V(S) = \mathcal{N}(S)$ a finitely generated S-semimodule. Then [10, Theorem 3.2] implies that A is invertible if and only if $A = D\left(\sum_{\sigma\in X} f_\sigma P_\sigma\right) + N$, where $X \subseteq S_n$, D is an invertible diagonal matrix, P_σ is a permutation matrix for every $\sigma \in X$, and $\sum_{\sigma\in X} f_\sigma = 1$ is an orthogonal decomposition of 1 and $N \in M_n(\mathcal{N}(S))$. So, let A be as above and choose $A' = D'\left(\sum_{\sigma'\in X'} f_{\sigma'} P_{\sigma'}\right) + N' \in GL_n(S)$. Denote $A_1 = D\left(\sum_{\sigma\in X} f_\sigma P_\sigma\right)$ and observe that $A = A_1(I + N_1)$ for some $N_1 \in M_n(\mathcal{N}(S))$ and similarly $A' = A_2(I + N_2)$ for some $N_2 \in M_n(\mathcal{N}(S))$. By Lemma 2.4 and by using [9, Lemma 5] similarly as above, we arrive at $\mathrm{per}(AA') = \mathrm{per}(D)\mathrm{per}(D')\mathrm{per}((I + N_1)(I + N_2))$. Now, observe that all the diagonal elements of the matrix $(I+N_1)(I+N_2)$ belong to the set $1+V(S)$ and all the off-diagonal elements belong to $V(S)$. Since $1+2V(S)^2 = 1$, we have $1+xy = 1-xy$ for all $x, y \in V(S)$ and furthermore $xyz = -xyz$ for all $x, y, z \in V(S)$. The fact that the only summands in the permanent of the matrix $(I+N_1)(I+N_2)$ that do not contain at least two factors from $V(S)$ come from multiplying the diagonal elements of the matrix, implies that $\mathrm{per}((I + N_1)(I + N_2)) = \det((I + N_1)(I + N_2)) = \det(I + N_1)\det(I + N_2)$. The same argument now yields $\det(I + N_i) = \mathrm{per}(I + N_i)$ for $i = 1, 2$. Finally, Lemma 2.4 implies that $\mathrm{per}(I + N_1)\mathrm{per}(D) = \mathrm{per}(A)$ and $\mathrm{per}(I + N_2)\mathrm{per}(D') = \mathrm{per}(A')$, so we have proved that $\mathrm{per}(AA') = \mathrm{per}(A)\mathrm{per}(A')$. □

Furthermore, we have some reasons to believe that the above theorem might hold in a more general setting. Thus, we have the following conjecture.

CONJECTURE. *Let $n \geq 2$ and S be a commutative semiring. Then* $\mathrm{per}(AB) = \mathrm{per}(A)\mathrm{per}(B)$ *for every $A, B \in GL_n(S)$ if and only if $1 + 2V(S)^2 = 1$.*

3. (Invertible) matrices with a specified permanent. In this section, we use the results of the previous section to estimate the number of different (invertible) matrices with a specified permanent.

DEFINITION 3.1. *Let S be a finite commutative semiring. For $a \in S$ we define $p_n(S, a) = |\{A \in GL_n(S); \mathrm{per}(A) = a\}|$.*

We now immediately have the following theorem.

THEOREM 3.2. *Assume that $n \geq 2$, S is a finite commutative semiring, and $a \in S$. Let $f_1+f_2+\ldots+f_k = 1$ be an orthogonal decomposition of 1 of maximal length.*

1. *If $V(S) = 0$, then*
$$p_n(S, a) = p_n(S, 1) = \begin{cases} 0, & a \notin S^*, \\ |S^*|^{n-1}(n!)^k, & a \in S^*. \end{cases}$$
2. *If $V(S) = \mathcal{N}(S)$ is a finitely generated S-semimodule and $1 + 2V(S)^2 = 1$, then*
$$p_n(S, a) = 0, \text{ if } a \notin S^*, \text{ and}$$
$$p_n(S, a) = p_n(S, 1) \geq |S^*|^{n-1}(n!)^k|\mathcal{N}(S)|^{n\max\{0,n-k\}}, \text{ if } a \in S^*.$$

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Proof. By Theorem 2.5, we know that in both the above cases, the permanent is multiplicative on the set $GL_n(S)$. So, choose $A \in GL_n(S)$. Then $\text{per}(A)\text{per}(A^{-1}) = 1$, so $a = \text{per}(A)$ is invertible in S. Furthermore, there exists an invertible diagonal matrix $D = \text{Diag}(a^{-1}, 1, 1, \ldots, 1)$ such that $\text{per}(DA) = 1$. Since multiplying with matrix D is a bijection mapping from the set of all invertible matrices with their permanent equal to a to the set of all invertible matrices with their permanent equal to 1, we have $p_n(S, a) = p_n(S, 1)$, therefore $p_n(S, a) = \frac{|GL_n(S)|}{|S^*|}$.

Suppose first that $V(S) = 0$, so S is an antiring. By [8, Theorem 1], every $A \in GL_n(S)$ is of the form $A = D\sum_{\sigma \in S_n} a_\sigma P_\sigma$, where D is an invertible diagonal matrix, P_σ are permutation matrices and $\sum_{\sigma \in S_n} a_\sigma$ is an orthogonal decomposition of 1 (where some elements a_σ may be equal to 0). Suppose that $D\sum_{\sigma \in S_n} a_\sigma P_\sigma = D'\sum_{\sigma \in S_n} a'_\sigma P_\sigma$ for some invertible diagonal matrix D', permutation matrices P_σ, and an orthogonal decomposition of $1 = \sum_{\sigma \in S_n} a'_\sigma$. By Lemma 2.3, we can multiply this equation with f_i for $i = 1, \ldots, k$ and observe that $a_\sigma = a'_\sigma$ for every $\sigma \in S_n$ and then also $D = D'$. By our assumption, the number of nonzero summands in this sum is at most k, so Lemma 2.3 implies that we have exactly $|GL_n(S)| = |S^*|^n(n!)^k$ invertible matrices (since every permutation matrix can appear with each of the k summands $f_1, \ldots, f_k$). Thus, the statement follows.

Suppose now that $V(S) = \mathcal{N}(S)$ and $1 + 2V(S)^2 = 1$. Then [10, Theorem 3.2] implies that A is invertible if and only if $A = D\left(\sum_{\sigma \in X} f_\sigma P_\sigma\right) + N$, where $X \subseteq S_n$, D is an invertible diagonal matrix, P_σ is a permutation matrix for every $\sigma \in X$, $\sum_{\sigma \in X} f_\sigma = 1$ is an orthogonal decomposition of 1 and $N \in M_n(\mathcal{N}(S))$. Suppose that $D\left(\sum_{\sigma \in X} f_\sigma P_\sigma\right) + N = D'\left(\sum_{\sigma \in X'} f'_\sigma P_\sigma\right) + N'$ for some invertible diagonal matrix D', $X' \subseteq S_n$, permutation matrices P_σ, an orthogonal decomposition of $1 = \sum_{\sigma \in X'} f'_\sigma$ and $N' \in M_n(\mathcal{N}(S))$. It is easy to see (by multiplying both sides of the equation by f_σ and f'_σ and applying Lemma 2.3) that $X = X'$ and $f_\sigma = f'_\sigma$ for every $\sigma \in X$. If $k < n$, then at least $n - k$ entries in each row of the matrix on the left side of the equation are nilpotents, and they have to be equal to the corresponding entries on the right side of the equation. This implies that at least $n(n - k)$ entries of N' are uniquely determined by the corresponding entries in N, therefore obtaining a different matrix by choosing different entries. Furthermore, by summing all entries in the i-th row of both matrices on the left and the right sides of the equation, we get $d_i + \sum_{j=1}^n N_{ij} = d'_i + \sum_{j=1}^n N'_{ij}$; therefore, $d'_i = d_i + \sum_{j=1}^n N_{ij} - N'_{ij}$ is uniquely determined for every $1 \le i \le n$. Thus, we again obtain a different matrix by choosing different entries for the matrix D'. This implies that $|GL_n(S)| \ge |S^*|^n(n!)^k|\mathcal{N}(S)|^{n\max\{0, n-k\}}$ and the statement follows. ∎

The following example shows that the bound in the second case of the Theorem 3.2 can be achieved.

Example 3.3. *Let R be a finite ring and $S = \mathbb{N} \cup Rx$ with $x^2 = 0$ and $a + bx = a$ and $a(bx) = (ab)x$ for every $a \in \mathbb{N}$ and $b \in R$. It can be readily verified that S is indeed a semiring with $V(S) = \mathcal{N}(S) = Rx$ a finitely generated S-semimodule and $1 + 2V(S)^2 = 1$. Notice that S does not contain any nontrivial idempotents and $|S^*| = 1$, so by Theorem 3.2(2), we have $p_n(S, 1) \ge n!|R|^{n(n-1)}$. On the other hand, $p_n(S, 1) = |GL_n(S)|$ and by [10, Theorem 3.2] we know that A is an invertible matrix in $M_n(S)$ if and only if $A = P + N$, where $P \in M_n(S)$ is a permutation matrix and $N \in M_n(\mathcal{N}(S))$. Since $1 + \mathcal{N}(S) = 1$, A is therefore invertible if and only if we can write $A = P + N$, where $P \in M_n(S)$ is a permutation matrix and $N \in M_n(\mathcal{N}(S))$ is such that $N_{ij} \neq 0$ if and only if $P_{ij} = 0$. Since this notation is obviously unique, we have $p_n(S, 1) = |GL_n(S)| = n!|\mathcal{N}(S)|^{n(n-1)} = n!|R|^{n(n-1)}$ and thus the bound is achieved.*

Let us finally explore the number of different (arbitrary, not necessarily invertible) matrices with a prescribed permanent.

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Definition 3.4. *Let S be a finite commutative semiring. For $a \in S$ we define* $P_n(S,a) = |\{A \in M_n(S); \operatorname{per}(A) = a\}|$.

We have the following proposition.

Proposition 3.5. *Let $n \geq 2$ and S a finite commutative semiring. Then*

1. *if S is entire and $V(S) = 0$ then*

$$P_n(S,0) = |S|^{n^2} - \sum_{i=0}^{n} \binom{n}{i} (-1)^i (|S|^{n-i} - 1)^n;$$

2. *if $V(S) = \mathcal{N}(S)$ then for every $a \in \mathcal{N}(S)$ we have*

$$P_n(S,a) \geq |S|^* \left(P_{n-1}(S,1) |\mathcal{N}(S)|^{n-1} (2|S|^{n-1} - |\mathcal{N}(S)|^{n-1}) + P_{n-1}(S,x)(2|S|^{n-1} - 1) \right).$$

Proof. 1. Since S is an entire antiring, the only way for the permanent of an $n \times n$ matrix to equal zero is that the matrix has either a zero row or a zero column. So, let us count the number of matrices in $M_n(S)$ with no zero rows or columns. Observe that there are $(|S|^n - 1)^n$ matrices that have all rows nonzero. Now, some of them of course may have some zero columns. Suppose therefore that we have at least i zero columns for some $i \in \{1, 2, \ldots, n\}$. We have $\binom{n}{i}$ possible ways to choose the i columns. But if we disregard the zero columns, there are $|S|^{n-i} - 1$ possible ways to choose the remaining elements in every (nonzero) row. Since there are n rows, this yields $(|S|^{n-i} - 1)^n$ matrices. Now, in this way, we may have counted some matrices (with more than i zero columns) multiple times, but the inclusion exclusion principle then yields that there are exactly $\sum_{i=0}^{n} \binom{n}{i} (-1)^i (|S|^{n-i} - 1)^n$ matrices in $M_n(S)$ with no zero rows or columns.

2. Suppose now that $V(S) = \mathcal{N}(S)$ and choose $a \in \mathcal{N}(S)$. Choose $x \in S^*$ and suppose $A_{n-1} \in M_{n-1}(S)$ is such a matrix that $\operatorname{per}(A_{n-1}) = x$. Choose any $y_2, \ldots, y_n \in \mathcal{N}(S)$ and any $z_2, \ldots, z_n \in S$. Suppose that $\alpha \in S$ and let $A = \begin{bmatrix} \alpha & y_2 & \cdots & y_n \\ z_2 & & & \\ \vdots & & A_{n-1} & \\ z_n & & & \end{bmatrix}$. Then $\operatorname{per}(A) = \alpha x + n$ for some $n \in \mathcal{N}(S)$, so $\operatorname{per}(A) = a$ for $\alpha = x^{-1}(a - n)$. Similarly, we have $\operatorname{per}(A') = \alpha x + n$ for matrix $A' = \begin{bmatrix} \alpha & z_2 & \cdots & z_n \\ y_2 & & & \\ \vdots & & A_{n-1} & \\ y_n & & & \end{bmatrix}$, so again $\alpha = x^{-1}(a-n)$ ensures that $\operatorname{per}(A') = a$. This means that we can obtain at least $\sum_{x \in S^*} P_{n-1}(S,x) |\mathcal{N}(S)|^{n-1} (2|S|^{n-1} - |\mathcal{N}(S)|^{n-1})$ matrices in this way. Observe that $P_{n-1}(S,x) \geq P_{n-1}(S,1)$ for every $x \in S^*$, since we can multiply the first row of any matrix with permanent 1 by x to obtain a matrix with permanent x (and this constitutes an injective mapping). Thus, we have at least $|S^*| P_{n-1}(S,1) |\mathcal{N}(S)|^{n-1} (2|S|^{n-1} - |\mathcal{N}(S)|^{n-1})$ matrices with permanent equal to a. On the other hand, we can also obtain matrices with permanent equal to a thusly: suppose that $B_{n-1} \in M_{n-1}(S)$ is such a matrix that $\operatorname{per}(B_{n-1}) = xa$ for some $x \in S^*$ and $y_2, \ldots, y_n$ are arbitrary elements from S. Then matrices $A = \begin{bmatrix} x^{-1} & y_2 & \cdots & y_n \\ 0 & & & \\ \vdots & & B_{n-1} & \\ 0 & & & \end{bmatrix}$ and

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$$A' = \begin{bmatrix} x^{-1} & 0 & \dots & 0 \\ y_2 & & & \\ \vdots & & B_{n-1} & \\ y_n & & & \end{bmatrix}$$ have their permanents equal to a and by a similar argument as above, we have at least $P_{n-1}(S,x)|S^*|(2|S|^{n-1}-1)$ such matrices. Since $x \in S^*$ and $\alpha \in \mathcal{N}(S)$, we have not yet counted any of these matrices above. Thus, the statement follows. ∎

Remark 3.6. *There is no known closed formula for the expression in* (1) *of Proposition* 3.5 *even in the case of the binary Boolean semiring (see* [13]*).*

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