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EXTREMAL SPECTRAL RADII OF UNIFORM SUPERTREES[*]

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Abstract. For a *hypergraph* $\mathcal{G} = (V, E)$ consisting of a nonempty vertex set $V = V(\mathcal{G})$ and an edge set $E = E(\mathcal{G})$, its *adjacency matrix* $\mathcal{A}_{\mathcal{G}} = [(\mathcal{A}_{\mathcal{G}})_{ij}]$ is defined as $(\mathcal{A}_{\mathcal{G}})_{ij} = \sum_{e\in E_{ij}} \frac{1}{|e|-1}$, where $E_{ij} = \{e \in E : i, j \in e\}$. The *spectral radius* of a hypergraph $\mathcal{G}$, denoted by $\rho(\mathcal{G})$, is the maximum modulus among all eigenvalues of $\mathcal{A}_{\mathcal{G}}$. In this paper, we represent some results on the spectral radius changing under some graphic structural perturbations. With these results, among all k-uniform ($k \geq 3$) supertrees with fixed number of vertices, the supertrees with the maximum, the second maximum, and the minimum spectral radius are completely determined, respectively.

Key words. Spectral radius, Supertree, Hypergraph.

AMS subject classifications. 05C50.

1. Introduction.

In the past 20 years, some different connectivity hypermatrices (or tensors) had been defined and been developed to explore spectral hypergraph theory [1–5, 9, 12–16, 18–20, 24, 27–30, 32, 35, 36]. Using different hypermatrices for general hypergraphs, many interesting spectral properties have been studied that many properties of spectral graph theory have been extended to spectral hypergraph theory. A lot of interesting results have emerged and the spectra of hypergraphs have been further studied [6–8, 10, 21, 22, 25, 34, 39–42]. In [2], A. Banerjee introduced an adjacency matrix and use its spectrum so that some spectral and structural properties of hypergraphs are revealed. In this paper, we go on studying the spectra of hypergraphs according to the adjacency matrix introduced in [2].

Now we recall some notations and definitions related to hypergraphs. For a set S, we denote by $|S|$ its cardinality. A *hypergraph* $\mathcal{G} = (V, E)$ consists of a nonempty vertex set $V = V(\mathcal{G})$ and an edge set $E = E(\mathcal{G})$, where each edge $e \in E(\mathcal{G})$ is a subset of $V(\mathcal{G})$ containing at least two vertices. The cardinality $n = |V(\mathcal{G})|$ is called the order; $m = |E(\mathcal{G})|$ is called the edge number of hypergraph $\mathcal{G}$. Denote by t-set a set with size (cardinality) t. We say that a hypergraph $\mathcal{G}$ is *uniform* if its every edge has the same size and call it k-*uniform* if its every edge has size k (i.e., every edge is a k-subset). It is known that a 2-uniform graph is always called a ordinary graph or graph for short.

For a hypergraph $\mathcal{G}$, we define $\mathcal{G} - e$ ($\mathcal{G} + e$) to be the hypergraph obtained from $\mathcal{G}$ by deleting the edge $e \in E(\mathcal{G})$ (by adding a new edge e if $e \notin E(\mathcal{G})$); for an edge subset $B \subseteq E(\mathcal{G})$, we define $\mathcal{G} - B$ to be the hypergraph obtained from $\mathcal{G}$ by deleting each edge $e \in B$; for a vertex subset $S \subseteq V(\mathcal{G})$, we define $\mathcal{G} - S$ to be the hypergraph obtained from $\mathcal{G}$ by deleting all the vertices in S and deleting the edges incident with any vertex in S. For two k-uniform hypergraphs $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$, we say the two graphs are

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isomorphic if there is a bijection f from V_1 to V_2, and there is a bijection g from E_1 to E_2 that maps each edge $\{v_1, v_2, \ldots, v_k\}$ to $\{f(v_1), f(v_2), \ldots, f(v_k)\}$.

In a hypergraph, two vertices are said to be *adjacent* if both of them are contained in an edge. Two edges are said to be *adjacent* if their intersection is not empty. An edge e is said to be *incident* with a vertex v if $v \in e$. The *neighbor set* of vertex v in hypergraph $\mathcal{G}$, denoted by $N_{\mathcal{G}}(v)$, is the set of vertices adjacent to v in $\mathcal{G}$. The *degree* of a vertex v in $\mathcal{G}$, denoted by $deg_{\mathcal{G}}(v)$ (or $deg(v)$ for short), is the number of the edges incident with v. For a hypergraph $\mathcal{G}$, among all of its vertices, we denote by $\Delta(\mathcal{G})$ (or Δ for short) the *maximal degree* and denote by $\delta(\mathcal{G})$ (or δ for short) the *minimal degree*, respectively. A vertex of degree 1 is called a *pendant vertex*. A *pendant edge* is an edge with at most one vertex of degree more than one and other vertices in this edge being all pendant vertices.

In a hypergraph, a *hyperpath* of length q (*q-hyperpath*) is defined to be an alternating sequence of vertices and edges $v_1e_1v_2e_2\cdots v_qe_qv_{q+1}$ such that (1) $v_1, v_2, \ldots, v_{q+1}$ are all distinct vertices; (2) $e_1, e_2, \ldots, e_q$ are all distinct edges; (3) $v_i, v_{i+1} \in e_i$ for $i = 1, 2, \ldots, q$; (4) $e_i \cap e_{i+1} = v_{i+1}$ for $i = 1, 2, \ldots, q-1$; (5) $e_i \cap e_j = \emptyset$ if $|i-j| \geq 2$. If there is no discrimination, a hyperpath is sometimes written as $e_1e_2\cdots e_{q-1}e_q$, $e_1v_2e_2\cdots v_qe_q$ or $v_1e_1v_2e_2\cdots v_qe_q$. A *hypercycle* of length q (*q-hypercycle*) $v_1e_1v_2e_2\cdots v_{q-1}e_{q-1}v_qe_qv_1$ is obtained from a hyperpath $v_1e_1v_2e_2\cdots v_{q-1}e_{q-1}v_q$ by adding a new edge e_q between v_1 and v_q where $e_q \cap e_1 = \{v_1\}$, $e_q \cap e_{q-1} = \{v_q\}$, $e_q \cap e_j = \emptyset$ if $j \neq 1, q-1$, and $|q-j| \geq 2$. The length of a hyperpath P (or a hypercycle C), denoted by $L(P)$ (or $L(C)$), is the number of the edges in P (or C). A hypergraph $\mathcal{G}$ is connected if there exists a hyperpath from v to u for all $v, u \in V$, and $\mathcal{G}$ is called *acyclic* if it contains no hypercycle.

Recall that a tree is an ordinary graph which is 2-uniform, connected, and acyclic. A *supertree* is similarly defined to be a hypergraph which is both connected and acyclic. Clearly, in a supertree, its each pair of the edges have at most one common vertex. Therefore, the edge number of a k-uniform supertree of order n is $m = \frac{n-1}{k-1}$.

Let $G = (V, E)$ be an ordinary graph (2-uniform). For every $k \geq 3$, the kth power of G, denoted by $G^k = (V^k, E^k)$, is defined as the k-uniform hypergraph with the edge set $E^k = \{e \cup \{v_{e_1}, v_{e_2}, \ldots, v_{e_{k-2}}\} : e \in E\}$ and the vertex set $V^k = V \cup (\cup_{e\in E}\{v_{e_1}, v_{e_2}, \ldots, v_{e_{k-2}}\})$, where $V \cap (\cup_{e\in E}\{v_{e_1}, v_{e_2}, \ldots, v_{e_{k-2}}\}) = \emptyset$, $\{v_{e_1}, v_{e_2}, \ldots, v_{e_{k-2}}\} \cap \{v_{f_1}, v_{f_2}, \ldots, v_{f_{k-2}}\} = \emptyset$ for $e \neq f$, $e, f \in E$. The kth power of an ordinary tree is called a *hypertree*. Obviously, a hypertree is a supertree.

Denote by $\mathcal{P}(n,k)$ the k-uniform hyperpath of order n. A k-uniform *superstar* of order n, denoted by $\mathcal{S}^*(n,k)$ (see Fig. 1), is a supertree in which all edges intersect at just one common vertex. A k-uniform *double hyperstar* of order n, denote by $\mathcal{S}(n,k;l_1,l_2)$ where $l_1, l_2 \geq 1$ (see Fig. 1), is a supertree obtained by attaching l_1 pendant edges at vertex u_1 of an edge e and attaching l_2 pendant edges at the other vertex u_2 of edge e, where $u_1 \neq u_2$.

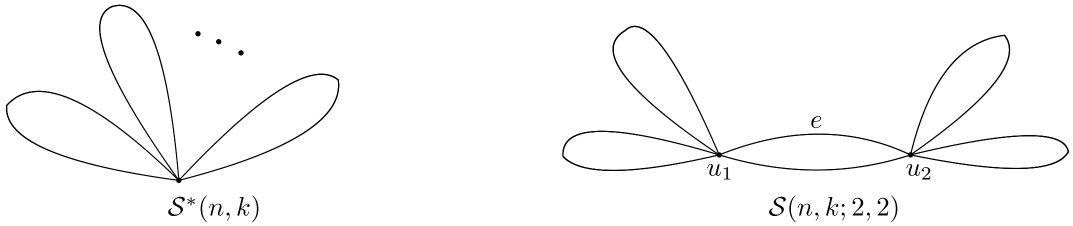

Fig. 1. $\mathcal{S}^*(n,k)$ and $\mathcal{S}(n;2,2)$.

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Let $E_{ij} = \{e \in E : i, j \in e\}$. The *adjacency matrix* $\mathcal{A}_{\mathcal{G}} = [(\mathcal{A}_{\mathcal{G}})_{ij}]$ of a hypergraph $\mathcal{G}$ is defined as:

$$(\mathcal{A}_{\mathcal{G}})_{ij} = \sum_{e \in E_{ij}} \frac{1}{|e| - 1}.$$

It is easy to find that $\mathcal{A}_{\mathcal{G}}$ is symmetric if there is no requirement for direction on hypergraph $\mathcal{G}$ and find that $\mathcal{A}_{\mathcal{G}}$ is very convenient to be used to investigate the spectum of a hypergraph even without the requirement for edge uniformity. The *spectral radius* $\rho(\mathcal{G})$ of a hypergraph $\mathcal{G}$ is defined to be the spectral radius $\rho(\mathcal{A}_{\mathcal{G}})$, which is the maximum modulus among all eigenvalues of $\mathcal{A}_{\mathcal{G}}$. In spectral theory of hypergraphs, the spectral radius is an index that attracts much attention due to its fine properties [3,4,7,8,17,20,22,25,33,35,37,41].

We assume that the hypergraphs throughout this paper are simple, that is, $e_i \neq e_j$ if $i \neq j$, and assume the hypergraphs throughout this paper are undirected. In this paper, we represent some results on the spectral radius changing under some graphic structural perturbations. With these results, among all k-uniform ($k \geq 3$) supertrees with fixed number of vertices, the supertrees with the maximum, the second maximum, and the minimum spectral radius are completely determined, respectively, getting the following result:

Theorem 1.1. *Let $\mathcal{G}$ be a k-uniform ($k \geq 3$) supertree of order n. Then $\rho(\mathcal{G}) \leq \rho(\mathcal{S}^*(n, k))$ with equality if and only if $\mathcal{G} \cong \mathcal{S}^*(n, k)$.*

Theorem 1.2. *Let $\mathcal{G}$ be a k-uniform ($k \geq 3$) supertree of order n and with $m(\mathcal{G}) \geq 3$ satisfying that $\mathcal{G} \ncong \mathcal{S}^*(n, k)$. Then $\rho(\mathcal{G}) \leq \rho(\mathcal{S}(n, k; \frac{n-1}{k-1} - 2, 1))$ with equality if and only if $\mathcal{G} \cong \mathcal{S}(n, k; \frac{n-1}{k-1} - 2, 1)$.*

Theorem 1.3. *Let $\mathcal{G}$ be a k-uniform ($k \geq 3$) supertree of order n. Then $\rho(\mathcal{P}(n, k)) \leq \rho(\mathcal{G})$ with equality if and only if $\mathcal{G} \cong \mathcal{P}(n, k)$.*

Corollary 1.4. *Suppose T^k ($k \geq 3$) of order n is the kth power of ordinary tree T. Then*

(1) *$\rho(T^k) \leq \mathcal{S}^*(n, k)$ with equality if and only if $T^k \cong \mathcal{S}^*(n, k)$.*

(2) *$\rho(\mathcal{P}(n, k)) \leq \rho(T^k)$ with equality if and only if $T^k \cong \mathcal{P}(n, k)$.*

The layout of this paper is as follows: Section 2 introduces some basic knowledge and working lemmas; Section 3 represents our results.

2. Preliminary. For the requirements in the narrations afterward, we need some prepares. For a hypergraph $\mathcal{G}$ with vertex set $\{v_1, v_2, \ldots, v_n\}$, and $X = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^T \in R^n$ on $\mathcal{G}$ is a vector that entry x_{v_i} is mapped to vertex v_i for $1 \leq i \leq n$.

From [26], by the famous Perron–Frobenius theorem, for $\mathcal{A}_{\mathcal{G}}$ of a connected uniform hypergraph $\mathcal{G}$ of order n, we know that there is one unique positive eigenvector $X = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^T \in R^n_{++}$ (R^n_{++} means the set of positive real vectors of dimension n) corresponding to $\rho(\mathcal{G})$, where $\sum_{i=1}^{n} x_{v_i}^2 = 1$ and each entry x_{v_i} is mapped to each vertex v_i for $1 \leq i \leq n$. We call such an eigenvector X the *principal eigenvector* of $\mathcal{G}$.

Let A be an irreducible nonnegative $n \times n$ real matrix (with every entry being real number) with spectral radius $\rho(A)$. The following extremal representation (Rayleigh quotient) will be useful:

$$\rho(A) = \max_{X \in R^n, X \neq 0} \frac{X^T A X}{X^T X},$$

and if a vector X satisfies that $\frac{X^T A X}{X^T X} = \rho(A)$, then $AX = \rho(A)X$.

Lemma 2.1. *Let A be an irreducible nonnegative square real matrix with order n and spectral radius ρ, $Y \in R_+^n \setminus \{0\}^n$ be a nonnegative vector (R_+^n means the set of nonnegative real vectors of dimension n, $\{0\}^n = \{(0,0,\dots,0)^T\}$). If $AY \geq \rho Y$, then $AY = \rho Y$.*

Proof. Note the relation between the spectral radius and Rayleigh quotient for an irreducible nonnegative square real matrix. It follows that $\frac{Y^TAY}{Y^TY} = \rho$, and $AY = \rho Y$. Thus, the result follows. This completes the proof. $\square$

Lemma 2.2. [38] *Let A be an irreducible nonnegative square symmetric real matrix with order n and spectral radius ρ, $Y \in R_+^n \setminus \{0\}^n$ be a nonnegative vector. If there exists $r \in R_+$ such that $AY \leq rY$, then $\rho \leq r$. Similarly, if there exists $r \in R_+$ such that $AY \geq rY$, then $\rho \geq r$.*

3. Main results. Let X be an eigenvector of a connected k-uniform hypergraph G. For the simplicity, we let $x_e = \sum\limits_{i<j, v_i, v_j \in e} x_{v_i}x_{v_j}$ for an edge $e = \{v_1, v_2, \dots, v_k\}$.

Lemma 3.1. *Let $e_1 = \{v_{1,1}, v_{1,2}, \dots, v_{1,k}\}$, $e_2 = \{v_{2,1}, v_{2,2}, \dots, v_{2,k}\}$, $\dots$, $e_j = \{v_{j,1}, v_{j,2}, \dots, v_{j,k}\}$ be some edges in a connected k-uniform hypergraph $\mathcal{G}$; $v_{u,1}, v_{u,2}, \dots, v_{u,t}$ be vertices in $\mathcal{G}$ that $t < k$. For $1 \leq i \leq j$, $\{v_{u,1}, v_{u,2}, \dots, v_{u,t}\} \nsubseteq e_i$, $e_i' = (e_i \setminus \{v_{i,1}, v_{i,2}, \dots, v_{i,t}\}) \cup \{v_{u,1}, v_{u,2}, \dots, v_{u,t}\}$ satisfying that $e_i' \notin E(\mathcal{G})$. Let $\mathcal{G}' = \mathcal{G} - \sum e_i + \sum e_i'$. If in the principal eigenvector X of $\mathcal{G}$, for $1 \leq i \leq j$, $x_{v_{i,1}} \leq x_{v_{u,1}}$, $x_{v_{i,2}} \leq x_{v_{u,2}}$, $\dots$, and $x_{v_{i,t}} \leq x_{v_{u,t}}$, then $\rho(\mathcal{G}') > \rho(\mathcal{G})$.*

Proof. Note that $X^T(\mathcal{A}_{\mathcal{G}'} - \mathcal{A}_{\mathcal{G}})X = \frac{2}{k-1}\sum(x_{e_i'} - x_{e_i}) \geq 0$. It follows that $\rho(\mathcal{G}') \geq \rho(\mathcal{G})$. If $\rho(\mathcal{G}') = \rho(\mathcal{G})$, then $\rho(\mathcal{G}') = X^T\mathcal{A}_{\mathcal{G}'}X = X^T\mathcal{A}_{\mathcal{G}}X = \rho(\mathcal{G})$. It follows that $\mathcal{A}_{\mathcal{G}'}X = \rho(\mathcal{G}')X = \rho(\mathcal{G})X = \mathcal{A}_{\mathcal{G}}X$. Without loss of generality, suppose $v_{u,1} \notin e_1$. Then $(\mathcal{A}_{\mathcal{G}'}X)_{v_{u,1}} - (\mathcal{A}_{\mathcal{G}}X)_{v_{u,1}} \geq \frac{x_{v_{1,k}}}{k-1} > 0$, which contradicts $\mathcal{A}_{\mathcal{G}'}X = \mathcal{A}_{\mathcal{G}}X$. Consequently, it follows that $\rho(\mathcal{G}') > \rho(\mathcal{G})$. This completes the proof. $\square$

Proof of Theorem 1.1. Suppose $\mathcal{T}$ is a k-uniform supertree of order n satisfying that $\rho(\mathcal{T}) = \max\{\rho(\mathcal{G}) : \mathcal{G}$ is a k-uniform supertree of order $n\}$. Let X be the principal eigenvector of $\mathcal{T}$ and $x_u = \max\{x_v : v \in V(\mathcal{T})\}$. Suppose that $\mathcal{T} \ncong \mathcal{S}^*(n,k)$. Then in $\mathcal{T}$, there exist edges not incident with vertex u. Suppose e' is not incident with vertex u. Note that $\mathcal{T}$ is connected. Then there is a hyperpath $P = ue_1v_1e_2v_2\cdots v_te'$ from u to e'. Let $e_2' = (e_2 \setminus \{v_1\}) \cup \{u\}$, and $\mathcal{T}' = \mathcal{T} - e_2 + e_2'$. Then by Lemma 3.1, it follows that $\rho(\mathcal{T}') > \rho(\mathcal{T})$, which contradicts the maximality of $\rho(\mathcal{T})$. Hence, it follows that $\mathcal{T} \cong \mathcal{S}^*(n,k)$. This completes the proof. $\square$

Proof of Theorem 1.2. Let $\Lambda = \{\mathcal{G} : \mathcal{G}$ be a k-uniform supertree of order n and $\mathcal{G} \ncong \mathcal{S}^*(n,k)\}$. Suppose $\mathcal{T} \in \Lambda$ satisfies that $\rho(\mathcal{T}) = \max\{\rho(\mathcal{G}) : \mathcal{G} \in \Lambda\}$. Let X be the principal eigenvector of $\mathcal{T}$ and $x_u = \max\{x_v : v \in V(\mathcal{T})\}$. Note that $\mathcal{T} \ncong \mathcal{S}^*(n,k)$, and $\mathcal{T}$ is connected. Then in $\mathcal{T}$, there exist a hyperpath $P = ue_1v_1e_2$.

Assertion. Except edges e_1, e_2, any one of other edges is incident with vertex u.

Otherwise, suppose one edge e_t is not incident with vertex u. Note that $\mathcal{T}$ is connected. Then there is a hyperpath $\mathcal{P} = ue_{a_1}v_{a_1}e_{a_2}v_{a_2}\cdots v_{a_t}e_t$ from u to e_t.

Claim 1. If $e_1 \in E(\mathcal{P})$, then $e_{a_1} = e_1$.

Otherwise, suppose $e_{a_i} = e_1$ where $1 < i \leq t$. Then $ue_{a_1}v_{a_1}e_{a_2}v_{a_2}\cdots v_{a_i}e_{a_i}u$ contains cycle, which contradicts that $\mathcal{T}$ is a supertree.

In the same way, we get the following Claim 2.

Claim 2. If $e_2 \in E(\mathcal{P})$, then $e_{a_1} = e_1$, $e_{a_2} = e_2$.

Therefore, there three cases to consider, which are (1) $e_{a_1} = e_1$, $e_2 \notin E(\mathcal{P})$; (2) $e_{a_1} = e_1$, $e_{a_2} = e_2$; and (3) $e_1 \notin E(\mathcal{P})$, $e_2 \notin E(\mathcal{P})$. For the case that $e_{a_1} = e_1$, $e_2 \notin E(\mathcal{P})$, let $e'_{a_2} = (e_{a_2} \setminus \{v_{a_1}\}) \cup \{u\}$, and $\mathcal{T}' = \mathcal{T} - e_{a_2} + e'_{a_2}$, where $\mathcal{T}' \in \Lambda$. Then by Lemma 3.1, it follows that $\rho(\mathcal{T}') > \rho(\mathcal{T})$, which contradicts the maximality of $\rho(\mathcal{T})$. In the same way, for the cases (2) and (3), we can get a supertree $\mathcal{T}'$ where $\mathcal{T}' \in \Lambda$, such that $\rho(\mathcal{T}') > \rho(\mathcal{T})$ which contradicts the maximality of $\rho(\mathcal{T})$. Thus, our assertion holds.

From the above assertion, it follows that $\mathcal{T} \cong \mathcal{S}(n, k; \frac{n-1}{k-1} - 2, 1)$. This completes the proof. $\square$

Lemma 3.2. [23] *Let A be an irreducible nonnegative square matrix with order n and spectral radius ρ. Let s_i^A be the ith row sum, $s_A = \min\{s_i^A : 1 \le i \le n\}$, and $S_A = \max\{s_i^A : 1 \le i \le n\}$. Then $s_A \le \rho \le S_A$ with either one equality if and only if A is regular (all of the row sums of A are equal).*

From Lemma 3.2, combining with hypergraph, we can get the following corollary naturally.

Corollary 3.3. *For a connected hypergraph $\mathcal{G}$, we have $\delta \le \rho(\mathcal{G}) \le \Delta$ with either one equality if and only if $\mathcal{G}$ is regular, where δ is the minimum degree and Δ is the maximum degree.*

Using Lemma 3.2, we can get an improvement for Lemma 2.2.

Lemma 3.4. *Let A be an irreducible nonnegative square symmetric real matrix with order n and spectral radius ρ, $y \in R_{++}^n$ be a positive vector. If there exists $r \in R_+$ such that $Ay \le ry$, then $\rho \le r$ with equality if and only if $Ay = ry$. Similarly, if there exists $r \in R_+$ such that $Ay \ge ry$, then $\rho \ge r$ with equality if and only if $Ay = ry$.*

Proof. Using Lemma 2.2 gets that $\rho \le r$ if $Ay \le ry$; $\rho \ge r$ if $Ay \ge ry$. Next, we prove the conclusion for $\rho = r$.

We first prove the conclusion that $\rho = r$ if and only if $Ay = ry$ under the condition $Ay \le ry$. Suppose $y = (y_1, y_2, \dots, y_n)^T$. Let

$$B = \begin{pmatrix} \frac{1}{y_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{y_2} & 0 & \vdots & 0 \\ \vdots & & & & \\ 0 & 0 & \vdots & 0 & \frac{1}{y_n} \end{pmatrix} A \begin{pmatrix} y_1 & 0 & 0 & \cdots & 0 \\ 0 & y_2 & 0 & \vdots & 0 \\ \vdots & & & & \\ 0 & 0 & \vdots & 0 & y_n \end{pmatrix}.$$

Denote by $\rho(B)$ the spectral radius of B. Note that the eigenvalues of B are the same to the eigenvalues of A; $\rho = r$ means $\rho(B) = r$. Note that $Ay \le ry$ means $S_B \le r$; $\rho(B) = r$ means all of the row sums of B equals r by Lemma 3.2, which implies that $Ay = ry$. As a result, it follows that under the condition that $Ay \le ry$, if $\rho = r$, then $Ay = ry$. Conversely, if $Ay = ry$, then all of the row sums of B equals r, and then $\rho(B) = r = \rho$.

In the same way, we get that $\rho = r$ if and only if $Ay = ry$ under the condition that $Ay \ge ry$. This completes the proof. $\square$

Lemma 3.5. (1) *Suppose $c > 0$, $d > 0$, $a - c > 0$, $b - d > 0$. If $\frac{a}{b} \ge \frac{c}{d}$, then $\frac{a-c}{b-d} \ge \frac{a}{b}$ with equality if and only if $\frac{a}{b} = \frac{c}{d}$. Moreover, if $\frac{a}{b} > \frac{c}{d}$, then $\frac{a-c}{b-d} > \frac{a}{b}$.*

(2) *Suppose $c > 0$, $d > 0$, $a - c > 0$, $b - d > 0$. If $\frac{a}{b} \ge \frac{c}{d}$, then $\frac{a+c}{b+d} \le \frac{a}{b}$ with equality if and only if $\frac{a}{b} = \frac{c}{d}$. Moreover, if $\frac{a}{b} > \frac{c}{d}$, then $\frac{a+c}{b+d} < \frac{a}{b}$.*

(3) *Suppose $\frac{a}{b} \ge 1$, $b > c > 0$. Then $\frac{a-c}{b-c} \ge \frac{a}{b}$.*

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Proof. (1) From $\frac{a}{b} \geq \frac{c}{d}$, it follows that $ab - bc \geq ab - ad$, which induces $\frac{a-c}{b-d} \geq \frac{a}{b}$. In the same way, we get that $\frac{a-c}{b-d} > \frac{a}{b}$ if $\frac{a}{b} > \frac{c}{d}$. Then (1) follows.

(2) is proved as (1). (3) is a corollary following from (1). This completes the proof. □

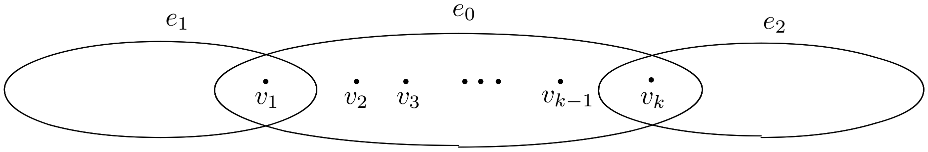

Fig. 2. e_0, e_1, e_2 in $\mathcal{G}$.

Lemma 3.6. *Let $\mathcal{G}$ be a hypergraph with spectral radius ρ, e_0, e_1, e_2 be three edges in $\mathcal{G}$ with $e_0 = \{v_1, v_2, \ldots, v_{k-1}, v_k\}$, satisfying that $deg_{\mathcal{G}}(v_2) = deg_{\mathcal{G}}(v_3) = \cdots deg_{\mathcal{G}}(v_{k-1}) = 1$ $(k \geq 3)$, $e_1 \cap e_0 = \{v_1\}$, $e_2 \cap e_0 = \{v_k\}$ (see Fig. 2). Let X be the principal eigenvector of hypergraph $\mathcal{G}$. Then $x_{v_2} = x_{v_3} = \cdots = x_{v_{k-1}} = \frac{x_{v_1}+x_{v_k}}{(k-1)\rho-(k-3)} < \min\{x_{v_1}, x_{v_k}\}$.*

Proof. For $2 \leq i \leq k-1$, we prove $x_{v_i} < \min\{x_{v_1}, x_{v_k}\}$ by contradiction. Suppose that $\min\{x_{v_1}, x_{v_k}\} = x_{v_1}$, and $x_{v_z} \geq \min\{x_{v_1}, x_{v_k}\}$ for some $2 \leq z \leq k-1$. Let $e_1' = (e_1 \setminus \{v_1\}) \cup \{v_z\}$ and $\mathcal{G}_1 = \mathcal{G} - e_1 + e_1'$. Using Lemma 3.1 gets $\rho(\mathcal{G}_1) > \rho(\mathcal{G})$. But it contradicts $\rho(\mathcal{G}_1) = \rho(\mathcal{G})$ because $\mathcal{G}_1 \cong \mathcal{G}$. As a result, for $2 \leq i \leq k-1$, it follows that $x_{v_i} < \min\{x_{v_1}, x_{v_k}\}$.

Note that $\rho x_{v_2} = \frac{1}{k-1}(x_{v_1} + x_{v_3} + \sum_{i=4}^{k} x_{v_i})$, $\rho x_{v_3} = \frac{1}{k-1}(x_{v_1} + x_{v_2} + \sum_{i=4}^{k} x_{v_i})$. It follows that $(\rho + \frac{1}{k-1})(x_{v_2} - x_{v_3}) = 0$. Note that $\rho > 1$ by Corollary 3.3. Then we get $x_{v_2} = x_{v_3}$. Proceeding like this, we get that $x_{v_2} = x_{v_3} = \cdots = x_{v_{k-1}}$. Thus from $\rho x_{v_2} = \frac{1}{k-1}((k-3)x_{v_3} + x_{v_1} + x_{v_k})$, it follows that $x_{v_2} = \frac{x_{v_1}+x_{v_k}}{(k-1)\rho-(k-3)}$. Thus, the result follows. This completes the proof. □

Simillar to Lemma 3.6, we get the following Lemma 3.7.

Lemma 3.7. *Let $\mathcal{G}$ be a hypergraph with spectral radius ρ, $e = \{u, v_1, v_2, \ldots, v_{k-1}\}$ be a pendant edge in $\mathcal{G}$ $(k \geq 2)$, where $deg_{\mathcal{G}}(u) \geq 2$. Then in the principal eigenvector X of $\mathcal{G}$, $x_{v_1} = x_{v_2} = \cdots = x_{v_{k-1}} = \frac{x_u}{(k-1)\rho-(k-2)} < x_u$.*

Lemma 3.8. *Suppose $\mathcal{G}$ is a connected hypergraph with spectral radius ρ and principal eigenvector X. e_1, e_2, e_3, e_4 are edges in $\mathcal{G}$, where $|e_1|, |e_2|, |e_3|, |e_4| \geq 3$, $e_1 \cap e_2 = \{v_1\}$, $e_1 \cap e_4 = \{v_2\}$, $e_2 \cap e_3 = \{v_3\}$, $deg_{\mathcal{G}}(v_1) = deg_{\mathcal{G}}(v_2) = deg_{\mathcal{G}}(v_3) = 2$, $deg_{\mathcal{G}}(v) = 1$ for $v \in (e_1 \cup e_2) \setminus \{v_1, v_2, v_3\}$.*

(1) *Let $e' \subset (e_1 \cup e_2)$ satisfy that $\{v_2, v_3\} \subseteq e'$, $e' \notin E(\mathcal{G})$. Let $\mathcal{G}_0 = \mathcal{G} - e_1 - e_2 + e'$ and $t = |e'|$ (see Fig. 3).*

(1.1) *If $t \geq \max\{|e_1|, |e_2|\}$, $x_{v_1} \geq x_{v_2}$, $x_{v_1} \geq x_{v_3}$, then $\rho(\mathcal{G}_0) \leq \rho(\mathcal{G})$ with equality if and only if $|e'| = |e_1| = |e_2|$ and $x_{v_1} = x_{v_2} = x_{v_3}$. Moreover, if $t > \max\{|e_1|, |e_2|\}$, $x_{v_1} \geq x_{v_2}$, $x_{v_1} \geq x_{v_3}$, then $\rho(\mathcal{G}_0) < \rho(\mathcal{G})$.*

(1.2) *If $t \leq \max\{|e_1|, |e_2|\}$, $x_{v_1} \leq x_{v_2}$, $x_{v_1} \leq x_{v_3}$, then $\rho(\mathcal{G}_0) \geq \rho(\mathcal{G})$ with equality if and only if $|e'| = |e_1| = |e_2|$ and $x_{v_1} = x_{v_2} = x_{v_3}$. Moreover, if $t < \max\{|e_1|, |e_2|\}$, $x_{v_1} \leq x_{v_2}$, $x_{v_1} \leq x_{v_3}$, then $\rho(\mathcal{G}_0) > \rho(\mathcal{G})$.*

(2) *Let $e_1' = (e_1 \setminus \{v_1\}) \cup \{u\}$, $e_2' = (e_2 \setminus \{v_1\}) \cup \{u\}$, $e' = \{v_1, u_1, u_2, \ldots, u_{t-2}, u\}$ where $u \notin V(\mathcal{G})$, $u_i \notin V(\mathcal{G})$ for $1 \leq i \leq t-2$, $\mathcal{G}_1 = \mathcal{G} - e_1 + e_1' + e'$, $\mathcal{G}_2 = \mathcal{G} - e_2 + e_2' + e'$ (see Fig. 3).*

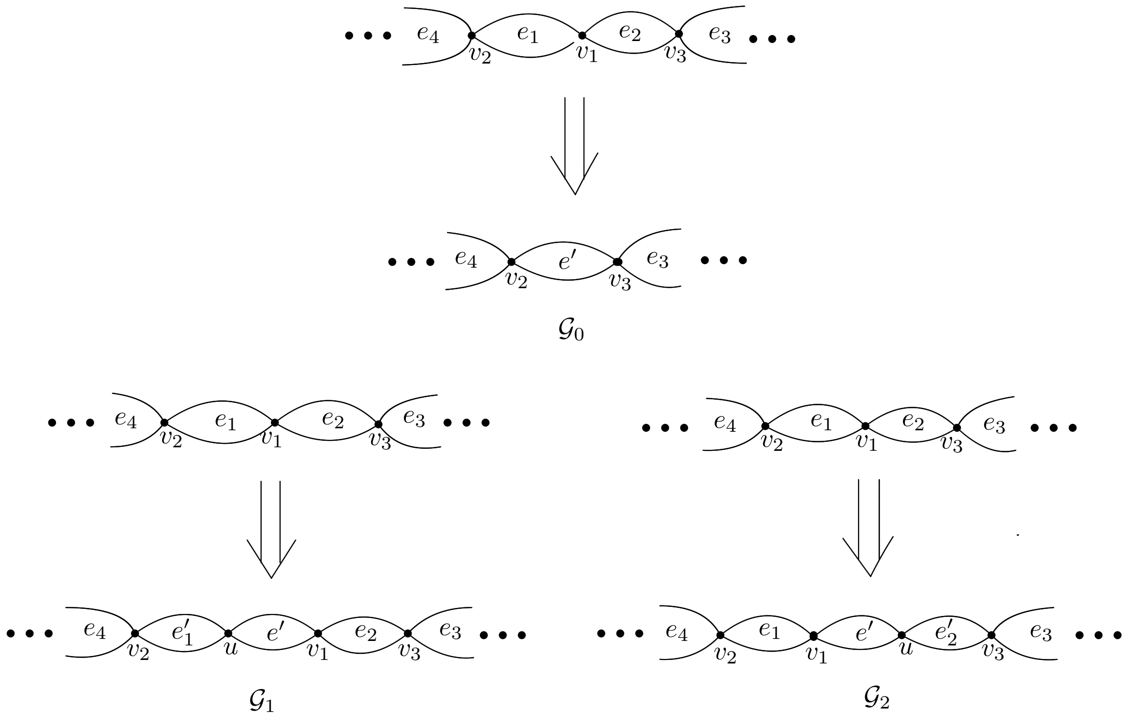

Fig. 3. $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$.

(2.1) *If* $t \le \min\{|e_1|, |e_2|\}$, $x_{v_1} \ge x_{v_2}$, $x_{v_1} \ge x_{v_3}$, *then* $\rho(\mathcal{G}_1) \ge \rho(\mathcal{G})$, $\rho(\mathcal{G}_2) \ge \rho(\mathcal{G})$ *with either equality holding if and only if* $|e'| = |e_1| = |e_2|$, $x_{v_1} = x_{v_2} = x_{v_3} = x_u$ *and* $x_z = x_\omega$ *for* $z, \omega \in (e_1 \cup e_2 \cup e') \setminus \{v_1, v_2, u\}$. *Moreover, if* $t < \min\{|e_1|, |e_2|\}$, $x_{v_1} \ge x_{v_2}$, $x_{v_1} \ge x_{v_3}$, *then* $\rho(\mathcal{G}_1) > \rho(\mathcal{G})$, $\rho(\mathcal{G}_2) > \rho(\mathcal{G})$.

(2.2) *If* $t \ge \max\{e_1, e_2\}$, $x_{v_1} \le x_{v_2}$, $x_{v_1} \le x_{v_3}$, *then* $\rho(\mathcal{G}_1) \le \rho(\mathcal{G})$, $\rho(\mathcal{G}_2) \le \rho(\mathcal{G})$ *with either equality holding if and only if* $|e'| = |e_1| = |e_2|$, $x_{v_1} = x_{v_2} = x_{v_3} = x_u$ *and* $x_z = x_\omega$ *for* $z, \omega \in (e_1 \cup e_2 \cup e') \setminus \{v_1, v_2, u\}$. *Moreover, if* $t > \max\{e_1, e_2\}$, $x_{v_1} \le x_{v_2}$, $x_{v_1} \le x_{v_3}$, *then* $\rho(\mathcal{G}_1) < \rho(\mathcal{G})$, $\rho(\mathcal{G}_2) < \rho(\mathcal{G})$.

Proof. (1.1) Suppose $e_1 = \{v_1, v_{\alpha(1,1)}, v_{\alpha(1,2)}, \ldots, v_{\alpha(1,j_1-2)}, v_2\}$, $e_2 = \{v_1, v_{\alpha(2,1)}, v_{\alpha(2,2)}, \ldots, v_{\alpha(2,j_2-2)}, v_3\}$. By Lemma 3.6, we have $x_{v_{\alpha(1,w)}} = x_{v_{\alpha(1,z)}} < \min\{x_{v_1}, x_{v_2}\}$ for $1 \le w < z \le j_1 - 2$, $x_{v_{\alpha(2,w)}} = x_{v_{\alpha(2,z)}} < \min\{x_{v_1}, x_{v_3}\}$ for $1 \le w < z \le j_2 - 2$. Let Y be a vector on $\mathcal{G}_0$ satisfying that

$$\begin{cases} y_v = \min\{x_z : z \in (e_1 \cup e_2) \setminus \{v_1, v_2, v_3\}\}, & v \in e' \setminus \{v_2, v_3\} \\ y_v = x_v, & \textit{others}. \end{cases}$$

Note that $|e'| \ge \max\{|e_1|, |e_2|\}$, $x(v_1) \ge x(v_2)$, $x(v_1) \ge x(v_3)$. Without loss of generality, suppose $\min\{x_v : v \in (e_1 \cup e_2)\} = x_{v_{\alpha(2,1)}}$. For $v \in (e' \setminus \{v_2, v_3\})$, noting that $deg_{\mathcal{G}_0}(v) = 1$ and $x_{v_{\alpha(2,1)}} < \min\{x_{v_1}, x_{v_3}\}$, we have

$$\begin{aligned} (\mathcal{A}_{\mathcal{G}_0} Y)_v = \frac{(t-3)y_v + y_{v_2} + y_{v_3}}{t-1} &= \frac{(t-3)x_{v_{\alpha(2,1)}} + x_{v_2} + x_{v_3}}{t-1} \\ &= \frac{(j_2-3)x_{v_{\alpha(2,1)}} + x_{v_2} + x_{v_3} + (t-j_2)x_{v_{\alpha(2,1)}}}{j_2 - 1 + t - j_2} \\ &\le \frac{(j_2-3)x_{v_{\alpha(2,1)}} + x_{v_1} + x_{v_3} + (t-j_2)x_{v_{\alpha(2,1)}}}{j_2 - 1 + t - j_2} \end{aligned}$$

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$$\begin{aligned}&\le \frac{(j_2-3)x_{v_{\alpha(2,1)}}+x_{v_1}+x_{v_3}}{j_2-1} && (by\ Lemma\ 3.5)\\ &=\rho x_{v_{\alpha(2,1)}}=\rho y_v.\end{aligned}$$

In the same way, we get

$$(\mathcal{A}_{\mathcal{G}_0}Y)_{v_2}=\frac{(t-2)x_{v_{\alpha(2,1)}}+y_{v_3}}{t-1}=\frac{(t-2)x_{v_{\alpha(2,1)}}+x_{v_3}}{t-1}\le \rho x_{v_2}=\rho y_{v_2};$$

$$(\mathcal{A}_{\mathcal{G}_0}Y)_{v_3}=\frac{(t-2)x_{v_{\alpha(2,1)}}+y_{v_2}}{t-1}=\frac{(t-2)x_{v_{\alpha(2,1)}}+x_{v_2}}{t-1}\le \rho x_{v_3}=\rho y_{v_3};$$

for $v\in(V(\mathcal{G}_0)\setminus e')$, $(\mathcal{A}_{\mathcal{G}_0}Y)_v=(\mathcal{A}_{\mathcal{G}}X)_v=\rho y_v$. By Lemma 2.2, it follows that $\rho(\mathcal{G}_0)\le\rho(\mathcal{G})$. Note that Y is positive. Combining Lemma 3.4, we find that if $\rho(\mathcal{G}_0)=\rho(\mathcal{G})$, then $\mathcal{A}_{\mathcal{G}_0}Y=\rho Y$. Thus, it follows that $|e'|=|e_1|=|e_2|$ and $x_{v_1}=x_{v_2}=x_{v_3}$. Conversely, if $|e'|=|e_1|=|e_2|$ and $x_{v_1}=x_{v_2}=x_{v_3}$, then it can be checked as above that for $v\in(e'\setminus\{v_2,v_3\})$, $(\mathcal{A}_{\mathcal{G}_0}Y)_v=\rho y_v$; $(\mathcal{A}_{\mathcal{G}_0}Y)_{v_2}=\rho y_{v_2}$; $(\mathcal{A}_{\mathcal{G}_0}Y)_{v_3}=\rho y_{v_3}$; for $v\in(V(\mathcal{G}_0)\setminus e')$, $(\mathcal{A}_{\mathcal{G}_0}Y)_v=(\mathcal{A}_{\mathcal{G}}X)_v=\rho y_v$. Thus, it follows that $\rho(\mathcal{G}_0)=\rho(\mathcal{G})$.

If $t>\max\{|e_1|,|e_2|\}$, $x_{v_1}\ge x_{v_2}$, $x_{v_1}\ge x_{v_3}$, combining Lemma 3.5, as the above proof, we get $\rho(\mathcal{G}_0)<\rho(\mathcal{G})$.

From the above narrations, then (1.1) follows as desired.

(1.2) Let Y be a vector on $\mathcal{G}_0$ satisfying that

$$\begin{cases} y(v)=\max\{x_z : z\in(e_1\cup e_2)\setminus\{v_1,v_2,v_3\}\}, & v\in e'\setminus\{v_2,v_3\}\\ y(v)=x_v, & others.\end{cases}$$

Then (1.2) is proved as (1.1).

(2.1) For both $\mathcal{G}_1$ and $\mathcal{G}_2$, let Y be a vector satisfying that

$$\begin{cases} y(u)=x_{v_1} & \\ y(v)=\max\{x_z : z\in(e_1\cup e_2)\setminus\{v_1,v_2,v_3\}\}, & v\in e'\setminus\{v_1,u\}\\ y(v)=x_v, & others.\end{cases}$$

Then (2.1) is proved as (1.1).

(2.2) For both $\mathcal{G}_1$ and $\mathcal{G}_2$, let Y be a vector satisfying that

$$\begin{cases} y(u)=x_{v_1} & \\ y(v)=\min\{x_z : z\in(e_1\cup e_2)\setminus\{v_1,v_2,v_3\}\}, & v\in e'\setminus\{v_1,u\}\\ y(v)=x_v, & others.\end{cases}$$

Then (2.2) is proved as (1.1).

Thus the result follows. This completes the proof. □

Lemma 3.9. *Let e be a new edge not containing in connected hypergraph $\mathcal{G}$. Let $\mathcal{G}'=\mathcal{G}+e$. If $\mathcal{G}'$ is also connected, then $\rho(\mathcal{G}')>\rho(\mathcal{G})$.*

Proof. Let X be the *principal eigenvector* of $\mathcal{G}$, Y be a vector on $\mathcal{G}'$ satisfying that

$$\begin{cases} y_v=x_v, & v\in V(\mathcal{G})\\ \\ y_v=0, & others.\end{cases}$$

Then $Y^T\mathcal{A}_{\mathcal{G}'}Y - X^T\mathcal{A}_{\mathcal{G}}X \geq 0$, $Y^TY = X^TX$. It follows that $\rho(\mathcal{G}') \geq \rho(\mathcal{G})$. Suppose that $\rho(\mathcal{G}') = \rho(\mathcal{G})$. Then $\rho(\mathcal{G}') = Y^T\mathcal{A}_{\mathcal{G}'}Y = X^T\mathcal{A}_{\mathcal{G}}X = \rho(\mathcal{G})$, and then Y is a principal eigenvector of $\mathcal{G}'$. If there exists $y_v = 0$, then we get a contradiction because the principal eigenvector of $\mathcal{G}'$ is positive by Perron–Frobenius theorem.

Suppose Y is positive next. Note that $Y = X$ if Y is positive. It follows that $V(\mathcal{G}') = V(\mathcal{G})$ now. Denote by $e = \{v_1, v_2, \ldots, v_k\}$. Then $e \subseteq V(\mathcal{G})$, and

$$\rho(\mathcal{G}')y_{v_1} = (\mathcal{A}_{\mathcal{G}'}Y)_{v_1} = (\mathcal{A}_{\mathcal{G}'}X)_{v_1} + \frac{1}{k-1}\sum_{i=2}^{k} x_{v_i} = \rho(\mathcal{G})x_{v_1} + \frac{1}{k-1}\sum_{i=2}^{k} x_{v_i} > \rho(\mathcal{G})x_{v_1},$$

which contradicts $\rho(\mathcal{G}') = \rho(\mathcal{G})$. As a result, we get that $\rho(\mathcal{G}') > \rho(\mathcal{G})$. This completes the proof. □

Denote by $\mathcal{G}(\mathcal{D}v; p, q; v_{p+q}\mathcal{H})$ the k-uniform connected hypergraph obtained from k-uniform hypergraph $\mathcal{D}$ and k-uniform hypergraph $\mathcal{H}$ by adding a pendant path P_1 with length p at vertex v of $\mathcal{D}$ and adding a path P_2 with length q between vertex v and vertex v_{p+q} of $\mathcal{H}$, where $\mathcal{D}$ and $\mathcal{H}$ are two disjoint hypergraphs, $V(P_1) \cap V(\mathcal{D}) = \{v\}$, $V(P_2) \cap V(\mathcal{D}) = \{v\}$, $V(P_2) \cap V(\mathcal{H}) = \{v_{p+q}\}$ (see two examples in Fig. 4). In particular, if $H = v_{p+q}$, we denote by $\mathcal{G}(\mathcal{D}v; p, q; v_{p+q})$ for $\mathcal{G}(\mathcal{D}v; p, q; v_{p+q}H)$ for short.

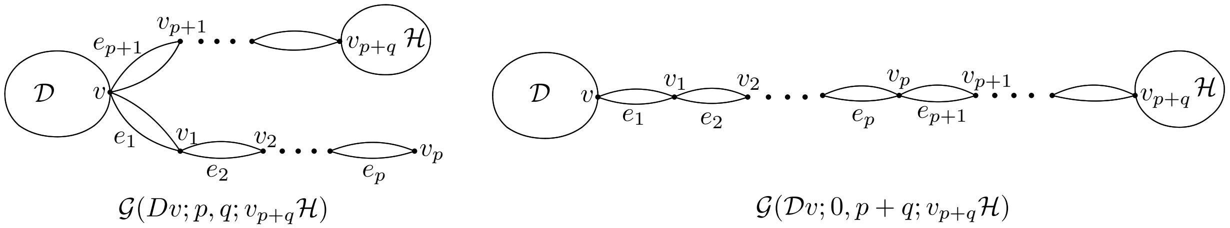

Fig. 4. $\mathcal{G}(Dv; p, q; v_{p+q}\mathcal{H})$ and $\mathcal{G}(\mathcal{D}v; 0, p+q; v_{p+q}\mathcal{H})$.

Lemma 3.10. *If* $p, q > 0$, *then* $\rho(\mathcal{G}(\mathcal{D}v; p, q; v_{p+q}\mathcal{H})) > \rho(\mathcal{G}(\mathcal{D}v; 0, p+q; v_{p+q}\mathcal{H}))$.

Proof. Let X be the principal eigenvector of the uniform hypergraph $\mathcal{G}((\mathcal{D}v; 0, p+q; v_{p+q}\mathcal{H}))$. Assume that in $\mathcal{D}$, the edges incident with v are $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\eta$. Let $e'_{p+1} = (e_{p+1}\backslash\{v_p\})\cup\{v\}$, $\mathcal{G}_1 = \mathcal{G}((\mathcal{D}v; 0, p+q; v_{p+q}\mathcal{H})) - e_{p+1} + e'_{p+1}$. If $x_v \geq x_{v_p}$, then $\rho(\mathcal{G}_1) > \rho(\mathcal{G}(\mathcal{D}v; 0, p+q; v_{p+q}\mathcal{H}))$ by Lemma 3.1. Let $\varepsilon'_i = (\varepsilon_i \setminus \{v\}) \cup \{v_p\}$ for $1 \leq i \leq \eta$, $\mathcal{G}_2 = \mathcal{G}((\mathcal{D}v; 0, p+q; v_{p+q}\mathcal{H})) - \sum_{i=1}^{\eta}\varepsilon_i + \sum_{i=1}^{\eta}\varepsilon'_i$. If $x_v \leq x_{v_p}$, then $\rho(\mathcal{G}_2) > \rho(\mathcal{G}(\mathcal{D}v; 0, p+q; v_{p+q}\mathcal{H}))$ by Lemma 3.1. Note that both G_1 and G_2 are isomorphic to $\mathcal{G}(\mathcal{D}v; p, q; v_{p+q}\mathcal{H})$. Thus, we get that $\rho(\mathcal{G}(\mathcal{D}v; p, q; v_{p+q}\mathcal{H})) > \rho(\mathcal{G}(\mathcal{D}v; 0, p+q; v_{p+q}\mathcal{H}))$. This completes the proof. □

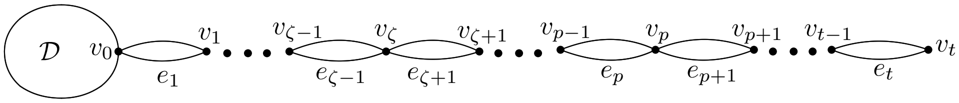

Fig. 5. $\mathcal{G}(\mathcal{D}v_0; 0, t; v_t)$.

Lemma 3.11. *Let X be the principal eigenvector of the uniform hypergraph $\mathcal{G}(\mathcal{D}v_0; 0, t; v_t)$. Denote by $\mathcal{P} = v_0e_1v_1e_2v_2\cdots e_tv_t$ the pendant path from vertex v_0 to vertex v_t in $\mathcal{G}(\mathcal{D}v_0; 0, t; v_t)$ and $e_i = \{v_{i-1}, v_{a(i,1)}, v_{a(i,2)}, \ldots, v_{a(i,k-2)}, v_i\}$ for $1 \leq i \leq t$ (see Fig. 5). Suppose $x_{v_p} = \max\{x_{v_i} : 0 \leq i \leq t\}$. Then*

(1) $p \leq t-1$.

Let $t = p+q$. Moreover, we have

(2) if $p > 0$, then $x_{v_i} \leq x_{v_{i+1}}$ for $0 \leq i \leq p-1$, $x_{v_i} \geq x_{v_{i+1}}$ for $p \leq i \leq t-1$; if $p = 0$, then $x_{v_i} \geq x_{v_{i+1}}$ for $0 \leq i \leq t-1$.

(3) if $p > 0$ and there exists $\omega \leq p$ and $\eta \leq q$ such that $x_{v_{p-\omega}} \geq x_{v_{p+\eta}}$, then

(3.1) if $\omega \leq \eta$, then $x_{v_{p-\omega+i}} \geq x_{v_{p+\eta-i}}$ for $0 \leq i \leq \omega$, $x_{v_{a(p-\omega+i,1)}} \geq x_{v_{a(p+\eta-i+1,1)}}$ for $1 \leq i \leq \omega$.

(3.2) if $\omega \geq \eta$, then $x_{v_{p-\omega+i}} \geq x_{v_{p+\eta-i}}$ for $0 \leq i \leq \eta-1$, $x_{v_j} = x_{v_p}$ for $p-\omega+\eta \leq j \leq p$, and $x_{v_{a(p-\omega+i,1)}} \geq x_{v_{a(p+\eta-i+1,1)}}$ for $1 \leq i \leq \eta$.

(4) if $p > 0$ and there exists $\omega \leq p$ and $\eta \leq q$ such that $x_{v_{p-\omega}} \leq x_{v_{p+\eta}}$, then

(4.1) if $\omega \leq \eta$, then $x_{v_{p-\omega+i}} \leq x_{v_{p+\eta-i}}$ for $0 \leq i \leq \omega-1$, $x_{v_j} = x_{v_p}$ for $p+1 \leq j \leq p+\eta-\omega$, and $x_{v_{a(p-\omega+i,1)}} \leq x_{v_{a(p+\eta-i+1,1)}}$ for $1 \leq i \leq \omega$.

(4.2) if $\omega \geq \eta$, then $x_{v_{p-\omega+i}} \leq x_{v_{p+\eta-i}}$ for $0 \leq i \leq \eta$, $x_{v_{a(p-\omega+i,1)}} \leq x_{v_{a(p+\eta-i+1,1)}}$ for $1 \leq i \leq \eta$.

(5) if $p > 0$ and there exists $\omega \leq p$, $\eta \leq q$ such that $x_{v_{p-\omega}} = x_{v_{p+\eta}}$, then

(5.1) if $\omega \leq \eta$, then $x_{v_{p-\omega+i}} = x_{v_{p+\eta-i}}$ for $0 \leq i \leq \omega-1$, $x_{v_j} = x_{v_p}$ for $p+1 \leq j \leq p+\eta-\omega$, and $x_{v_{a(p-\omega+i,1)}} = x_{v_{a(p+\eta-i+1,1)}}$ for $1 \leq i \leq \omega$.

(5.2) if $\omega \geq \eta$, then $x_{v_{p-\omega+i}} = x_{v_{p+\eta-i}}$ for $0 \leq i \leq \eta-1$, $x_{v_j} = x_{v_p}$ for $p-\omega+\eta \leq j \leq p$, $x_{v_{a(p-\omega+i,1)}} = x_{v_{a(p+\eta-i+1,1)}}$ for $1 \leq i \leq \eta$.

Proof. (1) By Lemma 3.7, it follows that $x_{v_t} < x_{v_{t-1}}$. Thus, $p \leq t-1$.

(2) Using Lemma 3.6 gets that for $1 \leq i \leq t-1$,

$$x_{v_{a(i,1)}} = x_{v_{a(i,2)}} = \cdots = x_{v_{a(i,k-2)}} < \min\{x_{v_{i-1}}, x_{v_i}\}.$$

Note that $\mathcal{G}(\mathcal{D}v_0; 0, t; v_t)$ is uniform and $p \leq t-1$.

Case 1. $p > 0$.

Claim. $x_{v_0} \leq x_{v_1} \leq x_{v_2} \leq \cdots \leq x_{v_p}$. If $p = 1$, this claim hold naturally.

For $p \geq 2$, we prove this claim by contradiction. Suppose that x_{v_z} $(0 \leq z \leq p-1)$ is the first vertex from 0 to $p-1$ such that $x_{v_z} > x_{v_{z+1}}$. Then there exists $z+1 \leq \zeta \leq p-1$ such that $x_{v_z} > x_{v_{z+1}} \geq \cdots \geq x_{v_\zeta} \leq x_{v_{\zeta+1}}$. Let $e' = \{v_\zeta, u_1, u_2, \ldots, u_{k-2}, u\}$, $e'_\zeta = (e_\zeta \setminus \{v_\zeta\}) \cup \{u\}$, where $u \notin V(\mathcal{G})$, $u_i \notin V(\mathcal{G})$ for $1 \leq i \leq t-2$, $\mathcal{G}_1 = \mathcal{G} - e_\zeta + e'_\zeta + e'$, $\mathcal{G}_2 = \mathcal{G}_1 - \{v_{a(t,1)}, v_{a(t,2)}, \ldots, v_{a(t,k-2)}, v_t\}$. By Lemma 3.8 and Lemma 3.9, we get that $\rho(\mathcal{G}_2) < \rho(\mathcal{G}_1) \leq \rho(\mathcal{G})$, which contradicts $\rho(\mathcal{G}) = \rho(\mathcal{G}_2)$ because $\mathcal{G}_2 \cong \mathcal{G}$. Thus, our claim holds.

In the same way, it is proved that $x_{v_p} \geq x_{v_{p+1}} \geq x_{v_{p+2}} \geq \cdots \geq x_{v_{t-1}}$. Combining Lemma 3.7, we get that $x_{v_p} \geq x_{v_{p+1}} \geq x_{v_{p+2}} \geq \cdots \geq x_{v_{t-1}} > x_{v_t}$.

Case 2. $p = 0$. As Case 1, it is proved that $x_{v_i} \geq x_{v_{i+1}}$ for $0 \leq i \leq t-1$. Thus, (2) follows.

(3) If $\omega \leq \eta$, we let Y be a vector on $\mathcal{G}(\mathcal{D}v_0; 0, t; v_t)$ satisfying that

$$\begin{cases} y_{v_{p-\omega+i}} = \max\{x_{v_{p-\omega+i}}, x_{v_{p+\eta-i}}\} & 0 \leq i \leq \eta; \\ y_{v_{a(p-\omega+i,j)}} = \max\{x_{v_{a(p-\omega+i,1)}}, x_{v_{a(p+\eta-i+1,1)}}\} & 1 \leq i \leq \omega, 1 \leq j \leq k-2; \\ y_v = x_v & \textit{others}. \end{cases}$$

As the proof of Lemma 3.8, it is proved that $\mathcal{A}(\mathcal{G}(\mathcal{D}v_0; 0, t; v_t))Y \geq \rho(\mathcal{G}(\mathcal{D}v_0; 0, t; v_t))Y$. Using Lemma 2.1 gets that $\mathcal{A}(\mathcal{G}(\mathcal{D}v_0; 0, t; v_t))Y = \rho(\mathcal{G}(\mathcal{D}v_0; 0, t; v_t))Y$. Note that $\mathcal{G}(\mathcal{D}v_0; 0, t; v_t)$ is connected. Then $\mathcal{A}(\mathcal{G}(\mathcal{D}v_0; 0, t; v_t))$ is irreducible. Consequently, it follows that the dimension of the eigenspace of the eigenvalue $\rho(\mathcal{G}(\mathcal{D}v_0; 0, t; v_t))$ is 1. Then $Y = lX$ for some $l > 0$. Then (3.1) follows.

(3.2), (4.1), and (4.2) are proved in the same way. (5) follows from (3) or (4).

This completes the proof. □

Lemma 3.12. *$L_0, L_1, L_2, \ldots, L_f$ ($f \geq 1$) are positive integers satisfying $L_1 \leq L_2 \leq \cdots \leq L_f \leq L_0 - 1$ and $\sum_{i=0}^{f} L_i = t$. For an integer $\mu > 0$, if $t - \mu \geq L_0$, then there exists some $1 \leq j \leq f$ such that $t - \mu > L_1 + \cdots + L_j$, but $L_0 + L_1 + \cdots + L_j > t - \mu$.*

Proof. If $\sum_{i=1}^{f} L_i < t - \mu$, then $t = \sum_{i=0}^{f} L_i > t - \mu$ follows naturally. Thus, the result holds.

Suppose $\sum_{i=1}^{f} L_i \geq t - \mu$. Note that $t - \mu \geq L_0$ and $L_1 \leq L_2 \leq \cdots \leq L_f \leq L_0 - 1$. Then $f \geq 2$ now, and there exists some $1 \leq g \leq f - 1$ such that $L_1 + \cdots + L_g + L_{g+1} \geq t - \mu$, but $L_1 + \cdots + L_g < t - \mu$. Note that $L_{g+1} \leq L_0 - 1$. Then it follows that $L_0 + L_1 + \cdots + L_g > t - \mu$, but $t - \mu > L_1 + \cdots + L_g$. This completes the proof. □

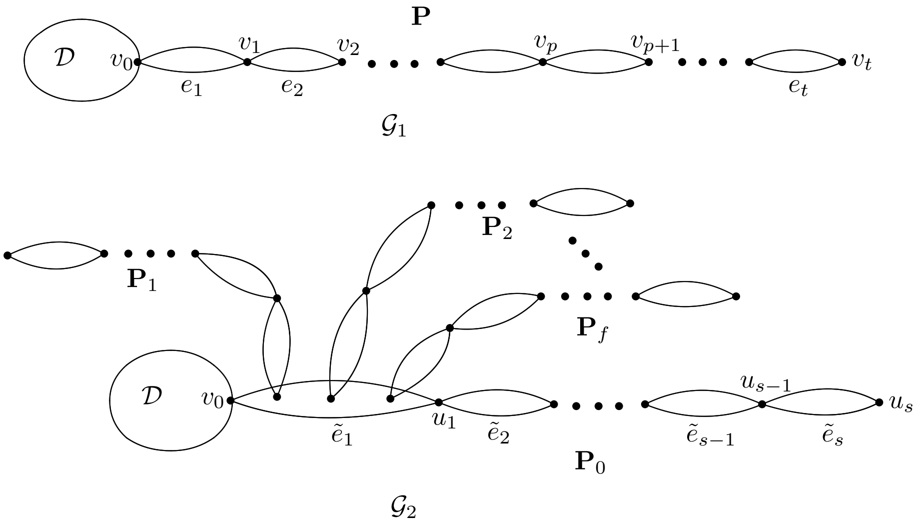

Fig. 6. $\mathcal{G}_1$, $\mathcal{G}_2$.

Lemma 3.13. *$\mathcal{D}$ is a k-uniform hypergraph where $v_0 \in V(\mathcal{D})$. Both $\mathbf{P} = v_0 e_1 v_1 e_2 v_2 \cdots e_t v_t$ and $\mathbf{P}_0 = v_0 \tilde{e}_1 u_1 \tilde{e}_2 u_2 \cdots \tilde{e}_s u_s$ are k-uniform hyperpaths where $\tilde{e}_1 = \{v_0, v_{\varphi(1,1)}, v_{\varphi(1,2)}, \ldots, v_{\varphi(1,k-2)}, u_1\}$, $V(\mathcal{D}) \cap V(\mathbf{P}) = \{v_0\}$, $V(\mathcal{D}) \cap V(\mathbf{P}_0) = \{v_0\}$. $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_f$ ($1 \leq f \leq k-2$) are k-uniform hyperpaths attached,*

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respectively, at vertices $v_{\varphi(1,1)}$, $v_{\varphi(1,2)}$, $\ldots$, $v_{\varphi(1,f)}$ *in* $\tilde{e}_1$ *satisfying* $1 \le L(\mathbf{P}_1) \le L(\mathbf{P}_2) \le \cdots \le L(\mathbf{P}_f) \le L(\mathbf{P}_0) - 1$, $\sum_{i=0}^{f} L(\mathbf{P}_i) = t$, $V(\mathcal{D}) \cap V(\mathbf{P}_i) = \emptyset$ *for* $1 \le i \le f$. *Let* $\mathcal{G}_1$ *be a k-uniform hypergraph of the form* $\mathcal{G}(\mathcal{D}v_0; 0, t; v_t)$ *consisting of* $\mathcal{D}$ *and* $\mathbf{P}$; $\mathcal{G}_2$ *be a k-uniform hypergraph consisting of* $\mathcal{D}$ *and* $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\ldots$, $\mathbf{P}_f$ *(see Fig. 6). Then* $\rho(\mathcal{G}_1) < \rho(\mathcal{G}_2)$.

Proof. For brevity, we denote by $L_i = L(\mathbf{P}_i)$ for $0 \le i \le f$. Without loss of generality, we suppose $f = 3$ next.

In $\mathcal{G}_1$, denote by $e_i = \{v_{i-1}, v_{a(i,1)}, v_{a(i,2)}, \ldots, v_{a(i,k-2)}, v_i\}$ for $1 \le i \le t-1$, $e_t = \{v_{t-1}, v_{a(t,1)}, v_{a(t,2)}, \ldots, v_{a(t,k-2)}, v_{a(t,k-1)}\}$ where $v_{a(t,k-1)} = v_t$. Assume that in $\mathcal{D}$, the edges incident with v_0 are ε_1, ε_2, $\ldots$, ε_η. Let X be the principal eigenvector of $\mathcal{G}_1$, and $x_{v_p} = \max\{v_i : 0 \le i \le t\}$. By Lemma 3.11, we know that $p \le t-1$. By Lemma 3.6 and Lemma 3.7, we know that $x_{v_{a(i,j)}} = x_{v_{a(i,z)}} < \min\{x_{v_{i-1}}, x_{v_i}\}$ for $2 \le j, z \le k-2$ where $1 \le i \le t-1$, $x_{v_{a(t,j)}} = x_{v_{a(t,z)}}$ for $2 \le j, z \le k-1$. By Lemma 3.11, we know that if $p > 0$, then $x_{v_i} \le x_{v_{i+1}}$ for $0 \le i \le p-1$; if $p \ge 0$, then $x_{v_i} \ge x_{v_{i+1}}$ for $p \le i \le t-1$.

Case 1. $p > 0$.

Subcase 1.1 $t-p \ge L_0$ (see Fig. 7). By Lemma 3.12, there exists $1 \le j \le 3$ such that $t-p > L_1+\cdots+L_j$, but $L_0 + L_1 + \cdots + L_j > t-p$. Without loss of generality, we suppose $j = 2$. Now $t - L_1 - L_2 > p$, $t - L_0 - L_1 - L_2 < p$. Note that $L_3 = t - L_0 - L_1 - L_2$. For brevity and convenience, we let $\xi = t - L_1$, $\eta = t - L_1 - L_2$.

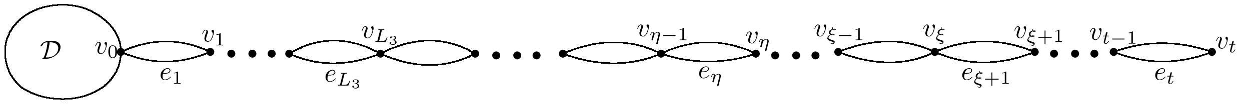

FIG. 7. $\mathcal{G}_1$ $(t - p \ge L_0)$.

Subcase 1.1.1 $x_{v_\eta} \ge x_{v_{L_3}}$. By Lemma 3.11, we know that $x_{v_{\eta-1}} \ge x_{v_{L_3+1}}$.

Subcase 1.1.1.1 $x_{v_0} \le x_{v_{a(\eta,1)}}$, $x_{v_{L_3}} \le x_{v_{a(\eta,2)}}$, $x_{v_\xi} \le x_{v_{a(\eta,3)}}$. Let $\varepsilon_i' = (\varepsilon_i \setminus \{v_0\}) \cup \{v_{a(\eta,1)}\}$ for $1 \le i \le \eta$, $e_{L_3}' = (e_{L_3} \setminus \{v_{L_3}\}) \cup \{v_{a(\eta,2)}\}$, $e_{\xi+1}' = (e_{\xi+1} \setminus \{v_\xi\}) \cup \{v_{a(\eta,3)}\}$; $\mathcal{S}_i = \sum_{v \in (\varepsilon_i \setminus \{v_0\})} x_v$ for $i = 1, 2, \ldots, \eta$, $\mathbb{S}_{L_3} = \sum_{v \in e_{L_3} \setminus \{v_{L_3}\})} x_v$, $\mathbb{S}_{\xi+1} = \sum_{v \in (e_{\xi+1} \setminus \{v_\xi\})} x_v$.

Let

$$\mathcal{G}_1' = \mathcal{G}_1 - \sum_{i=1}^{\eta} \varepsilon_i + \sum_{i=1}^{\eta} \varepsilon_i' - e_{L_3} + e_{L_3}' - e_{\xi+1} + e_{\xi+1}'.$$

Then

$$X^T \mathcal{A}(\mathcal{G}_1')X - X^T \mathcal{A}(\mathcal{G}_1)X = \frac{2}{k-1}\{(x_{v_{a(\eta,1)}} - x_{v_0}) \sum_{i=1}^{\eta} \mathcal{S}_i + (x_{v_{a(\eta,2)}} - x_{v_{L_3}})\mathbb{S}_{L_3} + (x_{v_{a(\eta,3)}} - x_{v_\xi})\mathbb{S}_{\xi+1}\} \ge 0.$$

It follows that $\rho(\mathcal{G}_1') \ge \rho(\mathcal{G}_1)$. Suppose $\rho(\mathcal{G}_1') = \rho(\mathcal{G}_1)$. Then $\rho(\mathcal{G}_1') = X^T \mathcal{A}(\mathcal{G}_1')X = X^T \mathcal{A}(\mathcal{G}_1)X = \rho(\mathcal{G}_1)$. Hence, X is also the principal eigenvector of $\mathcal{G}_1'$ and $\mathcal{A}(\mathcal{G}_1')X = \mathcal{A}(\mathcal{G}_1)X$. But a contradiction comes immediately because $(\mathcal{A}(\mathcal{G}_1')X)_{v_{a(\eta,1)}} > (\mathcal{A}(\mathcal{G}_1)X)_{v_{a(\eta,1)}}$. Thus, it follows that $\rho(\mathcal{G}_1') > \rho(\mathcal{G}_1)$. Note that $\mathcal{G}_1' \cong \mathcal{G}_2$. Then $\rho(\mathcal{G}_2) > \rho(\mathcal{G}_1)$.

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Subcase 1.1.1.2 $x_{v_0} > x_{v_{a(\eta,1)}}$, $x_{v_{L_3}} \leq x_{v_{a(\eta,2)}}$, $x_{v_{t-L_1}} \leq x_{v_{a(\eta,3)}}$. Let $e'_1 = (e_1 \setminus \{v_0\}) \cup \{v_{a(\eta,1)}\}$, $e'_\eta = (e_\eta \setminus \{v_{a(\eta,1)}\}) \cup \{v_0\}$, $e'_{L_3} = (e_{L_3} \setminus \{v_{L_3}\}) \cup \{v_{a(\eta,2)}\}$, $e'_{\xi+1} = (e_{\xi+1} \setminus \{v_\xi\}) \cup \{v_{a(\eta,3)}\}$; $\mathbb{S}_1 = \sum_{v \in (e_1 \setminus \{v_0\})} x_v$, $\mathbb{S}_\eta = \sum_{v \in (e_\eta \setminus \{v_{a(\eta,1)}\})} x_v$, $\mathbb{S}_{L_3} = \sum_{v \in e_{L_3} \setminus \{v_{L_3}\})} x_v$, $\mathbb{S}_{\xi+1} = \sum_{v \in (e_{\xi+1} \setminus \{v_\xi\})} x_v$. Note $x_{v_\eta} \geq x_{v_{t-L_0-L_1-L_2}} \geq x_{v_0}$, $x_{\eta-1} \geq x_{t-L_0-L_1-L_2+1} = x_{L_3+1}$. Using Lemma 3.6, we get $x_{v_{a(1,j)}} < x_{a(L_3+1,j)} < x_{v_{a(\eta,j)}} < \min\{x_{v_{\eta-1}}, x_{v_\eta}\}$ for $1 \leq j \leq k-2$. As a result, it follows that $\mathbb{S}_1 \leq \mathbb{S}_\eta$.

Let

$$\mathcal{G}'_1 = \mathcal{G}_1 - e_1 + e'_1 - e_\eta + e'_\eta - e_{L_3} + e'_{L_3} - e_{\xi+1} + e'_{\xi+1}.$$

Then

$$X^T \mathcal{A}(\mathcal{G}'_1) X - X^T \mathcal{A}(G_1) X = \frac{2}{k-1} \{ (x_{v_0} - x_{v_{a(\eta,1)}}) \mathbb{S}_\eta - (x_{v_0} - x_{v_{a(\eta,1)}}) \mathbb{S}_1 + (x_{v_{a(\eta,2)}} - x_{v_{L_3}}) \mathbb{S}_{L_3}$$

$$+ (x_{v_{a(\eta,3)}} - x_{v_\xi}) \mathbb{S}_{\xi+1} \} \geq 0.$$

Thus, it follows that $\rho(\mathcal{G}'_1) \geq \rho(\mathcal{G}_1)$. Note that $(\mathcal{A}(\mathcal{G}'_1) X)_{v_\eta} > (\mathcal{A}(\mathcal{G}_1) X)_{v_\eta}$ and $\mathcal{G}'_1 \cong \mathcal{G}_2$. As Subcase 1.1.1.1, we get that $\rho(\mathcal{G}_2) > \rho(\mathcal{G}_1)$.

Subcase 1.1.1.3 $x_{v_0} \leq x_{v_{a(\eta,1)}}$, $x_{v_{L_3}} > x_{v_{a(\eta,2)}}$, $x_{v_\xi} \leq x_{v_{a(\eta,3)}}$. Let $\varepsilon'_i = (\varepsilon_i \setminus \{v_0\}) \cup \{v_{a(\eta,1)}\}$ for $1 \leq i \leq \eta$, $e'_\eta = (e_\eta \setminus \{v_{a(\eta,2)}\}) \cup \{v_{L_3}\}$, $e'_{L_3+1} = (e_{L_3+1} \setminus \{v_{L_3}\}) \cup \{v_{a(\eta,2)}\}$, $e'_{\xi+1} = (e_{\xi+1} \setminus \{v_\xi\}) \cup \{v_{a(\eta,3)}\}$.

Let

$$G'_1 = G_1 - \sum_{i=1}^{\eta} \varepsilon_i + \sum_{i=1}^{\eta} \varepsilon'_i - e_{L_3+1} + e'_{L_3+1} - e_\eta + e'_\eta - e_{\xi+1} + e'_{\xi+1}.$$

As Subcase 1.1.1.1 and Subcase 1.1.1.2, we get that $\rho(\mathcal{G}'_1) > \rho(\mathcal{G}_1)$, and $\rho(\mathcal{G}_2) > \rho(\mathcal{G}_1)$.

In the same way, for the subcases: (i) $x_{v_0} \leq x_{v_{a(\eta,1)}}$, $x_{v_{L_3}} \leq x_{v_{a(\eta,2)}}$, $x_{v_\xi} > x_{v_{a(\eta,3)}}$; (ii) $x_{v_0} > x_{v_{a(\eta,1)}}$, $x_{v_{L_3}} > x_{v_{a(\eta,2)}}$, $x_{v_\xi} \leq x_{v_{a(\eta,3)}}$; (iii) $x_{v_0} > x_{v_{a(\eta,1)}}$, $x_{v_{L_3}} > x_{v_{a(\eta,2)}}$, $x_{v_\xi} > x_{v_{a(\eta,3)}}$; and (iv) $x_{v_0} \leq x_{v_{a(\eta,1)}}$, $x_{v_{L_3}} > x_{v_{a(\eta,2)}}$, $x_{v_\xi} > x_{v_{a(\eta,3)}}$; (v) $x_{v_0} > x_{v_{a(\eta,1)}}$, $x_{v_{L_3}} \leq x_{v_{a(\eta,2)}}$, $x_{v_\xi} > x_{v_{a(\eta,3)}}$, we get that $\rho(\mathcal{G}_2) > \rho(\mathcal{G}_1)$.

As a result, from the above narrations for Subcase 1.1.1 that $x_{v_\eta} \geq x_{v_{L_3}}$, we get $\rho(\mathcal{G}_2) > \rho(\mathcal{G}_1)$.

Subcase 1.1.2 $x_{v_\eta} < x_{v_{L_3}}$. By Lemma 3.11, we know that $x_{v_{\eta-1}} \leq x_{v_{L_3+1}}$. By considering the comparisons between x_{v_0} and $x_{v_{a(L_3+1,1)}}$, between x_{v_ξ} and $x_{v_{a(L_3+1,2)}}$, between x_{v_η} and $x_{v_{a(L_3+1,3)}}$, as Subcase 1.1.1, we get that $\rho(\mathcal{G}_2) > \rho(\mathcal{G}_1)$.

Subcase 1.2 $t - p < L_0$ (see Fig. 8). Let $\omega = L_1 + L_2$, $\varphi = L_1 + L_2 + L_3 + 1$. By considering the comparisons between x_{v_0} and $x_{v_{a(\varphi,1)}}$, between $x_{v_{L_1}}$ and $x_{v_{a(\varphi,2)}}$, between x_{v_ω} and $x_{v_{a(\varphi,3)}}$, as Subcase 1.1, we get that $\rho(\mathcal{G}_2) > \rho(\mathcal{G}_1)$.

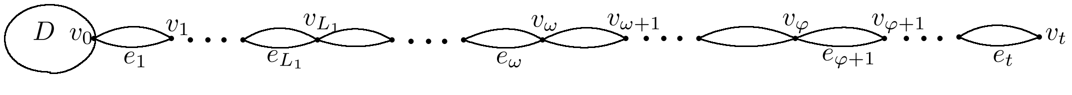

FIG. 8. $\mathcal{G}_1$ $(t - p < L_0)$.

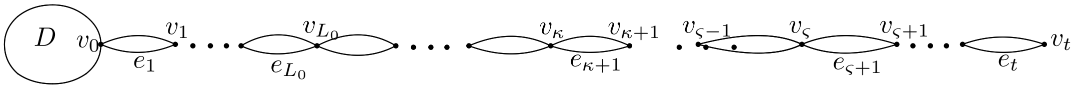

FIG. 9. $\mathcal{G}_1$ $(p = 0)$.

Case 2. $p = 0$ (see Fig. 9). Let $\kappa = L_0 + L_1$, $\varsigma = L_0 + L_1 + L_2$. By considering the comparisons between $x_{v_{a(1,1)}}$ and $x_{v_{L_0}}$, between $x_{v_{a(1,2)}}$ and x_{v_κ}, between $x_{v_{a(1,3)}}$ and x_{v_ς}, as Case 1, we get that $\rho(\mathcal{G}_2) > \rho(\mathcal{G}_1)$.

This completes the proof. ☐

Proof of Theorem 1.3. For k-uniform supertrees of order n, using Lemma 3.10 and Lemma 3.13 repeatedly gets the result. This completes the proof. ☐

Proof of Corollary 1.4. Note that $\mathcal{S}^*(n,k)$ is the kth power of the ordinary star $S^*(\frac{n-1}{k-1}+1,2)$, $\mathcal{P}(n,k)$ is the kth power of the ordinary path $P(\frac{n-1}{k-1}+1,2)$. Then the result follows from Theorem 1.1 and Theorem 1.3. This completes the proof. ☐

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Declarations

The authors declare no potential conflict of interests, no data was used for the research described in this article, and no dataset was generated or analyzed during this study.

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