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SDD_1 TENSORS AND B_1-TENSORS*

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Abstract. Strong $\mathcal{H}$-tensors play an important role in the fields of science and engineering. In this paper, we first propose a new subclass of strong $\mathcal{H}$-tensors that we call the class of SDD_1 tensors. We also prove that if a tensor is an SDD_1 tensor, it is a strong $\mathcal{H}$-tensor. As an application, a sufficient condition for an SDD_1 tensor to certify positive definiteness of even-order real symmetric tensors is proposed. Furthermore, we propose a new class of tensors by means of SDD_1 tensors, naming it B_1-tensors, and show that B-tensors are a subclass of it. Meanwhile, some properties of B_1-tensor were introduced. The numerical examples demonstrate the validity of our results.

Key words. $\mathcal{H}$-tensor, SDD_1 Tensor, real symmetric tensor, B_1-tensor

AMS subject classifications. 15A15, 15A18, 15A69.

1. Introduction. Let n and m be integers, and $N = \{1, 2, \ldots, n\}$ and $\mathbb{C}$(*resp.* $\mathbb{R}$) be the set of all complex(*resp.* real) numbers. We call $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ a complex(*resp.* real) tensor with order m and dimension n, if $a_{i_1 i_2 \cdots i_m} \in \mathbb{C}(resp.\ \mathbb{R})$, where $i_j = 1, 2, \ldots, n$ for $j = 1, 2, \ldots, m$. Let $\mathbb{C}^{[m,n]}$ (*resp.* $\mathbb{R}^{[m,n]}$)$(m, n \geq 2)$ be the set of all complex (*resp.* real) order m dimension n tensors. In fact, a vector is a tensor of order 1 and a matrix is a tensor of order 2. Tensors, as a high-order extension of matrices, have important applications in many scientific fields, such as signal and image processing, continuum physics, magnetic resonance imaging, high-order statistics [1, 13, 17].

For a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$, if there exists a complex number λ and a complex vector $x = (x_1, x_2, \ldots, x_n)^T \neq (0, 0, \ldots, 0)^T$ that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an eigenvalue of $\mathcal{A}$ and x is its corresponding eigenvector [7, 8, 16], where $\mathcal{A}x^{m-1}$ and $\lambda x^{[m-1]}$ are vectors, and whose ith components are

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m \in N} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m},\ and\ (x^{[m-1]})_i = x_i^{m-1}.$$

In particular, if λ and x are both real, then λ is called an $\mathcal{H}$-eigenvalue of $\mathcal{A}$ and x is its corresponding $\mathcal{H}$-eigenvector [16]. The $\mathcal{H}$-eigenvalues of tensors are widely used in data analysis, high-order Markov chains, and positive definiteness of even-order homogeneous polynomials [5, 14, 15]. In this paper, we will discuss its applications to the positive definiteness of even-order homogeneous polynomials. The notions of homogeneous polynomials and positive definiteness are given in the following.

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For an mth degree homogeneous polynomial of n variables, $f(x)$ can be usually denoted as:

$$f(x) \equiv \mathcal{A}x^m = \sum_{i_1,i_2,\ldots,i_m \in N} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$

where $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ and $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ is a symmetric tensor [15]. The tensor $\mathcal{A}$ is called symmetric if its elements are invariant under any permutation of indices $\{i_1, i_2, \ldots, i_m\}$ [16]. If m is even, and

$$f(x) > 0, \ \textit{for all } x \in \mathbb{R}^n, \, x \neq 0,$$

then we say that $f(x)$ is positive definite. The symmetric tensor $\mathcal{A}$ is called positive definite if $f(x)$ is positive definite [15].

The positive definiteness of homogeneous polynomials play a key role in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control [15, 19, 26]. Concretely, for the system $\dot{x} = g(x)$, if a multivariate polynomial $f(x)$ can be found such that $f(x)$ is positive definite and

$$\left(\frac{\partial f}{\partial x}\right)^T g(x) < 0, \ \forall x \in \mathbb{R}^n, \ x \neq 0,$$

then the system $\dot{x} = g(x)$ is asymptotically stable. Hence, ascertaining whether a multivariate polynomial $f(x)$ is positive definite for all real x is often crucial to the use of Lyapunov stability tests [15]. To solve the problem of determining the positive definiteness of homogeneous polynomial, Qi proposed in [16] that $f(x)$ is positive definite if and only if the real symmetric tensor $\mathcal{A}$ is positive definite, and Qi showed that even-order real symmetric tensor is positive definite if and only if all of its $\mathcal{H}$-eigenvalues are positive. However, it is hard to compute all $\mathcal{H}$-eigenvalues or the smallest $\mathcal{H}$-eigenvalue of $\mathcal{A}$ when m and n are very large. Subsequently, a verifiable sufficient condition for positive definite tensors was proposed if a tensor has special structures; see Theorem 1.2. Before that, we give the definition of strong $\mathcal{H}$-tensors.

Definition 1.1 (Ding, Qi, and Wei [2]). *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. If there is a positive vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ such that*

$$|a_{ii\cdots i}| x_i^{m-1} > \sum_{\substack{i_2,\ldots,i_m \in N \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| x_{i_2} \cdots x_{i_m}, \ \forall i \in N,$$

where $|a|$ is the modulus of $a \in \mathbb{C}$, then $\mathcal{A}$ is called a strong $\mathcal{H}$-tensor.

Theorem 1.2 (Li et al. [9]). *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ with $a_{kk\cdots k} > 0$ for all $k \in N$ and m be even. If $\mathcal{A}$ is a strong $\mathcal{H}$-tensor, then $\mathcal{A}$ is positive definite.*

With this theorem, for the positive definiteness of even-order real symmetric tensors, it is possible to first determine whether the given tensor is a strong $\mathcal{H}$-tensor. This article will propose a new subclass of strong $\mathcal{H}$-tensors, which are called SDD_1 tensors. Determining whether a given tensor is an SDD_1 tensor depends only on the elements of the tensor. This provides a sufficient condition for determining the positive definiteness of tensors. More criteria for judging whether a tensor is a strong $\mathcal{H}$-tensor have been widely proposed; see [3, 4, 10, 11, 12, 20, 21, 22, 23, 24, 25, 27].

We now give the structural arrangement of this paper. In Section 2, we will propose SDD_1 tensors, which are an extension of SDD_1 matrices. Meanwhile, we will prove that it is a subclass of strong $\mathcal{H}$-tensors.

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As an application, a sufficient condition will be presented for using SDD_1 tensors to determine the positive definiteness of even-order real symmetric tensors. In Section 3 , we propose B_1-tensors, which include B-tensors as a subclass. At the same time, some properties of B_1-tensors are introduced. Moreover, some numerical examples are presented to illustrate those new results. Finally, in Section 4 , a brief conclusion is given.

2. SDD_1 tensors.

In this section, a new class of tensors are proposed, which are called SDD_1 tensors. In particular, it is shown that SDD_1 tensors are strong $\mathcal{H}$-tensors. For the convenience of discussion, some notations, definitions, lemmas, and theorems are given in the following.

Calligraphic letters $\mathcal{A}$, $\mathcal{B}$, ... represent tensors; capital letters A, B, ... denote matrices; lowercase letters x, y, ... refer to vectors.

For a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$, we denote

$$\begin{aligned}
r_i(\mathcal{A}) &= \sum_{\substack{i_2 \cdots i_m \in N^{m-1} \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \cdots i_m}| = \sum_{i_2 \cdots i_m \in N^{m-1}} |a_{i i_2 \cdots i_m}| - |a_{ii\cdots i}|, \\
N_L &= N_L(\mathcal{A}) = \{i \in N : |a_{ii\cdots i}| \le r_i(\mathcal{A})\}, \\
N_U &= N_U(\mathcal{A}) = \{i \in N : |a_{ii\cdots i}| > r_i(\mathcal{A})\}, \\
N^{m-1} &= \{i_2 \cdots i_m : i_j \in N, j = 2, \ldots, m\}, \\
N^{m-1} \backslash S^{m-1} &= \{i_2 i_3 \cdots i_m : i_2 i_3 \cdots i_m \in N^{m-1} \ and \ i_2 i_3 \cdots i_m \notin S^{m-1}\}, \\
N_0^{m-1} &= N^{m-1} \backslash (N_L^{m-1} \cup N_U^{m-1}),
\end{aligned}$$

where $r_i(\mathcal{A})$ is weight of the off-diagonal entries of the ith row of the flattening $\mathcal{A}$. N_L is the set of indices where the modulus of the diagonal entry is less than or equal to the corresponding off-diagonal weight. N_U is the set of indices where the modulus of the diagonal entry is greater than the corresponding off-diagonal weight. N^{m-1} is the set of indices in which i_2 to i_m belong to N. $N^{m-1} \backslash S^{m-1}$ denotes the difference set of N^{m-1} and S^{m-1}, where i_2 to i_m belong to N and i_2 to i_m does not belong to S. N_0^{m-1} denotes the difference set of N^{m-1} and $N_L^{m-1} \cup N_U^{m-1}$, where i_2 to i_m belong partly to N_L and partly to N_U.

Definition 2.1 (Qi [16]). *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. $\mathcal{A}$ is called a diagonally dominant tensor if*

$$|a_{ii\cdots i}| \ge r_i(\mathcal{A}), \ \forall i \in N.$$

$\mathcal{A}$ is called a strictly diagonally dominant (SDD) tensor if all inequalities hold strictly.

Definition 2.2 (Kolda and Bader [6]). *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ and $X = diag(x_1, x_2, \ldots, x_n)$. If*

$$\mathcal{B} = (b_{i_1 i_2 \cdots i_m}) = \mathcal{A} X^{m-1},$$

where

$$b_{i_1 i_2 \cdots i_m} = a_{i_1 i_2 \cdots i_m} x_{i_2} \ldots x_{i_m}, \ i_j \in N, \ j \in \{1, 2, \ldots, m\},$$

then we call $\mathcal{B}$ the product of the tensor $\mathcal{A}$ and the matrix X.

Let us introduce a new class of tensors containing SDD tensors, by imposing to the rows that are not strictly diagonally dominant a weaker condition.

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DEFINITION 2.3. *A tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ *is called an* SDD_1 *tensor if*

$$|a_{i\cdots i}| > p_i(\mathcal{A}),\ \forall i \in N_L,$$

where

$$p_i(\mathcal{A}) = \sum_{\substack{i_2\cdots i_m \in N_L^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_U^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{\frac{r_j(\mathcal{A})}{|a_{jj\cdots j}|}\right\} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_0^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|.$$

Remark 2.4. Obviously, we have $p_i(\mathcal{A}) \leq r_i(\mathcal{A})$, so the SDD tensors form a subclass of the SDD_1 tensors.

In the following, we will give some lemmas, which will be used frequently in the subsequent proofs.

LEMMA 2.5 (Ding, Qi and Wei [2]). *If* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ *is an SDD tensor, then* $\mathcal{A}$ *is a strong* $\mathcal{H}$*-tensor.*

LEMMA 2.6 (Wang and Song [21]). *Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$. *If there exists a positive diagonal matrix* X *such that* $\mathcal{A}X^{m-1}$ *is a strong* $\mathcal{H}$*-tensor, then* $\mathcal{A}$ *is a strong* $\mathcal{H}$*-tensor.*

LEMMA 2.7. *A tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ *is an* SDD_1 *tensor if and only if* $|a_{i\cdots i}| > p_i(\mathcal{A})$ *for all* $i \in N$.

Proof. One implication is obvious. For the other one, let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ be an SDD_1 tensor. From Definition 2.3, we get that $|a_{i\cdots i}| > p_i(\mathcal{A})$ for any $i \in N_L$ and for any $i \in N_U$, from the definition of N_U and $p_i(\mathcal{A})$, we have $|a_{i\cdots i}| > r_i(\mathcal{A}) \geq p_i(\mathcal{A})$. Thus, there is $|a_{i\cdots i}| > p_i(\mathcal{A})$ for all $i \in N$. ∎

LEMMA 2.8. *If a tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ *is an* SDD_1 *tensor and* $N_L \neq \emptyset$, *then we have* $\sum_{\substack{i_2\cdots i_m \in N_U^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \neq 0$.

Proof. For a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$, if $\sum_{\substack{i_2\cdots i_m \in N_U^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| = 0$, then there is $r_i(\mathcal{A}) = p_i(\mathcal{A})$. Because $\mathcal{A}$ is SDD_1 tensor, then $|a_{i\cdots i}| > p_i(\mathcal{A}) = r_i(\mathcal{A})$ for all $i \in N_L$, which contradicts the definition of N_L. The proof is complete. ∎

In the following, we construct a positive diagonal matrix X that scales an SDD_1 tensor to an SDD tensor.

THEOREM 2.9. *If a tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ *is an* SDD_1 *tensor, then* $\mathcal{A}$ *is a strong* $\mathcal{H}$*-tensor.*

Proof. Let a tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ be an SDD_1 tensor, according to Lemma 2.7, $|a_{i\cdots i}| > p_i(\mathcal{A})$ for all $i \in N$. Hence, we have

$$|a_{i\cdots i}| - p_i(\mathcal{A}) > 0,\ \forall i \in N,$$

and

$$1 - \frac{p_i(\mathcal{A})}{|a_{ii\cdots i}|} > 0,\ \forall i \in N.$$

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Then there exists a positive number $\varepsilon > 0$ such that

$$0 < \varepsilon < \min \left\{ 1 - \frac{p_i(\mathcal{A})}{|a_{ii\cdots i}|}, \frac{|a_{i\cdots i}| - p_i(\mathcal{A})}{\sum\limits_{\substack{i_2\cdots i_m \in N_U^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|} \right\}, \tag{2.1}$$

if $\sum\limits_{\substack{i_2\cdots i_m \in N_U^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| = 0$, then the corresponding fraction is defined to be ∞. Construct a matrix $X = \operatorname{diag}(x_1, x_2, \ldots, x_n)$, where

$$x_i = \begin{cases} 1, & i \in N_L, \\ \left(\frac{p_i(\mathcal{A})}{|a_{ii\cdots i}|} + \varepsilon \right)^{\frac{1}{m-1}}, & i \in N_U. \end{cases}$$

From inequality (2.1), we can obtain that $\frac{p_i(\mathcal{A})}{|a_{ii\cdots i}|} + \varepsilon < \frac{p_i(\mathcal{A})}{|a_{ii\cdots i}|} + (1 - \frac{p_i(\mathcal{A})}{|a_{ii\cdots i}|}) = 1$, so $x_i \neq +\infty$, which shows that X is a positive diagonal matrix. Let $\mathcal{B} = (b_{i_1i_2\cdots i_m}) = \mathcal{A}X^{m-1}$, then $b_{i_1i_2\cdots i_m} = a_{i_1i_2\cdots i_m} x_{i_2} \cdots x_{i_m}$, $\forall i_j \in N$, $j \in \{1, 2, \ldots, m\}$.

Next, we will prove that $\mathcal{B}$ is an SDD tensor:

$$\begin{aligned}
r_i(\mathcal{B}) &= \sum_{\substack{i_2\cdots i_m \in N_L^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |b_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_U^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |b_{ii_2\cdots i_m}| + \sum_{i_2\cdots i_m \in N_0^{m-1}} |b_{ii_2\cdots i_m}| \\
&= \sum_{\substack{i_2\cdots i_m \in N_L^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_U^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \left(\frac{p_{i_2}(\mathcal{A})}{|a_{i_2\cdots i_2}|} + \varepsilon \right)^{\frac{1}{m-1}} \cdots \\
&\quad \left(\frac{p_{i_m}(\mathcal{A})}{|a_{i_m\cdots i_m}|} + \varepsilon \right)^{\frac{1}{m-1}} + \sum_{i_2\cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| x_{i_2} \cdots x_{i_m} \\
&\le \sum_{\substack{i_2\cdots i_m \in N_L^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_U^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \left(\frac{p_{i_2}(\mathcal{A})}{|a_{i_2\cdots i_2}|} + \varepsilon \right)^{\frac{1}{m-1}} \cdots \\
&\quad \left(\frac{p_{i_m}(\mathcal{A})}{|a_{i_m\cdots i_m}|} + \varepsilon \right)^{\frac{1}{m-1}} + \sum_{i_2\cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| \\
&\le \sum_{\substack{i_2\cdots i_m \in N_L^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_U^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| \left(\frac{r_{i_2}(\mathcal{A})}{|a_{i_2\cdots i_2}|} + \varepsilon \right)^{\frac{1}{m-1}} \cdots \\
&\quad \left(\frac{r_{i_m}(\mathcal{A})}{|a_{i_m\cdots i_m}|} + \varepsilon \right)^{\frac{1}{m-1}} + \sum_{i_2\cdots i_m \in N_0^{m-1}} |a_{ii_2\cdots i_m}| \\
&\le \sum_{\substack{i_2\cdots i_m \in N_L^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_U^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} \max_{j \in \{i_2,\cdots,i_m\}} \left\{ \frac{r_j(\mathcal{A})}{|a_{j\cdots j}|} + \varepsilon \right\} |a_{ii_2\cdots i_m}|
\end{aligned}$$

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$$\begin{aligned}&+\sum_{i_2\cdots i_m\in N_0^{m-1}}|a_{ii_2\cdots i_m}|\\&=p_i(\mathcal{A})+\varepsilon\sum_{\substack{i_2\cdots i_m\in N_U^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|.\end{aligned}$$

For any $i\in N_L$, according to Equation (2.1) and Lemma 2.8, we have

$$r_i(\mathcal{B})\le p_i(\mathcal{A})+\varepsilon\sum_{\substack{i_2\cdots i_m\in N_U^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|<p_i(\mathcal{A})+|a_{i\cdots i}|-p_i(\mathcal{A})=|a_{ii\cdots i}|=|b_{ii\cdots i}|.$$

And for any $i\in N_U$, we have

$$r_i(\mathcal{B})\le p_i(\mathcal{A})+\varepsilon\sum_{\substack{i_2\cdots i_m\in N_U^{m-1}\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|<p_i(\mathcal{A})+\varepsilon|a_{ii\cdots i}|=|b_{ii\cdots i}|.$$

Thus, we obtain $|b_{ii\cdots i}|>r_i(\mathcal{B})$ $(\forall i\in N)$. That is, $\mathcal{B}$ is a strictly diagonally dominant tensor so that it follows from Lemma 2.5 that $\mathcal{B}$ is a strong $\mathcal{H}$-tensor. Furthermore, $\mathcal{A}$ is a strong $\mathcal{H}$-tensor by Lemma 2.6. The proof is completed. ∎

The following corollary is easily obtained from Theorems 1.2 and 2.9.

COROLLARY 2.10. *Let* $\mathcal{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{R}^{[m,n]}$ *be an even-order symmetric tensor, and* $a_{pp\cdots p}>0$ *for all* $p\in N$. *If* $\mathcal{A}$ *is an* SDD_1 *tensor, then* $\mathcal{A}$ *is positive definite.*

Next, the relations between the various tensor classes described in this section are easily obtained from Theorem 2.9 and Remark 2.4, see Figure 1.

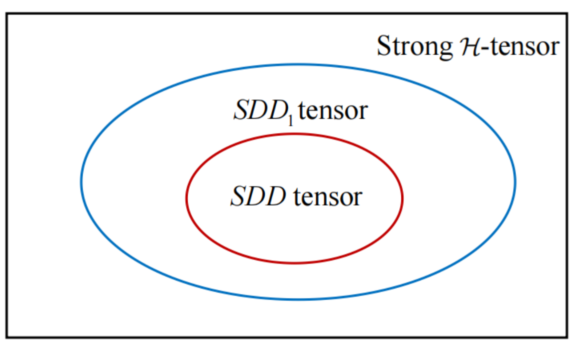

FIG. 1. *Relationships between some tensor classes.*

We present an example of determining whether a given tensor is a strong $\mathcal{H}$-tensor using the definition of the SDD_1 tensor.

EXAMPLE 2.1. *Let us consider tensor* $\mathcal{A}=(a_{ijk})=[A(1,:,:),A(2,:,:),A(3,:,:)]\in\mathbb{C}^{[3,3]}$, *where*

$$A(1,:,:)=\begin{pmatrix}6&0.5&0\\0&2&0\\0.5&1&20\end{pmatrix},\ A(2,:,:)=\begin{pmatrix}1&0&0\\0&20&0\\0&0&3\end{pmatrix},\ A(3,:,:)=\begin{pmatrix}1&0&0\\0&2&0\\1&0&20\end{pmatrix}.$$

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Obviously,

$$|a_{111}| = 6,\ r_1(\mathcal{A}) = 24,\ |a_{222}| = 20,\ r_2(\mathcal{A}) = 4,\ |a_{333}| = 20 \textit{ and } r_3(\mathcal{A}) = 4,$$

so $N_L = \{1\}$, $N_U = \{2, 3\}$. *By calculation, we obtain*

$$\max_{j\in N_U}\left\{\frac{r_j(\mathcal{A})}{|a_{jjj}|}\right\} = \frac{1}{5},$$

when $i \in N_L$, *we get*

$$\begin{aligned} p_1(\mathcal{A}) =& \sum_{\substack{i_2i_3\in N_L^2\\ \delta_{1i_2i_3}=0}} |a_{1i_2i_3}| + \sum_{i_2i_3\in N_U^2} \max_{j\in\{i_2,i_3\}}\left\{\frac{r_j(\mathcal{A})}{|a_{jjj}|}\right\}|a_{1i_2i_3}| + \sum_{i_2i_3\in N_0^2} |a_{1i_2i_3}| \\ =& 0 + \frac{23}{5} + 1 = \frac{28}{5} < 6 = |a_{111}|. \end{aligned}$$

Hence, $\mathcal{A}$ *satisfies the conditions of Theorem 2.9 so that* $\mathcal{A}$ *is a strong* $\mathcal{H}$*-tensor.*

In addition, an example of determining the positive definiteness of homogeneous polynomial of even degree is given.

Example 2.2. *Consider the following 4th-degree homogeneous polynomial*

$$f(x) = \mathcal{A}x^4 = 17x_1^4 + 21x_2^4 + 24x_3^4 + 19x_4^4 - 8x_1^3x_4 + 12x_1^2x_2x_3 - 12x_2x_3^2x_4 - 24x_1x_2x_3x_4,$$

where $x = (x_1, x_2, x_3, x_4)^T$. *Then we can obtain a symmetric tensor* $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,4]}$, *where*

$$\begin{aligned} a_{1111} = 17, a_{2222} = 21, a_{3333} = 24, a_{4444} = 19,\\ a_{1114} = a_{1141} = a_{1411} = a_{4111} = -2,\\ a_{1123} = a_{1132} = a_{1213} = a_{1312} = a_{1231} = a_{1321} = 1,\\ a_{2113} = a_{2131} = a_{2311} = a_{3112} = a_{3121} = a_{3211} = 1,\\ a_{3234} = a_{3243} = a_{3324} = a_{3342} = a_{3423} = a_{3432} = -1,\\ a_{2334} = a_{2343} = a_{2433} = a_{4233} = a_{4323} = a_{4332} = -1,\\ a_{1234} = a_{1243} = a_{1324} = a_{1342} = a_{1423} = a_{1432} = -1,\\ a_{2134} = a_{2143} = a_{2314} = a_{2341} = a_{2413} = a_{2431} = -1,\\ a_{3124} = a_{3142} = a_{3214} = a_{3241} = a_{3412} = a_{3421} = -1,\\ a_{4123} = a_{4132} = a_{4213} = a_{4231} = a_{4312} = a_{4321} = -1, \end{aligned}$$

and others are zeros. Then,

$$\begin{aligned} |a_{1111}| = 17,\ r_1(\mathcal{A}) = 18,\ |a_{2222}| = 21,\ r_2(\mathcal{A}) = 12,\\ |a_{3333}| = 24,\ r_3(\mathcal{A}) = 15,\ |a_{4444}| = 19,\ r_4(\mathcal{A}) = 11, \end{aligned}$$

so $N_L = \{1\}$, $N_U = \{2, 3, 4\}$. *By calculation, we obtain*

$$\max_{j\in N_U}\left\{\frac{r_j(\mathcal{A})}{|a_{jjjj}|}\right\} = \frac{5}{8},$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 40, pp. 551-563, August 2024.

when $i \in N_L$, we get

$$\begin{aligned} p_1(\mathcal{A}) = &\sum_{\substack{i_2i_3i_4 \in N_L^3 \\ \delta_{1i_2i_3i_4}=0}} |a_{1i_2i_3i_4}| + \sum_{i_2i_3i_4 \in N_U^3} \max_{j \in \{i_2,i_3,i_4\}} \left\{ \frac{r_j(\mathcal{A})}{|a_{jjjj}|} \right\} |a_{1i_2i_3i_4}| \\ &+ \sum_{i_2i_3i_4 \in N_0^3} |a_{1i_2i_3i_4}| \\ =& 0 + \frac{15}{4} + 12 = \frac{63}{4} < 17 = |a_{1111}|. \end{aligned}$$

Hence, $\mathcal{A}$ satisfies the conditions of SDD_1 tensor, by Theorem 2.9, $\mathcal{A}$ is a strong $\mathcal{H}$-tensor. Moreover, the diagonal element is greater than 0. Further, from Corollary 2.10, we have that $\mathcal{A}$ is positive definite, that is $f(x)$ is positive definite.

3. B_1-tensor.

At the beginning of this section, we first recall the definition of the B-tensor.

DEFINITION 3.1 (Song and Qi [18]). *A tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called a B-tensor if and only if for all $i \in N$,*

$$\sum_{i_2,\ldots,i_m \in N} a_{ii_2\cdots i_m} > 0,$$

and

$$\frac{1}{n^{m-1}} \left(\sum_{i_2,\ldots,i_m \in N} a_{ii_2\cdots i_m} \right) > a_{ij_2\cdots j_m}, \ \forall (j_2, \ldots, j_m) \neq (i, \ldots, i).$$

In this section, we will introduce a new tensor class inspired by SDD_1 tensors and call it the B_1-tensor, which includes B-tensors as its subclass. Further, we introduce some of its properties. For the convenience of discussion, we also first provide some notations, definitions, and theorems in the following.

Given a tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$, for each row i, denote

$$r_i^+(\mathcal{A}) = \max\{0, a_{ij_2\cdots j_m} : (j_2, \ldots, j_m) \neq (i, \ldots, i)\}. \tag{3.1}$$

Let $\mathcal{A}^+ = (b_{i_1i_2\cdots i_m})$ be the tensor defined as:

$$b_{i_1i_2\cdots i_m} = a_{i_1i_2\cdots i_m} - r_{i_1}^+(\mathcal{A}). \tag{3.2}$$

Clearly, $\mathcal{A}^+$ is a Z-tensor. A tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{C}^{[m,n]}$ is called a Z-tensor if and only if $a_{i_1i_2\cdots i_m} \leq 0$ for all $(i_1, \ldots, i_m) \neq (i, \ldots, i)$ [26].

In this section, we listed the basic definition of a B-tensor. In [18], it was also proved that a B-tensor can be characterized by the following equivalent definition.

DEFINITION 3.2 (Song and Qi [18]). *A tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called a B-tensor if and only if for each $i \in N$,*

$$\sum_{i_2\cdots i_m \in N^{m-1}} |a_{ii_2\cdots i_m}| > n^{m-1} r_i^+(\mathcal{A}),$$

that is,

$$(a_{ii\cdots i} - r_i^+(\mathcal{A})) > \sum_{(i_2,\ldots,i_m) \neq (i,\ldots,i)} (r_i^+(\mathcal{A}) - a_{ii_2\cdots i_m}) = r_i(\mathcal{A}^+).$$

Inspired by the SDD_1 tensor, in the following we give a new class of tensors and call it B_1-tensors.

Electronic Journal of Linear Algebra, ISSN 1081-3810
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Volume 40, pp. 551-563, August 2024.

DEFINITION 3.3. *A tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ *is a* B_1*-tensor if for all* $i \in N$,

$$a_{i\cdots i} - r_i^+(\mathcal{A}) > p_i(\mathcal{A}^+).$$

Next, we investigate some very interesting properties of B_1-tensors.

PROPOSITION 3.4. *If a tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ *is* B*-tensor, then* $\mathcal{A}$ *is a* B_1*-tensor.*

Proof. If $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is B-tensor, then by Definition 3.2,

$$a_{i\cdots i} - r_i^+(\mathcal{A}) > \sum_{\substack{i_2\cdots i_m \in N^{m-1} \\ \delta_{i i_2 \cdots i_m} = 0}} (r_i^+(\mathcal{A}) - a_{i i_2 \cdots i_m}) = r_i(\mathcal{A}^+).$$

However, $r_i(\mathcal{A}^+) \geq p_i(\mathcal{A}^+)$, and hence $a_{i\cdots i} - r_i^+(\mathcal{A}) > p_i(\mathcal{A}^+)$, that is, $\mathcal{A}$ is a B_1-tensor. ◻

Based on the definition of B_1-tensors, it is easy to obtain the following properties.

PROPOSITION 3.5. *Let tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ *be a* B_1*-tensor, then* $\mathcal{A}^+$ *is an* SDD_1 *tensor.*

PROPOSITION 3.6. *If tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ *is a* B_1*-tensor, then* $\mathcal{A}^+$ *has positive diagonal elements.*

Proof. If tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is a B_1-tensor, then we have $a_{i\cdots i} - r_i^+(\mathcal{A}) > p_i(\mathcal{A}^+) \geq 0$, i.e., $a_{i\cdots i} - r_i^+(\mathcal{A}) > 0$. The proof is complete. ◻

From Proposition 3.6, we can obtain the following corollary.

COROLLARY 3.7. *If tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ *is a* B_1*-tensor, then there must be* $a_{i\cdots i} > r_i^+(\mathcal{A})$.

PROPOSITION 3.8. *A tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ *is a* B_1*-tensor if and only if* $\mathcal{A}^+$ *is an* SDD_1 *tensor with positive diagonal entries.*

Proof. If tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is a B_1-tensor, then by Propositions 3.5 and 3.6, $\mathcal{A}^+$ is an SDD_1 tensor with positive diagonal entries. Conversely, if $\mathcal{A}^+$ is an SDD_1 tensor, then $|a_{i\cdots i} - r_i^+(\mathcal{A})| > p_i(\mathcal{A}^+)$. Since $\mathcal{A}^+$ has positive diagonal entries, that is, $a_{i\cdots i} - r_i^+(\mathcal{A}) > 0$, then we have $a_{i\cdots i} - r_i^+(\mathcal{A}) > p_i(\mathcal{A}^+)$. Therefore, $\mathcal{A}$ is B_1-tensor. ◻

PROPOSITION 3.9. *Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ *be a* Z*-tensor with positive diagonal entries. Then* $\mathcal{A}$ *is a* B_1*-tensor if and only if* $\mathcal{A}$ *is an* SDD_1 *tensor.*

Proof. Since $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is a Z-tensor, we have $r_i^+(\mathcal{A}) = 0$ for all $i \in N$, and $\mathcal{A}^+ = \mathcal{A}$. Then $\mathcal{A}$ is a B_1-tensor if and only if $a_{i\cdots i} - r_i^+(\mathcal{A}) > p_i(\mathcal{A}^+)$, that is, $a_{i\cdots i} > p_i(\mathcal{A})$; therefore, $\mathcal{A}$ is an SDD_1 tensor. Conversely, if $\mathcal{A}$ is an SDD_1 tensor, we have $|a_{i\cdots i}| > p_i(\mathcal{A})$. Since $\mathcal{A}$ has positive diagonal entries, then $a_{i\cdots i} > p_i(\mathcal{A})$. Thus, $a_{i\cdots i} - r_i^+(\mathcal{A}) > p_i(\mathcal{A}^+)$ can be obtained immediately. That is to say $\mathcal{A}$ is a B_1-tensor. ◻

The following corollary can be obtained from Proposition 3.9 and Theorem 2.9.

COROLLARY 3.10. *Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ *be a* Z*-tensor with positive diagonal entries. If* $\mathcal{A}$ *is a* B_1*-tensor, then* $\mathcal{A}$ *is a strong* $\mathcal{H}$*-tensor.*

PROPOSITION 3.11. $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ *is a* B_1*-tensor if and only if* $\mathcal{A}^+$ *is a* B_1*-tensor.*

Proof. By Proposition 3.8, $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is B_1-tensor if and only if $\mathcal{A}^+$ is an SDD_1 tensor with positive diagonal entries. Since $\mathcal{A}^+$ is a Z-tensor, according to Proposition 3.9, the conclusion follows immediately. ◻

Electronic Journal of Linear Algebra, ISSN 1081-3810
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PROPOSITION 3.12. *If $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a B_1-tensor, and $\mathcal{D} \in \mathbb{R}^{[m,n]}$ is a nonnegative diagonal tensor of the same order and dimension, then $\mathcal{A} + \mathcal{D}$ is a B_1-tensor.*

Proof. Let $\mathcal{D} = (d_{i_1 i_2 \cdots i_m})$, where

$$d_{ii_2 \cdots i_m} = \begin{cases} d_i, & (i_2, \ldots, i_m) = (i, \ldots, i), \\ 0, & \textit{otherwise}, \end{cases}$$

and $d_i \geq 0$. Let $\mathcal{C} = \mathcal{A} + \mathcal{D}$, where

$$c_{ii_2 \cdots i_m} = \begin{cases} a_{i \cdots i} + d_i, & (i_2, \ldots, i_m) = (i, \ldots, i), \\ a_{ii_2 \cdots i_m}, & \textit{otherwise}. \end{cases}$$

Then, $\mathcal{A}$ and $\mathcal{C}$ have the same nondiagonal elements so that $r_i^+(\mathcal{A}) = r_i^+(\mathcal{C})$.

Next, let us prove $c_{i \cdots i} - r_i^+(\mathcal{C}) > p_i(\mathcal{C}^+)$ for all $i \in N$, where $\mathcal{C}^+ = (v_{i_1 i_2 \cdots i_m})$ with $v_{ii_2 \cdots i_m} = c_{ii_2 \cdots i_m} - r_i^+(\mathcal{C})$.

Since $\mathcal{A}$ is a B_1-tensor, for all $i \in N$, we have

$$c_{i \cdots i} - r_i^+(\mathcal{C}) = a_{i \cdots i} + d_i - r_i^+(\mathcal{A}) \geq a_{i \cdots i} - r_i^+(\mathcal{A}) > p_i(\mathcal{A}^+) \geq 0. \tag{3.3}$$

Meanwhile, we have the following equation that holds:

$$\begin{aligned} r_i(\mathcal{C}^+) &= \sum_{\substack{i_2 \cdots i_m \in N^{m-1} \\ \delta_{ii_2 \cdots i_m} = 0}} |v_{ii_2 \cdots i_m}| = \sum_{\substack{i_2 \cdots i_m \in N^{m-1} \\ \delta_{ii_2 \cdots i_m} = 0}} |c_{ii_2 \cdots i_m} - r_i^+(\mathcal{C})| \\ &= \sum_{\substack{i_2 \cdots i_m \in N^{m-1} \\ \delta_{ii_2 \cdots i_m} = 0}} |a_{ii_2 \cdots i_m} - r_i^+(\mathcal{A})| = r_i(\mathcal{A}^+). \end{aligned}$$

Hence, for $\forall i \in N_L(\mathcal{C}^+)$, we have $|c_{i \cdots i} - r_i^+(\mathcal{C})| \leq r_i(\mathcal{C}^+)$, that is, $|a_{i \cdots i} + d_i - r_i^+(\mathcal{A})| \leq r_i(\mathcal{A}^+)$. We can immediately obtain that $|a_{i \cdots i} - r_i^+(\mathcal{A})| \leq r_i(\mathcal{A}^+)$. Therefore, $i \in N_L(\mathcal{A}^+)$. This indicates that $N_L(\mathcal{C}^+) \subset N_L(\mathcal{A}^+)$. The same method can illustrate that $N_U(\mathcal{A}^+) \subset N_U(\mathcal{C}^+)$.

For $\forall i \in N$,

$$\begin{aligned} & p_i(\mathcal{C}^+) \\ &= \sum_{\substack{i_2 \cdots i_m \in N_L^{m-1}(\mathcal{C}^+) \\ \delta_{ii_2 \cdots i_m} = 0}} |v_{ii_2 \cdots i_m}| + \sum_{\substack{i_2 \cdots i_m \in N_U^{m-1}(\mathcal{C}^+) \\ \delta_{ii_2 \cdots i_m} = 0}} \max_{j \in \{i_2, \cdots, i_m\}} \left\{ \frac{r_j(\mathcal{C}^+)}{|v_{jj \cdots j}|} \right\} |v_{ii_2 \cdots i_m}| \\ & \quad + \sum_{\substack{i_2 \cdots i_m \in N_0^{m-1}(\mathcal{C}^+) \\ \delta_{ii_2 \cdots i_m} = 0}} |v_{ii_2 \cdots i_m}| \\ &= \sum_{\substack{i_2 \cdots i_m \in N_L^{m-1}(\mathcal{C}^+) \\ \delta_{ii_2 \cdots i_m} = 0}} |c_{ii_2 \cdots i_m} - r_i^+(\mathcal{C})| + \sum_{\substack{i_2 \cdots i_m \in N_U^{m-1}(\mathcal{C}^+) \\ \delta_{ii_2 \cdots i_m} = 0}} \max_{j \in \{i_2, \cdots, i_m\}} \left\{ \frac{r_j(\mathcal{C}^+)}{|c_{jj \cdots j} - r_j^+(\mathcal{C})|} \right\} \\ & \quad |c_{ii_2 \cdots i_m} - r_i^+(\mathcal{C})| + \sum_{\substack{i_2 \cdots i_m \in N_0^{m-1}(\mathcal{C}^+) \\ \delta_{ii_2 \cdots i_m} = 0}} |c_{ii_2 \cdots i_m} - r_i^+(\mathcal{C})| \end{aligned}$$

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Volume 40, pp. 551-563, August 2024.

$$=\sum_{\substack{i_2\cdots i_m\in N_L^{m-1}(\mathcal{C}^+)\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}-r_i^+(\mathcal{A})|+\sum_{\substack{i_2\cdots i_m\in N_U^{m-1}(\mathcal{C}^+)\\ \delta_{ii_2\cdots i_m}=0}}\max_{j\in\{i_2,\cdots,i_m\}}\left\{\frac{r_j(\mathcal{A}^+)}{|a_{jj\cdots j}+d_j-r_j^+(\mathcal{A})|}\right\}$$
$$|a_{ii_2\cdots i_m}-r_i^+(\mathcal{A})|+\sum_{\substack{i_2\cdots i_m\in N_0^{m-1}(\mathcal{C}^+)\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}-r_i^+(\mathcal{A})|$$
$$\leq\sum_{\substack{i_2\cdots i_m\in N_L^{m-1}(\mathcal{A}^+)\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}-r_i^+(\mathcal{A})|+\sum_{\substack{i_2\cdots i_m\in N_U^{m-1}(\mathcal{A}^+)\\ \delta_{ii_2\cdots i_m}=0}}\max_{j\in\{i_2,\cdots,i_m\}}\left\{\frac{r_j(\mathcal{A}^+)}{|a_{jj\cdots j}-r_j^+(\mathcal{A})|}\right\}$$
$$|a_{ii_2\cdots i_m}-r_i^+(\mathcal{A})|+\sum_{\substack{i_2\cdots i_m\in N_0^{m-1}(\mathcal{A}^+)\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}-r_i^+(\mathcal{A})|$$
$$=p_i(\mathcal{A}^+).$$

Combining equation (3.3), there is $c_{i\cdots i}-r_i^+(\mathcal{C})>p_i(\mathcal{C}^+)$, and this proof is completed. ◻

Proposition 3.13. *If $\mathcal{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{R}^{[m,n]}$ is a B_1-tensor, then we can write $\mathcal{A}$ as $\mathcal{A}=\mathcal{B}+\mathcal{C}$, where $\mathcal{B}$ is a Z-tensor with positive diagonal entries and $\mathcal{C}$ is a nonnegative tensor with $c_{ii_2\cdots i_m}=c_i,\ c_i\geq 0,\ \forall i\in N$. In particular, if $\mathcal{A}$ is a B_1-tensor and Z-tensor, then $\mathcal{C}$ is a zero tensor.*

Proof. Suppose $\mathcal{B}=(b_{i_1i_2\cdots i_m})$, where $b_{ii_2\cdots i_m}=a_{ii_2\cdots i_m}-r_i^+(\mathcal{A})$. According to the definition of $r_i^+(\mathcal{A})$, it can be inferred that $b_{ii_2\cdots i_m}\leq 0,\ (i_2,\ldots,i_m)\neq(i,\ldots i)$. And since $\mathcal{A}$ is a B_1-tensor, $b_{i\cdots i}=a_{i\cdots i}-r_i^+(\mathcal{A})>p_i(\mathcal{B})\geq 0$. Then $\mathcal{B}$ is a Z-tensor with positive diagonal entries.

Let $\mathcal{C}=c_{ii_2\cdots i_m}$ with $c_{ii_2\cdots i_m}=r_i^+(\mathcal{A})$, obviously $\mathcal{C}$ is a nonnegative tensor with $c_i=r_i^+(\mathcal{A})$. In particular, if $\mathcal{A}$ is a B_1-tensor and Z-tensor, we have $r_i^+(\mathcal{A})=0$, then $\mathcal{C}$ is a zero tensor. ◻

Proposition 3.14. *Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{R}^{[m,n]}$ be a B_1-tensor, and let $\mathcal{C}=(c_{i_1i_2\cdots i_m})$ be a nonnegative tensor of the form $c_{ii_2\cdots i_m}=c_i$, then $\mathcal{A}+\mathcal{C}$ is a B_1-tensor.*

Proof. Let $\mathcal{B}=\mathcal{A}+\mathcal{C}$, where $\mathcal{B}=(b_{i_1i_2\cdots i_m})$. By Proposition 3.11, we need to prove that $\mathcal{B}^+=(d_{i_1i_2\cdots i_m})$ is a B_1-tensor. By definition, we can see that for all $i\in N$,

$$r_i^+(\mathcal{B})=r_i^+(\mathcal{A})+c_i.$$

On the other hand, for all $i,i_2,\ldots,i_m\in N$, we have

$$d_{ii_2\cdots i_m}=b_{ii_2\cdots i_m}-r_i^+(\mathcal{B})=(a_{ii_2\cdots i_m}+c_i)-(r_i^+(\mathcal{A})+c_i)=a_{ii_2\cdots i_m}-r_i^+(\mathcal{A}).$$

This means that $\mathcal{B}^+=\mathcal{A}^+$. Since $\mathcal{A}$ is a B_1-tensor, by Proposition 3.11, $\mathcal{A}^+$ is a B_1-tensor. The conclusion follows immediately. ◻

Proposition 3.15. *If $\mathcal{A}=(a_{i_1i_2\cdots i_m})\in\mathbb{R}^{[m,n]}$ is a B_1-tensor, then every principal sub-tensor of $\mathcal{A}$ is also a B_1-tensor.*

Proof. Let J be a nonempty subset of N with $|J|=r$, $\mathcal{C}=\mathcal{A}_r^J\in\mathbb{R}^{[m,r]}$ be the principal sub-tensor of $\mathcal{A}$ and $\mathcal{A}^+=(b_{i_1i_2\cdots i_m})$. Since $\mathcal{A}$ is a B_1-tensor, we have $a_{i\cdots i}-r_i^+(\mathcal{A})>p_i(\mathcal{A}^+)\geq 0$. Thus, $a_{i\cdots i}>r_i^+(\mathcal{A})\geq r_i^+(\mathcal{C})$. Next, let us prove $a_{i\cdots i}-r_i^+(\mathcal{C})>p_i(\mathcal{C}^+)$ for all $i\in J$. We have

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$$r_i(\mathcal{A}^+) = \sum_{\substack{i_2\cdots i_m \in N^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m} - r_i^+(\mathcal{A})| = \sum_{\substack{i_2\cdots i_m \in N^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(\mathcal{A}) - a_{ii_2\cdots i_m})$$
$$\geq \sum_{\substack{i_2\cdots i_m \in J^{m-1} \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(\mathcal{C}) - a_{ii_2\cdots i_m}) = r_i(\mathcal{C}^+).$$

For $\forall i \in N_L(\mathcal{C}^+)$, we have $0 \leq a_{i\cdots i} - r_i^+(\mathcal{C}) \leq r_i(\mathcal{C}^+) \leq r_i(\mathcal{A}^+)$, and hence $0 \leq a_{i\cdots i} - r_i^+(\mathcal{A}) \leq r_i(\mathcal{A}^+)$. Therefore, $i \in N_L(\mathcal{A}^+)$. This indicates that $N_L(\mathcal{C}^+) \subset N_L(\mathcal{A}^+)$. The same method can illustrate that $N_U(\mathcal{A}^+) \subset N_U(\mathcal{C}^+)$.
For $\forall i \in J$,

$$\begin{aligned}
& a_{i\cdots i} - r_i^+(\mathcal{C}) \\
\geq & a_{i\cdots i} - r_i^+(\mathcal{A}) > p_i(\mathcal{A}^+) \\
= & \sum_{\substack{i_2\cdots i_m \in N_L^{m-1}(\mathcal{A}^+) \\ \delta_{ii_2\cdots i_m}=0}} |b_{ii_2\cdots i_m}| + \sum_{\substack{i_2\cdots i_m \in N_U^{m-1}(\mathcal{A}^+) \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{ \frac{r_j(\mathcal{A}^+)}{|b_{jj\cdots j}|} \right\} |b_{ii_2\cdots i_m}| \\
& + \sum_{\substack{i_2\cdots i_m \in N_0^{m-1}(\mathcal{A}^+) \\ \delta_{ii_2\cdots i_m}=0}} |b_{ii_2\cdots i_m}| \\
= & \sum_{\substack{i_2\cdots i_m \in N_L^{m-1}(\mathcal{A}^+) \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(\mathcal{A}) - a_{ii_2\cdots i_m}) + \sum_{\substack{i_2\cdots i_m \in N_U^{m-1}(\mathcal{A}^+) \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{ \frac{r_j(\mathcal{A}^+)}{a_{jj\cdots j} - r_j^+(\mathcal{A})} \right\} \\
& (r_i^+(\mathcal{A}) - a_{ii_2\cdots i_m}) + \sum_{\substack{i_2\cdots i_m \in N_0^{m-1}(\mathcal{A}^+) \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(\mathcal{A}) - a_{ii_2\cdots i_m}) \\
\geq & \sum_{\substack{i_2\cdots i_m \in N_L^{m-1}(\mathcal{C}^+) \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(\mathcal{C}) - a_{ii_2\cdots i_m}) + \sum_{\substack{i_2\cdots i_m \in N_U^{m-1}(\mathcal{C}^+) \\ \delta_{ii_2\cdots i_m}=0}} \max_{j\in\{i_2,\cdots,i_m\}} \left\{ \frac{r_j(\mathcal{C}^+)}{a_{jj\cdots j} - r_j^+(\mathcal{A})} \right\} \\
& (r_i^+(\mathcal{C}) - a_{ii_2\cdots i_m}) + \sum_{\substack{i_2\cdots i_m \in N_0^{m-1}(\mathcal{C}^+) \\ \delta_{ii_2\cdots i_m}=0}} (r_i^+(\mathcal{C}) - a_{ii_2\cdots i_m}) \\
= & p_i(\mathcal{C}^+)
\end{aligned}$$

Overall, there is $a_{i\cdots i} - r_i^+(\mathcal{C}) > p_i(\mathcal{C}^+)$, and this proof is completed. ∎

4. Conclusions. In this paper, SDD_1 tensors were proposed, and it was proved that an SDD_1 tensor is a strong $\mathcal{H}$-tensor. Meanwhile, we give an application of SDD_1 tensor, which is determining the positive definiteness of even-order real symmetric tensors. Furthermore, the B_1-tensor was proposed by the SDD_1 tensor, and some properties of B_1-tensor were introduced. Some examples were given to illustrate those new results.

Availability of data and materials. Not applicable.

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