



SPECTRAL CONDITIONS FOR RAINBOW MATCHINGS OF BIPARTITE GRAPHS*

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Abstract. Let $\mathcal{G} = \{G_1, G_2, \dots, G_k\}$ be a family of subgraphs of complete bipartite graph $K_{a,b}$ for $k \leq a \leq b$. In this paper, we prove that, if the spectral radius $\lambda(G_i)$ of G_i satisfies $\lambda(G_i) \geq \sqrt{(k-1)b}$ for all $i \in [k]$, then \mathcal{G} contains a rainbow matching unless $G_1 = G_2 = \dots = G_k \cong K_{k-1,b} \cup \overline{K_{a-k+1}}$.

Key words. Spectral radius, Rainbow matching, Bipartite graph.

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1. Introduction. Let $G = (V(G), E(G))$ be a simple graph with the vertex set $V(G) = [n] = \{1, 2, \dots, n\}$ and edge set $E(G) \subseteq \binom{[n]}{2} = \{ij : i, j \in [n], i \neq j\}$. Denote the number of edges of G , called the size of G , by $e(G)$. For a vertex i , the degree $d_G(i)$ is the cardinality of the neighborhood $N_G(i) = \{j : ij \in E(G)\}$ of i in G . We write $G \setminus \{ij\}$ to denote the graph obtained from G by deleting the edge ij . For two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, the *union* and the *join* of G and H are defined by $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$ and

$$G \vee H = (V(G) \cup V(H), E(G) \cup E(H) \cup \{ij : i \in V(G), j \in V(H)\}),$$

respectively. A *component* of G is a maximal connected subgraph of G . We denote by K_a and \overline{K}_a the complete graph and the empty graph of order n . Denote the complete bipartite graph with two parts of orders a and b by $K_{a,b}$.

Let $\mathcal{G} = \{G_1, G_2, \dots, G_k\}$ be a family of graphs with the same vertex set. A *rainbow matching* of \mathcal{G} is a set of k independent edges from different G_i for each edge. We call \mathcal{G} contains a rainbow matching if such a set of edges exists.

Naturally, it is intriguing to explore the properties of graphs of \mathcal{G} that ensure the presence of a rainbow matching. On this topic, there are many results focusing the size of graphs, one of which was obtained by Akiyama and Frankl [1].

THEOREM 1.1. [1] *Let n, k be positive integers with $n \geq 2k$. Let $\mathcal{G} = \{G_1, G_2, \dots, G_k\}$ be a family of graphs on the same vertex set $[n]$. If*

$$e(G_i) > \max \left\{ \binom{2k-1}{2}, \binom{n}{2} - \binom{n-k+1}{2} \right\},$$

for all $i \in [k]$, then \mathcal{G} contains a rainbow matching.

In addition, rainbow matching of bipartite graphs is an important topic in graph theory. When \mathcal{G} is a subset of subgraphs of $K_{n,n}$, Aharoni and Howard [2] proved the following result.

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THEOREM 1.2. [2] Let $\mathcal{G} = \{G_1, G_2, \dots, G_k\}$ be a family of subgraphs of $K_{n,n}$. If

$$e(G_i) > (k - 1)n,$$

for all $i \in [k]$, then \mathcal{G} contains a rainbow matching.

Let $\mathcal{G} = \{G_1, G_2, \dots, G_k\}$ be a family of not necessarily distinct graphs with the same vertex set V . A graph H with vertex set V is defined as \mathcal{G} -rainbow if there exists a bijection $\phi : E(H) \rightarrow [k]$ such that $e \in E(G_{\phi(e)})$ for each $e \in E(H)$. An interesting question was proposed by Joos and Kim [6] as follows.

QUESTION 1.3. Let H be a graph with k edges, \mathbf{G} be a family of graphs, and $\mathcal{G} = \{G_1, G_2, \dots, G_k\}$ be collection of not necessarily distinct graphs with the same vertex set V such that $G_i \in \mathbf{G}$ for all $1 \leq i \leq k$. Which properties imposed on \mathbf{G} yield a \mathcal{G} -rainbow graph isomorphic to H ?

Let $A(G)$ be the adjacency matrix of G and $\lambda(G)$ be the largest eigenvalue of $A(G)$. Clearly, $\lambda(G)$ is the spectral radius of G . Recently, Guo et al. [4] considered the property concerning spectral radius of graphs when H is a matching of size k and obtained the following result.

THEOREM 1.4. [4] Let n, k be positive integers with $n \geq 2k$. Let $\mathcal{G} = \{G_1, G_2, \dots, G_k\}$ be a family of graphs on the same vertex set $[n]$. If

$$\lambda(G_i) \geq \max \left\{ 2k - 2, \frac{1}{2}(k - 2 + \sqrt{4(k - 1)n - 3k^2 + 4k}) \right\},$$

for all $i \in [k]$, then

- (1) for $2k \leq n < 3k - 1$, \mathcal{G} admits a rainbow matching unless $G_1 = G_2 = \dots = G_k \cong K_{2k-1} \cup \overline{K_{n-2k+1}}$;
- (2) for $n = 3k - 1$, \mathcal{G} admits a rainbow matching unless $G_1 = G_2 = \dots = G_k \cong K_{k-1} \vee \overline{K_{n-k+1}}$ or $G_1 = G_2 = \dots = G_k \cong K_{2k-1} \cup \overline{K_{n-2k+1}}$;
- (3) for $n > 3k - 1$, \mathcal{G} admits a rainbow matching unless $G_1 = G_2 = \dots = G_k \cong K_{k-1} \vee \overline{K_{n-k+1}}$.

In this paper, inspired by the above result, we consider the condition of spectral radius of graphs when \mathbf{G} is a family of bipartite graphs and H is a matching of size k and obtain the following result.

THEOREM 1.5. Let $\mathcal{G} = \{G_1, G_2, \dots, G_k\}$ be a family of subgraphs of $K_{a,b}$ with $k \leq a \leq b$. If

$$\lambda(G_i) \geq \sqrt{(k - 1)b},$$

for all $i \in [k]$, then \mathcal{G} contains a rainbow matching unless $G_1 = G_2 = \dots = G_k \cong K_{k-1,b} \cup \overline{K_{a-k+1}}$.

As a corollary, the following result can be viewed as a spectral version of Theorem 1.2.

COROLLARY 1.6. Let $\mathcal{G} = \{G_1, G_2, \dots, G_k\}$ be a family of subgraphs of $K_{n,n}$. If

$$\lambda(G_i) \geq \sqrt{(k - 1)n},$$

for all $i \in [k]$, then \mathcal{G} contains a rainbow matching unless $G_1 = G_2 = \dots = G_k \cong K_{k-1,n} \cup \overline{K_{n-k+1}}$.

2. Preliminaries. In this section, we recall some fundamental properties related to graph spectrum, which can be also found in [7] for details.

THEOREM 2.1. [7] If H is a proper subgraph of a connected graph G , then $\lambda(H) < \lambda(G)$.

DEFINITION 2.2. [7] Let A be an $n \times n$ real matrix whose rows and columns are indexed by $X = 1, 2, \dots, n$. We partition X into X_1, X_2, \dots, X_k in order and rewrite A according X_1, X_2, \dots, X_k as follows:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,k} \\ \vdots & \ddots & \vdots \\ A_{k,1} & \cdots & A_{k,k} \end{pmatrix},$$

where $A_{i,j}$ is the block of A formed by rows in X_i and the columns in X_j . Let $b_{i,j}$ denote the average row sum of $A_{i,j}$. Then the matrix $B = [b_{i,j}]$ will be called the quotient matrix of the partition of A . In particular, when the row sum of each block $A_{i,j}$ is constant, the partition is called an equitable partition.

THEOREM 2.3. [7] Let $A \geq 0$ be an irreducible square matrix and B be the quotient matrix of an equitable partition of A . Then the spectrum of A contains the spectrum of B and $\lambda(A) = \lambda(B)$.

The shifting technique is a very important and widely used tools in extremal set theory. For more background on the applications of shifting, we refer the reader to the survey by Frankl [3]. Now we introduce the notion of shifting and some of its fundamental properties.

For two integers i and j with $1 \leq i < j \leq n$ and an edge $e \in E(G)$, define the (i, j) -shift S_{ij} on e and G as follows:

$$S_{ij}(e) = \begin{cases} (e \setminus \{j\}) \cup \{i\}, & \text{if } i \notin e, j \in e \text{ and } (e \setminus \{j\}) \cup \{i\} \notin E(G), \\ e, & \text{otherwise.} \end{cases}$$

$$S_{ij}(G) = (V(G), \{S_{ij}(e) : e \in E(G)\}).$$

The relationship between the rainbow matching and the spectral radius of G and $S_{ij}(G)$ is as follows.

PROPOSITION 2.4. [5] If $\mathcal{G} = \{G_1, G_2, \dots, G_k\}$ contains no rainbow matching, so does $S_{ij}(\mathcal{G}) = \{S_{ij}(G_1), S_{ij}(G_2), \dots, S_{ij}(G_k)\}$.

PROPOSITION 2.5. [4] For any graph G and two integers i and j with $1 \leq i < j \leq n$,

$$\lambda(G) \leq \lambda(S_{ij}(G)),$$

with equality holds if and only if $G \cong S_{ij}(G)$.

We say that G is shifted on $[n_1, n_2]$ if $S_{ij}(G) = G$ or, equivalently, $N_G(j) \setminus \{i\} \subseteq N_G(i) \setminus \{j\}$, for all pair $\{i, j\}$ with $n_1 \leq i < j \leq n_2$.

3. The proof of Theorem 1.5.

We now prove Theorem 1.5. Suppose to the contrary that $\mathcal{G} = \{G_1, G_2, \dots, G_k\}$ contains no rainbow matching. We label the two parts of vertices of $K_{a,b}$ by

$$1, 2, \dots, a \text{ and } 1', 2' \dots, b'.$$

Denote by $S(G)$, the bipartite graph obtained from the bipartite graph G by a series of shifts and is invariant under all possible shifts on $[a]$ and $[b]'$.

By Proposition 2.4, $\{S(G_1), S(G_2), \dots, S(G_k)\}$ contains no rainbow matching too. Let $S(G_i) = S_i$ for $i \in [k]$. By Proposition 2.5, we have

$$(3.1) \quad \lambda(S_i) \geq \lambda(G_i) \geq \sqrt{(k-1)b}.$$

Let $m_j = \{j, (k - j + 1)'\}$ for $j \in [k]$. We first prove that $\{m_2, \dots, m_{k-1}\} \subseteq E(S_i)$ for each $i \in [k]$. Otherwise, if there exists $r \in [2, k - 1]$ such that $m_r \notin E(S_i)$. Then for any s with $r \leq s \leq a$ and t with $(k - r + 1)' \leq t' \leq b'$, we have $st' \notin E(S_i)$. Let $H_r = K_{[a],[b]}' \setminus \{s't' : s' \in [r, a], t' \in [k - r + 1, b]'\}$. It is clear that $S_i \subseteq H_r$, and hence, by Theorem 2.1, we obtain $\lambda(S_i) \leq \lambda(H_r)$.

For $r \in [2, k - 1]$, we have that H_r is connected. Then $V(H_r)$ can be partitioned into four parts as follows:

$$V(H_r) = \{V_1 = [r - 1], V_2 = [r, a], V_3 = [k - r]', V_4 = [k - r + 1, b]'\}.$$

Clearly, this partition is an equitable partition. Then its quotient matrix

$$B = \begin{pmatrix} 0 & 0 & k - r & b - k + r \\ 0 & 0 & k - r & 0 \\ r - 1 & a - r + 1 & 0 & 0 \\ r - 1 & 0 & 0 & 0 \end{pmatrix},$$

and then the characteristic polynomial of B

$$\begin{aligned} \phi(B, x) &= x^4 - ((r - 1)b + (k - r)(a - r + 1))x^2 \\ &\quad + (r - 1)(k - r)(a - r + 1)(b - k + r) \\ &= x^4 - px^2 + q, \end{aligned}$$

where

$$p = (r - 1)b + (k - r)(a - r + 1) > 0,$$

and

$$q = (r - 1)(k - r)(a - r + 1)(b - k + r) > 0.$$

For convenience, let $\lambda = \lambda(H_i)$. Then by Theorem 2.3, we have

$$\phi(B, \lambda) = \lambda^4 - p\lambda^2 + q = 0.$$

Let $\alpha_1 = \frac{1}{2} \left(p - \sqrt{p^2 - 4q} \right)$ and $\alpha_2 = \frac{1}{2} \left(p + \sqrt{p^2 - 4q} \right)$. Then we have $\lambda = \sqrt{\alpha_2} \geq \sqrt{\alpha_1} > 0$ and $p = \alpha_1 + \alpha_2$. Thus, we obtain $\lambda < \sqrt{p}$.

Moreover, we note that

$$\begin{aligned} p &= (r - 1)b + (k - r)(a - r + 1) \\ &\leq (r - 1)b + (k - r)(b - r + 1) \\ &= (k - 1)b - (r - 1)(k - r) \\ &< (k - 1)b. \end{aligned}$$

Therefore, $\lambda(S_i) \leq \lambda < \sqrt{p} < \sqrt{(k - 1)b}$ for $i \in [k]$, which contradicts Inequality (3.1). Hence, $\{m_2, \dots, m_{k-1}\} \subseteq E(S_i)$ for each $i \in [k]$. Recall that $\{S_1, S_2, \dots, S_k\}$ contains no rainbow matching. Thus, if there exist $j, j' \in [k]$ such that $j \neq j'$, $m_1 \in S_j$ and $m_k \in S_{j'}$, then $\{m_1, \dots, m_k\}$ is a rainbow matching of $\{S_1, S_2, \dots, S_k\}$, a contradiction. Thus, we may discuss the following three cases.

Case 1. There exists an integer $l \in [k]$ such that $m_1, m_k \in E(S_l) \setminus (\bigcup_{i \in [k] \setminus l} E(S_i))$.

We, without loss of generality, assume that $l = k$. Then $m_1 = \{1, k'\}$ and $m_k = \{k, 1'\} \notin E(S_i)$ for each $i \in [k-1]$. Clearly, S_i is a subgraph of $H_k = K_{[k-1],[b]'} \cup \overline{K_{a-k+1}}$. Since $m_1 \notin E(S_i)$ and $m_1 \in H_k$, we have that S_i is the proper subgraph of H_k . Therefore, for $i \in [k-1]$,

$$\lambda(G_i) \leq \lambda(S_i) < \lambda(H_k) = \sqrt{(k-1)b},$$

a contradiction.

Case 2. $m_1 \notin \bigcup_{i \in [k]} E(S_i)$.

Clearly, S_i is a subgraph of $H_1 = K_{[a],[k-1]'} \cup \overline{K_{b-k+1}}$ for each $i \in [k]$ in this case. We have

$$\sqrt{(k-1)b} \leq \lambda(G_i) \leq \lambda(S_i) \leq \lambda(H_1) = \sqrt{(k-1)a}.$$

If $a < b$, a contradiction is obvious. If $a = b$, we obtain $G_i \cong S_i \cong H_1 \cong H_k$ for each $i \in [k]$.

Case 3. $m_k \notin \bigcup_{i \in [k]} E(S_i)$.

Clearly, S_i is a subgraph of $H_k = K_{[k-1],[b]'} \cup \overline{K_{a-k+1}}$ for each $i \in [k]$ in this case. We have

$$\sqrt{(k-1)b} \leq \lambda(G_i) \leq \lambda(S_i) \leq \lambda(H_k) = \sqrt{(k-1)b},$$

and hence $G_i \cong S_i \cong H_k$ for each $i \in [k]$.

Thus, we have

$$\lambda(G_1) = \lambda(G_2) = \cdots = \lambda(G_k) = \sqrt{(k-1)b},$$

and

$$G_1 = G_2 = \cdots = G_k \cong K_{k-1,b} \cup \overline{K_{a-k+1}}.$$

This completes the proof.

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