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# THE ANGULAR SPECTRUM OF THE 3 × 3 COPOSITIVE CONE[*]

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**Abstract.** Given a vector $x$ and a closed convex cone $\mathcal{C}$ in an $n$-dimensional inner product space. If $x$ is not in the dual cone of $\mathcal{C}$, then the maximal angle between $x$ and $\mathcal{C}$ is greater than $\frac{\pi}{2}$. In this case, a formula regarding the maximal angle between $x$ and $\mathcal{C}$ is given in terms of the metric projection of $-x$ on $\mathcal{C}$. Critical angles between two convex cones that are greater than or equal to $\frac{\pi}{2}$ are shown to be Nash angles by using this formula. Furthermore, some properties of critical pairs of the cone that is the sum of the $n \times n$ positive semidefinite cone and the cone of all $n \times n$ symmetric nonnegative matrices are presented. Since the $n \times n$ copositive cone is the same as the sum of the $n \times n$ positive semidefinite cone and the cone of all $n \times n$ symmetric nonnegative matrices for $n \leq 4$, a detailed discussion on how to obtain the angular spectrum of the copositive cone of order 3 is given using the results proved in this paper.

**Key words.** Maximal angles, Critical angles, Nash angles, Copositive cone, Metric projection.

**AMS subject classifications.** 15B48, 52A20, 15A18, 52A40.

## 1. Introduction.

In a series of papers [6, 9, 11], Iusem and Seeger have laid the groundwork for the study of the angular structure of a convex cone in a finite dimensional inner product space such as the $n$-dimensional Euclidean space $\mathbb{R}^n$. They proposed three kinds of angles for a convex cone, namely the maximal angle, critical angles, and Nash angles. The concepts of the maximal angle, critical angles, and Nash angles have been extended to angles between two convex cones by Seeger and Sossa in [14, 15]. While there is only one maximal angle for a given pair of convex cones, there may have many critical angles and Nash angles. For convenience, we recall the definitions of these angles from [14, 15]. The reader is referred to these references for reasons why these angles are introduced.

Definition 1.1. Let $V$ be an inner product space. For $u \in V$ and $v \in V$, $\langle u, v\rangle$ is used to denote the inner product of $u$ and $v$, and $\|u\| = \sqrt{\langle u, u\rangle}$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be closed convex cones in $V$ with their dual cones denoted by $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$, that is, $\mathcal{C}_i^* = \{v \in V \mid \langle u, v\rangle \geq 0 \text{ for all } u \in \mathcal{C}_i\}$, $i = 1, 2$. Let $\bar{u} \in \mathcal{C}_1$ and $\bar{v} \in \mathcal{C}_2$ be two unit vectors.

(i) $(\bar{u}, \bar{v})$ is an antipodal pair if $\bar{u}$ and $\bar{v}$ achieve the maximal angle between $\mathcal{C}_1$ and $\mathcal{C}_2$, that is

$$\arccos\langle \bar{u}, \bar{v}\rangle = \theta_{max}(\mathcal{C}_1, \mathcal{C}_2) \equiv \max_{\substack{u \in \mathcal{C}_1, v \in \mathcal{C}_2 \\ u \neq 0, v \neq 0}} \arccos \frac{\langle u, v\rangle}{\|u\|\|v\|}.$$

(ii) The angle $\theta(\bar{u}, \bar{v}) = \arccos\langle \bar{u}, \bar{v}\rangle$ formed by a critical pair $(\bar{u}, \bar{v})$ is called a critical angle. That a pair $(\bar{u}, \bar{v})$ is critical means that $\bar{u} - \langle \bar{u}, \bar{v}\rangle \bar{v} \in \mathcal{C}_2^*$ and $\bar{v} - \langle \bar{u}, \bar{v}\rangle \bar{u} \in \mathcal{C}_1^*$. A proper critical pair is a critical pair with $|\langle \bar{u}, \bar{v}\rangle| \neq 1$. The set of all proper critical angles of $(\mathcal{C}_1, \mathcal{C}_2)$ is called the angular spectrum of $(\mathcal{C}_1, \mathcal{C}_2)$.

(iii) $(\bar{u}, \bar{v})$ is a Nash antipodal pair of $(\mathcal{C}_1, \mathcal{C}_2)$ if

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$\theta(\bar{u},\bar{v}) \geq \theta(\bar{u},v) \equiv \arccos\frac{\langle \bar{u},v\rangle}{\|v\|}$ for any $v \neq 0$ and $v \in \mathcal{C}_2$ and
$\theta(\bar{u},\bar{v}) \geq \theta(u,\bar{v}) \equiv \arccos\frac{\langle u,\bar{v}\rangle}{\|u\|}$ for any $u \neq 0$ and $u \in \mathcal{C}_1$.
The number $\theta(\bar{u},\bar{v})$ is then called a Nash angle of $(\mathcal{C}_1,\mathcal{C}_2)$.

To find the maximal angle, critical angles, and Nash angles together with pairs of vectors achieving these angles for a given convex cone becomes an interesting research topic. Not only do these angles reveal the angular structure of the convex cone, but also these angles provide information about other properties of the convex cone such as the degree of pointedness and degree of solidity [7, 10]. Iusem and Seeger conducted extensive studies on angular analysis for various cones. For example, a detailed angular analysis for polyhedral cones was given in [6, 9, 11], and angular analysis about elliptic cones and spectral cones were given in [7, 9]. Other work on angular analysis of cones includes numerical computation of critical angles in polyhedral cones in [3, 13], and cardinality of critical angles for certain cones in [3, 4, 8]. While there is extensive work on angular structures of polyhedral cones, elliptic cones, and spectral cones, there are limited results on copositive cones and cones that are the sum of a positive semidefinite cone and a cone of symmetric nonnegative matrices of the same order.

Before we introduce copositive cones and some other matrix cones, we will first need the following definition of a copositive matrix and a completely positive matrix.

Definition 1.2. An $n \times n$ real symmetric matrix $A$ is copositive if $x^\top Ax \geq 0$ for each $x \in \mathbb{R}^n_+$, where $\mathbb{R}^n_+$ is the set of all $n$-dimensional vectors whose entries are nonnegative. An $n \times n$ real symmetric matrix $A$ is completely positive if there exist $x_i \in \mathbb{R}^n_+$ for $i = 1, 2, \ldots, p$, such that $A = \sum_{i=1}^p x_i x_i^\top$.

We let $\mathcal{S}_n$ to represent the inner product space of $n \times n$ real symmetric matrices with the Euclidean inner product, which is defined by $\langle A, B\rangle = Tr(AB)$ for $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_n$. The $n \times n$ positive semidefinite cone is denoted by $\mathcal{P}_n$, and the cone of all $n \times n$ symmetric nonnegative matrices is denoted by $\mathcal{N}_n$. We use $\mathcal{COP}_n$ to represent the $n \times n$ copositive cone that is the set of all $n \times n$ copositive matrices, and $\mathcal{C}_n$ to represent the $n \times n$ completely positive cone that is the set of all $n \times n$ completely positive matrices. With respect to the Euclidean inner product, it is known that $\mathcal{P}_n$ and $\mathcal{N}_n$ are self-dual, $\mathcal{COP}_n$ and $\mathcal{C}_n$ are dual cones, and $\mathcal{P}_n + \mathcal{N}_n$ and the doubly nonnegative cone $\mathcal{P}_n \cap \mathcal{N}_n$ are dual to each other. We have the inclusions regarding these cones: $\mathcal{C}_n \subseteq \mathcal{P}_n \cap \mathcal{N}_n \subseteq \mathcal{P}_n + \mathcal{N}_n \subseteq \mathcal{COP}_n$. It is a classical result that for $n \leq 4$, $\mathcal{COP}_n = \mathcal{P}_n + \mathcal{N}_n$ and $\mathcal{C}_n = \mathcal{P}_n \cap \mathcal{N}_n$. For $n \geq 5$, $(\mathcal{P}_n \cap \mathcal{N}_n)\backslash\mathcal{C}_n$ and $\mathcal{COP}_n\backslash(\mathcal{P}_n + \mathcal{N}_n)$ are not empty. Moreover, a representation of $R \in \mathcal{P}_n + \mathcal{N}_n$ as $R = X + A$ with $X \in \mathcal{P}_n$ and $A \in \mathcal{N}_n$ may not be unique.

In [5], Hiriart-Urruty and Seeger proved that the maximal angle for the $2 \times 2$ copositive cone is $\frac{3\pi}{4}$. They remarked that intensive numerical experimentation with randomly generated pairs of copositive matrices of order 3 has shown that the maximal angle of the $3 \times 3$ copositive cone is also $\frac{3\pi}{4}$ even though a serious argument could not be provided at the time. They further conjectured that the maximal angle of the $n \times n$ copositive cone is $\frac{3\pi}{4}$. However, this conjecture was disproved in [2]. The disproof is based on a construction of sequences of positive semidefinite matrices $P_k$ and symmetric nonnegative matrices $N_k$ with $\lim_{k\to\infty} \arccos\frac{\langle P_k, N_k\rangle}{\|P_k\|\|N_k\|} = \pi$. It remains an open question to compute the maximal angle of the $n \times n$ copositive cone and the maximal angle for the cone that is the sum of the $n \times n$ positive semidefinite cone and the cone of $n \times n$ symmetric nonnegative matrices for $n > 2$ as indicated in [1]. It is also an open question to verify whether these maximal angles are always attained at a pair of a positive semidefinite matrix $P$ and a symmetric nonnegative matrix $N$.

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Compared to $\mathcal{COP}_n$, $\mathcal{P}_n$ and $\mathcal{N}_n$ are relatively easier to deal with. This statement is due to the fact that there are many available properties for $\mathcal{P}_n$ and $\mathcal{N}_n$. For example, facial structures of these two cones are easy to describe. Even though $\mathcal{P}_n$ and $\mathcal{N}_n$ have many properties, to find the maximal angle between $\mathcal{P}_n$ and $\mathcal{N}_n$ in general has not been done. In [2], only the maximal angles between $\mathcal{P}_n$ and $\mathcal{N}_n$ for $n = 2, 3, 4$ have been calculated. In [16], the maximal angle between $\mathcal{P}_n$ and $\mathcal{N}_n$ for $n = 5$ has been found. For $n \geq 6$, we have no idea so far what the maximal angle between $\mathcal{P}_n$ and $\mathcal{N}_n$ is. With these difficulties in mind, we do not expect to get the maximal angle of the $n \times n$ copositive cone easily in general. It is reasonable to start with small $n$. In this paper, after we establish some general results using metric projection on a cone, we prove many properties regarding critical pairs or Nash antipodal pairs of $\mathcal{P}_n + \mathcal{N}_n$. We use these established properties to find the angular spectrum of the copositive cone of order 3.

**2. Metric projection on a cone.** In this paper, we work mostly with the inner product space $\mathcal{S}_n$. When $n = 1$, $\mathcal{S}_n = \mathbb{R}$. It is trivial to consider angles in $\mathbb{R}$. Hence, we always assume that $n \geq 2$. As pointed out in [2] that every $n \times n$ symmetric matrix $A$ has a unique decomposition as a difference of two positive semidefinite matrices orthogonal to each other. In other words, $A = Q - P$, with $Q, P \in \mathcal{P}_n$ and $\langle Q, P\rangle = 0$. $Q$ is called the positive definite part of $A$ and $P$ is called the negative definite part of $A$. Similarly, every $A \in \mathcal{S}_n$ has a unique decomposition as a difference of two nonnegative matrices orthogonal to each other: $A = M - N$, with $M, N \in \mathcal{N}_n$ and $\langle M, N\rangle = 0$. $M$ is called the positive part of $A$ and $N$ is called the negative part of $A$. These two forms of decomposition of a matrix are very useful in computing the maximal angles between $\mathcal{P}_n$ and $\mathcal{N}_n$ for $n = 2, 3, 4$ in [2] and $n = 5$ in [16]. We next generalize this kind of decomposition to a convex cone in an inner product space and use the decomposition to find the maximal angle between a vector $x$ and a convex cone $\mathcal{C}$.

Let $\mathcal{C}$ be a closed convex cone in $V$, $\mathcal{C}^*$ be the dual cone of $\mathcal{C}$. Let $x \in V$ be given such that $x \neq 0$. We are interested in the maximal angle between $x$ and $\mathcal{C}$, which is $\theta_{max} = \max\limits_{y\in\mathcal{C}, y\neq 0} \arccos \frac{\langle x,y\rangle}{\|x\|\|y\|}$. If $x \in \mathcal{C}^*$, then $\langle x, y\rangle \geq 0$, which implies that the maximal angle is less than or equal to $\frac{\pi}{2}$. Since our goal is to find critical angles of some matrix cones, which are usually greater than $\frac{\pi}{2}$, we assume that $x \notin \mathcal{C}^*$.

For $x \in V$, we use $P_\mathcal{C}(x)$ to represent the metric projection of $x$ on $\mathcal{C}$, which is the point of $\mathcal{C}$ closest to $x$, i.e., $\|x - P_\mathcal{C}(x)\| = \min_{y\in\mathcal{C}} \|x - y\|$. Such a point $P_\mathcal{C}(x)$ is well defined as one can see from the related literature. Similarly, we use $P_{\mathcal{C}^*}(x)$ to represent the metric projection of $x$ on $\mathcal{C}^*$ so $\|x - P_{\mathcal{C}^*}(x)\| = \min_{y\in\mathcal{C}^*} \|x-y\|$. By the well-known Moreau decomposition Theorem [12], we know that $x = P_{\mathcal{C}^*}(x)-P_\mathcal{C}(-x)$, and $\langle P_{\mathcal{C}^*}(x), P_\mathcal{C}(-x)\rangle = 0$.

Theorem 2.1. *Let $\mathcal{C}$ be a closed convex cone in $V$ and $x \notin \mathcal{C}^*$. Then the angle between $x$ and $P_C(-x) = P_{\mathcal{C}^*}(x) - x$ is the maximal angle $\theta$ between $x$ and $\mathcal{C}$. Conversely, if $x$ and $y \in \mathcal{C}$ form the maximal angle between $x$ and $\mathcal{C}$, then there is an $\alpha > 0$ such that $y = \alpha P_C(-x)$. Moreover, $\cos\theta = -\frac{\|P_\mathcal{C}(-x)\|}{\|x\|}$.*

*Proof.* We let $y \in \mathcal{C}$ and $y \neq 0$. Since $P_{\mathcal{C}^*}(x) \in \mathcal{C}^*$ and $P_{\mathcal{C}^*}(x) - x \in \mathcal{C}$, we know

$$\begin{aligned}
\frac{\langle x, y\rangle}{\|x\|\|y\|} &= \frac{\langle P_{\mathcal{C}^*}(x) - P_\mathcal{C}(-x), y\rangle}{\|x\|\|y\|} \\
&\geq \frac{\langle -P_\mathcal{C}(-x), y\rangle}{\|x\|\|y\|} \text{ (because } y \in \mathcal{C} \text{ and } P_{\mathcal{C}^*}(x) \in \mathcal{C}^*) \\
&\geq -\frac{\|P_\mathcal{C}(-x)\|}{\|x\|} \text{ (because } |\langle P_\mathcal{C}(-x), y\rangle| \leq \|P_\mathcal{C}(-x)\|\|y\|) \qquad (2.1)
\end{aligned}$$

$$= \frac{\langle P_{\mathcal{C}^*}(x) - P_{\mathcal{C}}(-x), P_{\mathcal{C}}(-x)\rangle}{\|x\|\|P_{\mathcal{C}}(-x)\|} \text{ (because } \langle P_{\mathcal{C}^*}(x), P_{\mathcal{C}}(-x)\rangle = 0)$$
$$= \frac{\langle x, P_{\mathcal{C}}(-x)\rangle}{\|x\|\|P_{\mathcal{C}}(-x)\|},$$

showing that the maximal angle between $x$ and $\mathcal{C}$ is formed by $x$ and $P_{\mathcal{C}}(-x) = P_{\mathcal{C}^*}(x) - x$. The statement that $\cos\theta = -\frac{\|P_{\mathcal{C}}(-x)\|}{\|x\|}$ can be seen from (2.1) by setting $y = P_{\mathcal{C}}(-x)$.

Conversely, if $y \in \mathcal{C}$ and $x$ form the maximal angle between $x$ and $\mathcal{C}$, and there is no $\alpha > 0$ such that $y = \alpha P_C(-x)$, then we know $\langle P_{\mathcal{C}}(-x), y\rangle < \|P_{\mathcal{C}}(-x)\|\|y\|$. From the proof above, we know that

$$\frac{\langle x, y\rangle}{\|x\|\|y\|} = \frac{\langle P_{\mathcal{C}^*}(x) - P_{\mathcal{C}}(-x), y\rangle}{\|x\|\|y\|} \geq \frac{\langle -P_{\mathcal{C}}(-x), y\rangle}{\|x\|\|y\|} > -\frac{\|P_{\mathcal{C}}(-x)\|}{\|x\|} = \frac{\langle x, P_{\mathcal{C}}(-x)\rangle}{\|x\|\|P_{\mathcal{C}}(-x)\|},$$

which shows the angle formed by $x$ and $y$ is not the maximal angle between $x$ and $\mathcal{C}$. Therefore, there is an $\alpha > 0$ such that $y = \alpha P_C(-x)$. ∎

**Special cases:**

1. $C = \mathcal{P}_n$. If $A = Q - P$, where $Q$ is the positive definite part of $A$, and $P$ is the negative definite part of $A$, then the maximal angle between $A$ and $\mathcal{P}_n$ is the angle formed by $A$ and $P$. This result has appeared in [2].

2. $C = \mathcal{N}_n$. If $A = M - N$, where $M$ is the positive part of $A$, and $N$ is the negative part of $A$, then the maximal angle between $A$ and $\mathcal{N}_n$ is the angle formed by $A$ and $N$. This result has appeared in [2].

It is easy to see that every antipodal pair is a Nash antipodal pair. It is proved in [9] that every Nash antipodal pair is a critical pair. It is interesting to consider under what conditions a critical pair is a Nash antipodal pair for a given cone. In [7], Iusem and Seeger gave necessary and sufficient conditions for a proper critical angle of an elliptic cone to be a Nash angle. In general, a critical angle may not be a Nash angle. However, the following corollary shows that every obtuse critical angle is a Nash angle.

Corollary 2.2. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be closed convex cones in $V$. Suppose that $x \in \mathcal{C}_1$ and $y \in \mathcal{C}_2$ are unit vectors and form a critical angle between $\mathcal{C}_1$ and $\mathcal{C}_2$ that is greater than or equal to $\frac{\pi}{2}$. Then $x$ and $y$ form a Nash angle.*

*Proof.* We first assume that $x$ and $y$ form a critical angle that is $\frac{\pi}{2}$. In this case, we know that $\langle x, y\rangle = 0$, and hence, $x = x - \langle x, y\rangle y \in \mathcal{C}_2^*$. Therefore, the maximal angle between $x$ and $\mathcal{C}_2$ is at most $\frac{\pi}{2}$, which is achieved by $x$ and $y$. Similar, the maximal angle between $y$ and $\mathcal{C}_1$ is $\frac{\pi}{2}$, which is achieved by $x$ and $y$. Therefore, the angle formed by $x$ and $y$ is a Nash angle.

Now we assume that $x \in \mathcal{C}_1$ and $y \in \mathcal{C}_2$ are unit vectors and form a critical angle that is greater than $\frac{\pi}{2}$. Then we know that

$$x - \langle x, y\rangle y \in \mathcal{C}_2^*, \; y - \langle x, y\rangle x \in \mathcal{C}_1^*, \text{ and } \langle x, y\rangle < 0.$$

Since $x = (x - \langle x, y\rangle y) - (-\langle x, y\rangle y)$, $x - \langle x, y\rangle y \in \mathcal{C}_2^*$, $-\langle x, y\rangle y \in \mathcal{C}_2$, and $\langle x - \langle x, y\rangle y, y\rangle = 0$, by Moreau's decomposition Theorem [12], we obtain $-\langle x, y\rangle y = P_{\mathcal{C}_2}(-x)$. Hence, by Theorem 2.1, we know that $x$ and $-\langle x, y\rangle y$ form the maximal angle between $x$ and $\mathcal{C}_2$. Because $-\langle x, y\rangle > 0$, we get that $x$ and $y$ form the maximal angle between $x$ and $\mathcal{C}_2$. Similarly, we know that $x$ and $y$ form the maximal angle between $y$ and $\mathcal{C}_1$. Therefore, the angle formed by $x$ and $y$ is a Nash angle. ∎

Now we are ready to prove a theorem.

Theorem 2.3. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be closed convex cones in $V$. If $x \in \mathcal{C}_1$ and $y \in \mathcal{C}_2$ form a proper critical angle of $(\mathcal{C}_1, \mathcal{C}_2)$ that is greater than $\frac{\pi}{2}$, then $x$ is a positive multiple of the metric projection of $-y$ on $\mathcal{C}_1$, and $y$ is a positive multiple of the metric projection of $-x$ on $\mathcal{C}_2$. Conversely, if $x \neq 0$ is a positive multiple of the metric projection of $-y$ on $\mathcal{C}_1$, and $y \neq 0$ is a positive multiple of the metric projection of $-x$ on $\mathcal{C}_2$, then $x$ and $y$ form a critical angle of $(\mathcal{C}_1, \mathcal{C}_2)$ that is greater than $\frac{\pi}{2}$.*

*Proof.* Let $\theta$ be the critical angle formed by $x$ and $y$. Since $\theta > \frac{\pi}{2}$, we know that $\theta$ is also a Nash angle by Corollary 2.2. Hence, $\theta$ is the maximal angle between $x$ and $\mathcal{C}_2$. By Theorem 2.1, we know that $y$ is a positive multiple of the metric projection of $-x$ on $\mathcal{C}_2$. Similarly, we have that $x$ is a positive multiple of the metric projection of $-y$ on $\mathcal{C}_1$.

Conversely, if $x \neq 0$ is a positive multiple of the metric projection of $-y$ on $\mathcal{C}_1$, then by Theorem 2.1, we know that the angle formed by $x$ and $y$ is the maximal angle between $y$ and $\mathcal{C}_1$. Similarly, the angle formed by $x$ and $y$ is the maximal angle between $x$ and $\mathcal{C}_2$. Hence, the angle formed by $x$ and $y$ is a Nash angle of $(\mathcal{C}_1, \mathcal{C}_2)$. The conclusion that the angle is greater than $\frac{\pi}{2}$ follows from the fact that the cosine of the angle is less than 0 by Theorem 2.1. ∎

The results stated in [14, Corollary 2.2 ] and [11, Theorem 3] will be used many times in this paper. When two cones in Corollary 2.2 of [14] are identical, Corollary 2.2 in [14] becomes Theorem 3 in [11]. Hence, Theorem 3 in [11] is a special case of Corollary 2.2 in [14]. We now state Corollary 2.2 in [14] below as a proposition and simply point to this proposition when referencing Corollary 2.4 in [14] and Theorem 3 in [11] in the sequel.

Proposition 2.4. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two convex cones in $V$. Let $\theta$, $\psi \in (0, \pi)$ be conjugate angles, i.e., $\theta + \psi = \pi$. Then $\theta$ is a critical angle of $(\mathcal{C}_1, \mathcal{C}_2)$ if and only if $\psi$ is a critical angle of $(\mathcal{C}_1^*, \mathcal{C}_2^*)$.*

In [9], Iusem and Seeger split the angular spectrum of a convex cone $\mathcal{C}$ into two disjoint pieces, $\Omega_{nash}(\mathcal{C})$ that is the set of all Nash angles, and $\Omega_{ord}(\mathcal{C})$ that is the set of all proper critical angles that are not Nash angles. Iusem and Seeger further claimed that as a general rule $\Omega_{nash}(\mathcal{C})$ is usually very small compared with the full collection of proper critical angles and most proper critical angles are in $\Omega_{ord}(\mathcal{C})$. Such a statement is not true for some convex cones. We take $\mathcal{COP}_n$ as an example. We know the dual cone of $\mathcal{COP}_n$ is $\mathcal{C}_n$. Obviously, completely positive matrices are nonnegative and positive semidefinite, so the maximal angle of $\mathcal{C}_n$ is less than or equal to $\frac{\pi}{2}$. Indeed, the maximal angle of $\mathcal{C}_n$ is $\frac{\pi}{2}$ that can been seen easily by finding an antipodal pair. By Proposition 2.4, we know the smallest critical angle of $\mathcal{COP}_n$ is $\frac{\pi}{2}$, and hence, by Corollary 2.2, we know that each critical angle of $\mathcal{COP}_n$ is also a Nash angle. Similarly, we can show that all critical angles of $\mathcal{P}_n + \mathcal{N}_n$ are Nash angles.

We complete this section by providing two examples to show how the results proved in this section can be applied to find critical angles of some cones. We will compute the angular spectrum of $(\mathcal{P}_3, \mathcal{N}_3)$. We will show that obtuse critical angles of $(\mathcal{P}_3, \mathcal{N}_3)$ are also critical angles of $\mathcal{P}_3 + \mathcal{N}_3$. Inspired by the case for $n = 3$, we further discover a critical angle for $\mathcal{P}_n + \mathcal{N}_n$ and $\mathcal{COP}_n$ for $n \geq 2$.

Example 2.5. *To find the angular spectrum of $(\mathcal{P}_3, \mathcal{N}_3)$, we need to find all critical angles. However, by Proposition 2.4 and because of the fact that $\mathcal{P}_3^* = \mathcal{P}_3$ and $\mathcal{N}_3^* = \mathcal{N}_3$, we only need to consider those critical angles that are greater than or equal to $\frac{\pi}{2}$. It is easy to verify that $\frac{\pi}{2}$ is a critical angle of $(\mathcal{P}_3, \mathcal{N}_3)$ by checking that* $\left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$ *is a Nash antipodal pair. Therefore, we only need to consider*

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*obtuse critical angles. By Corollary 2.2, we know that those critical angles are also Nash angles. We let* $A = (a_{ij}) \in \mathcal{P}_3$ *and* $B = (b_{ij}) \in \mathcal{N}_3$ *form a critical angle of* $(\mathcal{P}_3, \mathcal{N}_3)$ *that is greater than* $\frac{\pi}{2}$*. Then A must have negative entries. By Theorem 2.1, we know that there is a* $\beta > 0$ *such that* $\beta B$ *is the negative part of* $A$*. We can simply set* $\beta = 1$*. Otherwise, we can rename* $\beta B$ *to be* $B$*. Note that A is positive semidefinite, so entries that might be negative are* $a_{12} = a_{21}$, $a_{13} = a_{31}$*, and* $a_{23} = a_{32}$*. If at most two out of* $a_{12} = a_{21}$, $a_{13} = a_{31}$*, and* $a_{23} = a_{32}$ *are negative, then without loss of generality we can assume that* $a_{12} = a_{21} \geq 0$*, and* $a_{23} = a_{32} < 0$*. We now can set* $B = \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & s \\ t & s & 0 \end{pmatrix}$ *with* $t = -a_{13}$ *if* $a_{13} < 0$ *and* $t = 0$ *if* $a_{13} \geq 0$*, and* $s = -a_{23}$*. Since A and B form the maximal angle between B and* $\mathcal{P}_3$*, we know that A is a positive multiple of the negative definite part of B. Because the rank of B is 2 and B is indefinite, we obtain the rank of A and the rank of the positive definite part of B must be 1. By multiplying an appropriate positive number and relabeling t and s if necessary, we can assume that* $A = \begin{pmatrix} t \\ s \\ -1 \end{pmatrix} \begin{pmatrix} t & s & -1 \end{pmatrix} = \begin{pmatrix} t^2 & ts & -t \\ st & s^2 & -s \\ -t & -s & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & s \\ t & s & 0 \end{pmatrix}$*. Noting that A is a positive multiple of the negative definite part of B, we know there is a* $\gamma > 0$ *such that* $B = C - \frac{A}{\gamma}$*, where C is the positive definite part of B. By comparing both sides of* $B = C - \frac{A}{\gamma}$ *and noticing that the rank of C is 1, we obtain* $\gamma C = \begin{pmatrix} t \\ s \\ 1 \end{pmatrix} \begin{pmatrix} t & s & 1 \end{pmatrix} = \begin{pmatrix} t^2 & ts & t \\ st & s^2 & s \\ t & s & 1 \end{pmatrix}$*, and moreover,* $\gamma = 2$*. Since* $Tr(AC) = 0$*, with a direct calculation, we obtain* $t^2 + s^2 = 1$*. By Remark 4.9 in [15], we know that* $A = \begin{pmatrix} t^2 & ts & -t \\ st & s^2 & -s \\ -t & -s & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & s \\ t & s & 0 \end{pmatrix}$*, for any* $s \geq 0$, $t \geq 0$*, and* $s^2 + t^2 = 1$ *achieve the maximal angle* $\frac{3\pi}{4}$ *between* $\mathcal{P}_3$ *and* $\mathcal{N}_3$*. Hence,* $\frac{3\pi}{4}$ *is a critical angle of* $(\mathcal{P}_3, \mathcal{N}_3)$*. Next we show* $\frac{3\pi}{4}$ *is also a critical angle of* $\mathcal{P}_3 + \mathcal{N}_3$*. Since B is the negative part of A,* $A + B$ *is both nonnegative and positive semidefinite, and* $\langle A + B, B\rangle = 0$*, we know that B is the metric projection of* $-A$ *on* $(\mathcal{P}_3 + \mathcal{N}_3)$ *by Moreau's decomposition Theorem. Let* $D = \frac{1}{2}\begin{pmatrix} t^2 & ts & t \\ st & s^2 & s \\ t & s & 1 \end{pmatrix}$*. Then it is easy to see that D is both positive semidefinite and nonnegative,* $B = D - \frac{A}{2}$*, and* $\langle A, D\rangle = 0$*. By Moreau's decomposition Theorem, we know that* $\frac{A}{2}$ *is the metric projection of* $-B$ *on* $(\mathcal{P}_3 + \mathcal{N}_3)$*. Therefore, A and B form a critical angle of* $(\mathcal{P}_3 + \mathcal{N}_3)$ *by Theorem 2.3.*

*Now if there is an obtuse critical angle of* $(\mathcal{P}_3, \mathcal{N}_3)$ *that is not the maximal angle, then it should be achieved by A and B with* $a_{12} = a_{21}$, $a_{13} = a_{31}$*, and* $a_{23} = a_{32}$ *all negative. Without loss of generality, we assume that* $a_{11} + a_{22} + a_{33} = 1$*. We also assume B is the negative part of A, so* $B = \begin{pmatrix} 0 & -a_{12} & -a_{13} \\ -a_{21} & 0 & -a_{23} \\ -a_{31} & -a_{32} & 0 \end{pmatrix}$*. We have a couple of observations regarding the matrix A.*

- *A cannot be a rank 1 matrix. Otherwise,* $A = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$*. We obtain* $0 > a_{12} = x_1 x_2$, $0 > a_{13} = x_1 x_3$*, and* $0 > a_{23} = x_2 x_3$*, which is impossible.*

- *$A$ cannot be a rank 3 matrix. Otherwise, we know $\alpha A$ for some $\alpha > 0$ is the negative definite part of $B$. So the positive definite part of $B$ must be* 0 *matrix, which contradicts the fact that $B$ is indefinite.*
- *Therefore, we know that the negative definite part of $B$, which is $\alpha A$ for some $\alpha > 0$, is a matrix with rank 2, and the positive definite part of $B$, which is $B + \alpha A$, is a matrix with rank 1.*

*Note that* $B + \alpha A = \begin{pmatrix} \alpha a_{11} & (\alpha-1)a_{12} & (\alpha-1)a_{13} \\ (\alpha-1)a_{21} & \alpha a_{22} & (\alpha-1)a_{23} \\ (\alpha-1)a_{31} & (\alpha-1)a_{32} & \alpha a_{33} \end{pmatrix}$ *and $B + \alpha A$ has rank 1, we know that $\alpha < 1$. Otherwise, if $\alpha > 1$, then $(\alpha-1)a_{12} < 0$, $(\alpha-1)a_{13} < 0$, and $(\alpha-1)a_{23} < 0$, which contradicts the fact that the rank of $B + \alpha A$ is 1, and if $\alpha = 1$, then $Tr(B + A, A) = 0$ implies that $a_{11} = a_{22} = a_{33} = 0$, which is impossible. From the fact that $B + \alpha A$ has rank 1, we have $\alpha^2 a_{11}a_{22} = (\alpha-1)^2 a_{12}^2$. Since $a_{12} < 0$, $1 > \alpha > 0$, and $a_{11} > 0$ and $a_{22} > 0$, we obtain $a_{12} = \frac{\alpha\sqrt{a_{11}a_{22}}}{\alpha-1}$. Similarly, we obtain $a_{13} = \frac{\alpha\sqrt{a_{11}a_{33}}}{\alpha-1}$ and $a_{23} = \frac{\alpha\sqrt{a_{22}a_{33}}}{\alpha-1}$. When we plug them into $B + \alpha A$, we obtain*

$$B + \alpha A = \begin{pmatrix} \alpha a_{11} & \alpha\sqrt{a_{11}a_{22}} & \alpha\sqrt{a_{11}a_{33}} \\ \alpha\sqrt{a_{11}a_{22}} & \alpha a_{22} & \alpha\sqrt{a_{22}a_{33}} \\ \alpha\sqrt{a_{11}a_{33}} & \alpha\sqrt{a_{22}a_{33}} & \alpha a_{33} \end{pmatrix} = \alpha \begin{pmatrix} \sqrt{a_{11}} \\ \sqrt{a_{22}} \\ \sqrt{a_{33}} \end{pmatrix} \begin{pmatrix} \sqrt{a_{11}} & \sqrt{a_{22}} & \sqrt{a_{33}} \end{pmatrix}.$$

*Since $B + \alpha A$ is the positive definite part of $B$, we know, from the above identity and the assumption that $a_{11} + a_{22} + a_{33} = 1$, that* $\begin{pmatrix} \sqrt{a_{11}} \\ \sqrt{a_{22}} \\ \sqrt{a_{33}} \end{pmatrix}$ *is an eigenvector of $B$ associated with the eigenvalue $\alpha$. Therefore, we have the following equations:*

$$\begin{pmatrix} 0 & -a_{12} & -a_{13} \\ -a_{21} & 0 & -a_{23} \\ -a_{31} & -a_{32} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{a_{11}} \\ \sqrt{a_{22}} \\ \sqrt{a_{33}} \end{pmatrix} = \alpha \begin{pmatrix} \sqrt{a_{11}} \\ \sqrt{a_{22}} \\ \sqrt{a_{33}} \end{pmatrix}.$$

*Hence, we have $-a_{12}\sqrt{a_{22}} - a_{13}\sqrt{a_{33}} = \alpha\sqrt{a_{11}}$. Note that $a_{12} = \frac{\alpha\sqrt{a_{11}a_{22}}}{\alpha-1}$ and $a_{13} = \frac{\alpha\sqrt{a_{11}a_{33}}}{\alpha-1}$. We get $\frac{a_{22}+a_{33}}{1-\alpha} = 1$, which shows that $a_{11} = \alpha$ due to the assumption that $a_{11} + a_{22} + a_{33} = 1$. Similarly, we have $a_{22} = \alpha$ and $a_{33} = \alpha$. Therefore, we get $\alpha = a_{11} = a_{22} = a_{33} = \frac{1}{3}$. We have*

$$A = \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 \end{pmatrix}.$$

*Obviously, $A$ is positive semidefinite, $B$ is nonnegative, and $B$ is the negative part of $A$. Now we show that $\frac{A}{3}$ is the negative definite part of $B$. We show this by the decomposition below*

$$B = \left(B + \frac{A}{3}\right) - \frac{A}{3} = \begin{pmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix} - \frac{1}{3} \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \end{pmatrix}.$$

*Obviously, $B + \frac{A}{3}$ is positive semidefinite. It is also easy to check that $\langle B + \frac{A}{3}, \frac{A}{3} \rangle = 0$. Therefore, $\frac{A}{3}$ is the negative definite part of $B$, showing by Theorem 2.1 that $A$ and $B$ form a critical angle with the value being determined by*

$$\theta = \arccos\left(-\frac{\|B\|}{\|A\|}\right) = \arccos\left(-\frac{\|A/3\|}{\|B\|}\right) = \arccos\left(-\frac{\sqrt{3}}{3}\right).$$

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*Using Proposition 2.4, we obtain the angular spectrum* $\left\{\frac{\pi}{4}, \pi - \arccos\left(-\frac{\sqrt{3}}{3}\right), \frac{\pi}{2}\ ,\ \arccos\left(-\frac{\sqrt{3}}{3}\right), \frac{3\pi}{4}\right\}$ *of* $(\mathcal{P}_3, \mathcal{N}_3)$.

*Moreover, since* $B + \frac{A}{3} \in \mathcal{P}_3 \cap \mathcal{N}_3$ *and* $A + B \in \mathcal{P}_3 \cap \mathcal{N}_3$*, we know that* $B$ *is the metric projection of* $-A$ *on* $\mathcal{P}_3 + \mathcal{N}_3$ *and* $\frac{A}{3}$ *is the metric projection of* $-B$ *on* $\mathcal{P}_3 + \mathcal{N}_3$*. Therefore, by Theorem 2.1 we know that* $A$ *and* $B$ *also form a critical angle of* $\mathcal{P}_3 + \mathcal{N}_3$*. Hence, in addition to find the angular spectrum of* $(\mathcal{P}_3, \mathcal{N}_3)$*, we also show in this example that* $\frac{3\pi}{4}$ *and* $\arccos\left(-\frac{\sqrt{3}}{3}\right)$ *are critical angles of* $\mathcal{P}_3 + \mathcal{N}_3$.

Inspired by this example, we discover a critical angle for the $n \times n$ copositive cone.

EXAMPLE 2.6. *Let* $C$ *be an* $n \times n$ *matrix with all entries being 1 and* $I_n$ *be the* $n \times n$ *identity matrix. Let* $A = nI_n - C$ *and* $B = C - I_n$*. It is obvious that* $C \in \mathcal{C}_n \subset \mathcal{P}_n \cap \mathcal{N}_n \subset \mathcal{P}_n$*,* $A \in \mathcal{P}_n \subset \mathcal{P}_n + \mathcal{N}_n \subset \mathcal{COP}_n$*, and* $B \in \mathcal{N}_n \subset \mathcal{P}_n + \mathcal{N}_n \subset \mathcal{COP}_n$*. Note that*

$$A = (n-1)I - B,\ B = \frac{n-1}{n}C - \frac{1}{n}A, \langle I,\ B\rangle = 0,\ \text{and}\ \langle C,\ A\rangle = 0. \tag{2.2}$$

*So* $B$ *is the negative part of* $A$ *and* $\frac{1}{n}A$ *is the negative definite part of* $B$*. By Theorem 2.3, we know that* $A$ *and* $B$ *form the maximal angle between* $A$ *and* $\mathcal{N}_n$*, and* $B$ *and* $A$ *form the maximal angle between* $B$ *and* $\mathcal{P}_n$*. Hence,* $A$ *and* $B$ *form a Nash angle, and therefore, a critical angle of* $(\mathcal{P}_n, \mathcal{N}_n)$*. Since* $\frac{\langle A,B\rangle}{\|A\|\|B\|} = -\frac{\sqrt{n}}{n}$*, we know* $\angle(A, B) = \arccos\left(-\frac{\sqrt{n}}{n}\right)$*. Moreover, since* $I \in \mathcal{C}_n$ *and* $C \in \mathcal{C}_n$*, we know from (2.2) that* $B$ *is the metric projection of* $A$ *on* $\mathcal{COP}_n$*, and* $A$ *is the metric projection of* $B$ *on* $\mathcal{COP}_n$*, we know that* $A$ *and* $B$ *form a critical angle of* $\mathcal{COP}_n$*. That* $A$ *and* $B$ *form a critical angle of* $\mathcal{P}_n + \mathcal{N}_n$ *can be proved similarly.*

*Therefore,* $A$ *and* $B$ *form a critical angle* $\arccos\left(-\frac{\sqrt{n}}{n}\right)$ *of* $(\mathcal{P}_n, \mathcal{N}_n)$*,* $\mathcal{P}_n + \mathcal{N}_n$*, and* $\mathcal{COP}_n$*, respectively.*

REMARK 2.7. If a critical angle $\theta$ of $(\mathcal{C}_1, \mathcal{C}_2)$ is acute, then by Proposition 2.4, $\pi - \theta$ is a critical angle of $(\mathcal{C}_1^*, \mathcal{C}_2^*)$ that is obtuse. By Corollary 2.2, we know that $\pi - \theta$ is a Nash angle of $(\mathcal{C}_1^*, \mathcal{C}_2^*)$. Therefore, to find the angular spectrum of $(\mathcal{C}_1, \mathcal{C}_2)$, we only need to find Nash angles of $(\mathcal{C}_1, \mathcal{C}_2)$ and $(\mathcal{C}_1^*, \mathcal{C}_2^*)$. This observation is especially useful when both $\mathcal{C}_1$ and $\mathcal{C}_2$ are self-dual cones, i.e., $\mathcal{C}_1 = \mathcal{C}_1^*$ and $\mathcal{C}_2 = \mathcal{C}_2^*$, as we have seen from Example 2.5.

## 3. Critical angles of $\mathcal{P}_n + \mathcal{N}_n$.

Now we consider critical angles of $\mathcal{P}_n + \mathcal{N}_n$. As we mentioned before, all critical angles of $\mathcal{P}_n + \mathcal{N}_n$ are Nash angles. Therefore, we can use the words critical angles and Nash angles interchangeably in this section. Note that there is a critical angle of $\mathcal{P}_n + \mathcal{N}_n$ that is $\frac{\pi}{2}$ achieved by $\begin{pmatrix} 1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-1)} \end{pmatrix}$ and $\begin{pmatrix} 0_{(n-1)\times(n-1)} & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & 1 \end{pmatrix}$, where $0_{p\times q}$ represents the $p \times q$ zero matrix. We only need to consider obtuse critical angles. In the rest of this paper, we simply use critical angles or Nash angles to represent critical angles or Nash angles that are greater than $\frac{\pi}{2}$. We prove the following simple result first.

PROPOSITION 3.1. *The angular spectrum of* $\mathcal{P}_m + \mathcal{N}_m$ *is a subset of the angular spectrum of* $\mathcal{P}_n + \mathcal{N}_n$ *for* $n \geq m$.

*Proof.* Suppose $\theta$ is a critical angle of $\mathcal{P}_m + \mathcal{N}_m$ formed by $R$, $S \in \mathcal{P}_m + \mathcal{N}_m$. Then $\theta$ is also a Nash angle of $\mathcal{P}_m + \mathcal{N}_m$. We now append $n - m$ columns and $n - m$ rows with all zero entries to $R$ and $S$ to get matrices $\bar{R}$, $\bar{S} \in \mathcal{P}_n + \mathcal{N}_n$. Then using the fact that $R$ and $S$ form a Nash angle of $\mathcal{P}_m + \mathcal{N}_m$, we can show that $\bar{R}$ and $\bar{S}$ form the maximal angle between $\bar{R}$ and the cone $\mathcal{P}_n + \mathcal{N}_n$, and $\bar{S}$ and $\bar{R}$ form the maximal angle between $\bar{S}$ and the cone $\mathcal{P}_n + \mathcal{N}_n$. Therefore, $\bar{R}$ and $\bar{S}$ form a Nash angle of $\mathcal{P}_n + \mathcal{N}_n$. It is straightforward that the

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angle formed by $R$ and $S$ is the same as the angle formed by $\bar{R}$ and $\bar{S}$. Therefore, $\theta$ is a critical angle of $\mathcal{P}_n + \mathcal{N}_n$. We complete the proof. ∎

We now use Theorem 2.1 to prove the following theorem, which will be used frequently in the discussion of Nash angles of $\mathcal{P}_n + \mathcal{N}_n$ for $n = 3$ in Section 4.

Theorem 3.2. *Suppose that $(R, S)$ is a Nash antipodal pair of $\mathcal{P}_n + \mathcal{N}_n$. Then there are $\alpha > 0$ and $\beta > 0$ such that $R$ is the metric projection of $-\beta S$ on $\mathcal{P}_n + \mathcal{N}_n$, and $S$ is the metric projection of $-\alpha R$ on $\mathcal{P}_n + \mathcal{N}_n$. Moreover, if $R = X + A$ and $S = Y + B$ with $X, Y \in \mathcal{P}_n$ and $A, B \in \mathcal{N}_n$, then $X$ is the negative definite part of $A + \beta S$, and $A$ is the negative part of $X + \beta S$, $Y$ is the negative definite part of $\alpha R + B$, and $B$ is the negative part of $\alpha R + Y$, and both $\alpha R + S$ and $R + \beta S$ are in $\mathcal{P}_n \cap \mathcal{N}_n$.*

*Proof.* Since $(R, S)$ is a Nash antipodal pair of $\mathcal{P}_n + \mathcal{N}_n$, $R$ and $S$ form the maximal angle between the matrix $R$ and the cone $\mathcal{P}_n + \mathcal{N}_n$. By Theorem 2.1, we know that there is an $\alpha > 0$ such that $S$ is the metric projection of $-\alpha R$ on $\mathcal{P}_n + \mathcal{N}_n$. Therefore,

$$\|\alpha R + S\| = \min_{\substack{P \in \mathcal{P}_n \\ N \in \mathcal{N}_n}} \|P + N + \alpha R\| = \min_{P \in \mathcal{P}_n} \|P + B + \alpha R\| = \min_{N \in \mathcal{N}_n} \|Y + N + \alpha R\|.$$

Hence, $Y$ is the metric projection of $-(B + \alpha R)$ on $\mathcal{P}_n$, i.e., the negative definite part of $B + \alpha R$, and $B$ is the metric projection of $-(Y + \alpha R)$ on $\mathcal{N}_n$, i.e., the negative part of $Y + \alpha R$. Moreover, we know $\alpha R + S$ is doubly nonnegative. Similarly, we can prove that there is a $\beta > 0$ such that $X$ is the negative definite part of $-(A + \beta S)$, $A$ is the negative part of $-(X + \beta S)$, and $R + \beta S$ is doubly nonnegative. ∎

Now we consider properties related to entries of matrices that form a Nash angle of $\mathcal{P}_n + \mathcal{N}_n$. Suppose that $R = X + A$ and $S = Y + B$ form a Nash angle of $\mathcal{P}_n + \mathcal{N}_n$ with $X, Y \in \mathcal{P}_n$ and $A, B \in \mathcal{N}_n$. Then $A$ and $B$ must be matrices with diagonal entries being 0. In fact, we have the following more general result.

Theorem 3.3. *Suppose that $(R, S)$ is a Nash antipodal pair of $\mathcal{P}_n + \mathcal{N}_n$. Let $R = X + A$ and $S = Y + B$ with $X, Y \in \mathcal{P}_n$ and $A, B \in \mathcal{N}_n$. Let $i_0 j_0$ be an index such that $x_{i_0 j_0} \geq 0$ and $y_{i_0 j_0} \geq 0$. Then $a_{i_0 j_0} = b_{i_0 j_0} = 0$.*

*Proof.* We prove it by contradiction. Suppose at least one of $a_{i_0 j_0}$ and $b_{i_0 j_0}$ is not zero. For example, we can assume that $a_{i_0 j_0} > 0$ and $b_{i_0 j_0} \geq 0$. If we let $\bar{A}$ be the matrix obtained by replacing the $i_0 j_0$-th and $j_0 i_0$-th entries of $A$ with 0 and let $\bar{R} = X + \bar{A}$, then by the fact that this Nash angle is obtuse we know

$$0 > \frac{\langle R, S \rangle}{\|R\| \|S\|} = \frac{\langle \bar{R}, S \rangle + 2 a_{i_0 j_0} s_{i_0 j_0}}{\|R\| \|S\|} > \frac{\langle \bar{R}, S \rangle}{\|\bar{R}\| \|S\|},$$

which shows the angle formed by $\bar{R}$ and $S$ is greater than the angle formed by $R$ and $S$ contradicting the assumption. Hence, we have $a_{i_0 j_0} = 0$ and $b_{i_0 j_0} = 0$. ∎

Corollary 3.4. *Suppose that $(R, S)$ is a Nash antipodal pair of $\mathcal{P}_n + \mathcal{N}_n$. Let $R = X + A$ and $S = Y + B$ with $X, Y \in \mathcal{P}_n$ and $A, B \in \mathcal{N}_n$. Let $i_0 j_0$ be an index such that $x_{i_0 j_0} > 0$ and $y_{i_0 j_0} \geq 0$. Then for any $x_{i_0 j_0} \geq \epsilon > 0$. $X - \epsilon E_{i_0 j_0}$ is not positive semidefinite. Here, $E_{i_0 j_0}$ represents the $n \times n$ symmetric matrix whose $i_0 j_0$-th and $j_0 i_0$-th entries are 1, and other entries are all 0.*

*Proof.* By Theorem 3.3, we know $a_{i_0 j_0} = b_{i_0 j_0} = 0$. If for some $\epsilon > 0$ with $x_{i_0 j_0} \geq \epsilon$, $X - \epsilon E_{i_0 j_0}$ is positive semidefinite, then $R = (X - \epsilon E_{i_0 j_0}) + (A + \epsilon E_{i_0 j_0})$. Let $\bar{X} = X - \epsilon E_{i_0 j_0}$ and $\bar{A} = A + \epsilon E_{i_0 j_0}$. Then $\bar{X} \in \mathcal{P}_n$ and $\bar{A} \in \mathcal{N}_n$. Moreover, $\bar{x}_{i_0 j_0} \geq 0$. Because we assume that $y_{i_0 j_0} \geq 0$, by Theorem 3.3, we know that $\bar{a}_{ij} = \epsilon = 0$, which contradicts that $\epsilon > 0$. Therefore, for any $x_{i_0 j_0} \geq \epsilon > 0$, $X - \epsilon E_{i_0 j_0}$ is not positive semidefinite. ∎

Theorem 3.5. *Suppose that* $(R, S)$ *is a Nash antipodal pair of* $\mathcal{P}_n + \mathcal{N}_n$. *Let* $R = X + A$ *and* $S = Y + B$ *with* $X, Y \in \mathcal{P}_n$ *and* $A, B \in \mathcal{N}_n$. *Let* $i_0j_0$ *be an index such that* $x_{i_0j_0} \leq 0$ *and* $y_{i_0j_0} \leq 0$. *Then either*

*(1)* $r_{i_0j_0} = 0$ *and* $s_{i_0j_0} = 0$, *or*

*(2)* $r_{i_0j_0}$ *and* $s_{i_0j_0}$ *have opposite algebraic signs.*

*Proof.* Suppose at least one of $r_{i_0j_0}$ and $s_{i_0j_0}$ is not zero. For example, we can assume that $r_{i_0j_0} \neq 0$. Then we need to show that $s_{i_0j_0} \neq 0$, and $r_{i_0j_0}$ and $s_{i_0j_0}$ have opposite algebraic signs. We prove it by contradiction. Suppose that $s_{i_0j_0} = 0$, or $s_{i_0j_0}$ has the same algebraic sign as $r_{i_0j_0}$. If we let $\bar{R}$ be the matrix obtained by replacing the $i_0j_0$-th and $j_0i_0$-th entries of $R$ with 0, then we know

$$0 > \frac{\langle R, S\rangle}{\|R\|\|S\|} = \frac{\langle \bar{R}, S\rangle + 2r_{i_0j_0}s_{i_0j_0}}{\|R\|\|S\|} > \frac{\langle \bar{R}, S\rangle}{\|\bar{R}\|\|S\|},$$

which shows the angle formed by $\bar{R}$ and $S$ is greater than the one formed by $R$ and $S$ contradicting the assumption. Hence, under the assumption that at least one of $r_{i_0j_0}$ and $s_{i_0j_0}$ is not zero, we prove that $r_{i_0j_0}$ and $s_{i_0j_0}$ have opposite algebraic signs. We complete our proof. ☐

We next show that if $R$ and $S$ form a Nash angle of $\mathcal{P}_n + \mathcal{N}_n$, and an entry in $R$ (or in $S$) is negative, then the corresponding entry in $S$ (or in $R$) must be positive.

Theorem 3.6. *Suppose that* $(R, S)$ *is a Nash antipodal pair of* $\mathcal{P}_n + \mathcal{N}_n$ *with* $R, S \in \mathcal{P}_n + \mathcal{N}_n$. *Let* $i_0j_0$ *be an index such that* $r_{i_0j_0} < 0$. *Then* $s_{i_0j_0} > 0$.

*Proof.* Let $R = X + A$ and $S = Y + B$ with $X, Y \in \mathcal{P}_n$ and $A, B \in \mathcal{N}_n$. Suppose to the contrary that $s_{i_0j_0} = y_{i_0j_0} + b_{i_0j_0} \leq 0$. Since $a_{i_0j_0} \geq 0$ and $b_{i_0j_0} \geq 0$, we obtain $x_{i_0j_0} < 0$ and $y_{i_0j_0} \leq 0$. By the assumption that $r_{i_0j_0} < 0$ and using Theorem 3.5, we know $s_{i_0j_0} > 0$, which is a contradiction. Therefore, $s_{i_0j_0} > 0$. ☐

Now we state and prove another theorem, which will be used frequently in the next section.

Theorem 3.7. *Suppose that* $(R, S)$ *is a Nash antipodal pair of* $\mathcal{P}_n + \mathcal{N}_n$. *Let* $R = X + A$ *and* $S = Y + B$ *with* $X, Y \in \mathcal{P}_n$ *and* $A, B \in \mathcal{N}_n$. *Let* $i_0j_0$ *be an index such that* $r_{i_0j_0} > 0$ *and* $s_{i_0j_0} < 0$. *Then* $b_{i_0j_0} = 0$. *Moreover,* $Y - \epsilon E_{i_0j_0}$ *is not positive semidefinite for any* $\epsilon > 0$.

*Proof.* Since $R + xE_{i_0j_0} = X + (A + xE_{i_0j_0}) \in \mathcal{P}_n + \mathcal{N}_n$ for $x \geq -a_{i_0j_0}$, and $S + yE_{i_0j_0} = Y + (B + yE_{i_0j_0}) \in \mathcal{P}_n + \mathcal{N}_n$ for $y \geq -b_{i_0j_0}$, and due to the assumption that $R$ and $S$ form a Nash angle of $\mathcal{P}_n + \mathcal{N}_n$, we know that the following optimization problem has an optimal solution at $x = 0$.

$$\min_x \frac{\langle R + xE_{i_0j_0}, S)\rangle}{\|R + xE_{i_0j_0}\|\|S\|}, \quad \text{subject to } x \geq -a_{i_0j_0}, \tag{3.3}$$

and the following optimization problem has an optimal solution at $y = 0$.

$$\min_y \frac{\langle R, S + yE_{i_0j_0}\rangle}{\|R\|\|S + yE_{i_0j_0}\|}, \quad \text{subject to } y \geq -b_{i_0j_0}. \tag{3.4}$$

We now show $a_{i_0j_0}b_{i_0j_0} = 0$. We prove it by contradiction. Suppose that $a_{i_0j_0} > 0$ and $b_{i_0j_0} > 0$. Then by the formulation (3.3), we have

$$\frac{d}{dx}\left(\frac{\langle R + xE_{i_0j_0}, S\rangle}{\|R + xE_{i_0j_0}\|\|S\|}\right)\bigg|_{x=0} = \frac{d}{dx}\left(\frac{\langle R, S\rangle + 2xs_{i_0j_0}}{\|S\|\sqrt{\|R\|^2 + 4xr_{i_0j_0} + 2x^2}}\right)\Bigg|_{x=0} = \frac{2s_{i_0j_0}\|R\| - \langle R, S\rangle \frac{2r_{i_0j_0}}{\|R\|}}{\|R\|^2\|S\|} = 0.$$

Therefore, we have

$$s_{i_0j_0}\|R\|^2 = r_{i_0j_0}\langle R, S\rangle. \tag{3.5}$$

Similarly, based on (3.4) we obtain

$$r_{i_0j_0}\|S\|^2 = s_{i_0j_0}\langle R, S\rangle. \tag{3.6}$$

By (3.5) and (3.6) and using the assumption that $r_{i_0j_0} > 0$ and $s_{i_0j_0} < 0$, we know that $\|R\|^2\|S\|^2 = \langle R, S\rangle^2$, which shows that $\frac{\langle R,S\rangle}{\|R\|\|S\|} = -1$, indicating that the Nash angle is $\pi$. Since $\mathcal{P}_n + \mathcal{N}_n$ is a pointed cone, we have a contradiction. Therefore, $a_{i_0j_0}b_{i_0j_0} = 0$.

Now, we show $b_{i_0j_0} = 0$. If not, then $b_{i_0j_0} > 0$ and $a_{i_0j_0} = 0$. In this case, (3.6) is still true, which is

$$x_{i_0j_0}\|S\|^2 = s_{i_0j_0}\langle R, S\rangle. \tag{3.7}$$

But (3.5) may not. We can modify the process as follows to get an inequality.

$$\frac{d}{dx}\left(\frac{\langle R + xE_{i_0j_0}, S\rangle}{\|R + xE_{i_0j_0}\|\|S\|}\right)\Bigg|_{x=0^+} = \frac{d}{dx}\left(\frac{\langle R, S\rangle + 2xs_{i_0j_0}}{\|S\|\sqrt{\|R\|^2 + 4xx_{i_0j_0} + 2x^2}}\right)\Bigg|_{x=0^+} = \frac{2s_{i_0j_0}\|R\| - \langle R, S\rangle\frac{2x_{i_0j_0}}{\|R\|}}{\|R\|^2\|S\|} \geq 0.$$

Hence,

$$s_{i_0j_0}\|R\|^2 \geq x_{i_0j_0}\langle R, S\rangle. \tag{3.8}$$

By (3.7) and (3.8) and knowing that $s_{i_0j_0} < 0$ and $x_{i_0j_0} = x_{i_0j_0} + a_{i_0j_0} > 0$, we obtain $\|R\|^2\|S\|^2 \leq \langle R, S\rangle^2$, which shows that the Nash angle is $\pi$. Since $\mathcal{P}_n + \mathcal{N}_n$ is a pointed cone, we have a contradiction. Therefore, $b_{i_0j_0} = 0$.

Now if $Y - \epsilon E_{i_0j_0}$ is positive semidefinite for some $\epsilon > 0$. Then we apply the result just been proved to $Y' = Y - \epsilon E_{i_0j_0}$ and $B' = B + \epsilon E_{i_0j_0}$, we obtain $\epsilon = 0$. Hence, $Y - \epsilon E_{i_0j_0}$ is not positive semidefinite for any $\epsilon > 0$. ☐

Similar results like Theorems 3.2-3.7 can be stated and proved for a Nash angle of $(\mathcal{P}_n + \mathcal{N}_n, \mathcal{N}_n)$. For convenience of reference, we state similar results to Theorems 3.3-3.7 for a Nash angle of $(\mathcal{P}_n + \mathcal{N}_n, \mathcal{N}_n)$ below. The proofs are omitted since they are almost the same as the proofs of Theorems 3.3-3.7.

THEOREM 3.8. *Suppose that $(R, B)$ is a Nash antipodal pair of $(\mathcal{P}_n + \mathcal{N}_n, \mathcal{N}_n)$ with $R \in \mathcal{P}_n + \mathcal{N}_n$ and $B \in \mathcal{N}_n$. Let $R = X + A$ with $X \in \mathcal{P}_n$ and $A \in \mathcal{N}_n$. Let $i_0j_0$ be an index such that $x_{i_0j_0} \geq 0$. Then $a_{i_0j_0} = b_{i_0j_0} = 0$.*

THEOREM 3.9. *Suppose that $(R, B)$ is a Nash antipodal pair of $(\mathcal{P}_n + \mathcal{N}_n, \mathcal{N}_n)$ with $R \in \mathcal{P}_n + \mathcal{N}_n$ and $B \in \mathcal{N}_n$. Let $R = X + A$ with $X \in \mathcal{P}_n$ and $A \in \mathcal{N}_n$. Let $i_0j_0$ be an index such that $x_{i_0j_0} \leq 0$. Then either*

*(1) $r_{i_0j_0} = 0$ and $b_{i_0j_0} = 0$, or*

*(2) $r_{i_0j_0} < 0$ and $b_{i_0j_0} > 0$.*

THEOREM 3.10. *Suppose that $(R, B)$ is a Nash antipodal pair of $(\mathcal{P}_n + \mathcal{N}_n, \mathcal{N}_n)$ with $R \in \mathcal{P}_n + \mathcal{N}_n$ and $B \in \mathcal{N}_n$. Let $i_0j_0$ be an index such that $r_{i_0j_0} < 0$. Then $b_{i_0j_0} > 0$.*

THEOREM 3.11. *Suppose that $(R, B)$ is a Nash antipodal pair of $(\mathcal{P}_n + \mathcal{N}_n, \mathcal{N}_n)$ with $R \in \mathcal{P}_n + \mathcal{N}_n$ and $B \in \mathcal{N}_n$. Let $R = X + A$ with $X \in \mathcal{P}_n$ and $A \in \mathcal{N}_n$. Let $i_0j_0$ be an index such that $r_{i_0j_0} < 0$ and $b_{i_0j_0} > 0$. Then $a_{i_0j_0} = 0$. Moreover, $X - \epsilon E_{i_0j_0}$ is not positive semidefinite for any $\epsilon > 0$.*

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## 4. Angular spectra of $\mathcal{P}_n + \mathcal{N}_n$ for $n = 2, 3$.

Although theorems proved in Section 3 are true for all $n$, it seems implausible to apply these theorems to find critical angles of $\mathcal{P}_n + \mathcal{N}_n$ in general. However, for small $n$, these theorems are helpful because of the smaller number of entries of matrices in $\mathcal{S}_n$. Specifically, in this section, we consider the cases for $n = 2, 3$ by taking an advantage of the smaller number of possible algebraic sign patterns of entries for matrices in $\mathcal{P}_n + \mathcal{N}_n$. We will obtain the angular spectrum of $\mathcal{P}_2 + \mathcal{N}_2$ and $\mathcal{P}_3 + \mathcal{N}_3$ and show that the maximal angle of $\mathcal{P}_n + \mathcal{N}_n$, $n = 2, 3$, is formed by a positive semidefinite matrix and a symmetric nonnegative matrix. We start with $n = 2$.

THEOREM 4.1. *If $\theta$ is a Nash angle of $\mathcal{P}_2 + \mathcal{N}_2$, then $\theta$ is also a Nash angle of $(\mathcal{P}_2, \mathcal{N}_2)$.*

*Proof.* Suppose $R$ and $S$ in $\mathcal{P}_2 + \mathcal{N}_2$ form a critical angle of $\mathcal{P}_2 + \mathcal{N}_2$. Then $R \in \mathcal{P}_2 \cup \mathcal{N}_2$ and $S \in \mathcal{P}_2 \cup \mathcal{N}_2$ by Proposition 2.1 in [5]. If both $R$ and $S$ are in $\mathcal{N}_2$, then we can set $R = \begin{pmatrix} 1 & a \\ a & 0 \end{pmatrix}$, $a \geq 0$, and $S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ to form the Nash angle $\frac{\pi}{2}$.

If $R$ is not in $\mathcal{N}_2$, then it must be in $\mathcal{P}_2$. $S$ must be in $\mathcal{N}_2$ by Theorem 3.6. In this case, we can set $R = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to form the Nash angle $\frac{3\pi}{4}$, which is also the maximal angle. ☐

COROLLARY 4.2. *The angular spectrum of $\mathcal{P}_2 + \mathcal{N}_2$ is $\{\frac{\pi}{2}, \frac{3\pi}{4}\}$.*

Now we consider the case $n = 3$. We first prove the following theorem.

THEOREM 4.3. *If $\theta$ is a critical angle of $(\mathcal{P}_3 + \mathcal{N}_3, \mathcal{N}_3)$, then $\theta$ is also a critical angle of $(\mathcal{P}_3, \mathcal{N}_3)$.*

*Proof.* Since the maximal angle between $\mathcal{P}_3 \cap \mathcal{N}_3$ and $\mathcal{N}_3$ is $\frac{\pi}{2}$, we know that all critical angles of $(\mathcal{P}_3 + \mathcal{N}_3, \mathcal{N}_3)$ are greater than or equal to $\frac{\pi}{2}$ by Proposition 2.4. Since $\theta = \frac{\pi}{2}$ is a critical angle of $(\mathcal{P}_3 + \mathcal{N}_3, \mathcal{N}_3)$ and $(\mathcal{P}_3, \mathcal{N}_3)$ achieved at $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we only need to consider the case that $\theta > \frac{\pi}{2}$. Suppose that $\theta$ is achieved at $R$ and $B$, where $R \in \mathcal{P}_3 + \mathcal{N}_3$ and $B \in \mathcal{N}_3$. Then obviously, $b_{ii} = 0$ for $i = 1, 2, 3$. To prove the theorem, we need to show that $R \in \mathcal{P}_3$. We consider three different cases in terms of the number of negative entries of $R$.

Case 1: $R$ has exactly two negative entries. We may assume that $r_{12} = r_{21} < 0$. Then we have $R = \begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$. Since $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \in \mathcal{P}_2 + \mathcal{N}_2$ and $r_{12} < 0$, by Proposition 2.1 in [5], we obtain $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \in \mathcal{P}_2$. Therefore, $X \equiv \begin{pmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{P}_3$ and $A \equiv \begin{pmatrix} 0 & 0 & r_{13} \\ 0 & 0 & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \in \mathcal{N}_3$. By Theorem 3.8, we have $r_{13} = r_{23} = r_{33} = 0$. Therefore, $R \in \mathcal{P}_3$.

Case 2. $R$ has exactly four negative entries. We may assume that $r_{12} = r_{21} < 0$ and $r_{23} = r_{32} < 0$. We let $R = X + A$ with $X \in \mathcal{P}_3$ and $A \in \mathcal{N}_3$. Then by Theorems 3.8 and 3.11, we know $a_{11} = a_{22} = a_{33} = a_{12} = a_{23} = 0$. If $x_{13} < 0$, then by the assumption that $r_{13} \geq 0$ and Theorem 3.9, we know $r_{13} = 0$. In this case, since all the off-diagonal entries of $R$ are nonpositive, by Corollary 3.7 in [5], we obtain $R \in \mathcal{P}_3$. If $x_{13} \geq 0$, then by Theorem 3.8, we get $a_{13} = 0$. Hence, $A = 0$ showing that $R \in \mathcal{P}_3$.

Case 3: $R$ has six negative entries, that is, all off-diagonal entries are negative. In this case, by Corollary 3.7 in [5], we know $R \in \mathcal{P}_3$.

Therefore, a critical angle between $\mathcal{P}_3+\mathcal{N}_3$ and $\mathcal{N}_3$ is also a critical angle between $\mathcal{P}_3$ and $\mathcal{N}_3$. ☐

The following two lemmas give a critical angle of $\mathcal{P}_3+\mathcal{N}_3$, which has not been calculated before to the best of our knowledge. It is interesting to point out that this critical angle is related to the Golden ratio $\frac{1+\sqrt{5}}{2}$.

Lemma 4.4. *Let* $\mathcal{M}\equiv\{R \mid R\in\mathcal{P}_3+\mathcal{N}_3, R$ *has exactly two negative entries*$\}$. *Then there exist* $R\in\mathcal{M}$ *and* $S\in\mathcal{M}$ *such that* $R$ *and* $S$ *form a critical angle* $\theta=\arccos\left(-\frac{2}{1+\sqrt{5}}\right)$ *of* $\mathcal{P}_3+\mathcal{N}_3$. *Moreover, if* $R_1\in\mathcal{M}$ *and* $S_1\in\mathcal{M}$ *form a critical angle* $\gamma$ *of* $\mathcal{P}_3+\mathcal{N}_3$, *then* $\gamma=\arccos\left(-\frac{2}{1+\sqrt{5}}\right)$.

*Proof.* Suppose that $R\in\mathcal{M}$ and $S\in\mathcal{M}$ form a critical angle of $\mathcal{P}_3+\mathcal{N}_3$, which is also a Nash angle. Without loss of generality, we let $s_{23}=s_{32}<0$, $s_{12}=s_{21}\geq 0$, $s_{13}=s_{31}\geq 0$. We have

$$S=\begin{pmatrix} s_{11} & s_{12} & s_{13}\\ s_{21} & s_{22} & s_{23}\\ s_{31} & s_{32} & s_{33}\end{pmatrix}=\begin{pmatrix} 0 & 0 & 0\\ 0 & s_{22} & s_{23}\\ 0 & s_{32} & s_{33}\end{pmatrix}+\begin{pmatrix} s_{11} & s_{12} & s_{13}\\ s_{21} & 0 & 0\\ s_{31} & 0 & 0\end{pmatrix}.$$

Like in the proof of Theorem 4.3, we have $\begin{pmatrix} 0 & 0 & 0\\ 0 & s_{22} & s_{23}\\ 0 & s_{32} & s_{33}\end{pmatrix}\in\mathcal{P}_3$ and $\begin{pmatrix} s_{11} & s_{12} & s_{13}\\ s_{21} & 0 & 0\\ s_{31} & 0 & 0\end{pmatrix}\in\mathcal{N}_3$. By Theorem 3.3, we further know $s_{11}=0$. Therefore, if we let $S=Y+B$ and relabel $s_{ij}$, we can set $Y=\begin{pmatrix} 0 & 0 & 0\\ 0 & y_{22} & y_{23}\\ 0 & y_{32} & y_{33}\end{pmatrix}\in\mathcal{P}_3$ and $B=\begin{pmatrix} 0 & b_{12} & b_{13}\\ b_{21} & 0 & 0\\ b_{31} & 0 & 0\end{pmatrix}\in\mathcal{N}_3$.

Now we consider algebraic signs of entries of $R$. Since we assume that $s_{23}<0$, we know that $r_{23}>0$ by Theorem 3.6. So either $r_{12}<0$ or $r_{13}<0$, but not both because $R\in\mathcal{M}$. Without loss of generality, we assume that $r_{12}<0$ and $r_{13}\geq 0$. We let $R=X+A$ with $X\in\mathcal{P}_3$ and $A\in\mathcal{N}_3$. Then using the same argument as the one we did for $Y$ and $B$ above, we can set $X=\begin{pmatrix} x_{11} & x_{12} & 0\\ x_{21} & x_{22} & 0\\ 0 & 0 & 0\end{pmatrix}\in\mathcal{P}_3$ and $A=\begin{pmatrix} 0 & 0 & a_{13}\\ 0 & 0 & a_{23}\\ a_{31} & a_{32} & 0\end{pmatrix}\in\mathcal{N}_3$. By Theorem 3.3, we further obtain $a_{13}=b_{13}=0$. Therefore, we know $X=\begin{pmatrix} x_{11} & x_{12} & 0\\ x_{21} & x_{22} & 0\\ 0 & 0 & 0\end{pmatrix}$, $A=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & a_{23}\\ 0 & a_{32} & 0\end{pmatrix}$, $Y=\begin{pmatrix} 0 & 0 & 0\\ 0 & y_{22} & y_{23}\\ 0 & y_{32} & y_{33}\end{pmatrix}$, and $B=\begin{pmatrix} 0 & b_{12} & 0\\ b_{21} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$. Moreover, by Theorem 3.7, we know $x_{12}=-\sqrt{x_{11}x_{22}}$ and $y_{23}=-\sqrt{y_{22}y_{33}}$. Because the angle formed between two vectors $u$ and $v$ is the same as the one formed between $\alpha u$ and $\beta v$ for any $\alpha>0$ and $\beta>0$, to make our discussion easy, we assume that $x_{11}=1$.

By Theorem 3.2, we know that $\beta S$ for some $\beta>0$ is the metric projection of $-R$ on the cone $\mathcal{P}_3+\mathcal{N}_3$. We simply assume that $\beta=1$. Otherwise, we can relabel $\beta S$ by $S$. So $S$ is the metric projection of $-R$ on the cone $\mathcal{P}_3+\mathcal{N}_3$. This shows by Theorem 3.2 that $B$ is the negative part of $R+Y$, which gives

$B = \begin{pmatrix} 0 & \sqrt{x_{22}} & 0 \\ \sqrt{x_{22}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Similarly, $Y$ is the negative definite part of $R + B$. We can represent $Y$ in terms of $x_{22}$ and $a_{23}$. We complete this by finding the negative eigenvalue of $R+B$ and eigenvector associated with this negative eigenvalue. Note that $R + B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_{22} & a_{23} \\ 0 & a_{23} & 0 \end{pmatrix}$. So the characteristic polynomial of $R + B$ is

$$det(R+B-\lambda I) = (1-\lambda)(\lambda^2 - x_{22}\lambda - a_{23}^2) = (1-\lambda)\left(\lambda - \frac{x_{22} + \sqrt{x_{22}^2 + 4a_{23}^2}}{2}\right)\left(\lambda - \frac{x_{22} - \sqrt{x_{22}^2 + 4a_{23}^2}}{2}\right).$$

Hence, the negative eigenvalue of $R + B$ is $\lambda = \frac{x_{22} - \sqrt{x_{22}^2 + 4a_{23}^2}}{2}$. An eigenvector associated with $\lambda$ is $\begin{pmatrix} 0 & \frac{\lambda}{a_{23}} & 1 \end{pmatrix}^\top$. Because $Y$ is the negative definite part of $R + B$, we have

$$Y = \frac{-\lambda}{1 + \left(\frac{\lambda}{a_{23}}\right)^2} \begin{pmatrix} 0 \\ \frac{\lambda}{a_{23}} \\ 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{\lambda}{a_{23}} & 1 \end{pmatrix} = \frac{-\lambda a_{23}^2}{\lambda^2 + a_{23}^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\lambda^2}{a_{23}^2} & \frac{\lambda}{a_{23}} \\ 0 & \frac{\lambda}{a_{23}} & 1 \end{pmatrix}.$$

Therefore, we get $y_{22} = \frac{-\lambda^3}{\lambda^2 + a_{23}^2}$, $y_{33} = \frac{-\lambda a_{23}^2}{\lambda^2 + a_{23}^2}$, and $y_{23} = \frac{-\lambda^2 a_{23}}{\lambda^2 + a_{23}^2}$.

Similarly, we know that $R$ is a positive multiple of the metric projection of $-S$ on the cone $\mathcal{P}_3 + \mathcal{N}_3$. If we let $\alpha R$, $\alpha > 0$, be the metric projection of $-S$ on the cone $\mathcal{P}_3 + \mathcal{N}_3$, then $\alpha A$ is the metric projection of $-(S + \alpha X)$ on the cone $\mathcal{N}_3$. So $\alpha A$ is the negative part of $S + \alpha X$, which shows that $\alpha = \frac{\lambda^2}{\lambda^2 + a_{23}^2}$. Moreover, we know that $\alpha X$ is the negative definite part of $S + \alpha A$.

Since

$$\alpha X = \frac{\lambda^2}{\lambda^2 + a_{23}^2} \begin{pmatrix} 1 \\ -\sqrt{x_{22}} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{x_{22}} & 0 \end{pmatrix} = \frac{\lambda^2(1 + x_{22})}{\lambda^2 + a_{23}^2} \begin{pmatrix} \frac{1}{\sqrt{1+x_{22}}} \\ \frac{-\sqrt{x_{22}}}{\sqrt{1+x_{22}}} \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+x_{22}}} & \frac{-\sqrt{x_{22}}}{\sqrt{1+x_{22}}} & 0 \end{pmatrix},$$

we know that $\frac{-\lambda^2(1+x_{22})}{\lambda^2+a_{23}^2}$ is the only negative eigenvalue of $S+\alpha A$ with $\begin{pmatrix} 1 & -\sqrt{x_{22}} & 0 \end{pmatrix}^\top$ being an eigenvector associated with it. Note that $S + \alpha A = \begin{pmatrix} 0 & \sqrt{x_{22}} & 0 \\ \sqrt{x_{22}} & \frac{-\lambda^3}{\lambda^2 + a_{23}^2} & 0 \\ 0 & 0 & \frac{-\lambda a_{23}^2}{\lambda^2 + a_{23}^2} \end{pmatrix}$. Therefore, we have the following system of linear equations:

$$\begin{pmatrix} \frac{\lambda^2(1+x_{22})}{\lambda^2+a_{23}^2} & \sqrt{x_{22}} & 0 \\ \sqrt{x_{22}} & \frac{-\lambda^3}{\lambda^2+a_{23}^2} + \frac{\lambda^2(1+x_{22})}{\lambda^2+a_{23}^2} & 0 \\ 0 & 0 & \frac{-\lambda a_{23}^2}{\lambda^2+a_{23}^2} + \frac{\lambda^2(1+x_{22})}{\lambda^2+a_{23}^2} \end{pmatrix} \begin{pmatrix} 1 \\ -\sqrt{x_{22}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which is the same as the following

$$\frac{\lambda^2(1 + x_{22})}{\lambda^2 + a_{23}^2} - x_{22} = 0, \tag{4.9}$$

$$\sqrt{x_{22}}\left(1+\frac{\lambda^3}{\lambda^2+a_{23}^2}-\frac{\lambda^2(1+x_{22})}{\lambda^2+a_{23}^2}\right)=0. \tag{4.10}$$

Note that $\lambda=\frac{x_{22}-\sqrt{x_{22}^2+4a_{23}^2}}{2}$, we get $a_{23}^2=\lambda(\lambda-x_{22})$, plugging it to (4.10) we get $(2\lambda-x_{22})+\lambda^2-\lambda(1+x_{22})=0$. Therefore, we have $(\lambda+1)(\lambda-x_{22})=0$. Since $\lambda<0$ and $x_{22}>0$, we obtain $\lambda=-1$. Now plugging $\lambda=-1$ to (4.9) and noticing that $a_{23}^2=\lambda(\lambda-x_{22})$, we have the following quadratic equation $x_{22}^2+x_{22}-1=0$. Since $x_{22}>0$, we have $x_{22}=\frac{-1+\sqrt{5}}{2}$. Therefore, $a_{23}^2=\lambda(\lambda-x_{22})=\frac{1+\sqrt{5}}{2}$. Hence, we have

$$X=\begin{pmatrix}1 & -\sqrt{\frac{-1+\sqrt{5}}{2}} & 0\\ -\sqrt{\frac{-1+\sqrt{5}}{2}} & \frac{-1+\sqrt{5}}{2} & 0\\ 0 & 0 & 0\end{pmatrix}, A=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \sqrt{\frac{1+\sqrt{5}}{2}}\\ 0 & \sqrt{\frac{1+\sqrt{5}}{2}} & 0\end{pmatrix} \text{ and}$$

$$Y=\begin{pmatrix}0 & 0 & 0\\ 0 & \left(\frac{1+\sqrt{5}}{2}\right)^{-2} & -\left(\frac{1+\sqrt{5}}{2}\right)^{-\frac{3}{2}}\\ 0 & -\left(\frac{1+\sqrt{5}}{2}\right)^{-\frac{3}{2}} & \left(\frac{1+\sqrt{5}}{2}\right)^{-1}\end{pmatrix}, B=\begin{pmatrix}0 & \sqrt{\frac{-1+\sqrt{5}}{2}} & 0\\ \sqrt{\frac{-1+\sqrt{5}}{2}} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

We can verify in a straightforward manner that $S$ is the metric projection of $-R$ on the cone $\mathcal{P}_3+\mathcal{N}_3$ by checking $R+S$ is doubly nonnegative and $\langle R+S,S\rangle=0$. We can also verify that $\left(\frac{1+\sqrt{5}}{2}\right)^{-2}R$ is the metric projection of $-S$ on the cone $\mathcal{P}_3+\mathcal{N}_3$ by checking $\left(\frac{1+\sqrt{5}}{2}\right)^{-2}R+S$ is doubly nonnegative and $\left\langle\left(\frac{1+\sqrt{5}}{2}\right)^{-2}R+S,R\right\rangle=0$. Therefore, $R$ and $S$ form a critical angle of $\mathcal{P}_3+\mathcal{N}_3$. We now find $\|R\|$ and $\|S\|$ below.

$$\|R\|=\sqrt{1+2x_{22}+x_{22}^2+2a_{23}^2}=\sqrt{1+2\times\frac{-1+\sqrt{5}}{2}+\left(\frac{-1+\sqrt{5}}{2}\right)^2+2\times\frac{1+\sqrt{5}}{2}}=\frac{\sqrt[4]{5}(1+\sqrt{5})}{2}.$$

$$\|S\|=\sqrt{2x_{22}+\left(\frac{1}{1+a_{23}^2}\right)^2+2\frac{a_{23}^2}{(1+a_{23}^2)^2}+\frac{a_{23}^4}{(1+a_{23}^2)^2}}=\sqrt{2\times\frac{-1+\sqrt{5}}{2}+\frac{1+2a_{23}^2+a_{23}^4}{(1+a_{23}^2)^2}}=\sqrt[4]{5}.$$

Therefore, by Theorem 2.1, we know

$$\cos\theta=-\frac{\alpha\|R\|}{\|S\|}=-\frac{\|S\|}{\|R\|}=-\frac{\sqrt[4]{5}}{\frac{\sqrt[4]{5}(1+\sqrt{5}}{2})}=-\frac{2}{1+\sqrt{5}},$$

which is the critical angle of $\mathcal{P}_3+\mathcal{N}_3$ formed by $R$ and $S$. Note that $0>-\frac{2}{1+\sqrt{5}}>-\frac{\sqrt{2}}{2}$, so this critical angle is between $\frac{\pi}{2}$ and $\frac{3\pi}{4}$. ∎

Lemma 4.5. *Let* $\mathcal{M}\equiv\{R\mid R\in\mathcal{P}_3+\mathcal{N}_3, R \text{ has exactly two negative entries}\}$, *and* $\mathcal{Q}\equiv\{R\mid R\in\mathcal{P}_3+\mathcal{N}_3, R \text{ has exactly four negative entries}\}$. *Then there exist* $R\in\mathcal{Q}$ *and* $S\in\mathcal{M}$ *such that* $R$ *and* $S$ *form a critical angle* $\theta=\arccos\left(-\frac{2}{1+\sqrt{5}}\right)$ *of* $\mathcal{P}_3+\mathcal{N}_3$. *Moreover, if* $R_1\in\mathcal{Q}$ *and* $S_1\in\mathcal{M}$ *form a critical angle* $\gamma$ *of* $\mathcal{P}_3+\mathcal{N}_3$, *then* $\gamma=\arccos\left(-\frac{2}{1+\sqrt{5}}\right)$.

*Proof.* Suppose that $R\in\mathcal{Q}$ and $S\in\mathcal{M}$ form a critical angle of $\mathcal{P}_3+\mathcal{N}_3$. Let $R=X+A$ and $S=Y+B$ with $X,Y\in\mathcal{P}_3$, and $A,B\in\mathcal{N}_3$. Without loss of generality, we assume that $r_{12}<0$ and $r_{13}<0$. Then by

Theorem 3.6 we have $s_{12} > 0$ and $s_{13} > 0$, and by Theorem 3.7 we know $a_{12} = 0$ and $a_{13} = 0$. We have now $X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix}$. As in the proof of Lemma 4.4, we can assume $x_{11} = 1$. Note that the only entries of $S$ that are negative are $s_{23} = s_{32}$ because of the assumption that $S \in \mathcal{M}$. Like in the proof of Theorem 4.3, we have $Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y_{22} & y_{23} \\ 0 & y_{32} & y_{33} \end{pmatrix}$, and $B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & 0 \\ b_{31} & 0 & 0 \end{pmatrix}$.

By Theorem 3.2, we know that $\beta S$ for some $\beta > 0$ is the metric projection of $-R$ on the cone $\mathcal{P}_3 + \mathcal{N}_3$. We simply assume that $\beta = 1$. So $S$ is the metric projection of $-R$ on the cone $\mathcal{P}_3 + \mathcal{N}_3$. This shows by Theorem 3.2 that $B$ is the negative part of $R + Y$, which gives $B = \begin{pmatrix} 0 & -x_{12} & -x_{13} \\ -x_{21} & 0 & 0 \\ -x_{31} & 0 & 0 \end{pmatrix}$. Similarly, $Y$ is the negative definite part of $R + B$. We can represent $Y$ in terms of $x_{22}$, $x_{23}$, $x_{33}$, and $a_{23}$. We complete this by finding the negative eigenvalue of $R + B$ and eigenvector associated with this negative eigenvalue. Note that $R + B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_{22} & r_{23} \\ 0 & r_{32} & x_{33} \end{pmatrix}$. So the characteristic polynomial of $R + B$ is

$$det(R + B - \lambda I) = (1 - \lambda)((x_{22} - \lambda)(x_{33} - \lambda) - r_{23}^2) = (1 - \lambda)(\lambda^2 - (x_{22} + x_{33})\lambda - r_{23}^2 + x_{22}x_{33}).$$

Hence, the negative eigenvalue is $\lambda = \frac{(x_{22}+x_{33})-\sqrt{(x_{22}+x_{33})^2+4r_{23}^2-4x_{22}x_{33}}}{2}$. An eigenvector associated with $\lambda$ is $\begin{pmatrix} 0 & \frac{-(x_{33}-\lambda)}{r_{23}} & 1 \end{pmatrix}^\top$. Because $Y$ is the negative definite part of $R + B$, we have

$$Y = \frac{-\lambda}{1 + \left(\frac{-(x_{33}-\lambda)}{r_{23}}\right)^2} \begin{pmatrix} 0 \\ \frac{-(x_{33}-\lambda)}{r_{23}} \\ 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{-(x_{33}-\lambda)}{r_{23}} & 1 \end{pmatrix} = \frac{-\lambda r_{23}^2}{(x_{33} - \lambda)^2 + r_{23}^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{(x_{33}-\lambda)^2}{r_{23}^2} & -\frac{x_{33}-\lambda}{r_{23}} \\ 0 & -\frac{x_{33}-\lambda}{r_{23}} & 1 \end{pmatrix}.$$

Therefore, we get

$$y_{22} = \frac{-\lambda(x_{33} - \lambda)^2}{(x_{33} - \lambda)^2 + r_{23}^2}, \; y_{33} = \frac{-\lambda r_{23}^2}{(x_{33} - \lambda)^2 + r_{23}^2}, \; y_{23} = \frac{\lambda(x_{33} - \lambda)r_{23}}{(x_{33} - \lambda)^2 + r_{23}^2}. \tag{4.11}$$

Similarly, we know that $R$ is a positive multiple of the metric projection of $-S$ on the cone $\mathcal{P}_3 + \mathcal{N}_3$. Let $\alpha R$, $\alpha > 0$, be the metric projection of $-S$ on the cone $\mathcal{P}_3 + \mathcal{N}_3$. Then $\alpha A$ is the negative part of $S + \alpha X$, which shows that $\alpha = \frac{-\lambda(x_{33}-\lambda)}{(x_{33}-\lambda)^2+r_{23}^2}$. Moreover, we know that $\alpha X$ is the negative definite part of $S + \alpha A = \begin{pmatrix} 0 & -x_{12} & -x_{13} \\ -x_{21} & y_{22} & y_{23} + \alpha a_{23} \\ -x_{31} & y_{23} + \alpha a_{32} & y_{33} \end{pmatrix}$. So $S + \alpha R$ is the positive definite part of $S + \alpha A$. Note that

$$S + \alpha R = \begin{pmatrix} \alpha & (\alpha - 1)x_{12} & (\alpha - 1)x_{13} \\ (\alpha - 1)x_{12} & \alpha x_{22} + y_{22} & 0 \\ (\alpha - 1)x_{13} & 0 & \alpha x_{33} + y_{33} \end{pmatrix}.$$

Since we assume that $r_{12} < 0$, we obtain $x_{12} < 0$, which indicates that $x_{22} > 0$ due to the assumption that $X \in \mathcal{P}_3$. Similarly we have $y_{22} > 0$. Therefore, we have $\alpha x_{22} + y_{22} > 0$, which together with the fact that $\alpha > 0$ shows that the first column and second column of $S + \alpha R$ are linearly independent. Hence, the rank of $S+\alpha R$ can only be 2 or 3. If the rank is 3, than the negative definite part of $S+\alpha A$ is zero matrix. So $\alpha X = 0$, which contradicts the assumption that $R \in \mathcal{Q}$. If $S+\alpha R$ has rank 2, then $\alpha X$ has rank 1. Recall that $x_{12} < 0$ and $x_{13} < 0$, we have $X = \begin{pmatrix} 1 \\ -\sqrt{x_{22}} \\ -\sqrt{x_{33}} \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{x_{22}} & -\sqrt{x_{33}} \end{pmatrix}$. Therefore, we have $x_{12} = -\sqrt{x_{22}}$, $x_{13} = -\sqrt{x_{33}}$, and $x_{23} = \sqrt{x_{22}x_{33}}$.

Since

$$\begin{aligned}\alpha X &= \frac{-\lambda(x_{33}-\lambda)}{(x_{33}-\lambda)^2 + r_{23}^2} \begin{pmatrix} 1 \\ -\sqrt{x_{22}} \\ -\sqrt{x_{33}} \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{x_{22}} & -\sqrt{x_{33}} \end{pmatrix} \\ &= \frac{-\lambda(x_{33}-\lambda)(1+x_{22}+x_{33})}{(x_{33}-\lambda)^2 + r_{23}^2} \begin{pmatrix} \frac{1}{\sqrt{1+x_{22}+x_{33}}} \\ -\frac{\sqrt{x_{22}}}{\sqrt{1+x_{22}+x_{33}}} \\ -\frac{\sqrt{x_{33}}}{\sqrt{1+x_{22}+x_{33}}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+x_{22}+x_{33}}} & -\frac{\sqrt{x_{22}}}{\sqrt{1+x_{22}+x_{33}}} & -\frac{\sqrt{x_{33}}}{\sqrt{1+x_{22}+x_{33}}} \end{pmatrix},\end{aligned}$$

we know that

$$\delta \equiv \frac{\lambda(x_{33}-\lambda)(1+x_{22}+x_{33})}{(x_{33}-\lambda)^2 + r_{23}^2}, \tag{4.12}$$

is the only negative eigenvalue of $S+\alpha A$ with $\begin{pmatrix} 1 & -\sqrt{x_{22}} & -\sqrt{x_{33}} \end{pmatrix}^\top$ being an eigenvector associated with it. Note that $S+\alpha A = \begin{pmatrix} 0 & \sqrt{x_{22}} & \sqrt{x_{33}} \\ \sqrt{x_{22}} & y_{22} & y_{23}+\alpha a_{23} \\ \sqrt{x_{33}} & y_{23}+\alpha a_{32} & y_{33} \end{pmatrix}$. Therefore, we have the following system of linear equations:

$$\begin{pmatrix} -\delta & \sqrt{x_{22}} & \sqrt{x_{33}} \\ \sqrt{x_{22}} & y_{22}-\delta & y_{23}+\alpha a_{23} \\ \sqrt{x_{33}} & y_{32}+\alpha a_{32} & y_{33}-\delta \end{pmatrix} \begin{pmatrix} 1 \\ -\sqrt{x_{22}} \\ -\sqrt{x_{33}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In view of (4.11) and (4.12), we have the following

$$\frac{\lambda(x_{33}-\lambda)(1+x_{22}+x_{33})}{(x_{33}-\lambda)^2 + r_{23}^2} + x_{22} + x_{33} = 0, \tag{4.13}$$

and

$$\begin{aligned}\sqrt{x_{22}} - \sqrt{x_{22}} &\left( \frac{-\lambda(x_{33}-\lambda)^2}{(x_{33}-\lambda)^2 + r_{23}^2} - \frac{\lambda(x_{33}-\lambda)(1+x_{22}+x_{33})}{(x_{33}-\lambda)^2 + r_{23}^2} \right) \\ &- \sqrt{x_{33}} \left( \frac{\lambda(x_{33}-\lambda)r_{23}}{(x_{33}-\lambda)^2 + r_{23}^2} + \frac{-\lambda a_{23}(x_{33}-\lambda)}{(x_{33}-\lambda)^2 + r_{23}^2} \right) = 0.\end{aligned} \tag{4.14}$$

Recall $\lambda = \frac{(x_{22}+x_{33}) - \sqrt{(x_{22}+x_{33})^2 + 4r_{23}^2 - 4x_{22}x_{33}}}{2}$. We have

$$r_{23}^2 = (\lambda - x_{22})(\lambda - x_{33}), \tag{4.15}$$

plugging it to (4.14) and using $x_{23} = \sqrt{x_{22}x_{33}}$, we get

$$1 + \frac{\lambda(x_{33}-\lambda)}{x_{22}+x_{33}-2\lambda} + \frac{\lambda(1+x_{22}+x_{33})}{x_{22}+x_{33}-2\lambda} - \frac{\lambda x_{33}}{x_{22}+x_{33}-2\lambda} = 0,$$

which gives

$$(x_{22} + x_{33} - 2\lambda) + \lambda(x_{33} - \lambda) + \lambda(1 + x_{22} + x_{33}) - \lambda x_{33} = 0.$$

Since $(x_{22} + x_{33} - 2\lambda) + \lambda(x_{33} - \lambda) + \lambda(1 + x_{22} + x_{33}) - \lambda x_{33} = (\lambda + 1)(x_{22} + x_{33} - \lambda)$, $\lambda < 0$, $x_{22} > 0$, and $x_{33} > 0$, we obtain $\lambda = -1$. Now plugging $\lambda = -1$ to (4.13) and noticing (4.15), we have the following quadratic equation $(x_{22} + x_{33})^2 + (x_{22} + x_{33}) - 1 = 0$. Since $x_{22} + x_{33} > 0$, we have $x_{22} + x_{33} = \frac{-1+\sqrt{5}}{2}$. Moreover, we know that

$$\alpha = \frac{-\lambda(x_{33} - \lambda)}{(x_{33} - \lambda)^2 + r_{23}^2} = \frac{-\lambda(x_{33} - \lambda)}{(x_{33} - \lambda)^2 + (x_{22} - \lambda)(x_{33} - \lambda)} = \frac{1}{x_{22} + x_{33} + 2} = \left(\frac{1 + \sqrt{5}}{2}\right)^{-2}.$$

Using the information above, we can verify in a straightforward manner that $S$ is the metric projection of $-R$ on the cone $\mathcal{P}_3 + \mathcal{N}_3$ by checking $R + S$ is doubly nonnegative and $\langle R + S, S\rangle = 0$. We can also verify that $\left(\frac{1+\sqrt{5}}{2}\right)^{-2} R$ is the metric projection of $-S$ on the cone $\mathcal{P}_3 + \mathcal{N}_3$ by checking $\left(\frac{1+\sqrt{5}}{2}\right)^{-2} R + S$ is doubly nonnegative and $\left\langle \left(\frac{1+\sqrt{5}}{2}\right)^{-2} R + S, R \right\rangle = 0$. We now find $\|R\|$ and $\|S\|$ below.

$$\begin{aligned}
\|R\| &= \sqrt{1 + 2x_{22} + 2x_{33} + x_{22}^2 + x_{33}^2 + 2r_{23}^2} \\
&= \sqrt{1 + 2x_{22} + 2x_{33} + x_{22}^2 + x_{33}^2 + 2(x_{22} + 1)(x_{33} + 1)} \\
&= \sqrt{1 + 2(x_{22} + x_{33}) + (x_{22} + x_{33})^2 + 2(x_{22} + x_{33}) + 2} \\
&= \sqrt{2\sqrt{5} + 1 + \left(\frac{-1 + \sqrt{5}}{2}\right)^2} \\
&= \frac{\sqrt[4]{5}(1 + \sqrt{5})}{2}.
\end{aligned}$$

Note that

$$y_{22}^2 + 2y_{23}^2 + y_{33}^2 = \left(\frac{-\lambda(x_{33} - \lambda)^2}{(x_{33} - \lambda)^2 + r_{23}^2}\right)^2 + 2\left(\frac{\lambda(x_{33} - \lambda)r_{23}}{(x_{33} - \lambda)^2 + r_{23}^2}\right)^2 + \left(\frac{-\lambda r_{23}^2}{(x_{33} - \lambda)^2 + r_{23}^2}\right)^2 = 1.$$

Hence,

$$\|S\| = \sqrt{2x_{22} + 2x_{33} + y_{22}^2 + 2y_{23}^2 + y_{33}^2} = \sqrt{2 \times \frac{-1 + \sqrt{5}}{2} + 1} = \sqrt[4]{5}.$$

Therefore, by Theorem 2.1, we know

$$\cos\theta = -\frac{\alpha\|R\|}{\|S\|} = -\frac{\|S\|}{\|R\|} = -\frac{\sqrt[4]{5}}{\frac{\sqrt[4]{5}(1+\sqrt{5})}{2}} = -\frac{2}{1 + \sqrt{5}},$$

which shows that a critical angle of $\mathcal{P}_3 + \mathcal{N}_3$ formed by $R \in \mathcal{Q}$ and $S \in \mathcal{M}$ is $\theta = \arccos\left(-\frac{2}{1+\sqrt{5}}\right)$. ☐

Now we are ready to give the angular spectrum of $\mathcal{COP}_3 = \mathcal{P}_3 + \mathcal{N}_3$. We have the following theorem.

THEOREM 4.6. *The angular spectrum of* $\mathcal{P}_3 + \mathcal{N}_3$ *is* $\left\{\frac{\pi}{2}, \arccos\left(-\frac{\sqrt{3}}{3}\right), \arccos\left(-\frac{2}{1+\sqrt{5}}\right), \frac{3\pi}{4}\right\}$.

*Proof.* Suppose $\theta$ is a critical angle of $\mathcal{P}_3 + \mathcal{N}_3$ formed by $R$ and $S$ with $R, S \in \mathcal{P}_3 + \mathcal{N}_3$. We consider three different cases.

Case 1: If no entry of $R$ and $S$ is negative, then $R \in \mathcal{N}_3$ and $S \in \mathcal{N}_3$. Therefore, the critical angle must be $\frac{\pi}{2}$ achieved at $R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Case 2: If one of $R$ and $S$ does not have negative entries and the other one has negative entries, then by Theorem 4.3, we know that $\theta$ is a critical angle of $(\mathcal{P}_3, \mathcal{N}_3)$. From Example 2.5, we know that either $\theta = \arccos\left(-\frac{\sqrt{3}}{3}\right)$ or $\theta = \frac{3\pi}{4}$.

Case 3: If both $R$ and $S$ have negative entries, then at least one off-diagonal entry of $R$ and at least one off-diagonal entry of $S$ must be negative. Because both $R$ and $S$ are symmetric, $R$ and $S$ must have at least two negative entries. Moreover, by Theorem 3.6, corresponding entries of $R$ and $S$ cannot be negative at the same time. This shows that one of $R$ and $S$ must have exactly two negative entries, and the other one should have two negative entries or four negative entries. By Lemmas 4.4 and 4.5, we know that $\theta = \arccos\left(-\frac{2}{1+\sqrt{5}}\right)$. □

Using Proposition 2.4, we obtain the angular spectrum of $\mathcal{C}_3 = \mathcal{P}_3 \cap \mathcal{N}_3$ stated below as a corollary.

Corollary 4.7. *The angular spectrum of* $\mathcal{P}_3 \cap \mathcal{N}_3$ *is* $\left\{\frac{\pi}{4}, \pi - \arccos\left(-\frac{2}{1+\sqrt{5}}\right),\ \pi - \arccos\left(-\frac{\sqrt{3}}{3}\right), \frac{\pi}{2}\right\}$.

Remark 4.8. From the angular spectrum of $\mathcal{P}_3 + \mathcal{N}_3$, we know that the maximal angle of $\mathcal{P}_3 + \mathcal{N}_3$ is $\frac{3\pi}{4}$. To the best of our knowledge, this is the first time a serious proof of the statement that the maximal angle of the $3 \times 3$ copositive cone is $\frac{3\pi}{4}$ is given.

## 5. Conclusions.

We have proved a theorem that can be used to find the maximal angle between a vector and a convex cone under the assumption that the maximal angle is obtuse. We have further proved that critical angles that are greater than or equal to $\frac{\pi}{2}$ are Nash angles. Since critical angles of copositive cones and critical angles of $\mathcal{P}_n + \mathcal{N}_n$ are greater than or equal to $\frac{\pi}{2}$, we know that all critical angles of copositive cones and critical angles of $\mathcal{P}_n + \mathcal{N}_n$ are Nash angles. As an application of the results proved in this paper, we have provided a detailed analysis on the angular spectrum of the $3 \times 3$ copositive cone. We have showed that the angular spectrum of the $3 \times 3$ copositive cone contains four angles, and the maximal angle of the $3 \times 3$ copositive cone is the same as the maximal angle $\frac{3\pi}{4}$ between $\mathcal{P}_3$ and $\mathcal{N}_3$, which to the best of our knowledge is the first time a serious proof has been given.

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