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CYCLE PRODUCTS AND EFFICIENT VECTORS IN RECIPROCAL MATRICES*

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Abstract. We focus on the relationship between Hamiltonian cycle products and efficient vectors for a reciprocal matrix A, to more deeply understand the latter. This facilitates a new description of the set of efficient vectors (as a union of convex subsets), greater understanding of convexity within this set and of order reversals in efficient vectors. A straightforward description of all efficient vectors for an n-by-n, column perturbed consistent matrix is given; it is the union of at most $(n-1)(n-2)/2$ convex sets.

Key words. Cycle products, Decision analysis, Efficient vector, Hamiltonian cycle, Pair-wise comparisons, Reciprocal matrix.

AMS subject classifications. 15B48, 05C20, 90B50, 91B06.

1. Introduction.

An n-by-n, entry-wise positive matrix $A = [a_{ij}]$ is called *reciprocal* if $a_{ji} = \frac{1}{a_{ij}}$ for $1 \leq i, j \leq n$. Let $\mathcal{PC}_n$ denote the set of all n-by-n reciprocal matrices. Such matrices represent independent, pair-wise, ratio comparisons among n alternatives. In a variety of models employing reciprocal matrices A, a cardinal ranking vector $w = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}^T \in \mathbb{R}^n_+$ (the entry-wise positive vectors in $\mathbb{R}^n$) is to be deduced from A, so that ratios from w approximate the ratio comparisons in A [3, 11, 12, 15, 23, 25, 26, 32].

Vector w is said to be *efficient* for $A \in \mathcal{PC}_n$ if, for $v \in \mathbb{R}^n_+$, $\left|A - vv^{(-T)}\right| \leq \left|A - ww^{(-T)}\right|$ implies that v is proportional to w. Here, $w^{(-T)}$ and $v^{(-T)}$ represent the entry-wise inverse of the transpose of vectors w and v, respectively, $\leq$ an entry-wise inequality and $|\cdot|$ the entry-wise absolute value. Denote the set of efficient vectors for reciprocal A by $\mathcal{E}(A)$. This is a connected [6] but not necessarily convex set. A cardinal ranking vector should be chosen from $\mathcal{E}(A)$, else there is a better approximating vector. It is now known that the entry-wise geometric convex hull of the columns of A is contained in $\mathcal{E}(A)$ [17]. In particular, every column of A is efficient [18], as well as the simple entry-wise geometric mean of all columns [6]. However, for $n \geq 4$, the right Perron vector [24] of A, the original proposal for the ranking vector [28, 29], may or may not be efficient (depending upon A). Classes of reciprocal matrices for which it is [1, 2, 14], and for which it is not [7, 20], have been identified. In [20], recent developments about the efficiency of the Perron vector were provided.

There is a graph theoretic method to decide whether $w \in \mathcal{E}(A)$. The graph $G(A, w)$ with vertex set $\{1, \ldots, n\}$ has an edge $i \to j$, $1 \leq i \neq j \leq n$, if and only if $w_i \geq a_{ij} w_j$. First in [6], and then in [17] in a simple matricial way, it was shown that $w \in \mathcal{E}(A)$ if and only if $G(A, w)$ is strongly connected. Since A is reciprocal, for every pair of distinct indices $i, j \in \{1, 2, \ldots, n\}$, at least one of the edges $i \to j$ or $j \to i$ is in $G(A, w)$. A directed graph with at least one of $\{i \to j,\ j \to i\}$ as an edge (and, perhaps both) is called *semi-complete* [5] (if it never happens that both occur, it is called a tournament [5, 27]); so $G(A, w)$ is always

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semi-complete. A semi-complete digraph is strongly connected if and only if it contains a full cycle [5], i.e. a Hamiltonian cycle. We add this fact here to the graph theoretic characterization of efficiency.

THEOREM 1.1. *Suppose that $A \in \mathcal{PC}_n$ and that $w \in \mathbb{R}^n_+$. The following are equivalent:*

(i) $w \in \mathcal{E}(A)$;
(ii) $G(A, w)$ *is a strongly connected digraph;*
(iii) $G(A, w)$ *contains a Hamiltonian cycle.*

If there is a $w \in \mathbb{R}^n_+$ such that $A = ww^{(-T)}$, then A is called *consistent* (otherwise, it is said to be *inconsistent*). Of course, a consistent matrix is reciprocal and $\mathcal{E}(A)$ just consists of positive multiples of w. Any matrix in $\mathcal{PC}_2$ is consistent. Throughout, we focus on the study of $\mathcal{E}(A)$ when $A \in \mathcal{PC}_n$ is inconsistent (and, thus, $n \geq 3$).

In [21], a way to generate all vectors in $\mathcal{E}(A)$ (inductively) was given. Here we follow a different approach to characterize $\mathcal{E}(A)$. We connect the Hamiltonian cycles in $G(A, w)$ with the Hamiltonian cycle products ≤ 1 from matrix A. By a (Hamiltonian) cycle in A we mean a sequence of n entries a_{ij} for which $i \to j$ are the edges of a Hamiltonian cycle; the product of these entries is the (Hamiltonian) cycle product. There are at most $\frac{(n-1)!}{2}$ of these products < 1, and exactly this number if no cycle product is 1. We show that $\mathcal{E}(A)$ is the union of at most $\frac{(n-1)!}{2}$ convex subsets. Each subset is associated with a Hamiltonian cycle product from $A < 1$ (when A is inconsistent). We give sufficient conditions for the convexity of $\mathcal{E}(A)$, and for the cone generated by the columns of A (i.e., the set of nonzero, linear combinations of the columns of A with nonnegative coefficients) to lie in $\mathcal{E}(A)$. The former implies the latter since any column of A lies in $\mathcal{E}(A)$ [18]. If the latter occurs, the efficiency of the Perron vector of A and of the left singular vector of A (i.e., the Perron vector of AA^T) follows, as these vectors belong to the cone generated by the columns [22]. In fact, if λ and β are the Perron eigenvalues of A and AA^T and w and v are corresponding (positive) eigenvectors, then $w = A\left(\frac{1}{\lambda}w\right)$ and $v = A\left(\frac{1}{\beta}A^T v\right)$.

Using our new description of the efficient vectors for a reciprocal matrix, we give necessary and sufficient conditions for the existence of an efficient vector associated with a Hamiltonian cycle to have no order reversals with entries along the cycle and show that, in any circumstance, there is always one efficient vector with at most one such order reversal. An efficient vector w for $A \in \mathcal{PC}_n$ is said to exhibit an *order reversal* at i, j if $w_i > w_j$ ($w_i < w_j$) when $a_{ij} < 1$ ($a_{ij} > 1$), $w_i = w_j$ when $a_{ij} \neq 1$, or $w_i \neq w_j$ when $a_{ij} = 1$. There is now a considerable literature on order reversal and vectors that minimize the number of order reversals (see, for example, [9, 13, 31]).

If one off-diagonal entry of a consistent matrix and its symmetrically placed entry are changed, so that the resulting A lies in $\mathcal{PC}_n$, then A is called a *simple perturbed consistent matrix*. In prior work [10], $\mathcal{E}(A)$ for any simple perturbed consistent matrix $A \in \mathcal{PC}_n$ was determined, and this set is defined by a finite system of linear inequalities on the entries of the vectors, implying that $\mathcal{E}(A)$ is convex for simple perturbed consistent matrices A. Since this class of matrices includes $\mathcal{PC}_3$, it follows that $\mathcal{E}(A)$ is convex for any $A \in \mathcal{PC}_3$. Recently, in [30], the authors illustrated geometrically how $\mathcal{E}(A)$ is the union of 3 convex sets in the 4-by-4 case. These facts follow from the main result in this paper (Theorem 2.6).

In [22], we have given examples of matrices obtained from consistent matrices by changing one column (and the corresponding row reciprocally), called *column perturbed consistent matrices*, for which the set of efficient vectors is not convex. However, it was shown that the cone generated by the columns of such matrices A is contained in $\mathcal{E}(A)$. Here we describe the set of efficient vectors for a column perturbed consistent matrix

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$A \in \mathcal{PC}_n$ and show that it is the union of at most $(n-1)(n-2)/2$ convex sets. This extends known results. In fact, the simple perturbed consistent matrices and some type of double perturbed consistent matrices are special cases of column perturbed consistent matrices. As mentioned above, the set of efficient vectors for the former was studied in [10] and is convex. The one for the latter was obtained in [16]. We give here an example illustrating that it may be not convex. We also present an example of a matrix A that is not a simple perturbed consistent matrix and for which $\mathcal{E}(A)$ is convex. In the more recent paper [19], we have described the efficient vectors for reciprocal matrices obtained from a consistent matrix by modifying a 3-by-3 principal submatrix and have provided a class of efficient vectors if the modified block is of size greater than 3.

A useful observation is that the set $\mathcal{PC}_n$ is closed under both positive diagonal similarity and permutation similarity (that is, monomial similarity). Fortunately, such transformations interface with efficient vectors in a natural way.

Lemma 1.2. [18] *Suppose that $A \in \mathcal{PC}_n$ and $w \in \mathcal{E}(A)$. Then, if D is an n-by-n positive diagonal matrix (P is an n-by-n permutation matrix), then $Dw \in \mathcal{E}(DAD^{-1})$ ($Pw \in \mathcal{E}(PAP^T)$).*

Note that the graphs $G(A, w)$ and $G(DAD^{-1}, Dw)$ coincide. From the lemma we have that, if S is an n-by-n monomial matrix, then $\mathcal{E}(SAS^{-1}) = S\mathcal{E}(A)$.

In the next section, we present a fundamental cycle theorem for efficient vectors of reciprocal matrices. In Section 3, we apply that theorem to give some results on order reversals in an efficient vector. Then, in Section 4 we apply the fundamental cycle theorem to study the efficient vectors for a column perturbed consistent matrix. Some conclusions are presented in Section 5.

2. The fundamental cycle theorem for efficient vectors of reciprocal matrices.

Here, we present a fundamental theorem, based upon cycles, that explains $\mathcal{E}(A)$ for $A \in \mathcal{PC}_n$. This permits insight into the nature of $\mathcal{E}(A)$, including a sufficient condition for convexity.

2.1. Auxiliary results.

Next, we note that, for a fixed A, the set of all w such that $G(A, w)$ share a common edge is convex.

Lemma 2.1. *Suppose that $A \in \mathcal{PC}_n$ and that $u, v \in \mathbb{R}^n_+$. If $G(A, u)$ and $G(A, v)$ share a common edge, that edge must also occur in $G(A, w)$ for any $w = tu + (1-t)v$, $t \in (0, 1)$.*

Proof. Suppose that $G(A, u)$ and $G(A, v)$ have a common edge $i \to j$. Then, $u_i \geq a_{ij}u_j$ and $v_i \geq a_{ij}v_j$, so that

$$\begin{aligned} w_i &= tu_i + (1-t)v_i \geq ta_{ij}u_j + (1-t)a_{ij}v_j \\ &= a_{ij}(tu_j + (1-t)v_j) = a_{ij}w_j. \end{aligned}$$

□

As an immediate consequence of Lemma 2.1 and Theorem 1.1, we have the following.

Lemma 2.2. *Suppose that $A \in \mathcal{PC}_n$ and that $u, v \in \mathbb{R}^n_+$. If $G(A, u)$ and $G(A, v)$ share a common Hamiltonian cycle, then the line segment joining u and v lies in $\mathcal{E}(A)$.*

The following lemma on Hamiltonian cycles in A along which all entries are 1 will be helpful.

Lemma 2.3. *Suppose that $A \in \mathcal{PC}_n$ is inconsistent and $\mathcal{C}$ is a Hamiltonian cycle in A. If all entries in A along $\mathcal{C}$ are 1, then there is a Hamiltonian cycle $\mathcal{C}'$ in A such that all entries in A along $\mathcal{C}'$ are ≤ 1 and there is at least one entry < 1.*

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Proof. Since a permutation similarity on A keeps the sets of entries in A along the Hamiltonian cycles the same, we may assume, without loss of generality, that $\mathcal{C}$ is the cycle $1 \to 2 \to 3 \to \cdots \to n-1 \to n \to 1$. Then, $a_{j,j+1} = a_{j+1,j} = a_{1n} = a_{n1} = 1$, for $j = 1, \ldots, n-1$. Since A is not consistent, it has an entry < 1. Let k (> 1) be the smallest integer such that either the k-th upper-diagonal or the k-th lower-diagonal of A has an entry < 1. Then, in one of such diagonals, one of the following occurs: (i) there is an entry < 1 followed by an entry ≤ 1; (ii) after some possible entries equal to 1, the entries alternate between < 1 and > 1, with the last entry < 1. Without loss of generality, we may assume that one of these situations occurs for the k-th upper-diagonal (as otherwise we may consider A^T instead of A). Note that $a_{pq} = 1$ for $|p - q| < k$.

Case (i) Suppose that $a_{i,i+k} < 1$ and $a_{i+1,i+1+k} \leq 1$ for some $i \in \{1, \ldots, n-k-1\}$. Let

$$\begin{gathered}\mathcal{C}' : i \to i+k \to i+1 \to i+2 \to \cdots \to i+k-1 \to i+k+1 \to \\ i+k+2 \to \cdots \to n \to 1 \to 2 \to \cdots \to i-1 \to i.\end{gathered}$$

Note that, if $k = 2$, then $a_{i+k-1,i+k+1} = a_{i+1,i+1+k} \leq 1$.

Case (ii) We consider two subcases.

Case (iia) Suppose that $k > 2$ and $a_{n-k,n} < 1$. Let

$$\mathcal{C}' : n-k \to n \to 1 \to 2 \to \cdots \to n-k-1 \to n-k+1 \to n-k+2 \to \cdots \to n-1 \to n-k.$$

Case (iib) Suppose that $k = 2$, $a_{n-2,n} < 1$, $a_{n-3,n-1} \geq 1$, $a_{n-4,n-2} \leq 1$, $a_{n-5,n-3} \geq 1$, etc. Let

$$\begin{aligned}&\mathcal{C}' : 1 \to 2 \to 4 \to \cdots \to n-2 \to n \to n-1 \to n-3 \to \cdots \to 3 \to 1, \text{ if } n \text{ is even,}\\ &\mathcal{C}' : 1 \to 3 \to 5 \to \cdots \to n-2 \to n \to n-1 \to n-3 \to \cdots \to 4 \to 2 \to 1, \text{ if } n \text{ is odd.}\end{aligned}$$

In each case, the cycle $\mathcal{C}'$ verifies the claim. ∎

2.2. Description of the set $\varepsilon_A(\mathcal{C})$. In what follows, given $A \in \mathcal{PC}_n$ and a Hamiltonian cycle $\mathcal{C}$, we denote by $\varepsilon_A(\mathcal{C})$ the set of vectors w such that $G(A, w)$ contains the cycle $\mathcal{C}$. Note that, by Theorem 1.1, $\varepsilon_A(\mathcal{C}) \subseteq \mathcal{E}(A)$ and, by Lemma 2.2, $\varepsilon_A(\mathcal{C})$ is convex. Next, we describe the sets $\varepsilon_A(\mathcal{C})$. We denote by $\pi_1(A)$ and $\pi_{<1}(A)$ the set of all Hamiltonian cycles for A with product ≤ 1 and < 1, respectively. Of course $\pi_{<1}(A) \subseteq \pi_1(A)$.

Suppose that $\mathcal{C}$ is the Hamiltonian cycle

$$\gamma_1 \to \gamma_2 \to \gamma_3 \to \cdots \to \gamma_n \to \gamma_1. \tag{2.1}$$

Then,

$$a_{\gamma_n\gamma_1} \prod_{i=1}^{n-1} a_{\gamma_i\gamma_{i+1}},$$

is the cycle product for $\mathcal{C}$ in A. We have that $w \in \varepsilon_A(\mathcal{C})$ if and only if

$$\begin{array}{lll} w_{\gamma_1} \geq a_{\gamma_1\gamma_2} w_{\gamma_2}, & w_{\gamma_2} \geq a_{\gamma_2\gamma_3} w_{\gamma_3}, & \ldots, \\ \ldots, & w_{\gamma_{n-1}} \geq a_{\gamma_{n-1}\gamma_n} w_{\gamma_n}, & w_{\gamma_n} \geq a_{\gamma_n\gamma_1} w_{\gamma_1}, \end{array} \tag{2.2}$$

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or, equivalently,

$$\begin{aligned} & w_{\gamma_1} \geq a_{\gamma_1\gamma_2} w_{\gamma_2} \geq \left(\prod_{i=1}^{2} a_{\gamma_i\gamma_{i+1}}\right) w_{\gamma_3} \geq, \\ & \cdots \geq \left(\prod_{i=1}^{n-1} a_{\gamma_i\gamma_{i+1}}\right) w_{\gamma_n} \geq \left(a_{\gamma_n\gamma_1} \prod_{i=1}^{n-1} a_{\gamma_i\gamma_{i+1}}\right) w_{\gamma_1}. \end{aligned} \tag{2.3}$$

For a matricial description of $\varepsilon_A(\mathcal{C})$, consider

$$P = \left[\begin{array}{c|ccc} 0 & & & \\ \vdots & & I_{n-1} & \\ 0 & & & \\ \hline 1 & 0 & \cdots & 0 \end{array}\right] \text{ and } S = \operatorname{diag}(a_{\gamma_1\gamma_2},\ a_{\gamma_2\gamma_3},\ \ldots,\ a_{\gamma_{n-1}\gamma_n},\ a_{\gamma_n\gamma_1}).$$

Then, (2.2) is equivalent to $SPw_\gamma \leq w_\gamma$, in which $w_\gamma = \begin{bmatrix} w_{\gamma_1} & w_{\gamma_2} & \cdots & w_{\gamma_n} \end{bmatrix}^T$. Thus, $w \in \varepsilon_A(\mathcal{C})$ if and only if w_γ satisfies the system of linear inequalities

$$(I_n - SP)w_\gamma \geq 0,\ w_\gamma > 0.$$

We say that a set of efficient vectors for A is a *singleton* if it only contains a single (positive) vector (and all its positive multiples).

LEMMA 2.4. *Suppose that $A \in \mathcal{PC}_n$ and that $\mathcal{C}$, as in (2.1), lies in $\pi_1(A)$. The set of positive solutions w to (2.2), when any $n-1$ inequalities are taken to be equalities, is a singleton. Moreover, a vector in each of the n singletons is extreme in $\varepsilon_A(\mathcal{C})$, so that $\varepsilon_A(\mathcal{C})$ is the cone generated by these n vectors.*

Proof. Let $k \in \{1, \ldots, n\}$. Suppose that all inequalities in (2.2) are taken to be equalities except the k-th one. First suppose that $k \neq n$. Multiplying the equalities, we get

$$\begin{aligned} \prod_{i=1, i\neq k}^{n} w_{\gamma_i} &= a_{\gamma_n\gamma_1} w_{\gamma_1} \prod_{i=1, i\neq k}^{n-1} \left(a_{\gamma_i\gamma_{i+1}} w_{\gamma_{i+1}}\right) \\ \Leftrightarrow w_{\gamma_{k+1}} &= \left(a_{\gamma_n\gamma_1} \prod_{i=1}^{n-1} a_{\gamma_i\gamma_{i+1}}\right) a_{\gamma_{k+1}\gamma_k} w_{\gamma_k}. \end{aligned}$$

Since

$$a_{\gamma_n\gamma_1} \prod_{i=1}^{n-1} a_{\gamma_i\gamma_{i+1}} \leq 1,$$

we get

$$w_{\gamma_{k+1}} \leq a_{\gamma_{k+1}\gamma_k} w_{\gamma_k} \Leftrightarrow w_{\gamma_k} \geq a_{\gamma_k\gamma_{k+1}} w_{\gamma_{k+1}},$$

implying that the k-th inequality is satisfied. If $k = n$, it can be seen in a similar way that the equalities taken from the first $n-1$ inequalities in (2.2) imply the n-th inequality. Clearly, taking one entry 1, each $n-1$ equalities in (2.2) determines a unique positive solution for w. So, we get n (some possibly equal) vectors in $\varepsilon_A(\mathcal{C})$ that are precisely the extreme points of the set. ∎

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Next we show an important fact about the structure of $\varepsilon_A(\mathcal{C})$.

LEMMA 2.5. *Suppose that $A \in \mathcal{PC}_n$ and that $\mathcal{C} \in \pi_1(A)$. Then, $\varepsilon_A(\mathcal{C})$ is a singleton if and only if the cycle product for $\mathcal{C}$ in A is 1. Moreover, if A is inconsistent and the product for $\mathcal{C}$ in A is 1, then $\varepsilon_A(\mathcal{C}) \subseteq \varepsilon_A(\mathcal{C}')$, for some $\mathcal{C}' \in \pi_{<1}(A)$.*

Proof. Since a diagonal similarity does not change any cycle product in A, and by a diagonal similarity on A any $n-1$ entries along $\mathcal{C}$ may be made 1 (and the remaining entry is ≤ 1, since $\mathcal{C} \in \pi_1(A)$), taking into account Lemma 1.2, we assume that this situation occurs in order to prove the result.

When the product for $\mathcal{C}$ in A is 1, all entries in A along $\mathcal{C}$ are 1. Then, from (2.2) (with $\mathcal{C}$ as in (2.1)), $w \in \varepsilon_A(\mathcal{C})$ if and only if, up to a positive factor, w is the vector of 1s, implying that $\varepsilon_A(\mathcal{C})$ is a singleton. When the product of $\mathcal{C}$ in A is < 1, by Lemma 2.4, there is a w satisfying (2.2) in which the first $n-1$ inequalities are taken to be equalities. Then, the n-th inequality is strict. Any vector w' obtained from w by a sufficiently small decrease of w_{γ_n}, and agreeing with w in all other components, still satisfies (2.2) (with w'_{γ_n} instead of w_{γ_n}). So $\varepsilon_A(\mathcal{C})$ is not a singleton. This completes the proof of the first claim.

Suppose that the product along $\mathcal{C}$ in A (inconsistent) is 1, in which case all entries in A along $\mathcal{C}$ are 1 (as we are assuming that there are $n-1$ such entries equal to 1). Thus, all vectors w in $\varepsilon_A(\mathcal{C})$ are constant. By Lemma 2.3, there is a Hamiltonian cycle $\mathcal{C}'$ in A such that all entries in A along $\mathcal{C}'$ are ≤ 1 and there is at least one entry < 1. For $\mathcal{C}'$ as in (2.1), it follows that w satisfies (2.2), implying that $w \in \varepsilon_A(\mathcal{C}')$. Since $\mathcal{C}' \in \pi_{<1}(A)$, the last claim follows. ◻

We observe that, if the product along a Hamiltonian cycle $\mathcal{C}$ in A is 1, then so is the product along the reverse cycle $\mathcal{C}^r$ of $\mathcal{C}$, and $\varepsilon_A(\mathcal{C}) = \varepsilon_A(\mathcal{C}^r)$.

2.3. The main result. Now, we may give the fundamental theorem on the cycle structure of reciprocal matrices and their efficient vectors. If $A \in \mathcal{PC}_n$ is consistent, then $\mathcal{E}(A)$ is the singleton generated by any column of A, otherwise, since each column is in $\mathcal{E}(A)$ and not all columns are proportional, $\mathcal{E}(A)$ is not a singleton. We assume that A is inconsistent.

THEOREM 2.6. *Suppose that $A \in \mathcal{PC}_n$ is inconsistent. If $w \in \mathcal{E}(A)$ and $\mathcal{C}$ is a Hamiltonian cycle in $G(A,w)$, then $\mathcal{C} \in \pi_1(A)$. On the other hand, if $\mathcal{C} \in \pi_1(A)$, then the set*

$$\varepsilon_A(\mathcal{C}) = \{w : G(A,w) \text{ contains } \mathcal{C}\},$$

is a nonempty, convex subset of $\mathcal{E}(A)$. Moreover,

$$\mathcal{E}(A) = \bigcup_{\mathcal{C} \in \pi_{<1}(A)} \varepsilon_A(\mathcal{C}). \tag{2.4}$$

Proof. If $w \in \mathcal{E}(A)$ and $\mathcal{C}$, as in (2.1), is a Hamiltonian cycle in $G(A,w)$, then $w \in \varepsilon_A(\mathcal{C})$. Hence, (2.3) holds and the inequality between the left most and the right most expressions imply

$$a_{\gamma_n\gamma_1} \prod_{i=1}^{n-1} a_{\gamma_i\gamma_{i+1}} \leq 1,$$

which means that $\mathcal{C} \in \pi_1(A)$.

Suppose that $\mathcal{C} \in \pi_1(A)$. The set $\varepsilon_A(\mathcal{C})$ is nonempty by Lemma 2.4. As, for $w \in \varepsilon_A(\mathcal{C})$, $G(A,w)$ contains the cycle $\mathcal{C}$, by Theorem 1.1, $w \in \mathcal{E}(A)$. Also, note that, as all such $G(A,w)$ contain $\mathcal{C}$, the set is convex by

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Lemma 2.2 (alternatively, (polyhedral) convexity follows from the fact that $w \in \varepsilon_A(\mathcal{C})$ is determined by the finite system of linear inequalities (2.2)). Since, by Theorem 1.1, each $w \in \mathcal{E}(A)$ lies in $\varepsilon_A(\mathcal{C})$ for some $\mathcal{C}$, and, by the first part of the proof, $\mathcal{C} \in \pi_1(A)$, it follows that

$$\mathcal{E}(A) = \bigcup_{\mathcal{C} \in \pi_1(A)} \varepsilon_A(\mathcal{C}). \tag{2.5}$$

Now (2.4) follows taking into account the last claim in Lemma 2.5. This conclusion also follows from the following alternate argument. Since $\mathcal{E}(A)$ is connected [6] and is not a singleton (as A is inconsistent), it follows that any singleton $\varepsilon_A(\mathcal{C})$ in the union in (2.5) should be contained in a non-singleton $\varepsilon_A(\mathcal{C}')$, which, by Lemma 2.5, implies that $\mathcal{C}' \in \pi_{<1}(A)$. □

Note that $\pi_{<1}(A)$ has no more than $\frac{(n-1)!}{2}$ elements and has less if there are Hamiltonian cycles from A with product 1. If $n = 3$, then $\frac{(n-1)!}{2} = 1$ which means that $\mathcal{E}(A) = \varepsilon_A(\mathcal{C})$ for some cycle $\mathcal{C} \in \pi_1(A)$ (in fact, $\mathcal{C} \in \pi_{<1}(A)$ if A is inconsistent), which is another way to see that $\mathcal{E}(A)$ is convex for $A \in \mathcal{PC}_3$.

We next give some consequences of Theorem 2.6. Since the results can be trivially verified for consistent matrices, we state them for general reciprocal matrices.

Corollary 2.7. *Suppose $A \in \mathcal{PC}_n$. If there exists a Hamiltonian cycle that lies in $G(A, w)$ for every $w \in \mathcal{E}(A)$, then $\mathcal{E}(A)$ is convex. In particular, this holds if there is just one Hamiltonian cycle product < 1 in A.*

Observe that the first claim in Corollary 2.7 is also a consequence of Lemma 2.2.

Corollary 2.8. *Suppose $A \in \mathcal{PC}_n$. If all columns of A are in $\varepsilon_A(\mathcal{C})$ for some $\mathcal{C} \in \pi_1(A)$, then the cone generated by the columns of A is contained in $\mathcal{E}(A)$. In particular, the Perron vector and the singular vector of A are efficient for A.*

It is known that, in general, $\mathcal{E}(A)$ is not closed under entry-wise geometric mean [18]. However, we note that each subset $\varepsilon_A(\mathcal{C})$ is (in fact, it is closed under entry-wise weighted geometric means), because it is defined by inequalities (2.2).

Corollary 2.9. *Suppose $A \in \mathcal{PC}_n$. If there exists a Hamiltonian cycle that lies in $G(A, w)$ for every $w \in \mathcal{E}(A)$, then $\mathcal{E}(A)$ is closed under entry-wise weighted geometric means.*

We give next an example illustrating the previous results.

Example 2.10. *Let*

$$A = \begin{bmatrix} 1 & 2 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 2 & 1 \\ 1 & \frac{1}{2} & 1 & 2 \\ 2 & 1 & \frac{1}{2} & 1 \end{bmatrix}.$$

Note that there are 2 cycle products in A equal to 1,

$$\mathcal{C}_1 : 1 \to 2 \to 4 \to 3 \to 1,$$
$$\mathcal{C}_2 : 1 \to 3 \to 2 \to 4 \to 1,$$

and their reverses, and the cycle product < 1

$$\mathcal{C}_3 : 1 \to 4 \to 3 \to 2 \to 1.$$

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By Theorem 2.6, $\mathcal{E}(A) = \varepsilon_A(\mathcal{C}_3)$, and, so, $\mathcal{E}(A)$ is convex. In particular, the Perron vector and the singular vector of A are efficient for A. Also, the set $\mathcal{E}(A)$ is closed under entry-wise weighted geometric means. Note that all columns of A lie in $\varepsilon_A(\mathcal{C}_3)$, as they are efficient for A. Using (2.2), we can see that

$$\mathcal{E}(A) = \varepsilon_A(\mathcal{C}_3) = \left\{ w \in \mathbb{R}^4_+ : w_1 \geq \frac{1}{2} w_4 \geq \frac{1}{4} w_3 \geq \frac{1}{8} w_2 \geq \frac{1}{16} w_1 \right\}.$$

We also have

$$\begin{aligned} \varepsilon_A(\mathcal{C}_1) &= \left\{ w \in \mathbb{R}^4_+ : w_1 = 2w_2 = 2w_4 = w_3 \right\}, \\ \varepsilon_A(\mathcal{C}_2) &= \left\{ w \in \mathbb{R}^4_+ : w_1 = w_3 = \frac{1}{2} w_2 = \frac{1}{2} w_4 \right\}, \end{aligned}$$

which can be verified to be contained in $\varepsilon_A(\mathcal{C}_3)$, as expected. We observe that A is not a simple perturbed consistent matrix, as it is nonsingular. Thus, this example illustrates that the convexity of $\mathcal{E}(A)$ may occur for other matrices A than the simple perturbed consistent matrices.

3. Order reversals. Here we give some consequences of Theorem 2.6 regarding the existence of order reversals in an efficient vector. Of course, if A is consistent, no efficient vector for A exhibits an order reversal, as $\frac{w_i}{w_j} = a_{ij}$ for all i, j.

THEOREM 3.1. *Suppose that $A \in \mathcal{PC}_n$ is inconsistent and $\mathcal{C} \in \pi_1(A)$. Then, there is a $w \in \varepsilon_A(\mathcal{C})$ that exhibits no order reversal with entries of A along $\mathcal{C}$ if and only if either there is an entry of $A > 1$ along $\mathcal{C}$ or the cycle product is 1. Otherwise, there is a $w \in \varepsilon_A(\mathcal{C})$ with exactly 1 order reversal along $\mathcal{C}$.*

Proof. Suppose that there is a $w \in \varepsilon_A(\mathcal{C})$ that exhibits no order reversal with entries of A along $\mathcal{C}$ and all entries along $\mathcal{C}$ are ≤ 1, with at least one inequality being strict. Then, $\frac{w_i}{w_j}$ would be ≤ 1 for the efficient vector w and all i, j along the cycle, with at least one inequality being strict, implying

$$1 = \prod_{i,j \text{ along } \mathcal{C}} \frac{w_i}{w_j} < 1,$$

a contradiction. So, an entry > 1 or all entries 1 along the cycle is necessary.

Suppose there is an entry > 1 or all entries are 1 along $\mathcal{C}$. For $\mathcal{C}$ as in (2.1), suppose, without loss of generality, that $a_{\gamma_n \gamma_1} \geq 1$, with $a_{\gamma_n \gamma_1} = 1$ if and only if all entries along the cycle are 1. Now, use the inequalities (2.2) and take the first $n-1$ of them to be equalities, so as to define w, taking $w_1 = 1$ (Lemma 2.4). The last inequality (which is implied by the first $n-1$ inequalities) then ensures that there is no order reversal at γ_n, γ_1. In fact, $a_{\gamma_n \gamma_1} > 1$ implies $\frac{w_{\gamma_n}}{w_{\gamma_1}} > 1$. Also, $a_{\gamma_n \gamma_1} = 1$ implies, by hypothesis, all entries of A along the cycle equal to 1, and thus, by the $n-1$ equalities, $\frac{w_{\gamma_{j+1}}}{w_{\gamma_j}} = 1$, $1 \leq j \leq n-1$. Then, $\frac{w_{\gamma_n}}{w_{\gamma_1}} = 1$. There is no order reversal elsewhere along the cycle because of the equalities. The same construction ensures at most one order reversal without the assumption of an entry > 1 or all entries 1 along $\mathcal{C}$. ∎

Theorem 3.1 concerns the existence of an efficient vector for A with at most one order reversal along a cycle $\mathcal{C} \in \pi_1(A)$. The vector may exhibit order reversals at positions in the matrix not along the cycle.

4. Efficient vectors for column perturbed consistent matrices. In this section, we study the efficient vectors for column perturbed consistent matrices [20, 22]. Based on Lemma 1.2 and the following observation, we may assume that these matrices have a simple form. By J_k we denote the k-by-k matrix with all entries equal to 1.

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LEMMA 4.1. *If $B \in \mathcal{PC}_n$ is a column perturbed consistent matrix, then B is monomially similar to a matrix of the form*

$$A = \begin{bmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & & & & \\ \frac{1}{a_{13}} & & & & \\ \vdots & & J_{n-1} & & \\ \frac{1}{a_{1n}} & & & & \end{bmatrix} \in \mathcal{PC}_n. \tag{4.6}$$

Proof. Suppose that B is obtained from the consistent matrix $ww^{(-T)}$ by modifying, say, row and column i. Then, for $D^{-1} = \text{diag}(w)$, we have that DBD^{-1} has all entries equal to 1, except those in row and column i. Then, DBD^{-1} is permutationally similar to a reciprocal matrix with all entries equal to 1, except those in the first row and column. ∎

Before we give the description of the efficient vectors for a column perturbed consistent matrix, we illustrate it with an example.

EXAMPLE 4.2. *Let*

$$A = \begin{bmatrix} 1 & \frac{1}{5} & \frac{1}{4} & 2 & 3 \\ 5 & 1 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 & 1 \\ \frac{1}{2} & 1 & 1 & 1 & 1 \\ \frac{1}{3} & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We have no cycle products equal to 1, *since $a_{1i}a_{j1} \neq 1$ for $i \neq j$. So, we have exactly* 12 *cycle products* < 1. *They can be written as*

$$\mathcal{C}_\gamma : 1 \to \gamma_2 \to \gamma_3 \to \gamma_4 \to \gamma_5 \to 1,$$

with $a_{1\gamma_2}a_{\gamma_5 1} < 1$. Then,

$$\varepsilon_A(\mathcal{C}_\gamma) = \{w : w_1 \geq a_{1\gamma_2}w_{\gamma_2} \wedge w_{\gamma_2} \geq w_{\gamma_3} \geq w_{\gamma_4} \geq w_{\gamma_5} \wedge w_{\gamma_5} \geq a_{\gamma_5 1}w_1\}.$$

We have that γ can be one of the following permutations of $\{2,3,4,5\}$ (we present them in a convenient order):

$$\begin{array}{llllll} \gamma^{(1)} : 2453, & \gamma^{(2)} : 2543, & \gamma^{(3)} : 2354, & \gamma^{(4)} : 2534, & \gamma^{(5)} : 2345, & \gamma^{(6)} : 2435, \\ \gamma^{(7)} : 3254, & \gamma^{(8)} : 3524, & \gamma^{(9)} : 3245, & \gamma^{(10)} : 3425, & \gamma^{(11)} : 4235, & \gamma^{(12)} : 4325. \end{array}$$

Then,

$$\begin{aligned} \varepsilon_A(\mathcal{C}_{\gamma^{(1)}}) \cup \varepsilon_A(\mathcal{C}_{\gamma^{(2)}}) &= \left\{w \in \mathbb{R}^5_+ : a_{21}w_1 \geq w_2 \geq w_4, w_5 \geq w_3 \geq a_{31}w_1\right\}, \\ \varepsilon_A(\mathcal{C}_{\gamma^{(3)}}) \cup \varepsilon_A(\mathcal{C}_{\gamma^{(4)}}) &= \left\{w \in \mathbb{R}^5_+ : a_{21}w_1 \geq w_2 \geq w_3, w_5 \geq w_4 \geq a_{41}w_1\right\}, \\ \varepsilon_A(\mathcal{C}_{\gamma^{(5)}}) \cup \varepsilon_A(\mathcal{C}_{\gamma^{(6)}}) &= \left\{w \in \mathbb{R}^5_+ : a_{21}w_1 \geq w_2 \geq w_3, w_4 \geq w_5 \geq a_{51}w_1\right\}, \\ \varepsilon_A(\mathcal{C}_{\gamma^{(7)}}) \cup \varepsilon_A(\mathcal{C}_{\gamma^{(8)}}) &= \left\{w \in \mathbb{R}^5_+ : a_{31}w_1 \geq w_3 \geq w_2, w_5 \geq w_4 \geq a_{41}w_1\right\}, \\ \varepsilon_A(\mathcal{C}_{\gamma^{(9)}}) \cup \varepsilon_A(\mathcal{C}_{\gamma^{(10)}}) &= \left\{w \in \mathbb{R}^5_+ : a_{31}w_1 \geq w_3 \geq w_2, w_4 \geq w_5 \geq a_{51}w_1\right\}, \\ \varepsilon_A(\mathcal{C}_{\gamma^{(11)}}) \cup \varepsilon_A(\mathcal{C}_{\gamma^{(12)}}) &= \left\{w \in \mathbb{R}^5_+ : a_{41}w_1 \geq w_4 \geq w_2, w_3 \geq w_5 \geq a_{51}w_1\right\}. \end{aligned}$$

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By Theorem 2.6,

$$\mathcal{E}(A) = \bigcup_{i=1}^{6} \left(\varepsilon_A(\mathcal{C}_{\gamma^{(2i-1)}}) \cup \varepsilon_A(\mathcal{C}_{\gamma^{(2i)}})\right).$$

Note that each set $\varepsilon_A(\mathcal{C}_{\gamma^{(2i-1)}}) \cup \varepsilon_A(\mathcal{C}_{\gamma^{(2i)}})$ is convex. So, in this case $\mathcal{E}(A)$ is the union of 6 convex sets.

For $A \in \mathcal{PC}_n$ and $i, j \in \{1, \dots, n\}$, $i \neq j$, let

$$\varepsilon_{ij}(A) = \left\{w \in \mathbb{R}^n_+ : a_{i1}w_1 \geq w_i \geq w_k \geq w_j \geq a_{j1}w_1,\ k \neq 1, i, j\right\}.$$

Since w in $\varepsilon_{ij}(A)$ is defined by a finite number of linear inequalities in its (positive) entries, we have the following.

Lemma 4.3. *For $A \in \mathcal{PC}_n$ and $i, j \in \{1, \dots, n\}$, $i \neq j$, $\varepsilon_{ij}(A)$ is convex.*

Lemma 4.4. *Let $i, j \in \{2, \dots, n\}$, $i \neq j$. If $A \in \mathcal{PC}_n$ is as in (4.6) with $a_{1i}a_{j1} < 1$, then $\varepsilon_{ij}(A) \subseteq \mathcal{E}(A)$. In particular, $\varepsilon_{ij}(A)$ is the subset of $\mathcal{E}(A)$ that comes from the cycles in $\pi_{<1}(A)$ of the form $1 \to i \to \cdots \to j \to 1$.*

Proof. Denote by $\mathcal{C}_{ij}$ the set of all Hamiltonian cycles $\mathcal{C}$ of the form $1 \to i \to \gamma_3 \to \cdots \to \gamma_{n-1} \to j \to 1$ (there are $(n-3)!$). Suppose that $a_{1i}a_{j1} < 1$, so that the product in A for any cycle $\mathcal{C}$ in $\mathcal{C}_{ij}$ is < 1. We have

$$\varepsilon_A(\mathcal{C}) = \left\{w \in \mathbb{R}^n_+ : a_{i1}w_1 \geq w_i \geq w_{\gamma_3} \geq \cdots \geq w_{\gamma_{n-1}} \geq w_j \geq a_{j1}w_1\right\}.$$

Then,

$$\varepsilon_{ij}(A) = \bigcup_{\mathcal{C} \in \mathcal{C}_{ij}}^{n} \varepsilon_A(\mathcal{C}).$$

Since, by Theorem 2.6, $\varepsilon_A(\mathcal{C}) \subseteq \mathcal{E}(A)$, for each $\mathcal{C} \in \mathcal{C}_{ij}$, the claim follows. ◻

We give next the main result of this section.

Theorem 4.5. *If $A \in \mathcal{PC}_n$ is inconsistent of the form (4.6), then*

$$\mathcal{E}(A) = \bigcup_{(i,j) \in \mathcal{N}} \varepsilon_{ij}(A), \tag{4.7}$$

for $\mathcal{N} = \{(i, j) : i, j \in \{2, \dots, n\}$ and $a_{1i}a_{j1} < 1\}$.

Proof. Let $\mathcal{C}$ be a Hamiltonian cycle in A. By Lemma 4.4, $\varepsilon_{ij}(A)$, with $(i, j) \in \mathcal{N}$, is contained in $\mathcal{E}(A)$, proving the inclusion $\supseteq$ in (4.7). On the other hand, by Theorem 2.6, if $w \in \mathcal{E}(A)$, then $w \in \varepsilon_A(\mathcal{C})$ for some $\mathcal{C} \in \pi_{<1}(A)$. Then, for $\mathcal{C} : 1 \to i \to \gamma_3 \to \cdots \to \gamma_{n-1} \to j \to 1$, the cycle product for $\mathcal{C}$ in A is $a_{1i}a_{j1} < 1$. By Lemma 4.4, $\varepsilon_A(\mathcal{C}) \subseteq \varepsilon_{ij}(A)$. ◻

The set $\mathcal{N}$ has at most $\frac{(n-1)(n-2)}{2}$ elements and has exactly this number if $a_{12}, \dots, a_{1n}$ are pair-wise distinct. In fact, the number of elements in $\mathcal{N}$ is the number of pairs (i, j) with $i, j \in \{2, \dots, n\}$, $j > i$, and $a_{1i} \neq a_{1j}$.

Corollary 4.6. *If $B \in \mathcal{PC}_n$ is a general inconsistent column perturbed consistent matrix, then $\mathcal{E}(B)$ is the union of (at most) $\frac{(n-1)(n-2)}{2}$ convex sets.*

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Proof. By previous observations, there is an n-by-n monomial matrix S such that $A = S^{-1}BS$ is as in (4.6). By Lemma 1.2, $\mathcal{E}(B) = S\mathcal{E}(A)$. Thus, by Theorem 4.5,

$$\mathcal{E}(B) = \bigcup_{(i,j)\in\mathcal{N}} S\varepsilon_{ij}(A),$$

with $\mathcal{N}$ as in the theorem. Since, by Lemma 4.3, $\varepsilon_{ij}(A)$ is convex then $S\varepsilon_{ij}(A)$ is convex [22]. ☐

We finally show that the description of the efficient vectors for a simple perturbed consistent matrix given in [10] is a straightforward consequence of the results developed in this paper. Suppose that $A \in \mathcal{PC}_n$, $n > 2$, is as in (4.6), with $a_{12} = \cdots = a_{1,n-1} = 1$ and $a_{1n} > 1$ (if $a_{1n} = 1$, A is consistent), which can be assumed for the purpose of studying the efficient vectors for a simple perturbed consistent matrix, by Lemma 1.2. Then, the Hamiltonian cycles whose products from A are < 1 are of the form $\mathcal{C} : 1 \to \ell \to \gamma_3 \to \cdots \to \gamma_{n-1} \to n \to 1$, $\ell = 2, \ldots, n-1$. We have

$$\varepsilon_{\ell n}(A) = \left\{ w \in \mathbb{R}^n_+ : w_1 \geq w_\ell \geq w_k \geq w_n \geq a_{n1}w_1,\ k \neq 1, \ell, n \right\}.$$

Then,

$$\bigcup_{\ell=2}^{n-1} \varepsilon_{\ell n}(A) = \left\{ w \in \mathbb{R}^n_+ : w_1 \geq w_k \geq w_n \geq a_{n1}w_1,\ k \neq 1, n \right\}. \tag{4.8}$$

By Theorem 4.5, $\mathcal{E}(A)$ is the set (4.8), as claimed in [10].

Though the set of efficient vectors for a simple perturbed consistent matrix is convex, when the reciprocal matrix is double perturbed, that is, is obtained from a consistent matrix by changing two pairs of reciprocal entries, non-convexity may occur.

Example 4.7. *Let $0 < \alpha < \beta < 1$ and*

$$A = \begin{bmatrix} 1 & \alpha & \beta & 1 \\ \frac{1}{\alpha} & 1 & 1 & 1 \\ \frac{1}{\beta} & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

By Theorem 4.5,

$$\mathcal{E}(A) = \varepsilon_{23}(A) \cup \varepsilon_{24}(A) \cup \varepsilon_{34}(A),$$

with

$$\varepsilon_{23}(A) = \left\{ w \in \mathbb{R}^4_+ : \frac{1}{\alpha}w_1 \geq w_2 \geq w_4 \geq w_3 \geq \frac{1}{\beta}w_1 \right\},$$
$$\varepsilon_{24}(A) = \left\{ w \in \mathbb{R}^4_+ : \frac{1}{\alpha}w_1 \geq w_2 \geq w_3 \geq w_4 \geq w_1 \right\},$$
$$\varepsilon_{34}(A) = \left\{ w \in \mathbb{R}^4_+ : \frac{1}{\beta}w_1 \geq w_3 \geq w_2 \geq w_4 \geq w_1 \right\}.$$

We have

$$u = \begin{bmatrix} 1 & \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha} \end{bmatrix}^T \in \varepsilon_{23}(A),$$
$$v = \begin{bmatrix} 1 & 1 & \frac{1}{\beta} & 1 \end{bmatrix}^T \in \varepsilon_{34}(A).$$

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However,

$$u+v=\left[\begin{array}{cccc} 2 & 1+\frac{1}{\alpha} & \frac{1}{\alpha}+\frac{1}{\beta} & 1+\frac{1}{\alpha} \end{array}\right]^T \notin \mathcal{E}(A),$$

implying that $\mathcal{E}(A)$ *is not convex.*

5. Conclusions. Examples of reciprocal matrices for which the set of efficient vectors is not convex are known, for example those for which the right Perron vector is not efficient. Here we have described the set of efficient vectors for a reciprocal matrix A as a union of at most $\frac{(n-1)!}{2}$ convex sets. Each of these sets corresponds to a Hamiltonian cycle in A whose product of the entries is smaller than 1 (if the matrix is inconsistent). Our characterization has allowed us to describe the set of efficient vectors for a reciprocal matrix obtained from a consistent matrix by modifying one column (row) as a union of at most $\frac{(n-1)(n-2)}{2}$ convex sets. In particular, the known characterization of the efficient vectors for a reciprocal matrix obtained from a consistent one by modifying one pair of reciprocal entries [10] followed in a straightforward way. In this case, the set of efficient vectors is convex. We also identified efficient vectors for A with at most one order reversal at positions along the associated cycle.

The results obtained here may be helpful in a better understanding of the convexity of the set of efficient vectors for a reciprocal matrix and of the existence of rank reversals in these vectors. They might also be applied to the study of the possible efficient vectors for reciprocal matrices with unspecified entries.

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