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# TRACIAL NUMERICAL RANGES AND LINEAR DEPENDENCE OF OPERATORS*

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**Abstract.** Linear dependence of two Hilbert space operators is expressed in terms of equality in modulus of certain sesquilinear and quadratic forms associated with the operators. The forms are based on generalized numerical ranges.

**Key words.** Hilbert space, Linear operators, Linear dependence, Generalized numerical range.

**AMS subject classifications.** 15A03, 47A12, 47A99.

**1. Introduction and main result.** Let $\mathcal{H}$ be a complex Hilbert space with the inner product $\langle\cdot,\cdot\rangle$, and let $L(\mathcal{H})$ be the algebra of linear bounded operators on $\mathcal{H}$. It will be assumed without further notice that $\dim\mathcal{H} \geq 2$. Denote by $\operatorname{Tr} X$ the trace of a trace-class operator $X \in L(\mathcal{H})$. We let $\mathbb{R}$ and $\mathbb{C}$ stand for the real and complex field, respectively.

Given an operator $T \in L(\mathcal{H})$, to what extent is it determined by its numerical range $W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H},\ \|x\| = 1\}$? In some rare situations, the numerical range alone can be used to classify a special type of operator. For instance, the $W(T) = \{\mu\}$ if and only if $T = \mu I$; $W(T) \subseteq \mathbb{R}$ if and only if $T = T^*$; $W(T) \subseteq [0,\infty)$ if and only if $T$ is positive semidefinite. On the other hand, it is a standard result that an operator on a complex Hilbert space is completely determined with the quadratic form that defines its numerical range. Based on applications in preserver problems and elsewhere, we asked in [5] to what extent an operator is determined if only partial information is known about the quadratic form. More precisely, we showed that, given a number $q \in [0,1]$, the operators $A$ and $B$ satisfy $|\langle Ax, y\rangle| = |\langle Bx, y\rangle|$ for every pair

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of normalized vectors $x, y \in \mathcal{H}$ with $\langle x, y\rangle = q$ only if $A = \mu B + \nu I$ or $A = \mu B^* + \nu I$ for some scalars $\mu, \nu$ with $|\mu| = 1$. In effect, this covers the modulus of quadratic form of classical numerical range (with $q = 1$) as well as of its generalization, the $q$-numerical range, defined by $W_q(T) := \{\langle Tx, y\rangle : \|x\| = 1 = \|y\|,\ \langle x, y\rangle = q\}$.

There are many more generalizations of classical numerical range which are extensively studied (see [7] for a survey). Two examples are the $k$-numerical range $W_k(T) := \{\sum_{i=1}^{k}\langle Tx_i, x_i\rangle : \langle x_i, x_j\rangle = \delta_{ij}\}$ and the $c$-numerical range for a summable sequence $c = \big(c_i\big)_{i\in\mathsf{N}}$ given by $W_c := \{\sum_i c_i\langle Tx_i, x_i\rangle : \langle x_i, x_j\rangle = \delta_{ij}\}$. The common extension of all these three types of numerical ranges is the $C$-numerical range, defined for a trace-class operator $C$ by $W_C(T) = \{\mathrm{Tr}(CUTU^*) : UU^* = I = U^*U\}$. For example, the $q$-numerical range equals the $C$-numerical range given by a rank-one operator $C = q\langle\cdot, y\rangle y + \sqrt{1-q^2}\langle\cdot, z\rangle y$ for a fixed orthonormal pair $(y, z)$. In light of this, our result [5] is about $C$-numerical ranges for rank-one operators $C$. Presently, we study the same kind of problem for general normal trace-class or finite rank $C$; see Theorem 1.5 below.

The following conjecture was formulated in [5]. It will be convenient to use the marker $\Delta_X = 1$ if $\mathrm{Tr}\,X = 0$; $\Delta_X = 0$ if $\mathrm{Tr}\,X \neq 0$; here $X$ is a trace-class operator.

Conjecture 1.1.[1] Suppose $C \in L(\mathcal{H})$ is a non-scalar trace-class operator. Then two operators $A, B \in L(\mathcal{H})$ have the property

$$|\,\mathrm{Tr}(CU^*AU)| = |\,\mathrm{Tr}(CU^*BU)|, \qquad \forall\ \text{unitary}\ \ U \in L(\mathcal{H}) \tag{1.1}$$

if and only if one of the following conditions holds:

(1) $C$ and $C^*$ are linearly dependent, and either $A = \mu B + \nu\Delta_C I$ or $A = \mu B^* + \nu\Delta_C I$ for some $\mu, \nu \in \mathbb{C}$, $|\mu| = 1$;
(2) $C$ and $C^*$ are linearly independent, and $A = \mu B + \nu\Delta_C I$ for some $\mu, \nu \in \mathbb{C}$, $|\mu| = 1$.

Note that we have $\dim\mathcal{H} \geq 2$ in Conjecture 1.1 because of the hypothesis that $C$ is non-scalar. The example below shows that this hypothesis is vital.

Example 1.2. If $\dim\mathcal{H} < \infty$ and $C = zI$, $z \in \mathbb{C}\setminus\{0\}$, then (1.1) is equivalent to $|\,\mathrm{Tr}\,A| = |\,\mathrm{Tr}\,B|$. There is not much to say in this situation.

The "if" part of Conjecture 1.1 is clear. Indeed, assume for example that (1) holds with $A = \mu B^* + \nu\Delta_C I$, $\mu, \nu \in \mathbb{C}$, $|\mu| = 1$. Then necessarily $C = \alpha C^*$ for some

[1]We use this opportunity to correct an inaccuracy in the formulation of [5, Conjecture 6.6]; $C = C^*$ (resp., $C \neq C^*$) was used there in place of "$C$ and $C^*$ are linearly dependent" (resp., "$C$ and $C^*$ are linearly independent").

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$|\alpha| = 1$ and moreover $\nu\Delta_C \neq 0$ only if $\operatorname{Tr} C = 0$. Therefore,

$$\begin{aligned}\operatorname{Tr}(CU^*AU) &= \mu \operatorname{Tr}(CU^*B^*U) = \mu\alpha \operatorname{Tr}(C^*U^*B^*U) \\ &= \mu\alpha\overline{\operatorname{Tr}(U^*BUC)} = \mu\alpha\overline{\operatorname{Tr}(CU^*BU)}\end{aligned}$$

for every unitary $U \in L(\mathcal{H})$, so (1.1) holds.

Conjecture 1.1 was proven in [5] for the case when $C$ has rank one. However, the conjecture generally fails if $C$ has rank larger than one, as the following examples show.

Example 1.3. Assume $\dim \mathcal{H} < \infty$, and let the operators $A, B, C$ have the following properties:

(a) $C, C^*, I$ are linearly dependent;
(b) $C, C^*$ are linearly independent;
(c) $B, B^*, I$ are linearly independent, and $\operatorname{Tr}(B) = 0$;
(d) $A = \mu B^* + \nu\Delta_C I$ for some $\mu, \nu \in \mathbb{C}$, $|\mu| = 1$.

If Conjecture 1.1 would hold, then we necessarily have $C = \alpha C^* + \beta I$, $|\alpha| = 1$, $\beta \in \mathbb{C}$ (the hypothesis that $C$ is non-scalar is used here). Since $\Delta_C \cdot (\operatorname{Tr} C) = 0$, we now obtain

$$\begin{aligned}\operatorname{Tr}(CU^*AU) &= \operatorname{Tr}(CU^*(\mu B^* + \nu\Delta_C I)U) \;=\; \mu \operatorname{Tr}(CU^*B^*U) \\ &= \mu\alpha \operatorname{Tr}(C^*U^*B^*U) \;=\; \mu\alpha\overline{\operatorname{Tr}(CU^*BU)}.\end{aligned}$$

Thus, (1.1) holds, but clearly neither (1) nor (2) holds. The property (c) is used to preclude the possibility that $A = \mu' B + \nu'\Delta_C I$ for some $\mu', \nu' \in \mathbb{C}$, $|\mu'| = 1$.

Note that if $C$ is non-scalar of rank-one, then it is easy to see that $C, C^*, I$ are linearly dependent only if $C, C^*$ are, and the situation of Example 1.3 cannot occur in this case.

Example 1.4. Let $\dim \mathcal{H} = 2$ (we identify $\mathcal{H}$ with $\mathbb{C}^2$), and

$$A = \begin{bmatrix} -1-i & 0 \\ 1 & -i \end{bmatrix}, \quad B = \begin{bmatrix} 1 & i \\ 0 & 1+i \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}.$$

A computation shows that for every unitary $U$, we have

$$\operatorname{Tr}(CU^*AU) = \overline{\operatorname{Tr}(CU^*BU)}, \tag{1.2}$$

and so (1.1) holds. Indeed, to prove (1.2) we use the fact that every unitary $U \in \mathbb{C}^{2\times 2}$ has the form

$$U = \begin{bmatrix} \cos t & e^{i\xi}\sin t \\ -e^{-i\xi}\sin t & \cos t \end{bmatrix} \cdot \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}, \quad \text{for some } \xi, t \in [0, 2\pi),\ \ p, q \in \mathbb{C},\ \ |p| = |q| = 1.$$

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Then $\mathrm{Tr}(CU^*AU)$ is computed to be equal to

$$\sin^2 t - (\sin\xi + \cos\xi)\sin t\cos t - i(1 + \sin^2 t + (\sin\xi - \cos\xi)\sin t\cos t),$$

and similarly,

$$\mathrm{Tr}(CU^*BU) = \sin^2 t - (\sin\xi + \cos\xi)\sin t\cos t + i(1 + \sin^2 t + (\sin\xi - \cos\xi)\sin t\cos t).$$

Note that $C, C^*, I$ are linearly dependent, $C, C^*$ are linearly independent, $B, B^*, I$ are linearly independent, and $A = iB^* + (-1 - 2i)I$. Note also that $A$ is not of the form $A = \mu B + \nu\Delta_C I$ or $A = \mu B^* + \nu\Delta_C I$ for any $\mu, \nu \in \mathbb{C}$.

We mention in passing that a related problem to characterize pairs of operators $A, B \in L(\mathcal{H})$ for which

$$\mathrm{Tr}(CU^*AU) = \mathrm{Tr}(CU^*BU) \qquad \forall \text{ unitary } U \in L(\mathcal{H}) \tag{1.3}$$

holds, has been resolved in [5]. Namely, assuming $C$ is non-scalar trace-class, (1.3) holds if and only if either (1) $\mathrm{Tr}\, C \neq 0$ and $A = B$, or (2) $\mathrm{Tr}\, C = 0$ and $A - B$ is scalar.

In view of these examples, it is of interest to find out whether or not (1.1) implies that either $A = \mu B + \nu I$ or $A = \mu B^* + \nu I$ for some $\mu, \nu \in \mathbb{C}$, $|\mu| = 1$. We prove that this is indeed the case for finite rank operators and for normal trace-class operators.

An operator $X \in L(\mathcal{H})$ is said to be *essentially selfadjoint* if there is $\nu \in \mathbb{C}$ such that $X - \nu I$ is a scalar multiple of a selfadjoint operator. Elementary calculations show that $X$ is essentially selfadjoint if and only if either one of the following equivalent statements holds:

(a) $X, X^*, I$ are linearly dependent;
(b) $X = \mu X^* + \nu I$ for some $\mu, \nu \in \mathbb{C}$, with $|\mu| = 1$ and $\mathrm{Re}\,(\nu\mu^{-1/2}) = 0$;
(c) $X$ is normal with spectrum on a straight line.

THEOREM 1.5. (1) *Assume $C$ is a non-scalar trace-class operator which is finite rank or normal. If* (1.1) *holds for $A, B \in L(\mathcal{H})$, then either $A = \mu B + \nu I$ or $A = \mu B^* + \nu I$ for some $\mu, \nu \in \mathbb{C}$, $|\mu| = 1$.*

(2) *If in addition $C$ is normal, and $B$, $A$, and $C$ are not essentially selfadjoint, then*

$$A = \mu B + \nu\Delta_C I \quad \text{or} \quad A = \mu B^* + \nu\Delta_C I \quad \text{for some } \mu, \nu \in \mathbb{C},\ |\mu| = 1. \tag{1.4}$$

It is easy to see (in view of the first part of Theorem 1.5) that under the hypotheses of the theorem, $A$ and $B$ either are both essentially selfadjoint or both are not essentially selfadjoint.

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Note that

$$\mathrm{Tr}(V^*CV \cdot U^*AU) = \mathrm{Tr}(C \cdot (UV^*)^* A(UV^*))$$

for all unitary $V \in L(\mathcal{H})$. Thus, we may replace $C$ by any operator which is unitarily similar to $C$ in Conjecture 1.1 and Theorem 1.5. We will use this observation in the proof of Theorem 1.5.

The following notation will be used throughout: $\mathbb{C}^{m\times n}$ stands for the vector space of $m \times n$ complex matrices, with $\mathbb{C}^{m\times 1}$ simplified to $\mathbb{C}^m$; $\mathrm{diag}\,(X_1,\dots,X_p) = X_1 \oplus X_2 \oplus \cdots \oplus X_p$ is the block diagonal matrix with the diagonal blocks $X_1,\dots,X_p$ (in this order). We denote by $E_{ij}$ the matrix (or operator with respect to a fixed orthonormal basis) having 1 in the $(i,j)$th position and zeros elsewhere; $\mathbf{e}_j$ stands for the unit coordinate vector with 1 in the $j$th position and zeros elsewhere. Thus, $E_{ij} = \mathbf{e}_i\mathbf{e}_j^*$.

Upon completion of our paper, we learned that Professor Fangyan Lu has studied Conjecture 1.1 independently with a different approach [9].

We conclude the introduction with a short overview of the next sections. Sections 2, 3, and 4 are preparatory for the proof of Theorem 1.5. There, we recall basic properties of real-analytic functions, study properties of linear dependence of operators on the whole space vs these properties on subspaces of fixed dimension (these results are of independent interest), and provide some information on $C$-numerical ranges. In Sections 5 and 6, we prove Theorem 1.1 for the cases when $C$ is normal and when $C$ is finite rank, respectively. In latter case, the proof is reduced to a finite-dimensional $\mathcal{H}$, and then proceeds by induction on the dimension of $\mathcal{H}$. Finally, in the last short section, we indicate an extension of Theorem 1.5 to a larger class of operators $C$.

**2. Preliminaries on real-analytic functions.** Here, we collect several well-known facts on real-analytic functions to be used in the sequel.

Let $W \subseteq \mathbb{R}^k$ be open subset. A function $f = f(x_1,\dots,x_k) : W \to \mathbb{C}$ is said to be *real-analytic* if for each point of $W$, there is a polydisc contained in $W$ with positive radii such that $f$ equals its Taylor series on this polydisc. Clearly, $f$ is real-analytic if and only if the real and imaginary parts of $f$ are real-analytic.

Proposition 2.1.

(a) *If $f = f(x_1,\dots,x_k) : W \to \mathbb{C}$ is real-analytic, then so are $\overline{f}$ and $|f|^2 = f\overline{f}$.*
(b) *The zero set of any real-analytic function with the connected domain of definition $W$ is either equal to $W$ or its complement is dense in $W$.*
(c) *A product of two nonzero real-analytic functions is itself nonzero real-analytic.*

For part (b), see e.g., [11, I §3 Lemma 3.2]; (c) obviously follows from (b).

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A subset $M \subseteq \mathbb{C}^n$ is a *real-analytic manifold* if it has an open cover with charts $(f_\alpha(W_\alpha), f_\alpha)$, with $W_\alpha \subseteq \mathbb{R}^k$ open and $f_\alpha : W_\alpha \to f(W_\alpha)$ a homeomorphism onto a relatively open subset $f(W_\alpha) \subseteq M$ such that $(f_\beta)^{-1} \circ f_\alpha : W_\alpha \cap f_\alpha^{-1}(f_\beta(W_\beta)) \to \mathbb{R}^k$ is real-analytic. If $M$ is a real-analytic manifold, then a map $F : M \to \mathbb{C}$ is *real-analytic* if $F \circ f_\alpha : W_\alpha \to \mathbb{C}$ is real-analytic for every index $\alpha$ [10, pp. 54].

It is well-known (see e.g., [12]) that the group $\mathcal{U}_n$ of unitaries in $\mathbb{C}^{n\times n}$ is a compact real-analytic manifold. Moreover, it is pathwise connected with real-analytic paths (which take the form $t \mapsto e^{i(tH_1+(1-t)H_2)}$ for appropriate hermitian $H_1, H_2$).

Proposition 2.2. *Given two fixed vectors* $\mathbf{a}, \mathbf{b}$*, the map* $U \mapsto \mathbf{a}^*U\mathbf{b}$*,* $U \in \mathcal{U}_n$*, is real-analytic. Also, given two matrices* $C$ *and* $A$*, the function* $U \mapsto \mathrm{Tr}(CUAU^*)$*,* $U \in \mathcal{U}_n$*, is real-analytic.*

It easily follows from Proposition 2.1 that if $F : M \to \mathbb{C}$ is a nonzero real-analytic map on a compact, real-analytic, pathwise connected manifold $M$, then the set of points where $F$ does not vanish is dense in $M$.

**3. Local vs global linear dependence of operators.** In this section, we prove results concerning local (i.e., restricted to proper subspaces) vs global linear dependence of operators that will be used in the proof of Theorem 1.5, and are of independent interest. We will consider the following properties of two operators $A, B \in L(\mathcal{H})$ frequently in our subsequent discussion.

- **(P1)** There exist a unimodular number $\mu$ and some complex number $\nu$ such that $B = \mu A + \nu I$ or $B = \mu A^* + \nu I$.
- **(P2)** There exist a unimodular number $\mu$ and some complex number $\nu$ such that $B = \mu A + \nu I$.

Theorem 3.1. *Fix a positive integer* $k > 1$*. Suppose* $\dim \mathcal{H} \geq 2$ *and let* $A, B \in L(\mathcal{H})$*.*

(1) *Assume that for every rank-k orthogonal projection* $P$*, the compressions* $A' = PAP|_{\mathrm{Im}\,P}$ *and* $B' = PBP|_{\mathrm{Im}\,P}$ *of* $A$ *and* $B$ *onto* $\mathrm{Im}\,P$ *have property* **(P1)***. Then* $A$ *and* $B$ *have property* **(P1)***.*

(2) *Assume that for every rank-k orthogonal projection* $P$*, the compressions* $A' = PAP|_{\mathrm{Im}\,P}$ *and* $B' = PBP|_{\mathrm{Im}\,P}$ *of* $A$ *and* $B$ *onto* $\mathrm{Im}\,P$ *have property* **(P2)***. Then* $A$ *and* $B$ *have property* **(P2)***.*

We indicate an immediate corollary of Theorem 3.1.

Corollary 3.2. *Fix cardinalities* $\aleph'$*,* $\aleph''$ *such that* $\aleph' + \aleph''$ *coincides with the dimension (= cardinality of an orthonormal basis) of* $\mathcal{H}$ *and* $\aleph'' \geq 2$*.*

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(1) *Assume that for every orthogonal projection* $P \in L(\mathcal{H})$ *with the image of dimension* $\aleph''$ *and the kernel of dimension* $\aleph'$*, the compressions* $A' = PAP|_{\operatorname{Im} P}$ *and* $B' = PBP|_{\operatorname{Im} P}$ *of* $A$ *and* $B$ *onto* $\operatorname{Im} P$ *have property* **(P1)**. *Then* $A$ *and* $B$ *have property* **(P1)**.

(2) *Assume that for every orthogonal projection* $P \in L(\mathcal{H})$ *with the image of dimension* $\aleph''$ *and the kernel of dimension* $\aleph'$*, the compressions* $A' = PAP|_{\operatorname{Im} P}$ *and* $B' = PBP|_{\operatorname{Im} P}$ *of* $A$ *and* $B$ *onto* $\operatorname{Im} P$ *have property* **(P2)**. *Then* $A$ *and* $B$ *have property* **(P2)**.

*Proof.* Indeed, the hypotheses of part (1) of Corollary 3.2 imply that for every rank-two orthogonal projection $Q$ the compressions of $A$ and $B$ to the range of $Q$ have property **(P1)**. Now apply Theorem 3.1. The proof of part (2) is analogous. ∎

For the proof of Theorem 3.1, we need a lemma (presented in greater generality than is needed in this paper.) Denote by $\operatorname{Gr}_k(\mathbb{C}^n)$ (Grassmannian) the set of all $k$-dimensional subspaces of $\mathbb{C}^n$ with the standard topology.

LEMMA 3.3. *Let* $A_1, \ldots, A_q \in \mathbb{C}^{n\times n}$. *Fix an integer* $k$, $1 \le k \le n$. *Then either* $PA_1P, \ldots, PA_qP$ *are linearly dependent (over* $\mathbb{C}$*) for every rank* $k$ *orthogonal projection* $P$*, or the set of* $k$*-dimensional subspaces* $\mathcal{M} \subseteq \mathbb{C}^n$ *such that* $P_{\mathcal{M}}A_1P_{\mathcal{M}}, \ldots, P_{\mathcal{M}}A_qP_{\mathcal{M}}$ *are linearly independent, where* $P_{\mathcal{M}} \in \mathbb{C}^{n\times n}$ *stands for the orthogonal projection onto* $\mathcal{M}$*, is dense in* $\operatorname{Gr}_k(\mathbb{C}^n)$.

*Proof.* We assume $k^2 \ge q$ (if $q > k^2$, then $P_{\mathcal{M}}A_1P_{\mathcal{M}}, \ldots, P_{\mathcal{M}}A_qP_{\mathcal{M}}$ are always linearly dependent).

We consider $\operatorname{Gr}_k(\mathbb{C}^n)$ as a manifold with the standard charts $\{C_{i_1,\ldots,i_k}\}$, where

$$C_{i_1,\ldots,i_k} := \left\{ \text{Column space of } \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ x_j \in \mathbb{C}^{1\times k},\ j = 1, 2, \ldots, n,\ \begin{bmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_k} \end{bmatrix} = I_k \right\}.$$

Here, $\{i_1, \ldots, i_k\}$ is a selection of indices $i_1, \ldots, i_k \in \{1, 2, \ldots, n\}$ such that $i_1 < i_2 < \cdots < i_k$. Then $\operatorname{Gr}_k(\mathbb{C}^n)$ is a real-analytic manifold whose charts are parametrized by $2k(n-k)$ real variables $t_1, \ldots, t_{2k(n-k)}$ that represent the real and imaginary parts of the $x_j$'s for $j \notin \{i_1, \ldots, i_k\}$.

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Fix a chart $C_{i_1,\dots,i_k}$. Applying the Gram-Schmidt orthogonalization to the columns of

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

we obtain an orthonormal basis in the subspace $\mathcal{M}$ spanned by these orthonormal columns, which we temporarily denote by $c_1,\dots,c_k$. Then, $P_{\mathcal{M}} = c_1c_1^* + \cdots + c_kc_k^*$. Note that the orthonormal basis $c_1,\dots,c_k$ is a real-analytic function of $t_1,\dots,t_{2k(n-k)}$ (as readily follows from the formulas for the Gram-Schmidt orthogonalization), and same then holds for the projections $P_{\mathcal{M}}$. So, we have

$$P_{\mathcal{M}}A_1P_{\mathcal{M}} = B_1,\ \dots,\ P_{\mathcal{M}}A_qP_{\mathcal{M}} = B_q,$$

where $B_1,\dots,B_q$ are $n\times n$ matrices whose entries are analytic functions of real variables $t_1,\dots,t_{2k(n-k)}$ (as well as functions of the entries of $A_1,\dots,A_q$ which are assumed to be fixed). We write the entries of each $B_j$ as a $n^2$-component column vector (in some fixed order of the entries), and collect these column vectors in a $n^2\times q$ matrix $Z$. Clearly, $P_{\mathcal{M}}A_1P_{\mathcal{M}},\dots,P_{\mathcal{M}}A_qP_{\mathcal{M}}$ are linearly dependent if and only if

$$\mathrm{Re}\,(\det Q_1) = 0,\ \mathrm{Im}\,(\det Q_1) = 0,\ \dots,\ \mathrm{Re}\,(\det Q_s) = 0,\ \mathrm{Re}\,(\det Q_s) = 0, \tag{3.1}$$

where $Q_1,\dots,Q_s$ are all $q\times q$ submatrices of $Z$. The equations (3.1) are of the form

$$f_1(t_1,\dots,t_{2k(n-k)}) = 0,\ \dots,\ f_{2s}(t_1,\dots,t_{2k(n-k)}) = 0, \tag{3.2}$$

where $f_1,\dots,f_{2s}$ are real valued real-analytic functions of $t_1,\dots,t_{2k(n-k)}$. Note that the solutions of equations (3.2) are exactly the zeros of $F = |f_1|^2+\cdots+|f_{2s}|^2$, which is an analytic function of real variables $(t_1,\dots,t_{2k(n-k)})$. By Proposition 2.1, either the solution set of (3.2) comprises all of $\mathbb{R}^{2k(n-k)}$, or the complement of the solution set is dense in $\mathbb{R}^{2k(n-k)}$. In the former case, using the property that intersection of any two charts is open and non-empty in either one of the two charts, we obtain that $PA_1P,\dots,PA_qP$ are linearly dependent for every rank $k$ orthogonal projection $P$ (see [11, I §1, Remark 1.20]), and in the latter case analogously we obtain that the set of subspaces $\mathcal{M}$ for which $P_{\mathcal{M}}A_1P_{\mathcal{M}},\dots,P_{\mathcal{M}}A_qP_{\mathcal{M}}$ are linearly independent, is dense in $\mathrm{Gr}_k\,(\mathbb{C}^n)$. $\square$

For convenience, we state also the following easily verified fact.

LEMMA 3.4. *The following statements are equivalent for $A\in L(\mathcal{H})$:*

(1) *$A$ is scalar;*

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(2) *Every nonzero* $x \in \mathcal{H}$ *is an eigenvector of* $\mathcal{H}$*;*
(3) *The compression of* $A$ *to any* 2*-dimensional subspace is scalar.*

*Proof.* Obviously, (1) implies (2) and (3). Assume (2) holds. Choose linearly independent $x$, $y$ in $\mathcal{H}$, and let $Ax = \lambda_x x$, $Ay = \lambda_y y$ ($\lambda_x, \lambda_y \in \mathbb{C}$). Since $x + y$ is also an eigenvector of $A$, we easily obtain $\lambda_x = \lambda_y$. Thus, all eigenvalues of $A$ are the same, and (1) holds. If (3) holds, then (2) holds as well, otherwise for some nonzero $x$, the compression of $A$ to the 2-dimensional subspace spanned by $x$ and $Ax$ would not be scalar. ◻

*Proof of Theorem 3.1.* Evidently, we need only to prove the case $k = 2$. It will be assumed therefore for the rest of the proof that $k = 2$. We also assume $\dim \mathcal{H} > k$ (if $\dim \mathcal{H} = k$, the result is trivial).

We dispose first of the easy case when $A$ (or $B$) is scalar. If $A$ is scalar, then by Lemma 3.4 (the equivalence of (1) and (3)) it follows that $B$ is scalar as well. Thus, we assume that neither $A$ nor $B$ are scalar.

*Proof of Statement (2).* Let $x, y$ be an orthonormal pair in $\mathcal{H}$, and let $\mathcal{M} = \operatorname{Span}\{x, y\}$. Then $\langle Ay, x\rangle$ (resp., $\langle By, x\rangle$) is the $(1, 2)$ entry in the matrix representation of $P_{\mathcal{M}} A P_{\mathcal{M}}|_{\mathrm{ImP}_{\mathcal{M}}}$ (resp., $P_{\mathcal{M}} B P_{\mathcal{M}}|_{\mathrm{ImP}_{\mathcal{M}}}$) with respect to the basis $\{x, y\}$. Since $P_{\mathcal{M}} A P_{\mathcal{M}}|_{\mathrm{ImP}_{\mathcal{M}}}$ and $P_{\mathcal{M}} B P_{\mathcal{M}}|_{\mathrm{ImP}_{\mathcal{M}}}$ satisfy the property **(P2)**, we have

$$|\langle Ay, x\rangle| = |\langle By, x\rangle|.$$

Since the orthonormal pair $\{x, y\}$ is arbitrary, the result follows from [5, Theorem 2.2].

*Proof of Statement (1).* Assume first that $\dim \mathcal{H} = n < \infty$.

We consider several cases.

*Case 1. For some rank-*2 *orthogonal projection* $P$*, the compressions of* $B, A^*, I$ *to the range of* $P$ *are linearly independent.*

Then by Lemma 3.3, the set

$$\mathrm{Gr}_0 := \{\mathcal{M} \in \mathrm{Gr}_2(\mathcal{H}) \,:\, \text{compressions of } B, A^*, I \text{ to } \mathcal{M} \text{ are linearly independent}\}$$

is dense in $\mathrm{Gr}_2(\mathcal{H})$. By the hypotheses of Theorem 3.1, we have

$$P_{\mathcal{M}} B P_{\mathcal{M}}|_{\mathcal{M}} = \mu P_{\mathcal{M}} A P_{\mathcal{M}}|_{\mathcal{M}} + q I_{\mathcal{M}}, \quad \forall \mathcal{M} \in \mathrm{Gr}_0, \tag{3.3}$$

where the unimodular number $\mu = \mu(\mathcal{M})$ and $q = q_{\mathcal{M}} \in \mathbb{C}$ depend on $\mathcal{M}$. If $\mathcal{M} \in \mathrm{Gr}_2(\mathcal{H}) \setminus \mathrm{Gr}_0$, then select a sequence $\{\mathcal{M}_m\}_{m=1}^{\infty}$ such that $\mathcal{M}_m \in \mathrm{Gr}_0$ and

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$\lim_{m\to\infty} \mathcal{M}_m = \mathcal{M}$ (equivalently, $\lim_{m\to\infty} P_{\mathcal{M}_m} = P_{\mathcal{M}}$), and upon selecting a convergent subsequence of $\{\mu(\mathcal{M}_m)\}_{m=1}^{\infty}$, it is easy to see that (3.3) holds also for $\mathcal{M}$. Thus, $A$ and $B$ have property **(P2)** in view of Statement (2), and the proof is completed in Case 1.

*Case 2. For some rank-*2 *orthogonal projection $P$, the compressions of $B, A, I$ to the range of $P$ are linearly independent.*

Then we argue as in the Case 1, replacing $B$ with $B^*$.

*Case 3. For all rank-*2 *orthogonal projections $P$, the compressions of $B, A, I$ to the range of $P$ are linearly dependent, and the compressions of $B, A^*, I$ to the range of $P$ are linearly dependent.*

Since $A$ and $B$ are not scalar, by Lemma 3.4, there exist $\mathcal{M}', \mathcal{M}'' \in \mathrm{Gr}_2(\mathcal{H})$ such that the compressions of $A$ and $I$ to $\mathcal{M}'$, as well as the compressions of $B$ and $I$ to $\mathcal{M}''$, are linearly independent. By a slight adaptation of the proof of Lemma 3.3, the set

$$\begin{aligned}\mathrm{Gr}_1 := \{\mathcal{M} \in \mathrm{Gr}_2(\mathcal{H}) \,:\, &\text{compressions of } A, I \text{ to } \mathcal{M} \text{ are linearly independent,}\\ &\text{as well as those of } B, I\}\end{aligned}$$

is dense in $\mathrm{Gr}_2(\mathcal{H})$. Pick $\mathcal{M}_0 \in \mathrm{Gr}_1$, and denote by $B', A', I'$ the compressions of $B, A, I$, respectively, to $\mathcal{M}_0$. We then have, in view of the hypotheses of Case 3 and selection of $\mathcal{M}_0$,

$$B' = aA' + bI', \quad B' = cA'^* + dI', \quad a, b, c, d \in \mathbb{C},$$

where $a \neq 0$, $c \neq 0$. Then

$$A' = (c/a)A'^* + ((d-b)/a)I'. \tag{3.4}$$

Taking adjoints, we have

$$A'^* = \overline{(c/a)}A' + \overline{((d-b)/a)}I',$$

or, solving for $A'$,

$$A' = \overline{(a/c)}A'^* - \overline{((d-b)/c)}I'.$$

Comparing with (3.4), it follows from linear independence of $A'$ and $I'$ that $|a| = |c|$. On the other hand, since $A'$ and $B'$ satisfy the property **(P1)**, at least one of the numbers $a$ and $c$ is unimodular, hence both are. Thus, for every $\mathcal{M}_0 \in \mathrm{Gr}_1$, the compressions of $A$ and $B$ to $\mathcal{M}_0$ satisfy the property **(P2)**. Now argue as in the proof of Case 1 to obtain that the compressions of $A$ and $B$ to any 2-dimensional

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subspace of $\mathcal{H}$ have property **(P2)**, and application of Statement (2) of Theorem 3.1 completes the proof of Case 3.

This completes the proof of Statement (1) for finite-dimensional $\mathcal{H}$.

Now assume $\mathcal{H}$ is infinite-dimensional. By replacing, if necessary, $A$ with $A^*$ we easily deduce, from the above considerations, that all the compressions of $A$ and $B$ onto any 2-dimensional subspace of $\mathcal{H}$ simultaneously have property **(P2)**. Indeed, otherwise, there would be rank-two projections $P, Q$ such that the compressed triples $(A', I', B') = (PAP|_{\operatorname{Im} P},\, P|_{\operatorname{Im} P},\, PBP|_{\operatorname{Im} P})$ and $((A'')^*, I'', B'') = (QA^*Q|_{\operatorname{Im} Q},\, Q|_{\operatorname{Im} Q},\, QBQ|_{\operatorname{Im} Q})$ are linearly independent. Consider the compressions $A'''$ and $B'''$ to the finite-dimensional subspace $\mathcal{H}''' = \operatorname{Im} P + \operatorname{Im} Q$. Clearly, they still satisfy the assumption (1) of Theorem 3.1, with $k = 2$. Hence, by the above argument, either $A'''$ and $B'''$ enjoy property **(P2)** or else $(A''')^* = (A^*)'''$ and $B'''$ enjoy property **(P2)**. Either case contradicts the erroneous assumption that $A' = PA'''P|_{\operatorname{Im} P}, I', B' = PB'''P|_{\operatorname{Im} P}$ and $(A'')^* = Q(A''')^*Q|_{\operatorname{Im} Q}, I'', B'' = QB'''Q|_{\operatorname{Im} Q}$ are two linearly independent triples. The result now follows from the already proven Statement (2). ◻

**4. $C$-numerical range.** In what follows, we will use the concept of the $C$-*numerical range* of an operator $X \in L(\mathcal{H})$ defined as follows:

$$W_C(X) := \{\operatorname{Tr}(CU^*XU) : U \text{ is unitary}\}.$$

Lemma 4.1. *If $C \in L(\mathcal{H})$ is trace-class, then the closure $\operatorname{cl} W_C(X)$ of $W_C(X)$ is star-shaped for every $X \in L(\mathcal{H})$; moreover, if $\operatorname{Tr} C = 0$, then zero is a star-center of $\operatorname{cl} W_C(X)$.*

*Proof.* In the case $\mathcal{H}$ is finite-dimensional, the result is proved in [2]. Now assume $\mathcal{H}$ is infinite-dimensional. Let $\{C_m\}_{m=1}^{\infty}$ be a sequence of finite rank operators such that $\lim_{m\to\infty} C_m = C$ in the trace norm, denoted $\|\cdot\|_1$, and $\operatorname{Tr} C_m = \operatorname{Tr} C$ for all $m = 1, 2, \ldots$ By a result of Jones [4], $\operatorname{cl} W_{C_m}(X)$ is star-shaped with a star-center at $(\operatorname{Tr} C)z_0$, where $z_0$ is any element in the essential numerical range of $X$. Arguing by contradiction, assume $(\operatorname{Tr} C)z_0$ is not a star-center of $\operatorname{cl} W_C(X)$, and let $d > 0$ be the distance from some point $y_0 := \alpha(\operatorname{Tr} C)z_0 + (1-\alpha)y$, where $y \in \operatorname{cl} W_C(X)$ and $0 < \alpha < 1$, to the closure of $W_C(X)$. Using the standard norm inequalities

$$\begin{aligned} &|\operatorname{Tr}(CU^*XU) - \operatorname{Tr}(C_mU^*XU)| \le \|C_mU^*XU - CU^*XU\|_1 \\ &\le \|C_m - C\|_1 \|U^*XU\| = \|C_m - C\|_1 \|X\|, \quad \forall \text{ unitary } U, \end{aligned} \tag{4.1}$$

we see that there is a sequence $\{y_m\}_{m=1}^{\infty}$ such that $y_m \in \operatorname{cl} W_{C_m}(X)$ and $\lim_{m\to\infty} y_m = y$. By [4], $\alpha(\operatorname{Tr} C)z_0 + (1-\alpha)y_m \in \operatorname{cl} W_{C_m}(X)$, and obviously

$$\lim_{m\to\infty} (\alpha(\operatorname{Tr} C)z_0 + (1-\alpha)y_m) = y_0.$$

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So, there exists a sequence of unitary operators $\{U_m\}_{m=1}^{\infty}$ such that

$$\lim_{m\to\infty}(\mathrm{Tr}(C_m U_m^* X U_m)) = y_0.$$

Now, using (4.1) again, we have

$$\lim_{m\to\infty} |\,\mathrm{Tr}(C U_m^* X U_m) - \mathrm{Tr}(C_m U_m^* X U_m)| = 0,$$

a contradiction with the choice of $y_0$. $\square$

LEMMA 4.2. *Assume $C \in L(\mathcal{H})$ is a trace-class operator. Then:*

(a) *$W_C(X)$ is a nondegenerate line segment, with or without one or both endpoints, if and only if both $C$ and $X$ are essentially selfadjoint non-scalar operators.*
(b) *$W_C(X)$ is a singleton if and only if $C$ or $X$ is a scalar operator.*

*Proof.* Part (b) follows from [5, Theorem 6.1]. For the case of finite-dimensional $\mathcal{H}$, part (a) is stated in [7, property (7.3.a)]; a proof (again in finite dimensions) is found in [6].

Consider now part (a) for the case of infinite-dimensional $\mathcal{H}$. By the definition of essentially selfadjoint operators, as well as part (b), the "if" statement is easily verified. We prove the "only if" statement. Thus, assume that $W_C(X)$ is a nondegenerate line segment. By (b), we know that $C$ and $X$ are non-scalars.

Suppose first that $C$ has finite rank. Recall that the operator $X$ is essentially selfadjoint if and only if $X, X^*, I$ are linearly dependent, which, by using (2) of Corollary 3.2 on $(A, B) = (X^*, X)$, is equivalent to the fact that all the compressions of $X$ to $k$-dimensional subspaces of $\mathcal{H}$ are essentially selfadjoint; here $k \geq 2$ is a fixed integer. Assume erroneously that some finite-dimensional compression of $X$ is not essentially selfadjoint. Since the rank of $C$ is finite, we can then find a suitable choice of orthonormal basis in $\mathcal{H}$ so that $C$ and $X$ have operator matrices $C = C_1 \oplus 0$ and $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ with $C_1, X_{11} \in \mathbb{C}^{k\times k}$, where $k \geq 2$ is fixed, and $X_{11}$ is not essentially selfadjoint. Now, $W_C(X) = W_{C_1}(X_{11})$ and by [7, property (7.3.a)] we know that $X_{11}$ must be essentially selfadjoint, a desired contradiction. Suppose now $C$ has infinite rank. Then let $V \in L(\mathcal{H})$ be a unitary such that $V^*CV - C$ is of finite rank and non-scalar (choose $V$ so that $V - I$ is of finite rank). We have

$$\mathrm{Tr}((V^*CV - C)U^*XU) = \mathrm{Tr}(C\cdot(VU^*)X(UV^*)) - \mathrm{Tr}(CU^*XU)$$

for every unitary $U \in L(\mathcal{H})$. Therefore, $W_{V^*CV-C}(X)$ is contained in a line, and is in fact a nondegenerate line segment (because $V^*CV - C$ and $X$ are non-scalars). By the already proved case of part (a), we obtain that $X$ is again essentially selfadjoint.

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To prove that $C$ is essentially selfadjoint, we repeat the arguments in the preceding paragraph with the roles of $X$ and $C$ interchanged. ◻

Using Lemma 4.1 we can prove another (easy) case of Conjecture 1.1:

PROPOSITION 4.3. *Assume $C \in L(\mathcal{H})$ is non-scalar trace-class, and* (1.1) *holds for $A, B \in L(\mathcal{H})$. If one of $A$ and $B$ is scalar, then Conjecture* 1.1 *holds true for $A$ and $B$.*

*Proof.* Say, $B = \alpha I$, $\alpha \in \mathbb{C}$. Then $W_C(A)$ is contained in the circle of radius $|\alpha \operatorname{Tr} C|$ centered at the origin. Since the closure of $W_C(A)$ is star-shaped by Lemma 4.1, we must have that $W_C(A)$ is a singleton. But then $A$ is scalar by Lemma 4.2(b). Thus $A = \beta I$, where $\beta \in \mathbb{C}$ satisfies $|\beta \operatorname{Tr} C| = |\alpha \operatorname{Tr} C|$. Obviously, at least one of the two conditions (1), (2) in Conjecture 1.1 holds. ◻

**5. Proof of Theorem 1.5, the case of normal $C$.** Throughout this section, it will be assumed that $C \in L(\mathcal{H})$ is a non-scalar trace class normal operator (not necessarily of finite rank).

**5.1. Proof of the first part of Theorem 1.5.** We divide the proof into two cases: one for finite-dimensional $\mathcal{H}$ and the other for infinite-dimensional. We start with the finite-dimensional case.

LEMMA 5.1. *Let $n \geq 2$ be an integer. If* (1.1) *holds for matrices $A, B \in \mathbb{C}^{n\times n}$, then $A = \mu B + \nu I$ or $A = \mu B^* + \nu I$ for some $\mu, \nu \in \mathbb{C}$, $|\mu| = 1$.*

*Proof.* Induction on rank of $C$. For rank $C = 1$, this was proven in [5]. Assume the lemma holds for every normal non-scalar $C$ of rank at most $k$. If $k = n$, then there is nothing to prove. If $k < n$, pick any normal non-scalar $C \in \mathbb{C}^{n\times n}$ with rank $k + 1 \leq n$. Assume first $n = 2$. Then, $C$ has two distinct eigenvalues and we may clearly pick one, name it $\gamma$ such that $C' := C - \gamma I$ is normal, with rank $C' \leq k$, and moreover $\operatorname{Tr} C' \neq 0$. Then,

$$\operatorname{Tr}(CUXU^*) = \operatorname{Tr}(C'UXU^* + \gamma UXU^*) = \operatorname{Tr}\big(C'U\big(X + ((\operatorname{Tr} C')^{-1}\gamma \operatorname{Tr} X)I\big)U^*\big),$$

for every $X \in \mathbb{C}^{n\times n}$ and for every unitary $U \in \mathbb{C}^{n\times n}$. So, from identity (1.1), we derive that for $A' = A + ((\operatorname{Tr} C')^{-1}\gamma \operatorname{Tr} A)I$ and $B' = B + ((\operatorname{Tr} C')^{-1}\gamma \operatorname{Tr} B)I$, it holds

$$|\operatorname{Tr}(C'UA'U^*)| = |\operatorname{Tr}(C'UB'U^*)|, \qquad \forall \text{ unitary } U \in \mathbb{C}^{n\times n}.$$

By induction, $A'$ and $B'$ enjoy property **(P1)**.

Assume now $n > 2$. Let $c_1, \ldots, c_{k+1}$ be all nonzero eigenvalues of $C$ counted with multiplicities. Arguing by contradiction, it is easy to see that there is at least one index $j$ such that $c_1 + \cdots + c_{j-1} + c_{j+1} + \cdots + c_{k+1} \neq 0$. Fix a unimodular vector

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$\mathbf{x} \in \mathbb{C}^n$. By applying unitary similarity to $C$ we may assume $C\mathbf{x} = c_j\mathbf{x}$, where $c_j$ is chosen so that $c_1 + \cdots + c_{j-1} + c_{j+1} + \cdots + c_{k+1} \neq 0$. Now suppose $A, B \in \mathbb{C}^{n\times n}$ are such that $|\operatorname{Tr}(CU^*AU)| = |\operatorname{Tr}(CU^*BU)|$ for all unitary $U$. With respect to the orthogonal decomposition $\mathbb{C}^n = \operatorname{Span}\{\mathbf{x}\} \oplus (\operatorname{Span}\{\mathbf{x}\})^\perp$, write

$$C = \begin{bmatrix} c_j & 0 \\ 0 & C' \end{bmatrix}; \qquad A = \begin{bmatrix} \langle A\mathbf{x}, \mathbf{x}\rangle & * \\ * & A' \end{bmatrix}; \qquad B = \begin{bmatrix} \langle B\mathbf{x}, \mathbf{x}\rangle & * \\ * & B' \end{bmatrix}.$$

We may assume that $C'$ is not scalar (otherwise $\operatorname{rank}(C-\gamma I) = 1$ and $\operatorname{Tr}(C-\gamma I) \neq 0$ for some $\gamma \in \mathbb{C}$, and we can repeat the arguments of the case $n = 2$). We take the unitaries $U$ in the block diagonal form

$$U = \begin{bmatrix} 1 & 0 \\ 0 & \widehat{U} \end{bmatrix};$$

here $\widehat{U}$ is any unitary on $(\operatorname{Span}\{\mathbf{x}\})^\perp$. Now

$$\operatorname{Tr}(CU^*AU) = c_j\langle A\mathbf{x}, \mathbf{x}\rangle + \operatorname{Tr}(C'\widehat{U}^*A'\widehat{U}) = \operatorname{Tr}(C'\widehat{U}^*(A' + c_j\langle A\mathbf{x}, \mathbf{x}\rangle(\operatorname{Tr} C')^{-1}I)\widehat{U}),$$

and similarly for $B$. Thus,

$$|\operatorname{Tr}(C'\widehat{U}^*(A' + c_j\langle A\mathbf{x}, \mathbf{x}\rangle(\operatorname{Tr} C')^{-1}I)\widehat{U})| = |\operatorname{Tr}(C'\widehat{U}^*(B' + c_j\langle B\mathbf{x}, \mathbf{x}\rangle(\operatorname{Tr} C')^{-1}I)\widehat{U}|.$$

Since this holds for all unitaries $\widehat{U}$, by the induction hypothesis the operators $A'$ and $B'$ have property **(P1)**. In view of the arbitrariness of $\mathbf{x}$, by Theorem 3.1, $A$ and $B$ have property **(P1)** as well, and we are done.

It remains to consider infinite-dimensional $\mathcal{H}$. We consider two cases separately.

*Case 1. $C$ has distinct eigenvalues $c_1, c_2$ such that $c_1 + c_2 \neq 0$.* Let $x_1, x_2$ be any orthonormal pair of vectors in $\mathcal{H}$. Applying a suitable unitary similarity to $C$, we may assume that $x_1, x_2$ are eigenvectors of $C$ corresponding to the eigenvalues $c_1, c_2$, respectively. Write operators as $2\times 2$ block matrices with respect to the orthogonal decomposition $\mathcal{H} = (\operatorname{Span}\{x_1, x_2\})^\perp \oplus (\operatorname{Span}\{x_1, x_2\})$:

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & C' \end{bmatrix}; \quad A = \begin{bmatrix} A_1 & * \\ * & A' \end{bmatrix}; \quad B = \begin{bmatrix} B_1 & * \\ * & B' \end{bmatrix}; \quad C' = \operatorname{diag}(c_1, c_2).$$

We restrict ourselves to consider unitaries $U$ having the block diagonal form

$$U = \begin{bmatrix} I & 0 \\ 0 & \widehat{U} \end{bmatrix};$$

here $\widehat{U}$ is any unitary on $\operatorname{Span}\{x_1, x_2\}$. Note that our hypothesis guarantees that $C'$ is not a scalar operator. Now

$$\operatorname{Tr}(CU^*AU) = \operatorname{Tr}(C_1A_1) + \operatorname{Tr}(C'\widehat{U}^*A'\widehat{U}) = \operatorname{Tr}(C'\widehat{U}^*(A' + \operatorname{Tr}(C_1A_1)(\operatorname{Tr} C')^{-1}I)\widehat{U}),$$

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and similarly for $B$. Then, the assumptions of the theorem give

$$|\operatorname{Tr}(C'\widehat{U}^*(A' + \operatorname{Tr}(C_1A_1)(\operatorname{Tr} C')^{-1}I)\widehat{U})| = |\operatorname{Tr}(C'\widehat{U}^*(B' + \operatorname{Tr}(C_1B_1)(\operatorname{Tr} C')^{-1}I)\widehat{U})|.$$

Since this holds for all unitaries $\widehat{U}$, by Lemma 5.1, the operators $A'$ and $B'$ have property **(P1)**. In view of the arbitrariness of $x_1, x_2$, all compressions of $A$ and $B$ to 2-dimensional subspaces have property **(P1)**, and by Corollary 3.2, $A$ and $B$ have property **(P1)**.

*Case 2. There is no pair of distinct eigenvalues of $C$ that sum up to a nonzero number.* It is easy to see that $C$ must have exactly two distinct eigenvalues $a$ and $-a$ (the case when $C$ has all eigenvalues equal is excluded by the hypothesis that $C$ is non-scalar). Because $C$ is of trace-class, both $a$ and $-a \neq a$ have finite multiplicities. Then $\operatorname{Ker} C = 0$ implies $\dim \mathcal{H} < \infty$, and Lemma 5.1 applies. □

**5.2. Proof of the second part of Theorem 1.5.** Here, we prove (1.4) under additional hypotheses that $C, C^*, I$ are linearly independent and $A$ and $B$ are not essentially selfadjoint. It will be convenient to have a lemma first.

LEMMA 5.2. *Let $B, C \in L(\mathcal{H})$ be such that $C$ is a trace-class normal operator and $B$ and $C$ are not essentially selfadjoint. If $\gamma \in \mathbb{C}$ is such that $|\operatorname{Tr}(CU^*BU) + \gamma| = |\operatorname{Tr}(CU^*B^*U)|$ for all unitary operators $U \in L(\mathcal{H})$, then $\gamma = 0$.*

*Proof.* We necessarily have $\dim \mathcal{H} \geq 3$ as every normal operator in $L(\mathbb{C}^2)$ is essentially selfadjoint.

With respect to a suitable orthogonal decomposition of $\mathcal{H}$, we may assume that $C = C_1 \oplus C_2$ is such that

$$C_1 = \operatorname{diag}(c_1, c_2), \quad C_2 = \operatorname{diag}(c_3, C_3),$$

where $c_1, c_2, c_3 \in \mathbb{C}$ and $c_1 - c_2 = r \neq 0$. We also assume that $c_1 - c_2 = 1$. Otherwise, replace $(C, \gamma)$ by $(C/r, \gamma/r)$. Also, we assume that

$$c_2 - \overline{c_2} - c_3 + \overline{c_3} \neq 0; \tag{5.1}$$

this choice of $c_3$ is possible in view of non-essential selfadjointness of $C$ . By Corollary 3.2 (indeed, the pair of operators $B^*, B$ does not have property **(P2)**, therefore there exists a 2-dimensional compression of $B^*, B$ that does not have property **(P2)**), we may replace $B$ by $V^*BV$ for a suitable unitary $V \in L(\mathcal{H})$ and assume that $B = \begin{bmatrix} B_1 & * \\ * & B_2 \end{bmatrix}$ so that $B_1 \in \mathbb{C}^{2\times 2}$ is not essentially selfadjoint. Then the trace condition of the lemma implies that

$$|\operatorname{Tr}(C(U^* \oplus I)B(U \oplus I)) + \gamma| = |\operatorname{Tr}(C(U^* \oplus I)B^*(U \oplus I))|$$

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for any unitary $U \in \mathbb{C}^{2\times 2}$. Let $E_{11}$ be the rank-one operator with 1 in the top left corner and zeros elsewhere (with respect to the same orthogonal decomposition of $\mathcal{H}$). It follows that

$$\begin{aligned}
&|(c_1 - c_2)\operatorname{Tr}(E_{11}U^*B_1U) + c_2 \operatorname{Tr} B_1 + \operatorname{Tr}(C_2B_2) + \gamma| \\
&= |\operatorname{Tr}(CU^*BU) + \gamma| \\
&= |\operatorname{Tr}(C_1U^*B_1^*U) + \operatorname{Tr}(C_2B_2^*)| \\
&= |(c_1 - c_2)\operatorname{Tr}(E_{11}U^*B_1^*U) + c_2 \operatorname{Tr} B_1^* + \operatorname{Tr}(C_2B_2^*)|. \qquad (5.2)
\end{aligned}$$

Let

$$c_2 \operatorname{Tr} B_1 + \operatorname{Tr}(C_2B_2) = f + ig, \quad c_2 \operatorname{Tr} B_1^* + \operatorname{Tr}(C_2B_2^*) = f' + ig', \quad \text{and} \quad \gamma = \alpha + i\beta,$$

where $f, g, f', g', \alpha, \beta$ are real. If $\operatorname{Tr}(E_{11}U^*B_1U) = x + iy$, then $\operatorname{Tr}(E_{11}U^*B_1^*U) = x - iy$. Together with the assumption that $c_1 - c_2 = 1$, equality (5.2) becomes

$$|(x + iy) + (f + ig) + (\alpha + i\beta)| = |(x - iy) + (f' + ig')|,$$

or equivalently,

$$(x + f + \alpha)^2 + (y + g + \beta)^2 = (x + f')^2 + (g' - y)^2. \qquad (5.3)$$

Since $B_1$ is not essentially selfadjoint, the set of numbers $x + iy = \operatorname{Tr}(E_{11}U^*B_1U)$ is just the numerical range $W(B_1)$ of $B_1$, which has non-empty interior. Thus, (5.3) holds for infinitely many $x + iy_0$ for a fixed $y_0$ and infinitely many $x_0 + iy$ for a fixed $x_0$. Thus, comparing the coefficients of $x$ and $y$, we have

$$(f + \alpha, g + \beta) = (f', -g'). \qquad (5.4)$$

We can assume that

$$B = \begin{bmatrix} (b_{ij})_{i,j=1}^3 & * \\ * & * \end{bmatrix}$$

and $B_1$ is in triangular form. Since $B_1$ is not essentially selfadjoint, we see that $B_1 = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}$ and $b_{12} \neq 0$. Thus, $b_{11}$ is an interior point of $W(B_1)$. For any nonzero $\varepsilon_1$ with sufficiently small modulus, there is $U_1 \in \mathbb{C}^{2\times 2}$ such that $\widehat{B} = (U_1^* \oplus I)B(U_1 \oplus I)$ has diagonal entries $b_{11} + \varepsilon_1, b_{22} - \varepsilon_1$ with $b_{22} - \varepsilon_1 \neq b_{33}$. Then we can find a unitary $U_2 \in \mathbb{C}^{2\times 2}$ such that

$$\widetilde{B} = ([1] \oplus U_2^* \oplus I)\widehat{B}([1] \oplus U_2 \oplus I)$$

has its first three diagonal entries equal to $b_{11} + \varepsilon_1$, $b_{22} - \varepsilon_2$, $b_{33} - \varepsilon_3$ with nonzero $\varepsilon_2, \varepsilon_3$ satisfying $\varepsilon_2 + \varepsilon_3 = \varepsilon_1$. Thus, we can choose nonzero $\varepsilon_j$ for $j = 1, 2, 3$ such that for $\widetilde{B} = \begin{bmatrix} \widetilde{B}_1 & * \\ * & \widetilde{B}_2 \end{bmatrix}$ the following hold:

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(a) the matrix $\widetilde{B}_1 \in \mathbb{C}^{2\times 2}$ is still not essentially selfadjoint;
(b) if

$$\begin{aligned}\widetilde{f} + i\widetilde{g} &:= c_2 \operatorname{Tr} \widetilde{B}_1 + \operatorname{Tr}(C_2 \widetilde{B}_2) \ = \ c_2 \operatorname{Tr} B_1 + \operatorname{Tr}(C_2 B_2) + c_2(\varepsilon_1 - \varepsilon_2) - c_3\varepsilon_3 \\ &= \ f + ig + c_2(\varepsilon_1 - \varepsilon_2) - c_3\varepsilon_3, \end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\widetilde{f}' + i\widetilde{g}' &:= c_2 \operatorname{Tr} \widetilde{B}_1^* + \operatorname{Tr}(C_2 \widetilde{B}_2^*) = c_2 \operatorname{Tr} B_1^* + \operatorname{Tr}(C_2 B_2^*) + c_2(\overline{\varepsilon_1} - \overline{\varepsilon_2}) - c_3\overline{\varepsilon_3} \\ &= \ f' + ig' \ + \ c_2(\overline{\varepsilon_1} - \overline{\varepsilon_2}) - c_3\overline{\varepsilon_3}, \end{aligned} \tag{5.6}$$

where $\widetilde{f}, \widetilde{g}, \widetilde{f}', \widetilde{g}'$ are real, then adding $\alpha + i\beta$ to (5.5) and subtracting the complex conjugate of (5.6) yields (in view of (5.4))

$$\begin{aligned}(\widetilde{f} + i\widetilde{g}) + (\alpha + i\beta) - (\widetilde{f}' - i\widetilde{g}') &= (c_2 - \overline{c_2})(\varepsilon_1 - \varepsilon_2) - (c_3 - \overline{c_3})\varepsilon_3 \\ &= (c_2 - \overline{c_2} - c_3 + \overline{c_3})\varepsilon_3 \neq 0.\end{aligned}$$

((a) is possible because the set of non-essentially selfadjoint matrices is open, and (b) is possible in view of (5.1).) Consequently,

$$(\widetilde{f} + \alpha, \widetilde{g} + \beta) \neq (\widetilde{f}', -\widetilde{g}'). \tag{5.7}$$

Now, similar to the derivation of equalities (5.2), (5.3), (5.4), if $\operatorname{Tr}(E_{11}U^*\widetilde{B}_1U) = x + iy \in W(\widetilde{B}_1)$, then $\operatorname{Tr}(E_{11}U^*\widetilde{B}_1^*U) = x - iy$, and

$$|(x + iy) + (\widetilde{f} + i\widetilde{g}) + (\alpha + i\beta)| = |(x - iy) + (\widetilde{f}' + i\widetilde{g}')|.$$

Thus, we have $(\widetilde{f} + \alpha, \widetilde{g} + \beta) = (\widetilde{f}', -\widetilde{g}')$, contradicting (5.7). ☐

*Proof of part 2 of Theorem 1.5.* By the first part of the theorem, we assume that (1.1) holds, and in addition $A = \mu B + \nu I$ or $A = \mu B^* + \nu I$ for some $\mu, \nu \in \mathbb{C}$, $|\mu| = 1$, and $\operatorname{Tr}(C) \neq 0$. We consider two cases separately:

(1) $A = \mu B + \nu I$ holds;
(2) $A = \mu B^* + \nu I$ holds.

*Case (1).* Under the hypotheses of Case (1), we have

$$|x + z| = |x|, \qquad \forall \ \ x \in W_C(B), \tag{5.8}$$

where $z = \nu\mu^{-1} \operatorname{Tr} C$. Arguing by contradiction, suppose $z \neq 0$. Then the set of complex numbers $\Gamma_z = \{x : |x + z| = |x|\}$ is a line and $W_C(B) \subseteq \Gamma_z$. But $B$ is assumed to be not essentially selfadjoint, a contradiction with Lemma 4.2(a). Therefore, $z = 0 = \nu$ and $A = \mu B$.

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**ELA**



*Case (2).* We have

$$\operatorname{Tr}(CU^*AU) = \mu \operatorname{Tr}(CU^*B^*U) + \nu \operatorname{Tr}(C),$$

and therefore by (1.1),

$$|\operatorname{Tr}(CU^*B^*U) + \nu\mu^{-1} \operatorname{Tr}(C)| = |\operatorname{Tr}(CU^*BU)|$$

for every unitary $U$. By Lemma 5.2, we must have $\nu = 0$, as required. □

## 6. Proof of Theorem 1.5, the case of finite rank $C$.

**6.1. Preliminary results.** In this subsection, we present several lemmas needed for the proof.

The following was proven by Brešar and Šemrl [1, Theorem 2.4].

LEMMA 6.1. *Let $U$ and $V$ be vector spaces over an infinite field $F$, char$F \neq 2$, and let $R_i : U \to V$ , $i = 1, 2, 3$, be linear operators. Then the following two statements are equivalent:*

(i) *The vectors $R_1u$, $R_2u$, and $R_3u$ are linearly dependent for every $u \in U$.*
(ii) *One of $(a)$-$(d)$ holds:*
   (a) *$R_1, R_2, R_3$ are linearly dependent;*
   (b) *there exist $v, w \in V$ such that $R_iU \in \operatorname{Span}\{v, w\}$, $i = 1, 2, 3$;*
   (c) *there exist linearly independent vectors $v_1, v_2, v_3 \in V$, $3 \times 3$ invertible matrices $Q_1$ and $Q_2$, a linear mapping $R$ from $U$ into the space of all $3\times3$ skew-symmetric matrices such that $R_i : u \mapsto \sum_{k=1}^{3}[Q_1(Ru)Q_2]_{ki}v_k$, $i = 1, 2, 3$, where $[Q_1(Ru)Q_2]_{ki}$ stands for the $(k, i)$ entry of the matrix $Q_1(Ru)Q_2$;*
   (d) *there exists an idempotent $P : V \to V$ of rank one such that*

$$\dim \operatorname{Span}\{(I_V - P)R_1, (I_V - P)R_2, (I_V - P)R_3\} = 1.$$

*Here, $I_V$ denotes the identity operator on $V$.*

REMARK 6.2. Lemma 6.1 will be applied on at least 3-dimensional $V = U = \mathbb{C}^n$, and operators $R_1 = A$, $R_2 = B$, $R_3 = I$, the identity operator. Then options $(b), (c)$ are not possible because both $(b)$ and $(c)$ imply that $\operatorname{rank} R_i \le 2$. The conclusion is that either $A, B, I$ are linearly dependent or, under (d), $A = \lambda_A I + \mathbf{x}\mathbf{f}^*$ and $B = \lambda_B I + \mathbf{x}\mathbf{g}^*$ for some vectors $\mathbf{x}, \mathbf{f}, \mathbf{g} \in \mathbb{C}^n$ and scalars $\lambda_A, \lambda_B$.

LEMMA 6.3. *Let $n \geq 2$ and let the nonzero vectors $\mathbf{a}, \mathbf{c}_1, \mathbf{c}_2, \mathbf{b} \in \mathbb{C}^n$ be such that $\mathbf{c}_1^*U\mathbf{a} = 0$ implies $\mathbf{c}_2^*U\mathbf{b} = 0$ for every unitary $U \in \mathbb{C}^{n\times n}$. Then, there exists a unitary $V$ such that $\mathbf{c}_1 \in \operatorname{Span}\{V\mathbf{a}\}$ and $\mathbf{c}_2 \in \operatorname{Span}\{V\mathbf{b}\}$. Moreover, if $n = 2$ then $\mathbf{c}_1$ and $\mathbf{c}_2$*

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*are linearly dependent or orthogonal, and if $n \geq 3$ then $\mathbf{c}_1$ and $\mathbf{c}_2$ are always linearly dependent.*

*Proof.* There exist unitary $W_1, W_2$ such that $W_1\mathbf{c}_1 = \lambda\mathbf{e}_1$ and $W_2\mathbf{a} = \mu\mathbf{e}_1$ for some nonzero scalars $\lambda, \mu$, where $\mathbf{e}_1$ belongs to the standard basis of $\mathbb{C}^n$. We may assume the two vectors $\mathbf{c}_1$ and $\mathbf{a}$ are already collinear with $\mathbf{e}_1$, otherwise we would regard unitary $W_1^* U W_2$ in place of $U$, and thus replace $(\mathbf{c}_1, \mathbf{a}; \mathbf{c}_2, \mathbf{b})$ with $(W_1\mathbf{c}_1, W_2\mathbf{a}; W_1\mathbf{c}_2, W_2\mathbf{b})$. Using unitary matrices that fix $\mathbf{e}_1$, we may further assume $\mathbf{b} = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2$ and $\mathbf{c}_2 = \gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2 + \gamma_3\mathbf{e}_3$ for some scalars $\beta_i, \gamma_j$ where we agreed upon that $\gamma_3$ is absent when $n = 2$. Use the unitaries $U_t := \begin{bmatrix} 0 & 1 \\ e^{it} & 0 \end{bmatrix} \oplus I_{n-2}$ for $t \in \mathbb{R}$. Clearly, $\mathbf{c}_1^* U_t \mathbf{a} = \overline{\lambda}\mu e^{it}\mathbf{e}_1^*\mathbf{e}_2 = 0$ for every $t$, hence also

$$0 = \mathbf{c}_2^* U_t \mathbf{b} = \overline{\gamma_2} e^{it}\beta_1 + \overline{\gamma_1}\beta_2$$

for every $t$. This is possible only if

$$\overline{\gamma}_2\beta_1 = 0 = \overline{\gamma}_1\beta_2.$$

If $n = 2$, we have from $\mathbf{b} \neq 0 \neq \mathbf{c}_2$ that either $\beta_1 = 0 = \gamma_1$ or $\gamma_2 = 0 = \beta_2$. In each case, $\mathbf{b}$ is a scalar multiple of $\mathbf{c}_2$, and both are either orthogonal to $\mathbf{c}_1$ (equivalently, to $\mathbf{a}$) or are collinear with $\mathbf{a}$.

If $n \geq 3$, we also use unitaries

$$U_t' := \begin{bmatrix} 0 & 0 & 1 \\ 0 & e^{it} & 0 \\ 1 & 0 & 0 \end{bmatrix} \oplus I_{n-3}, \qquad t \in \mathbb{R},$$

to derive additionally $\overline{\gamma_3}\beta_1 + e^{it}\overline{\gamma_2}\beta_2 = 0$ for every $t \in \mathbb{R}$, which further gives

$$\overline{\gamma_3}\beta_1 = 0 = \overline{\gamma_2}\beta_2.$$

Combined with the previously obtained identities gives either (i) $\beta_1 = 0$ which forces $\gamma_1 = 0 = \gamma_2$, or (ii) $\beta_2 = 0$ which forces $\gamma_3 = 0 = \gamma_2$. The second option gives that $\mathbf{b}$ and $\mathbf{c}_2$ are both collinear with $\mathbf{a}$ and $\mathbf{c}_1$. The first option is contradictory, because then, a unitary

$$U = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \oplus I_{n-3}$$

would satisfy $\mathbf{c}_1^* U \mathbf{a} = 0 \neq \mathbf{c}_2^* U \mathbf{b}$. Clearly, the unitary matrix $V := W_1^* W_2$ finishes the proof. $\square$

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LEMMA 6.4. *Let $n \geq 2$ and suppose $\mathbf{x}, \mathbf{y}, \mathbf{x}_2, \mathbf{y}_2, \mathbf{x}_3, \mathbf{y}_3 \in \mathbb{C}^n$ are nonzero vectors. If $\mathbf{x}^* U \mathbf{y} = 0$ implies $(\mathbf{x}_2^* U \mathbf{y}_2) \cdot (\mathbf{x}_3^* U \mathbf{y}_3) = 0$ for every unitary $U \in \mathbb{C}^{n \times n}$, then there exists an index $i \in \{2, 3\}$ such that already $\mathbf{x}^* U \mathbf{y} = 0$ implies $\mathbf{x}_i^* U \mathbf{y}_i = 0$ for all unitary $U$.*

*Proof.* Without loss of generality, we can assume that $\mathbf{x}, \mathbf{y}$ are both collinear with $\mathbf{e}_1$, otherwise we replace $U$ by $V^* U W$ for suitably chosen unitaries $V, W$. Assume erroneously that there is no such index. Then, there would exist unitary $U_1, U_2$ such that $\mathbf{x}^* U_1 \mathbf{y} = 0 = \mathbf{x}^* U_2 \mathbf{y}$ and $(\mathbf{x}_2^* U_1 \mathbf{y}_2) \cdot (\mathbf{x}_3^* U_2 \mathbf{y}_3) \neq 0$.

We will show that there exists a real-analytic path $f : [0, 1] \to \mathcal{U}_n$, which connects $U_1$ with $U_2$ in the set of those unitaries that satisfy $\mathbf{x}^* U \mathbf{y} = 0$. Once we verify this, the assumptions of the Lemma would imply $(\mathbf{x}_2^* f(t) \mathbf{y}_2) \cdot (\mathbf{x}_3^* f(t) \mathbf{y}_3) = 0$ for every $0 \leq t \leq 1$. This would contradict Proposition 2.1(c).

To verify the existence of the path with the above properties, we start by choosing $f_1(t) = (1-t)U_1 + t e^{-i\alpha(t)} U_2$ where $\alpha : [0, 1] \to \mathbb{R}$ is any real-analytic function such that $\alpha(1) = 0$ and $e^{i\alpha(1/2)} \notin \mathrm{Sp}\,(-U_1^{-1} U_2)$. Then, $f_1(t)$, $0 \leq t \leq 1$ is never singular because otherwise,

$$\frac{e^{i\alpha(t)}}{t} U_1^{-1} f_1(t) = \frac{1-t}{t} e^{i\alpha(t)} I + U_1^{-1} U_2$$

would be singular and hence $\frac{1-t}{t} e^{i\alpha(t)}$ would be an eigenvalue of a unitary $-U_1^{-1} U_2$. As the eigenvalues of unitary matrix are unimodular, this would imply that $t = 1/2$, a contradiction. Hence, $f_1(t)$ is invertible matrix for $0 \leq t \leq 1$. Then, the Gram-Schmidt orthogonalization performed on columns of $f_1(t)$ gives a real-analytic function $f(t)$ that connects $U_1$ and $U_2$ in the set of unitaries. Due to

$$\mathbf{x}^* f(t) \mathbf{y} \in \mathbb{C} \mathbf{e}_1^* f_1(t) \mathbf{e}_1 = \mathbb{C}(1-t) \mathbf{e}_1^* U_1 \mathbf{e}_1 + t e^{-i\alpha(t)} \mathbf{e}_1^* U_2 \mathbf{e}_1 = 0 + 0 = 0,$$

the constructed path has all the desired properties. ◻

LEMMA 6.5. *Let $n \geq 3$ and let $C \in \mathsf{C}^{n \times n}$ be a non-scalar matrix. Then there exists a unitary $U \in \mathsf{C}^{n \times n}$ such that for $UCU^* := \begin{bmatrix} c_{11} & \mathbf{c}_{12}^* \\ \mathbf{c}_{21} & \widehat{C} \end{bmatrix}$ the following hold:*

- *Column vectors $\mathbf{c}_{12}, \mathbf{c}_{21} \in \mathbb{C}^{n-1}$ are both nonzero.*
- *When $n = 3$, $\mathbf{c}_{12}$ and $\mathbf{c}_{21}$ are not orthogonal.*
- *$\widehat{C}$ is a non-scalar matrix with $\mathrm{Tr}\,\widehat{C} \neq 0$.*

*Proof.* Let us first find $U$ such that $\mathbf{c}_{12}, \mathbf{c}_{21}$ are nonzero. Since $C$ is non-scalar, there exists a normalized vector $\mathbf{x}$ such that $\mathbf{x}$ and $C\mathbf{x}$ are linearly independent. Write $C\mathbf{x} = \alpha \mathbf{x} + \beta \mathbf{y}$ where normalized $\mathbf{y}$ is orthogonal to $\mathbf{x}$, and enlarge it to an orthonormal basis $(\mathbf{z}_1 = \mathbf{x}, \mathbf{z}_2 = \mathbf{y}, \mathbf{z}_3, \ldots, \mathbf{z}_n)$. Clearly, $\mathbf{z}_2^* C \mathbf{z}_1 = \beta \neq 0$. Consequently,

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there exists a unitary $V_1$, which maps $(\mathbf{z}_1,\dots,\mathbf{z}_n)$ onto the standard basis, and for which $\mathbf{e}_2^*(V_1CV_1^*)\mathbf{e}_1 \neq 0$. Considering orthonormal basis $(\mathbf{z}_2,\mathbf{z}_1,\mathbf{z}_3,\dots,\mathbf{z}_n)$ in place of $(\mathbf{z}_1,\dots,\mathbf{z}_n)$ we further see that for some unitary $V_2$, we have $\mathbf{e}_1^*(V_2CV_2^*)\mathbf{e}_2 \neq 0$. There exists hermitian $H_k$ such that $V_k = e^{iH_k}$, $k = 1, 2$. Note that

$$t \mapsto \mathbf{e}_1^* \exp(i(tH_1 + (1-t)H_2)) \cdot C \cdot \exp(-i(tH_1 + (1-t)H_2))\mathbf{e}_2$$

and

$$t \mapsto \mathbf{e}_2^* \exp(i(tH_1 + (1-t)H_2)) \cdot C \cdot \exp(-i(tH_1 + (1-t)H_2))\mathbf{e}_1$$

are two real-analytic nonzero functions of $t \in [0,1]$. By Proposition 2.1(c) we can find some $t = t_0 \in [0,1]$ such that both functions are nonzero. Consequently, the unitary $V_3 = e^{i(t_0H_1+(1-t_0)H_2)}$ forces $(\mathbf{e}_2^*V_3CV_3^*\mathbf{e}_1)\cdot(\mathbf{e}_1^*V_3CV_3^*\mathbf{e}_2) \neq 0$. Clearly, we can assume $V_3 = I$ for the rest of the proof.

We next achieve that also $\operatorname{Tr}\widehat{C} \neq 0$. Since $C$ is non-scalar, its numerical range is not a singleton. So, there exists a unitary $V_4$ such that $\mathbf{e}_1^*V_4CV_4^*\mathbf{e}_1 \neq \operatorname{Tr} C$. With this $V_4$, we have

$$\operatorname{Tr}(C) = \operatorname{Tr}(V_4CV_4^*) = \mathbf{e}_1^*V_4CV_4^*\mathbf{e}_1 + \operatorname{Tr}\big((I-E_{11})V_4UV_4^*(I-E_{11})\big),$$

so that $\operatorname{Tr}\widehat{C} \neq 0$. Again, writing $V_3 = e^{iH_3}$ and $V_4 = e^{iH_4}$ for hermitian $H_3 = 0$, $H_4$, and forming a real-analytic function

$$f : t \mapsto \exp(i(tH_3 + (1-t)H_4))C\exp(-i(tH_3 + (1-t)H_4))$$

we find that the two functions $t \mapsto (\mathbf{e}_2^*f(t)\mathbf{e}_1)\cdot(\mathbf{e}_1^*f(t)\mathbf{e}_2)$, and $t \mapsto \operatorname{Tr}(I-E_{11})f(t)$, which are both nonzero real-analytic functions of $t$, are simultaneously nonzero at some $t = t_0 \in [0,1]$. Hence, with the unitary $V_5 := \exp(i(t_0H_3 + (1-t_0)H_4))$ we have $\mathbf{c}_{12}, \mathbf{c}_{21} \neq 0$ and $\operatorname{Tr}\widehat{C} \neq 0$. Again we can assume $V_5 = I$. Since $C$ is non-scalar, there exists a permutation matrix $V_6$ such that the lower-right $(n-1)\times(n-1)$ block of $V_6CV_6^*$ is non-scalar. Again, the real-analytic path that connects $V_5$ with $V_6$ in the set of unitaries must contain a unitary $V_7$ such that $V_7CV_7^*$ satisfies all the claims, with the sole exception that, when $n = 3$, $\mathbf{c}_{12}, \mathbf{c}_{21}$ might be orthogonal.

So, suppose $n = 3$. If $\mathbf{c}_{12}, \mathbf{c}_{21} \in \mathbb{C}^2$ are not orthogonal, then we are done. If they are orthogonal, we can use unitary $V_8 = [1] \oplus \widehat{V}_8 \in \mathbb{C} \oplus \mathbb{C}^{2\times 2}$ such that the two off-diagonal blocks of $C' := V_8CV_8^*$ equal $\mathbf{c}_{12}^*\widehat{V}_8^* = \lambda\mathbf{e}_1^* \neq 0$ and $\widehat{V}_8\mathbf{c}_{21} = \mu\mathbf{e}_2 \neq 0$, respectively. We then use a unitary of the form

$$V_9 = \begin{bmatrix} \cos t & e^{i\phi}\sin t \\ -\sin t & e^{i\phi}\cos t \end{bmatrix} \oplus [1], \qquad t, \phi \in \mathbb{R},$$

to achieve that the corresponding off-diagonal vectors $\mathbf{c}'_{12}$ and $\mathbf{c}'_{21}$ of

$$V_9C'V_9^* := \begin{bmatrix} c'_{11} & (\mathbf{c}'_{12})^* \\ \mathbf{c}'_{21} & C'_{22} \end{bmatrix}$$

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are not orthogonal vectors. In fact, if the entries of $C'$ at positions $(2,3)$ and $(3,2)$ are both nonzero we can set $t = \frac{\pi}{2}$, and if either of the entries $(2,3)$ or $(3,2)$ is zero, we use $t = \frac{\pi}{4}$ and appropriate $\phi \in \mathbb{R}$. Having found a unitary $V_{10} = V_9V_8$ such that the two side blocks of $V_{10}CV_{10}^*$ are not orthogonal, we connect $V_{10}CV_{10}^*$ with a real-analytic path to $V_7CV_7^*$ and complete the proof as before. ☐

We will also need a well known result (see e.g., [3]) on rational functions that take unimodular values on the unit circle:

LEMMA 6.6. *If a rational function $r(\lambda) = \frac{p(\lambda)}{q(\lambda)}$, where $p(\lambda)$ and $q(\lambda)$ are polynomials, satisfies $|r(e^{i\xi})| = 1$ for every $\xi \in \mathbb{R}$, then there exists a unimodular number $\mu$ and integers $d \geq 0$ and $k$ such that*

$$r(\lambda) = \mu\lambda^k \frac{a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_d\lambda^d}{\overline{a}_d + \overline{a}_{d-1}\lambda + \overline{a}_{d-2}\lambda^2 + \cdots + \overline{a}_0\lambda^d}, \quad a_0, \ldots, a_d \in \mathbb{C}, \tag{6.1}$$

*where the numerator and denominator in* (6.1) *have no zeros in common, and $a_0 \neq 0$, $a_d \neq 0$.*

*Proof.* For the reader's convenience, we supply a proof. We may clearly assume that numerator, $p(\lambda)$ and denominator, $q(\lambda)$ share no common zeros. Let $B(\lambda) = \lambda^s \prod_{i=1}^m \frac{\alpha_i - \lambda}{1-\overline{\alpha_i}\lambda}\frac{|\alpha_i|}{\alpha_i}$ be a Blaschke product containing all the zeros of denominator of $r(\lambda)$ which lie inside the unit disc (no zero lies on the boundary, because $|p(e^{i\xi})| = |q(e^{i\xi})|$ implies that every zero on the boundary is removable). Then, $r(\lambda)B(\lambda)$ is a rational function, unimodular on the boundary of a unit disc and without poles inside unit disc. Hence, it is holomorphic inside the unit disc, and $|r(e^{i\xi})B(e^{i\xi})| = 1$. Therefore, also

$$\lim_{\rho \nearrow 1} \int_{-\pi}^{\pi} |\ln|r(\rho e^{i\xi})B(\rho e^{i\xi})||\, d\xi = 0.$$

By [13, Exercise 17.22, p. 353] we obtain that $r(\lambda)B(\lambda)$ is a Blaschke product, up to a unimodular constant. Therefore, $r(\lambda) = \mu B_1(\lambda)/B(\lambda)$ is a quotient of two Blaschke products, up to unimodular constant $\mu$. Observe that, in Blaschke product, the zeros of numerator lie inside the unit disc while the zeros of denominator lie outside it. Hence, numerator and denominator in $B_1(\lambda)$ and in $B(\lambda)$ share no zeros in common. Moreover, if numerators of $B_1(\lambda)$ and $B(\lambda)$ share a common factor, say $\lambda-\alpha$, then also denominators of $B_1(\lambda)$ and $B(\lambda)$ share a common factor $1 - \overline{\alpha}\lambda$. We may cancel out such factors to obtain that $r(\lambda) = \mu\widetilde{B}_1(\lambda)/\widetilde{B}(\lambda)$, where $\widetilde{B}_1(\lambda)$ and $\widetilde{B}(\lambda)$ are again Blaschke products but with no factors in common. So $\widetilde{B}_1(\lambda)/\widetilde{B}(\lambda)$ is irreducible. Now observe that each Blaschke product may be written as $r_0(\lambda)/\overline{\mu^m r_0(1/\mu)}$, $\mu := \overline{\lambda}$, where $m$ is the degree of its numerator. Finally, $a_0 \neq 0$, $a_d \neq 0$ can be guaranteed by adjusting $k$, if necessary. ☐

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**6.2. Inductive step and basis for induction.** When $\dim \mathcal{H} < \infty$, the operators are represented by matrices, and we prove Theorem 1.5 for finite dimensional $\mathcal{H}$ by induction on the size $n$ of matrices. The lemma below is the inductive step.

LEMMA 6.7. *Suppose the first part of Theorem* 1.5 *holds for every non-scalar* $2 \times 2$ *matrix* $C$. *Let* $n \geq 3$. *Assume*

$$|\operatorname{Tr}(CUAU^*)| = |\operatorname{Tr}(CUBU^*)|, \qquad \forall \;\; \textit{unitary} \;\; U \in \mathbb{C}^{n\times n} \tag{6.2}$$

*holds for a fixed non-scalar* $C \in \mathbb{C}^{n\times n}$, *and fixed* $A, B \in \mathbb{C}^{n\times n}$. *Then* $A$ *and* $B$ *have property* **(P1)**.

*Proof.* In view of Theorem 3.1, it suffices to show that for every corank-one projection $P$, the compressions $PAP$ and $PBP$ have property **(P1)**.

There exists a unitary similarity $U_P$ such that $U_P P U_P^* = I - E_{11}$. We may assume that already $P = I - E_{11}$, otherwise we would regard the matrices

$$(U_P P U_P^*;\, U_P A U_P^*,\, U_P B U_P^*,\, U_P C U_P^*)$$

in place of $(P; A, B, C)$. This reduction is possible because of

$$\begin{aligned}\operatorname{Tr}(U_P C U_P^* \cdot U(U_P X U_P^*)U^*) &= \operatorname{Tr}(C U_P^* \cdot U(U_P X U_P^*)U^* U_P)\\ &= \operatorname{Tr}(C \cdot (U_P^* U U_P) X (U_P^* U U_P)^*)\end{aligned}$$

for every unitary $U$.

Using Proposition 4.3, we may (and do) assume that both $A$ and $B$ are non-scalar. It is easy to see that then there exists a unitary $U' \in \mathbb{C}^{n\times n}$ such that writing

$$U'A(U')^* = \begin{bmatrix} a_{11} & \mathbf{a}_{12}^* \\ \mathbf{a}_{21} & A_{22} \end{bmatrix}, \qquad U'B(U')^* = \begin{bmatrix} b_{11} & \mathbf{b}_{12}^* \\ \mathbf{b}_{21} & B_{22} \end{bmatrix}$$

with respect to decomposition $\mathbb{C}^n = \mathbb{C} \oplus \mathbb{C}^{n-1}$, we have that

$$\mathbf{a}_{12} \neq 0, \quad \mathbf{a}_{21} \neq 0, \quad \mathbf{b}_{12} \neq 0, \quad \mathbf{b}_{21} \neq 0. \tag{6.3}$$

Indeed, by Propositions 2.2 and 2.1(c), we need only show that $\mathbf{a}_{12} \neq 0$, $\mathbf{a}_{21} \neq 0$ for some unitary $U'$. By Lemma 3.4 the proof is reduced to the case of $2 \times 2$ matrices, in which case elementary calculations (using the assumed hypothesis that $A$ is not scalar) yield the result. Replacing $A$ and $B$ with $U'A(U')^*$ and $U'B(U')^*$, respectively, we assume in the sequel that

$$A = \begin{bmatrix} a_{11} & \mathbf{a}_{12}^* \\ \mathbf{a}_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \mathbf{b}_{12}^* \\ \mathbf{b}_{21} & B_{22} \end{bmatrix}, \quad \mathbf{a}_{12} \neq 0, \quad \mathbf{a}_{21} \neq 0, \quad \mathbf{b}_{12} \neq 0, \quad \mathbf{b}_{21} \neq 0. \tag{6.4}$$

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Denote temporarily

$$X_{21}(U) := (I - E_{11})U^*XUE_{11}, \quad X_{12}(U) := E_{11}U^*XU(I - E_{11}), \quad X \in \mathbb{C}^{n\times n},$$

and assume, to continue, that for every unitary $U$, the two matrices

$$A_{21}(U) \quad \text{and} \quad B_{21}(U) \tag{6.5}$$

are linearly dependent (some or both could also be zero) and that the same holds for

$$A_{12}(U) \quad \text{and} \quad B_{12}(U). \tag{6.6}$$

Multiplying both matrices in (6.5) on the left with $U$ and on the right with $U^*$, we see that this is equivalent to the fact that

$$(I - \mathbf{x}\mathbf{x}^*)A\mathbf{x}\mathbf{x}^* = \big(A\mathbf{x} - (\mathbf{x}^*A\mathbf{x})\mathbf{x}\big)\mathbf{x}^* \quad \text{and} \quad (I - \mathbf{x}\mathbf{x}^*)B\mathbf{x}\mathbf{x}^* = \big(B\mathbf{x} - (\mathbf{x}^*B\mathbf{x})\mathbf{x}\big)\mathbf{x}^*$$

are linearly dependent for every unit vector $\mathbf{x} \in \mathsf{C}^n$. This implies that $A\mathbf{x}, I\mathbf{x}, B\mathbf{x}$ are linearly dependent for every vector $\mathbf{x}$. By Remark 6.2, either $A, I, B$ are linearly dependent or else $A = \lambda_A I + \mathbf{x}\mathbf{f}^*$ and $B = \lambda_B I + \mathbf{x}\mathbf{g}^*$. In the second case we use the same arguments on the conjugate transpose of (6.6), to see that $A^*, I, B^*$ are also locally linearly dependent. Thus, by the same Remark 6.2, $\mathbf{f} = \mu\mathbf{g}$, $\mu \in \mathsf{C}$, and so $A, B, I$ are linearly dependent.

Likewise we argue when $A_{21}^*(V)$ and $B_{21}(V)$ as well as $A_{12}^*(V)$ and $B_{12}(V)$ are linearly dependent for every unitary $V$; in this case, $A^*, B, I$ are linearly dependent.

In the sequel, we can thus assume that there exists unitaries $U_1 = e^{iH_1}, U_2 = e^{iH_1}$ for some Hermitian $H_1, H_2$ so that at least one of the following four conditions hold (indeed, if all four conditions fail, then we are in the situation taken care of in one of the two preceding paragraphs):

($\alpha$) $A_{21}(U_1)$ and $B_{21}(U_1)$ are linearly independent, and $A_{21}^*(U_2)$ and $B_{21}(U_2)$ are linearly independent;
($\beta$) $A_{12}(U_1)$ and $B_{12}(U_1)$ are linearly independent, and $A_{21}^*(U_2)$ and $B_{21}(U_2)$ are linearly independent;
($\gamma$) $A_{12}(U_1)$ and $B_{12}(U_1)$ are linearly independent, and $A_{12}^*(U_2)$ and $B_{12}(U_2)$ are linearly independent;
($\delta$) $A_{21}(U_1)$ and $B_{21}(U_1)$ are linearly independent, and $A_{12}^*(U_2)$ and $B_{12}(U_2)$ are linearly independent.

We will consider only the case ($\alpha$); other cases can be dealt with similarly. Actually we may assume that $U_1 = U_2$. Namely, since linear independence of two matrices is equivalent to nonvanishing of at least one of a certain finite collection of 2–by–2 minors, which are polynomials in coefficients of both matrices, we can then find a

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unitary $U_A = e^{i(tH_1+(1-t)H_2)}$ for some $t \in [0,1]$ such that the corresponding minors do not vanish. Having found $U_A$, we may assume that already

$$(I - E_{11})AE_{11}, \quad (I - E_{11})BE_{11} \quad \text{are linearly independent,} \tag{6.7}$$

$$(I - E_{11})A^*E_{11}, \quad (I - E_{11})BE_{11} \quad \text{are linearly independent,} \tag{6.8}$$

and simultaneously (6.4) holds, otherwise we replace $(A,B)$ by $(V^*AV,\, V^*BV)$ for some suitable unitary $V$.

Now we choose a unitary $U_C$ so that $U_C C U_C^*$ satisfies the claims in Lemma 6.5. Since

$$\operatorname{Tr}(U_C C U_C^* \cdot UXU^*) = \operatorname{Tr}(C(U_C^* U)X(U_C^* U)^*), \quad X \in \mathbb{C}^{n\times n},$$

we can also assume with no loss of generality that already $C$ satisfies the claims of Lemma 6.5. In particular, with $C = \begin{bmatrix} c_{11} & \mathbf{c}_{12}^* \\ \mathbf{c}_{21} & \widehat{C} \end{bmatrix}$ we have that $\widehat{C}$ is non-scalar with nonzero trace. Moreover, $\mathbf{c}_{12}, \mathbf{c}_{21}$ are nonzero and, if $n = 3$, they are not orthogonal. Now we use unitaries $U = [e^{i\phi}] \oplus \widehat{U}$ to derive that $|\operatorname{Tr}(CUAU^*)| = |\operatorname{Tr}(CUBU^*)|$ is equivalent to

$$\begin{aligned} &\big|\operatorname{Tr}\big(\widehat{C}\widehat{U}(A_{22} + c_{11}a_{11}(\operatorname{Tr}\widehat{C})^{-1}I)\widehat{U}^*\big) + e^{-i\phi}\mathbf{c}_{12}^*\widehat{U}\mathbf{a}_{21} + e^{i\phi}(\widehat{U}\mathbf{a}_{12})^*\mathbf{c}_{21}\big| \\ &= \big|\operatorname{Tr}\big(\widehat{C}\widehat{U}\big(B_{22} + c_{11}b_{11}(\operatorname{Tr}\widehat{C})^{-1}I\big)\widehat{U}^*\big) + e^{-i\phi}\mathbf{c}_{12}^*\widehat{U}\mathbf{b}_{21} + e^{i\phi}(\widehat{U}\mathbf{b}_{12})^*\mathbf{c}_{21}\big|. \end{aligned}$$

Multiply both sides with $1 = |e^{i\phi}|$ and rewrite into

$$\begin{aligned} &\big|\mathbf{c}_{12}^*\widehat{U}\mathbf{a}_{21} + e^{i\phi}\operatorname{Tr}\big(\widehat{C}\widehat{U}(A_{22} + c_{11}a_{11}(\operatorname{Tr}\widehat{C})^{-1}I)\widehat{U}^*\big) + e^{2i\phi}(\widehat{U}\mathbf{a}_{12})^*\mathbf{c}_{21}\big| \\ &= \big|\mathbf{c}_{12}^*\widehat{U}\mathbf{b}_{21} + e^{i\phi}\operatorname{Tr}\big(\widehat{C}\widehat{U}(B_{22} + c_{11}b_{11}(\operatorname{Tr}\widehat{C})^{-1}I)\widehat{U}^*\big) + e^{2i\phi}(\widehat{U}\mathbf{b}_{12})^*\mathbf{c}_{21}\big|. \end{aligned} \tag{6.9}$$

Clearly there exists unitary $\widehat{U}$ with

$$\mathbf{c}_{12}^*\widehat{U}\mathbf{a}_{21} \neq 0, \quad (\widehat{U}\mathbf{a}_{12})^*\mathbf{c}_{21} \neq 0, \quad \mathbf{c}_{12}^*\widehat{U}\mathbf{b}_{21} \neq 0, \quad (\widehat{U}\mathbf{b}_{12})^*\mathbf{c}_{21} \neq 0, \tag{6.10}$$

because all left hand sides in inequalities (6.10) are nonzero real-analytic functions of $\widehat{U} \in \mathcal{U}_{n-1}$. In the following, we will restrict $\widehat{U}$ to the open dense (in the real-analytic manifold $\mathcal{U}_{n-1}$) subset $\Omega$ of those unitary $\widehat{U}$ for which (6.10) holds.

With each fixed $\widehat{U} \in \Omega$, the equality (6.9) takes the form

$$|p_{\widehat{U}}(e^{i\xi})| = |q_{\widehat{U}}(e^{i\xi})|, \qquad \xi \in \mathbb{R}, \tag{6.11}$$

where $p_{\widehat{U}}(\lambda)$ and $q_{\widehat{U}}(\lambda)$ are quadratic and at most quadratic polynomials, respectively, and $p_{\widehat{U}}(0) = \mathbf{c}_{12}^*\widehat{U}\mathbf{a}_{21} \neq 0$. At each fixed $\widehat{U} \in \Omega$ we have three possibilities as regards

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their quotient $r_{\widehat{U}}(\lambda) := \frac{p_{\widehat{U}}(\lambda)}{q_{\widehat{U}}(\lambda)}$, namely: (i) $r_{\widehat{U}}(\lambda) = \mu_{\widehat{U}}$ is constant, (ii) $r_{\widehat{U}}(\lambda)$ is a linear rational function, i.e. $p_{\widehat{U}}(\lambda)$ and $q_{\widehat{U}}(\lambda)$ share a common zero, and (iii) $r_{\widehat{U}}(\lambda)$ is a quadratic rational function, i.e. $p_{\widehat{U}}(\lambda)$ and $q_{\widehat{U}}(\lambda)$ share no common zero.

By Lemma 6.6, we have under (i) that $r_{\widehat{U}}(\lambda) = \mu_{\widehat{U}}$, $|\mu_{\widehat{U}}| = 1$, is constant. Comparing the coefficients at $\lambda$, we get from equation (6.9) that

$$\left|\operatorname{Tr}\big(\widehat{C}\widehat{U}(A_{22} + c_{11}a_{11}(\operatorname{Tr}\widehat{C})^{-1}I)\widehat{U}^*\big)\right|^2 = \left|\operatorname{Tr}\big(\widehat{C}\widehat{U}(B_{22} + c_{11}b_{11}(\operatorname{Tr}\widehat{C})^{-1}I)\widehat{U}^*\big)\right|^2 \tag{6.12}$$

By the same Lemma 6.6, under (iii) we have

$$r_{\widehat{U}}(\lambda) = \mu_{\widehat{U}}\lambda^k \frac{a_0 + a_1\lambda + a_2\lambda^2}{\overline{a}_2 + \overline{a}_1\lambda + \overline{a}_0\lambda^2} \qquad (|\mu_{\widehat{U}}| = 1),$$

(clearly, in Lemma 6.6, $d \le 2$, otherwise, $r_{\widehat{U}}(\lambda)$ would have more than two zeros, counted with multiplicities). Moreover, $p_{\widehat{U}}(0) \neq 0$ implies $r_{\widehat{U}}(0) \neq 0$ which forces $k \le 0$, and hence $d = 2$. Actually, $k = 0$, otherwise $r_{\widehat{U}}(\lambda)$ would have at least three poles in the complex plane, including the pole at the origin, which is not possible in view of the form of $q_{\widehat{U}}(\lambda)$. It is easy to see that

$$p_{\widehat{U}}(\lambda) = \alpha(a_0 + a_1\lambda + a_2\lambda^2), \quad q_{\widehat{U}}(\lambda) = \beta(\overline{a}_2 + \overline{a}_1\lambda + \overline{a}_0\lambda^2)$$

for some (nonzero) constants $\alpha$ and $\beta$. Now (6.11) gives $|\alpha| = |\beta|$, and we obtain (6.12) again. Both sides of (6.12) are real-analytic functions of $\widehat{U}$ (Propositions 2.1 and 2.2), on the real-analytic, pathwise connected manifold $\mathcal{U}_{n-1}$. Thus (Proposition 2.1(c)), the two sides are either equal identically or they differ on an open dense subset of unitaries. In the first case, and in view of Theorem 3.1(a), we are done by induction on $n$. In the second case, possibility (ii) holds on an open dense subset of unitaries (because (i) and (iii) imply equation (6.12), which presently holds only outside some open dense subset of the unitaries). We show this contradicts the assumption that $C$ satisfies Lemma 6.5.

Now, polynomial $p_{\widehat{U}}(\lambda)$ is of degree two, and shares a common zero with $q_{\widehat{U}}(\lambda)$ if and only if its leading coefficient is nonzero, and the resultant between $p_{\widehat{U}}(\lambda)$ and $q_{\widehat{U}}(\lambda)$ vanishes. Since the resultant of $p_{\widehat{U}}(\lambda)$ and $q_{\widehat{U}}(\lambda)$ is a polynomial in their coefficients, which themselves are real-analytic functions of $\widehat{U}$, we see that the resultant vanishes identically (otherwise (ii) would not hold on a dense subset). Consequently, $p_{\widehat{U}}(\lambda)$ is of degree two, does not vanish at $\lambda = 0$, and differs from any scalar multiple of $q_{\widehat{U}}(\lambda)$ but shares a common zero with it for every $\widehat{U}$ from a dense subset $\Omega_1$ of unitaries. Hence, at fixed $\widehat{U} \in \Omega_1$ we can write

$$p_{\widehat{U}}(\lambda) = (a + \lambda)(c + b\lambda), \qquad q_{\widehat{U}}(\lambda) = (a + \lambda)(d + f\lambda)$$

for some nonzero scalars $a, b, c, d, f$ which depend on coefficients of the two polynomials. Due to (6.11), we must have

$$|c + be^{i\phi}| = |d + fe^{i\phi}|, \quad \forall \quad \text{real} \quad \phi.$$

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Since $p_{\widehat{U}}$ and $q_{\widehat{U}}$ are not scalar multiple of each other, a straightforward computation shows that $(d,f) = \mu(\overline{b},\overline{c})$ for some unimodular $\mu = \mu_{\widehat{U}} \in \mathbb{C}$. This gives that

$$p_{\widehat{U}}(\lambda) = b\lambda^2 + (ab+c)\lambda + ac, \quad q_{\widehat{U}}(\lambda) = \mu_{\widehat{U}}(\overline{c}\lambda^2 + \left(\overline{b} + a\overline{c}\right)\lambda + a\overline{b}).$$

Comparing the coefficients of $\lambda$ in $p_{\widehat{U}}(\lambda)$, $q_{\widehat{U}}(\lambda)$, and in (6.9) we get that, for $\widehat{U} \in \Omega_1$,

$$b = (\widehat{U}\mathbf{a}_{12})^*\mathbf{c}_{21}, \quad \mu_{\widehat{U}} a\overline{b} = \mathbf{c}_{12}^*\widehat{U}\mathbf{b}_{21}, \quad ac = \mathbf{c}_{12}^*\widehat{U}\mathbf{a}_{21}, \quad \mu_{\widehat{U}}\overline{c} = (\widehat{U}\mathbf{b}_{12})^*\mathbf{c}_{21}. \tag{6.13}$$

Now,

$$\frac{\mathbf{c}_{12}^*\widehat{U}\mathbf{b}_{21}}{\mathbf{c}_{21}^*(\widehat{U}\mathbf{a}_{12})} = \frac{\mu_{\widehat{U}} a\overline{b}}{\overline{b}} = \mu_{\widehat{U}} a = \frac{a}{\overline{\mu_{\widehat{U}}}} = \frac{ac}{\overline{\mu_{\widehat{U}}\overline{c}}} = \frac{\mathbf{c}_{12}^*\widehat{U}\mathbf{a}_{21}}{\mathbf{c}_{21}^*(\widehat{U}\mathbf{b}_{12})},$$

which we rewrite into

$$\left(\mathbf{c}_{12}^*\widehat{U}\mathbf{b}_{21}\right) \cdot \left(\mathbf{c}_{21}^*\widehat{U}\mathbf{b}_{12}\right) = \left(\mathbf{c}_{12}^*\widehat{U}\mathbf{a}_{21}\right) \cdot \left(\mathbf{c}_{21}^*\widehat{U}\mathbf{a}_{12}\right). \tag{6.14}$$

By Propositions 2.2 and 2.1(c), the above identity holds for any unitary matrix $\widehat{U} \in \mathbb{C}^{(n-1)\times(n-1)}$.

Given a unitary $\widehat{U}$ such that $\mathbf{c}_{12}^*\widehat{U}\mathbf{a}_{21} = 0$ then at least one among $\mathbf{c}_{12}^*\widehat{U}\mathbf{b}_{21}$ and $\mathbf{c}_{21}^*\widehat{U}\mathbf{b}_{12}$ vanishes. By Lemma 6.4, we have two options:

*Option 1.* $\mathbf{c}_{12}^*\widehat{U}\mathbf{a}_{21} = 0$ always implies $\mathbf{c}_{21}^*\widehat{U}\mathbf{b}_{12} = 0$. Then, by Lemma 6.3, we see that there is a unitary $\widehat{V}$ such that $\mathbf{c}_{12} \in \operatorname{Span}\{\widehat{V}\mathbf{a}_{21}\}$ and $\mathbf{c}_{21} \in \operatorname{Span}\{\widehat{V}\mathbf{b}_{12}\}$, and either $\mathbf{c}_{12}, \mathbf{c}_{21} \in \mathbb{C}^{n-1}$ must be linearly dependent for $n \geq 4$ or, if $n = 3$, they are either linearly dependent or orthogonal. The second option (when $n = 3$) contradicts Lemma 6.5 for $C$. The first option implies $\mathbf{a}_{21}$ and $\mathbf{b}_{12}$ are linearly dependent, so $(I - E_{11})A^*E_{11}$ and $(I - E_{11})BE_{11}$ are linearly dependent, a contradiction with (6.8).

*Option 2.* $\mathbf{c}_{12}^*\widehat{U}\mathbf{a}_{21} = 0$ always implies $\mathbf{c}_{12}^*\widehat{U}\mathbf{b}_{21} = 0$. As $\widehat{U}$ runs over all unitaries, this implies that every vector $\mathbf{x}$, orthogonal to $\mathbf{a}_{21}$, is also orthogonal to $\mathbf{b}_{21}$, and so $\mathbf{a}_{21}$ and $\mathbf{b}_{21}$ are linearly dependent. Thus, we may divide (6.14) by $\mathbf{c}_{12}^*\widehat{U}\mathbf{a}_{21}$ on both sides and deduce by the same arguments that $\mathbf{a}_{12}$ and $\mathbf{b}_{12}$ are also linearly dependent, contradicting (6.7). ∎

Our last lemma gives the basis of the induction to prove Theorem 1.5 for finite-dimensional $\mathcal{H}$.

Lemma 6.8. *Suppose* $2 \times 2$ *matrices* $A, B, C$ *satisfy* (6.2) *with* $C$ *non-scalar. Then* $A, B$ *enjoy property* (**P1**).

*Proof.* Since the case $\operatorname{rank} C = 1$ was already proven in [5], we only need to consider the option when $\operatorname{rank} C = 2$.

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*Case 1. C is diagonalizable.* Let $\gamma$ be an eigenvalue of $C$. Then, $C' := C - \gamma I$ is of rank-one, and $\operatorname{Tr} C' \neq 0$. Then,

$$\operatorname{Tr}(CUXU^*) = \operatorname{Tr}(C'UXU^* + \gamma UXU^*) = \operatorname{Tr}\big(C'U\big(X + ((\operatorname{Tr} C')^{-1}\gamma \operatorname{Tr} X)I\big)U^*\big),$$

for every $X \in \mathbb{C}^{2\times 2}$. So, from identity (6.2) we derive that for

$$A' := A + ((\operatorname{Tr} C')^{-1}\gamma \operatorname{Tr} A)I, \qquad B' := B + ((\operatorname{Tr} C')^{-1}\gamma \operatorname{Tr} B)I,$$

it holds

$$|\operatorname{Tr}(C'UA'U^*)| = |\operatorname{Tr}(C'UB'U^*)|, \qquad \forall \text{ unitary } U \in \mathbb{C}^{2\times 2}.$$

By [5], $A'$ and $B'$ enjoy property **(P1)**, and we are done.

*Case 2. C is nondiagonalizable.* By multiplying both sides in (6.2) with a suitable positive scalar and using unitary similarity on $C$, we may assume without loss of generality that $C = \gamma I_2 + E_{12}$ for some nonzero $\gamma$. Also, replacing $(A, B)$ by $(\mu_1 A, \mu_2 B)$ for some suitable unimodular complex numbers $\mu_1$ and $\mu_2$, we may assume that both $\gamma \operatorname{Tr} A = 2\alpha\gamma$ and $\gamma \operatorname{Tr} B = 2\beta\gamma$ are nonnegative. Then, any unitary $U \in \mathbb{C}^{2\times 2}$ satisfies

$$|\operatorname{Tr}(CUAU^*)| = |2\alpha\gamma + \operatorname{Tr}(E_{12}UAU^*)|.$$

Note that the off-diagonal entries of $UAU^*$ and of $U(A - \alpha I_2)U^*$ are the same. Since $A - \alpha I_2$ has trace zero, we can find a unitary $V$ such that

$$V(A - \alpha I_2)V^* = \begin{bmatrix} 0 & \mathsf{a}_2 e^{it} \\ \mathsf{a}_1 & 0 \end{bmatrix}$$

with $\mathsf{a}_1 \geq \mathsf{a}_2 \geq 0$. Clearly, $\mathsf{a}_1, \mathsf{a}_2$ are the singular values of $A - \alpha I_2$. Thus, the maximal modulus of the $(2,1)$ entry of $UAU^*$, i.e. of $(UAU^*)_{21} = (U(A - \alpha I_2)U^*)_{21}$ is $\mathsf{a}_1$. As a result,

$$|\operatorname{Tr}(CUAU^*)| \leq |2\alpha\gamma| + |\operatorname{Tr}(E_{12}UAU^*)| \leq 2\alpha\gamma + \mathsf{a}_1. \tag{6.15}$$

Since the inequality holds for every unitary $U$ and since the equality in (6.15) is also possible (say, when $U = V$), we have that the right hand side satisfies $2\alpha\gamma + \mathsf{a}_1 = r_C(A)$ with

$$r_C(X) := \max\{|\operatorname{Tr}(CUXU^*)| : U \text{ unitary}\}$$

the $C$-numerical radius of $X \in \mathbb{C}^{2\times 2}$. Moreover, the equality in (6.15) holds only if $|(UAU^*)_{21}| = |(U(A - \alpha I_2)U^*)_{21}| = \mathsf{a}_1$ in which case $U(A - \alpha I_2)U^*$ has zero diagonal. Similarly,

$$|\operatorname{Tr}(CUAU^*)| \leq 2\beta\gamma + \mathsf{b}_1 = r_C(B),$$

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where $\mathsf{b}_1$ is the largest singular value of $B - \beta I_2$, and the equality holds only if $|(UBU^*)_{21}| = |(U(B-\beta I_2)U^*)_{21}| = \mathsf{b}_1$, in which case $U(B-\beta I_2)U^*$ has zero diagonal.

Suppose $\alpha\gamma \geq \beta\gamma \geq 0$. Otherwise, interchange the roles of $A$ and $B$. Replacing $(A, B)$ by $(VAV^*, VBV^*)$ for a suitable unitary $V \in \mathbb{C}^{2\times 2}$, we may assume that $A = \begin{bmatrix} \alpha & \alpha_{12} \\ \mathsf{a}_1 & \alpha \end{bmatrix}$. Then

$$|\operatorname{Tr}(CB)| = |\operatorname{Tr}(CA)| = r_C(A) = r_C(B)$$

implies that $B - \beta I_2$ has zero diagonal and, for its $(2,1)$ entry, $|B_{21}| = \mathsf{b}_1$. We assume in the sequel $\mathsf{a}_1 > 0$, else $A$ is scalar and we are done by Proposition 4.3. Now, for $D = \operatorname{diag}(1, e^{i\xi})$,

$$|2\alpha\gamma + \mathsf{a}_1 e^{i\xi}| = |\operatorname{Tr}(CDAD^*)| = |\operatorname{Tr}(CDBD^*)| = |2\beta\gamma + B_{21}e^{i\xi}|, \quad \xi \in [0, 2\pi).$$

This implies one of the three options (taking into account $\mathsf{a}_1 > 0$): (1) $2\alpha\gamma = 2\beta\gamma = 0$ and $\mathsf{a}_1 = |B_{21}|$; (2) $2\alpha\gamma \neq 2\beta\gamma > 0$, $B_{21} = 2\alpha\gamma$ and $\mathsf{a}_1 = 2\beta\gamma$; (3) $B_{21} = \mathsf{a}_1$ and $2\alpha\gamma = 2\beta\gamma > 0$.

*Subcase 1.* Assume that $\alpha\gamma = \beta\gamma$. If $\beta \neq 0$, then $B_{21} = \mathsf{a}_1$. If $\beta = 0$, then $\beta\gamma = \alpha\gamma = 0$, and we may replace $B$ by $\mu_3 B$ for a suitable unimodular complex number $\mu_3$ and assume that $B_{21} = \mathsf{a}_1$ also in this case (note that this transformation does not change $\operatorname{Tr} B$). For $U = (\cos\xi)I_2 + \sin\xi(e^{is}E_{12} - e^{-is}E_{21})$ with $\xi, s \in [0, 2\pi)$, we have

$$|\mathsf{a}_1 \cos^2\xi - \alpha_{12}e^{-i2s}\sin^2\xi| = |2\alpha\gamma + \mathsf{a}_1\cos^2\xi - \alpha_{12}e^{-i2s}\sin^2\xi| = |\operatorname{Tr}(CUAU^*)|$$

$$= |\operatorname{Tr}(CUBU^*)| = |2\alpha\gamma + \mathsf{a}_1\cos^2\xi - B_{12}e^{-i2s}\sin^2\xi| = |\mathsf{a}_1\cos^2\xi - B_{12}e^{-i2s}\sin^2\xi|.$$

We conclude that $\alpha_{12} = B_{12}$.

*Subcase 2.* Assume that $\alpha\gamma > \beta\gamma$. Then $2\beta\gamma = \mathsf{a}_1 > 0$ and $B_{21} = 2\alpha\gamma = \mathsf{b}_1$ (the second equality follows from $|B_{21}| = \mathsf{b}_1$). For $U = E_{12} + E_{21}e^{is}$ with $s \in [0, 2\pi)$,

$$|2\alpha\gamma + \alpha_{12}e^{is}| = |\operatorname{Tr}(CUAU^*)| = |\operatorname{Tr}(CUBU^*)| = |2\beta\gamma + B_{12}e^{is}|.$$

A straightforward calculation using the equality

$$(2\alpha\gamma + \alpha_{12}e^{is})(\overline{2\alpha\gamma + \alpha_{12}e^{is}}) = (2\beta\gamma + B_{12}e^{is})(\overline{2\beta\gamma + B_{12}e^{is}}), \quad s \in [0, 2\pi),$$

shows that $\mathsf{b}_1 = 2\alpha\gamma = \mu_2\overline{B_{12}}$ and $\mathsf{a}_1 = 2\beta\gamma = \mu_2^{-1}\alpha_{12}$. Thus, there is $\nu \in [0, 2\pi)$ such that

$$A = \alpha I_2 + 2\beta\gamma(E_{21} + e^{i\nu}E_{12}) \quad \text{and} \quad B = \beta I_2 + 2\alpha\gamma(E_{21} + e^{i\nu}E_{12}).$$

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Now for the unitary $U = \cos\xi I_2 + \sin\xi(e^{is}E_{12} - e^{-is}E_{21})$ with $\xi, s \in [0, 2\pi)$, we have

$$2\,|(\alpha\gamma + \beta\gamma\cos^2\xi) - \beta\gamma e^{i(\nu-2s)}\sin^2\xi| = |\operatorname{Tr}(CUAU^*)|$$

$$= |\operatorname{Tr}(CUBU^*)| = 2|(\beta\gamma + \alpha\gamma\cos^2\xi) - \alpha\gamma e^{i(\nu-2s)}\sin^2\xi|.$$

We conclude that $\alpha = \beta$, which is a contradiction. □

**6.3. Proof of Theorem 1.5, assuming $C$ is finite rank.** If $\mathcal{H}$ is finite-dimensional, we argue inductively on the dimension. Lemma 6.8 is the basis, while Lemma 6.7 is the inductive step. If $\mathcal{H}$ is infinite-dimensional, we reduce to the finite-dimensional case as follows. Assume erroneously that $A, B$ do not have property **(P1)**. Then, already some 2-dimensional compression of $A$, $B$ does not have property **(P1)**. Since also $\operatorname{rank} C < \infty$, we can find a unitary operator $U$ such that $UCU^* = C_1 \oplus 0$ where $C_1$ acts on finite-dimensional subspace $\mathcal{H}' \subseteq \mathcal{H}$ and the compressions of $UAU^*$ and $UBU^*$ to $\mathcal{H}'$ do not satisfy property **(P1)**. This contradicts the already proven result for finite-dimensional $\mathcal{H}'$. □

**7. A more general class of operators $C$.** The techniques used to prove Theorem 1.5 allow us to extend the result to a more general class (although less succinctly defined) of operators $C$. Namely, *assume that a trace-class operator $C \in L(\mathcal{H})$ has an orthogonally reducing invariant subspace $\mathcal{M}$ such that the restriction $C|_{\mathcal{M}}$ is non-scalar, either normal or finite rank, and has nonzero trace; if* (1.1) *holds for two operators $A, B \in L(\mathcal{H})$, then $A, B$ must satisfy $A = \mu B + \nu I$ or $A = \mu B^* + \nu I$ for some $\mu, \nu \in \mathbb{C}$, $|\mu| = 1$.* The proof follows the pattern of Subsection 5.1.

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REFERENCES

[1] M. Brešar and P. Šemrl. On locally linearly dependent operators and derivations. *Trans. Amer. Math. Soc.*, 351(3):1257–1275, 1999.
[2] W.-S. Cheung and N.-K. Tsing. The $C$-numerical range of matrices is star-shaped. *Linear and Multilinear Algebra*, 41:245–250, 1996.
[3] C. Glader. Minimal degree rational unimodular interpolation on the unit circle. *Electron. Trans. Numer. Anal.*, 30:88–106, 2008.
[4] M.S. Jones. A note on the shape of the generalized $C$-numerical range. *Linear and Multilinear Algebra*, 31:81–84, 1992.
[5] B. Kuzma, G. Lešnjak, C.-K. Li, T. Petek, and L. Rodman. Conditions for linear independence of two operators. *Oper. Theory Adv. Appl.*, 202:411–434, 2010.
[6] C.-K. Li. The $C$-convex matrices. *Linear and Multilinear Algebra*, 21:303–312, 1987.

**Electronic Journal of Linear Algebra ISSN 1081-3810**
**A publication of the International Linear Algebra Society**
**Volume 22, pp. 22-52, January 2011**



[7] C.-K. Li. $C$-numerical ranges and $C$-numerical radii. *Linear and Multilinear Algebra*, 37:51–82, 1994.

[8] C.-K. Li and Y.-T. Poon. Some results on the $c$-numerical range. *Five Decades As a Mathematician and Educator*, World Sci. Publ., River Edge, NJ, 247–258, 1995.

[9] F. Lu. Linear dependence of operators characterized by trace functionals. *Linear Algebra Appl.*, 434:343–355, 2011.

[10] R. Narasimhan. *Analysis on Real and Complex Manifolds*. North-Holland Publishing Co., Amsterdam, 1985.

[11] R.M. Range. *Holomorphic Functions and Integral Representations in Several Complex Variables*. Springer, New York, 1986.

[12] W. Rossmann. *Lie Groups. An Introduction Through Linear Groups*. Oxford University Press, Oxford, 2002.

[13] W. Rudin. *Real and Complex Analysis*. McGraw-Hill, New York, 1987.