

MINIMUM SEMIDEFINITE RANK OF OUTERPLANAR GRAPHS AND THE TREE COVER NUMBER*

FRANCESCO BARIOLI[†], SHAUN M. FALLAT[‡], LON H. MITCHELL[§], AND
SIVARAM K. NARAYAN[¶]

Abstract. Let $G = (V, E)$ be a multigraph with no loops on the vertex set $V = \{1, 2, \dots, n\}$. Define $S_+(G)$ as the set of symmetric positive semidefinite matrices $A = [a_{ij}]$ with $a_{ij} \neq 0$, $i \neq j$, if $ij \in E(G)$ is a single edge and $a_{ij} = 0$, $i \neq j$, if $ij \notin E(G)$. Let $M_+(G)$ denote the maximum multiplicity of zero as an eigenvalue of $A \in S_+(G)$ and $\text{mr}_+(G) = |G| - M_+(G)$ denote the minimum semidefinite rank of G . The tree cover number of a multigraph G , denoted $T(G)$, is the minimum number of vertex disjoint simple trees occurring as induced subgraphs of G that cover all of the vertices of G . The authors present some results on this new graph parameter $T(G)$. In particular, they show that for any outerplanar multigraph G , $M_+(G) = T(G)$.

Key words. Minimum rank graph, Maximum multiplicity, Minimum semidefinite rank, Outerplanar graphs, Tree cover number.

AMS subject classifications. 05C50, 15A03, 15A18.

1. Introduction. Let $G = (V, E)$ be an undirected graph with no loops but possibly multiple edges (a multigraph) with vertex set $V = \{1, 2, \dots, n\}$. A matrix $A = [a_{ij}]$ is *combinatorially symmetric* when $a_{ij} = 0$ if and only if $a_{ji} = 0$. Given a multigraph G , we say G is the graph of the combinatorially symmetric matrix $A = [a_{ij}]$ when

- $a_{ij} \neq 0$ whenever $i \neq j$ and i and j are adjacent by a single edge, and
- $a_{ij} = 0$ whenever $i \neq j$ and i and j are not adjacent.

Note that the entry a_{ij} for $i \neq j$ is not restricted if $ij \in E$ is a multiple edge and that the main diagonal entries of A play no role in determining G . Define $S(G, \mathbb{F})$ as the set of all n -by- n matrices that are real symmetric if $\mathbb{F} = \mathbb{R}$ or complex Hermitian if $\mathbb{F} = \mathbb{C}$

*Received by the editors on January 28, 2010. Accepted for publication on December 17, 2010.
Handling Editor: Bryan L. Shader.

[†]Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga TN, 37403 (francesco-barioli@utc.edu).

[‡]Department of Mathematics and Statistics, University of Regina, Regina, SK, Canada (sfallat@math.uregina.ca). Research supported in part by an NSERC Discovery research grant.

[§]Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, VA 23284-2014 (lmitchell2@vcu.edu).

[¶]Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859 (naray1sk@cmich.edu). Thanks to Central Michigan University for its support during his sabbatical leave in Fall of 2009 and MSRI for providing travel support to BIRS.

whose graph is G , and $S_+(G, \mathbb{F})$ the corresponding subsets of positive semidefinite (psd) matrices. When $A \in S(G, \mathbb{F})$, there is no restriction on the diagonal entries. If $A \in S_+(G, \mathbb{F})$ the diagonal entry $a_{ii} \geq 0$ and if there is a nonzero off-diagonal entry in the i th row of A , then $a_{ii} > 0$. By $M(G, \mathbb{F})$ (resp., $M_+(G, \mathbb{F})$) we denote the largest possible nullity of any matrix $A \in S(G, \mathbb{F})$ (resp., $A \in S_+(G, \mathbb{F})$). The smallest possible rank of any matrix $A \in S(G, \mathbb{F})$ is the *minimum rank* of G , denoted $\text{mr}(G, \mathbb{F})$, or alternately $\text{hmr}(G)$ if $\mathbb{F} = \mathbb{C}$ is the *Hermitian minimum rank* of G . The smallest possible rank of any matrix $A \in S_+(G, \mathbb{F})$ is denoted $\text{mr}_+(G, \mathbb{F})$, or alternately $\text{msr}(G)$ for $\mathbb{F} = \mathbb{C}$ is the *minimum semidefinite rank* of G . From the definitions above, it follows that $M(G, \mathbb{F}) + \text{mr}(G, \mathbb{F}) = M_+(G, \mathbb{F}) + \text{mr}_+(G, \mathbb{F}) = |G|$ where $|G|$ denotes the order of G . Moreover, it is also evident that, for any graph G ,

$$\text{mr}(G, \mathbb{F}) \leq \text{mr}_+(G, \mathbb{F}) \text{ and } M(G, \mathbb{F}) \geq M_+(G, \mathbb{F}).$$

Many of the results of this paper hold for both $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$. When this occurs, we will either use \mathbb{F} to mean either field, or will omit the field reference entirely from the notation of the preceding paragraph.

Of recent interest is to gain a better understanding of the relationship between the minimum rank and minimum semidefinite rank of graphs, and it does appear that techniques from one problem may be adapted to the other and vice-versa. For example, making use of the *zero-forcing number* of a graph (see [5, 10]). Also, of interest is to gain a better understanding of the possible connections between $\text{mr}_+(G, \mathbb{F})$ or $M_+(G, \mathbb{F})$ and various graph parameters. One existing connection that has been proved is that $\text{mr}_+(G, \mathbb{F})$ is exactly the clique cover number of the graph G (i.e., the minimum number of cliques needed to cover all of the edges of G), whenever G is a chordal graph (see [3]).

In the case of conventional minimum rank, it has long been known that for trees, the minimum rank of G is precisely the order of G minus the path cover number of G . The *path cover number* of a simple graph G , $P(G)$, is the minimum number of vertex disjoint paths occurring as induced subgraphs of G that cover all of the vertices of G . Consequently, it is known that $M(T, \mathbb{R}) = P(T)$ for every (simple) tree T (see [9]).

This result was extended to the case of unicyclic graphs in [1], where it was proved that $M(U, \mathbb{R}) = P(U)$ or $P(U) - 1$, for any unicyclic graph U , and both cases were characterized. More recently, Sinkovic has shown that for a (simple) outerplanar graph G , $M(G, \mathbb{R}) \leq P(G)$ and has given a family of outerplanar graphs for which equality holds [11]. Unfortunately, it is known that M and P are not comparable in general (see [1]). Furthermore, M_+ and P are not comparable in general.

For the graph in Figure 1.1, it can be shown that $M = M_+ = 3$ and $T = P = 2$. On the other hand, if G is the 5-sun (see Figure 4.2), it can be shown that $M = M_+ = T = 2$ and $P = 3$.



FIG. 1.1. Comparing M_+ , M and P .

Our goal is to study the minimum semidefinite rank of all outerplanar graphs. Along these lines, it is known that the minimum semidefinite rank of trees on n vertices is $n-1$ (see [12]), and that for any unicyclic graph on n vertices, the minimum semidefinite rank is then $n-2$. Thus, it is clear that the minimum semidefinite rank of both trees and unicyclic graphs has little to do with the path cover number, but it does appear to be connected with a different, new, parameter.

The *tree cover number* of G , denoted $T(G)$, is the minimum number of vertex disjoint simple trees occurring as induced subgraphs of G that cover all of the vertices of G . We emphasize that the induced trees in a tree cover of a multigraph G must be simple trees. We can similarly extend the definition of path cover number to multigraphs as the minimum number of vertex disjoint simple paths occurring as induced subgraphs of G that cover all of the vertices of G . Certainly, it is the case that $T(G) \leq P(G)$. Furthermore, it is clear that the tree cover number of a simple tree is one and the tree cover number of a simple unicyclic graph is two. Thus, in both of these cases we have $M_+(G) = T(G)$.

EXAMPLE 1.1. The graphs in Figure 1.2 give examples of the tree cover number. In (A), the tree cover number is three. In (B), the temptation is to view the three single edges as one tree covering all vertices, but such a tree is not induced, so the tree cover number is two. (A) also illustrates Lemma 3.1.

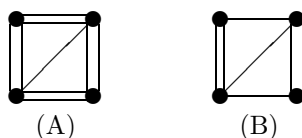


FIG. 1.2. Calculating the tree cover number: Examples.

In Section 3, we prove our main result that for any outerplanar (multi)graph G , $M_+(G) = T(G)$, and in Section 4, we discuss the relationship between $T(G)$, $M(G)$ and $P(G)$ for both outerplanar graphs and more general classes of graphs. In Section 2, we include and discuss some relevant background material in graph theory, vector representations of a multigraph, and orthogonal removal of a vertex v from a multigraph G .

2. Preliminaries.

2.1. Graph theory. A subgraph of G induced by $R \subset V$, denoted $G[R]$, has the vertex set R and edge set consisting of those edges of G where both vertices are elements of R . If $R = V(G) \setminus \{v\}$, we denote $G[R]$ by $G - v$. The *neighborhood* of a vertex v of a graph G , denoted $N(v)$, is the set of vertices of G adjacent to v . The *closed neighborhood* of a vertex v , $N[v]$, is $N(v) \cup \{v\}$. We define the *simple-neighborhood* of a vertex v , denoted $N_1(v)$, to be those vertices $u \in N(v)$ such that u is adjacent to v by a single edge. In a simple graph, $N_1(v) = N(v)$ for every vertex v .

DEFINITION 2.1. A vertex v is called *singly-isolated* in G if $N_1(v)$ is an empty set.

The *degree* of a vertex v in G , $d_G(v)$, is the cardinality of $N(v)$. Note that, in multigraphs, $d_G(v)$ may be strictly less than the number of edges incident to v . If $d_G(v) = 1$, then v is said to be a *pendant* vertex. Also, a vertex v is called *simplicial* if $N(v)$ is a clique.

DEFINITION 2.2. A graph is *outerplanar* if it has a crossing-free embedding in the plane such that all vertices are on the same face.

Equivalently, an outerplanar graph has no subgraph homeomorphic to K_4 or $K_{2,3}$ (see [13, p. 256]). The following observation is well-known in graph theory and can be easily proved:

REMARK 2.3. Every outerplanar graph G has a vertex v with $d_G(v) \leq 2$.

Further, every subgraph of an outerplanar graph is outerplanar. In addition, observe that any tree or unicyclic graph is an outerplanar graph.

2.2. Vector representations of G . A set of vectors $\vec{V} = \{\vec{v}_1, \dots, \vec{v}_n\}$ in \mathbb{F}^m is a *vector representation* of the multigraph G when $\langle \vec{v}_i, \vec{v}_j \rangle \neq 0$ if i and j are joined by a single edge and $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ if i and j are not adjacent. Let $X = [\vec{v}_1 \ \dots \ \vec{v}_n]$ be the matrix whose columns are vectors from \vec{V} . Then X^*X is a psd matrix, called the *Gram matrix* of \vec{V} , with regard to the usual inner product of \mathbb{F}^m , and, by construction, $A = X^*X \in S_+(G, \mathbb{F})$. By $\text{rank } \vec{V}$, we mean the dimension of $\text{Span}(\vec{V})$. Since any psd matrix $A \in M_n(\mathbb{F})$ may be factored as Y^*Y for some $Y \in M_n(\mathbb{F})$ with $\text{rank } A = \text{rank } Y$ [7], each psd matrix is the Gram matrix of a suitable set of vectors. Therefore, the smallest m for which there exists a vector representation of G in \mathbb{F}^m is equal to $\text{mr}_+(G, \mathbb{F})$.

2.3. Orthogonal removal. Given a vector representation \vec{V} of G , with $\vec{v} \neq 0$ representing vertex v , replace each vector $\vec{w} \in \vec{V}$ with the orthogonal projection

$$\vec{w} - \frac{\langle \vec{v}, \vec{w} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}$$

to yield a set of vectors denoted $\vec{V} \ominus \vec{v}$. It is easy to see $\text{rank } \vec{V} = \text{rank}(\vec{V} \ominus \vec{v}) + 1$. A graph corresponding to $\vec{V} \ominus \vec{v}$, denoted $G \ominus v$, is defined as follows: in the induced subgraph $G - v$ of G , between any $u, w \in N(v)$, add $e - 1$ edges, where e is the sum of the number of edges between u and v and the number of edges between w and v . The analysis above also establishes $M_+(G) \leq M_+(G \ominus v)$. Notice that $M_+(G \ominus v) - M_+(G)$ may be arbitrarily large. For example, $M_+(K_{2,n}) = 2$ when $n \geq 2$. If n is large, then orthogonally removing a vertex v of $G = K_{2,n}$ such that $d_G(v) = n$ gives $M_+(G \ominus v) = M_+(K_{n+1}) = n$. However, we have the following result, already shown when $\mathbb{F} = \mathbb{C}$ [3, Lemma 3.4], and whose proof remains valid when $\mathbb{F} = \mathbb{R}$:

LEMMA 2.4. *Suppose v is a simplicial vertex of a connected multigraph G that is not singly isolated. Then $M_+(G) = M_+(G \ominus v)$.*

For a vertex of degree at most two, we may arrive at the same conclusion.

LEMMA 2.5. *If G is a multigraph, v is a vertex of G that is not singly isolated, and $d_G(v) \leq 2$, then $M_+(G) = M_+(G \ominus v)$.*

Proof. Suppose v is a vertex that is not singly-isolated and $d_G(v) \leq 2$, so that $d_G(v)$ is either one or two. If v is pendant (i.e., $d_G(v) = 1$), then $G \ominus v = G - v$, and $\text{mr}_+(G) = \text{mr}_+(G \ominus v) + 1$ is well known [2, 4]. When $d_G(v) = 2$, if the vertices $u, w \in N(v) = N_1(v)$ are not adjacent in G , then $\text{mr}_+(G) = \text{mr}_+(G \ominus v) + 1$, as the proof in the complex case [8, Proposition 2.4] also works over the reals. If $d_G(v) = 2$ and $u, w \in N(v)$ are adjacent in G , then v is simplicial, and the result follows from Lemma 2.4.

In the remaining case where $d_G(v) = 2$, $u, w \in N(v) \neq N_1(v)$ are not adjacent in G , we show that $M_+(G) = M_+(G \ominus v)$. Note that u and w will be adjacent by multiple edges in $G \ominus v$. Let \vec{X}' be a vector representation of $G \ominus v$ such that $\text{mr}_+(G \ominus v, \mathbb{F}) = \text{rank } \vec{X}'$. Let \vec{u}', \vec{w}' be the vectors in \vec{X}' corresponding to vertices $u, w \in N(v)$. If $\langle \vec{u}', \vec{w}' \rangle \neq 0$, then we are in the case considered above [8, Proposition 2.4]. Suppose $\langle \vec{u}', \vec{w}' \rangle = 0$, uv is a single edge and wv is a multiple edge. Then let \vec{v} be a unit vector orthogonal to every vector in \vec{X}' and define $\vec{u} = \vec{u}' + \vec{v}$. Then $(\vec{X}' \setminus \{\vec{u}'\}) \cup \{\vec{u}, \vec{v}\}$ is a vector representation \vec{X} of G (with \vec{v} representing vertex v and \vec{u} representing vertex u) and \vec{X}' is the vector representation derived from \vec{X} by the orthogonal removal of \vec{v} . Therefore, $\text{mr}_+(G, \mathbb{F}) \leq \text{rank } \vec{X} = \text{rank } \vec{X}' + 1 = \text{mr}_+(G \ominus v, \mathbb{F}) + 1$, or equivalently, $M_+(G) = M_+(G \ominus v)$. \square

3. Outerplanar graphs. In this section, we demonstrate our main observation, that for an outerplanar graph G , $M_+(G) = T(G)$.

We begin by considering the relationship between the tree cover number of a graph and the graph obtained by vertex deletion and orthogonal removal.

LEMMA 3.1. *If v is a singly-isolated vertex in a multigraph G , then $T(G - v) = T(G) - 1$.*

Proof. Any tree cover of G must consist of a tree cover of $G - v$ and the tree $\{v\}$. \square

PROPOSITION 3.2. *If v is a vertex of a multigraph G that is not singly-isolated, and $N_1(v)$ induces (in G) a complete graph or has exactly two vertices that are not adjacent, then $T(G \ominus v) \geq T(G)$.*

Proof. Let $\{T_1, \dots, T_k\}$ be a tree cover of $G \ominus v$. From the definition of orthogonal removal of a vertex, the subgraph of $G \ominus v$ induced by the vertices of $N(v)$ is complete. By our assumption on $N_1(v)$, $(G \ominus v)[N(v)]$ contains at most one single edge and that single edge must belong to the induced subgraph $(G \ominus v)[N_1(v)]$. Thus, there exists at most one tree T_i in the tree cover such that T_i covers two vertices of $N(v)$. If T_i covers at most one vertex in $N(v)$, then, since orthogonal removal only alters edges among vertices in $N(v)$, T_i will remain as a simple induced tree in G . If no tree covers two vertices of $N(v)$, let $u \in N_1(v)$, and without loss of generality, let T_1 cover u . If there exists a tree T_1 that covers two vertices $u, w \in N(v)$, then uw must be a single edge in $G \ominus v$ that belongs to T_1 and, as a result, $u, w \in N_1(v)$ and u and w are not adjacent in G . In either case, define $T'_1 = G[V(T_1) \cup \{v\}]$. Then T'_1 is still an induced simple tree, since in the second case if there were a cycle using uv and vw , then it must come from a cycle of T_1 in $G \ominus v$ using uw . Now, $\{T'_1, T_2, \dots, T_k\}$ is a tree cover of G with the same cardinality. \square

For a vertex of degree at most two, we can say even more.

PROPOSITION 3.3. *If v is a vertex of a multigraph G such that v is not singly isolated in G and $d_G(v) \leq 2$, then $T(G \ominus v) = T(G)$.*

Proof. From Proposition 3.2 we may deduce that $T(G \ominus v) \geq T(G)$. Let $\{T_1, \dots, T_k\}$ be a tree cover of G . Without loss of generality, let v be covered by T_1 . If $d_G(v) = 1$ and $N(v) = N_1(v) = \{u\}$, only the edge uv is changed in $G \ominus v$, and it either does not belong to T_1 or is a pendant edge, so that $T'_1 = (G \ominus v)[V(T_1) \setminus \{v\}]$, T_2, \dots, T_k are induced simple trees (T'_1 possibly empty) that cover $G \ominus v$. If $N(v) = \{u, w\}$ and u and w are not covered by the same tree, $T'_1 = (G \ominus v)[V(T_1) \setminus \{v\}]$, T_2, \dots, T_k are induced simple trees (T'_1 possibly empty) that cover $G \ominus v$, since only the number of edges between u and w can change passing from G to $G \ominus v$. Suppose then that

$N(v) = \{u, w\}$ and u and w are covered by the same tree T_j . If $T_j = T_1$, then u and w are not adjacent in G and $N(v) = N_1(v)$ so that $T'_1 = (G \ominus v)[V(T_1) \setminus \{v\}]$, T_2, \dots, T_k are again induced simple trees that cover $G \ominus v$ (for T'_1 , any induced cycle in $G \ominus v$ using uw would come from a cycle of T_1 in G using uv and vw). Finally, suppose $T_j \neq T_1$, so that $T_1 = \{v\}$, and u and w are adjacent by a single edge in G . Consider T_j . Removing the single edge uw from T_j leaves two connected components, which are also induced simple trees. Label these T'_1 and T'_j . Then $T'_1, T_2, \dots, T_{j-1}, T'_j, T_{j+1}, \dots, T_k$ are induced simple trees that cover $G \ominus v$. In each case, the exhibited tree cover of $G \ominus v$ has cardinality at most k . \square

We are now in a position to establish our main observation.

THEOREM 3.4. *If G is a multigraph that is outerplanar, then $M_+(G) = T(G)$.*

Proof. We will induct on the order of G . When $|G| = 1$, the result is clear. If $|G| = k$, from Remark 2.3, G has a vertex v with $d_G(v) \leq 2$.

Suppose v is singly-isolated. Then $G - v$ is outerplanar and

$$M_+(G) = M_+(G - v) + 1 = T(G - v) + 1 = T(G)$$

using the induction hypothesis and Lemma 3.1.

If v is not singly-isolated, then we apply Proposition 3.3. Since $G \ominus v$ is outerplanar (this follows easily since the degree of v is at most two, so that orthogonally removing v is either deleting an isolated vertex, deleting a pendent vertex, or essentially edge contraction), using the induction hypothesis, in addition to Lemma 2.5, we get that

$$M_+(G) = M_+(G \ominus v) = T(G \ominus v) = T(G). \quad \square$$

Combining Theorem 3.4 with Sinkovic's result on the path cover number of outerplanar graphs gives:

COROLLARY 3.5. *For any outerplanar graph G , we have*

$$T(G) = M_+(G) \leq M(G) \leq P(G).$$

Specializing to the case of trees or unicyclic graphs we then have:

COROLLARY 3.6. *If G is a tree, then $M_+(G) = T(G) = 1$, and if G is a unicyclic graph, then $M_+(G) = T(G) = 2$.*

As a by-product of Theorem 3.4 we actually have a mechanism for computing $T(G)$ for any outerplanar graph G . Starting with a given outerplanar graph G on n

vertices, we will produce a sequence of n vertex removals (either conventional vertex deletion or orthogonal removal), say v_1, v_2, \dots, v_n . If k of the vertices from the sequence v_1, v_2, \dots, v_n were the result of a conventional vertex removal, then we have

$$M_+(G) = T(G) = k.$$

This follows since the only time M_+ is changed during the sequence of removals is when a vertex is simply deleted (assuming the vertex – at that stage – is singly isolated), and the increment is always by 1. Moreover, since both $M_+(G)$ and $T(G)$ are well-defined it also follows that every such sequence of vertices will have exactly k conventional vertex removals.

4. More on the tree cover number. Given the interesting connection established above between M_+ and T for outerplanar graphs, and the fact that $T(G)$ is a new parameter developed here, we investigate further the properties of $T(G)$ for general graphs.

We begin by demonstrating that M_+ and T need not be equal in general.

EXAMPLE 4.1. Note that strict inequality is possible in the conjectured inequality $M_+(G) \geq T(G)$. For example, for the Möbius ladder ML_8 (see Figure 4.1), $\text{mr}_+(ML_8) = 5$ [10], hence $M_+(ML_8) = 3$ but the tree cover number $T(ML_8) = 2$.

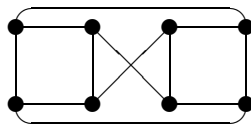


FIG. 4.1. The Möbius ladder ML_8 .

The Möbius ladder above in Figure 4.1 satisfies $M_+ > T$. So we ask the natural question as to whether there exists a graph for which $T > M_+$? At present, we know of no such graph, and as a result, we conjecture that $M_+(G) \geq T(G)$, for any multigraph G .

PROPOSITION 4.2. *If G is a chordal multigraph, then $M_+(G) \geq T(G)$.*

Proof. We will induct on the order of G . If $|G| = 1$ the result is clear. Let $|G| = k$. Since G is chordal, G has a simplicial vertex v . Suppose v is singly-isolated. Then $G - v$ is chordal and $M_+(G) = M_+(G - v) + 1 \geq T(G - v) + 1 = T(G)$ using the induction hypothesis and Lemma 3.1. If v is not singly-isolated, then $M_+(G) = M_+(G \ominus v) \geq T(G \ominus v) \geq T(G)$ by Lemma 2.4, the induction hypothesis, and Proposition 3.2. \square

A consequence to the above Proposition is: If G is a chordal multigraph, then $|G| \geq T(G) + cc(G)$. It is interesting to note that while this inequality involves graph parameters, it came as a result of considering minimum semidefinite rank.

PROPOSITION 4.3. *If $G = (L \cup R, E)$ is a simple connected bipartite graph with $n = |L| \geq |R| = m$ and $M_+(G) = m$, then $M_+(G) \geq T(G)$.*

Proof. Label the vertices of R as v_1, \dots, v_m . Choose a tree cover of cardinality m as follows. Let S_1 be the star induced by $N[v_1]$, and for $1 < i \leq m$, let S_i be the star induced by

$$N[v_i] \setminus \left(\bigcup_{j=1}^{i-1} V(S_j) \right). \quad \square$$

For example, simple bipartite graphs for which $|\bigcap_{v \in R} N(v)| \geq |R|$ meet the conditions of Proposition 4.3 [6, Proposition 2.2].

Recall that, if G_1 and G_2 are disjoint graphs, the *union* and the *join* of G_1 and G_2 , denoted respectively by $G_1 \cup G_2$ and $G_1 \vee G_2$, are the graphs defined by

$$\begin{aligned} V(G_1 \cup G_2) &= V(G_1 \vee G_2) = V(G_1) \cup V(G_2); \\ E(G_1 \cup G_2) &= E(G_1) \cup E(G_2); \\ E(G_1 \vee G_2) &= E(G_1) \cup E(G_2) \cup E, \end{aligned}$$

where E consists of all the edges (u, v) with $u \in V(G_1)$, $v \in V(G_2)$. It is evident that $T(G_1 \cup G_2) = T(G_1) + T(G_2)$ and that $M_+(G_1 \cup G_2) = M_+(G_1) + M_+(G_2)$. Hence, if the inequalities $T(G_i) \leq M_+(G_i)$, $i = 1, 2$ hold, then the corresponding inequality also holds for their union. As it turns out we can make a similar statement about the join of two graphs.

PROPOSITION 4.4. *Suppose G and H are two graphs without isolated vertices that satisfy $T(G) \leq M_+(G)$ and $T(H) \leq M_+(H)$. Then*

$$T(G \vee H) \leq M_+(G \vee H).$$

Proof. We have assumed that both G and H have no isolated vertices, so that the equality (see [6])

$$\text{mr}_+(G \vee H) = \max\{\text{mr}_+(G), \text{mr}_+(H)\},$$

holds. In this case we then have

$$\begin{aligned}
 T(G \vee H) &\leq T(G) + T(H) \\
 &\leq M_+(G) + M_+(H) \\
 &= |G| - \text{mr}_+(G) + |H| - \text{mr}_+(H) \\
 &= |G| + |H| - (\text{mr}_+(G) + \text{mr}_+(H)) \\
 &\leq |G| + |H| - (\max\{\text{mr}_+(G), \text{mr}_+(H)\}) \\
 &= |G| + |H| - \text{mr}_+(G \vee H) \\
 &= M_+(G \vee H). \quad \square
 \end{aligned}$$

We may also apply similar reasoning to the case of vertex sums of graphs. Let G_1, \dots, G_h be disjoint graphs. For each i , we select a vertex $v_i \in V(G_i)$ and join all G_i 's by identifying all v_i 's as a unique vertex v . The resulting graph is called the *vertex-sum* at v of the graphs G_1, \dots, G_h .

THEOREM 4.1. *Let G be vertex-sum at v of graphs G_1 and G_2 . Assume that $T(G_1) \leq M_+(G_1)$ and $T(G_2) \leq M_+(G_2)$. Then*

$$T(G) \leq M_+(G).$$

Proof. It is straightforward to verify that $T(G) \leq T(G_1) + T(G_2) - 1$, since the vertex sum of two trees is again a tree. Thus, it follows that

$$\begin{aligned}
 T(G) &\leq T(G_1) + T(G_2) - 1 \\
 &\leq M_+(G_1) + M_+(G_2) - 1 \\
 &= |G_1| - \text{mr}_+(G_1) + |G_2| - \text{mr}_+(G_2) - 1 \\
 &= |G_1| + |G_2| - 1 - (\text{mr}_+(G_1) + \text{mr}_+(G_2)) \\
 &= M_+(G) \quad (\text{by [2]}). \quad \square
 \end{aligned}$$

So, any such graph for which $T > M_+$ will need to be at least 2-connected.

What about the connection between T and P for general graphs? It is clear that $T \leq P$, but when does $T = P$ for a given graph? For example, if G is outerplanar and $P = T$, then we may conclude that

$$P(G) = T(G) = M_+(G) = M(G).$$

However, even for outerplanar graphs, we need not expect equality between the parameters T and P . Consider the example of the 5-sun, see Figure 4.2. In this case, $T = 2$ but $P = 3$. Even if we restrict further to the 2-connected case, then there

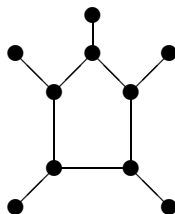


FIG. 4.2. *5-sun*.

does exist an outerplanar 2-connected graph for which $T < P$. Consider the graph in Figure 4.3. For this graph, it can be verified that $P = M = 4$ and that $T = M_+ = 3$.

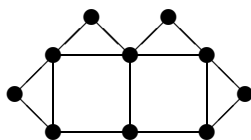


FIG. 4.3. *2-connected outerplanar graph with $P > T$.*

Acknowledgment. This research began at the Banff International Research Station during a Research in Teams program in September of 2009, and the authors thank BIRS for their support.

REFERENCES

- [1] F. Barioli, S. Fallat, and L. Hogben. On the difference between the maximum multiplicity and path cover number for tree-like graphs. *Linear Algebra Appl.*, 409:13–31, 2005.
- [2] M. Booth, P. Hackney, B. Harris, C.R. Johnson, M. Lay, L.H. Mitchell, S.K. Narayan, A. Pascoe, K. Steinmetz, B.D. Sutton, and W. Wang. On the minimum semidefinite rank of a simple graph. Submitted.
- [3] M. Booth, P. Hackney, B. Harris, C.R. Johnson, M. Lay, L.H. Mitchell, S.K. Narayan, A. Pascoe, K. Steinmetz, B.D. Sutton, and W. Wang. On the minimum rank among positive semidefinite matrices with a given graph. *SIAM J. Matrix Anal. Appl.*, 30(2):731–740, 2008.
- [4] S. Fallat and L. Hogben. The minimum rank of symmetric matrices described by a graph: a survey. *Linear Algebra Appl.*, 426:558–582, 2007.
- [5] AIM Minimum Rank-Special Graphs Work Group. Zero forcing sets and the minimum rank of graphs. *Linear Algebra Appl.*, 428:1628–1648, 2008.
- [6] P. Hackney, B. Harris, M. Lay, L.H. Mitchell, S.K. Narayan, and A. Pascoe. Linearly independent vertices and minimum semidefinite rank. *Linear Algebra Appl.*, 431:1105–1115, 2009.

- [7] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1990.
- [8] Y. Jiang, L.H. Mitchell and S.K. Narayan. Unitary matrix digraphs and minimum semidefinite rank. *Linear Algebra Appl.*, 428:1685–1695, 2008.
- [9] C.R. Johnson and A.L. Duarte. The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree. *Linear and Multilinear Algebra*, 46:139–144, 1999.
- [10] L.H. Mitchell, S.K. Narayan, and A.M. Zimmer. Lower bounds in minimum rank problems. *Linear Algebra Appl.*, 432:430–440, 2010.
- [11] J. Sinkovic. Maximum nullity of outerplanar graphs and the path cover number. *Linear Algebra Appl.*, 432:2052–2060, 2010.
- [12] H. van der Holst. Graphs whose positive semi-definite matrices have nullity at most two. *Linear Algebra Appl.*, 375:1–11, 2003.
- [13] D.B. West. *Introduction to Graph Theory*. Prentice Hall Inc., Upper Saddle River, NJ, 1996.