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MORE ON POLYNOMIAL BEZOUTIANS WITH RESPECT TO A GENERAL BASIS*

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Abstract. We use unified algebraic methods to investigate the properties of polynomial Bezoutians with respect to a general basis. Not only can three known results be easily verified, but also some new properties of polynomial Bezoutians are obtained. Nonsymmetric Lyapunov-type equations of polynomial Bezoutians are also discussed. It turns out that most properties of classical Bezoutians can be analogously generalized to the case of polynomial Bezoutians in the framework of algebraic methods.

Key words. Polynomial Bezoutians, General basis, Algebraic methods.

AMS subject classifications. 15A57.

1. Introduction. We denote by $\mathbb{C}_n[x]$ the linear space of complex polynomials with degree at most $n-1$. Let $\pi(x) = (1, x, \ldots, x^{n-1})$ and $Q(x) = (Q_0(x), Q_1(x), \ldots, Q_{n-1}(x))$ with $\deg Q_k(x) = k$ be vectors of the standard power basis and the general polynomial basis of $\mathbb{C}_n[x]$, respectively. Given a pair of polynomials $p(x)$ and $q(x)$ with $\deg p(x) = n$, $\deg q(x) \le n$, we call matrices $B(p,q) = (b_{ij})$ and $B_Q(p,q) = (c_{ij})$ determined by the bilinear form

$$R(x,y) = \frac{p(x)q(y) - p(y)q(x)}{x-y} = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} b_{ij}x^i y^j = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} c_{ij}Q_i(x)Q_j(y) \tag{1.1}$$

the (*classical*) *Bezoutian* and the *polynomial Bezoutian* of $p(x)$ and $q(x)$ with respect to $\pi(x)$ and $Q(x)$, respectively. It is easily seen that the sequence $Q(x)$ includes the standard power basis, Chebyshev polynomials, Newton polynomials and polynomial sequences of interpolatory type [17] as its special cases. These special polynomials appear frequently in approximation theory and interpolation problems.

The study of Bezoutians has a long history and has been an active field of research. Such matrices occur in a large variety of areas in pure and applied mathematics. For

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example, they often have connections with some structured matrices, such as Hankel, Toeplitz, and Vandermonde matrices, etc., and therefore have a lot of significant characteristic properties. A more detailed expansion can be found in the books of Heinig and Rost [9] and Lancaster and Tismenetsky [11]. On the other hand, Bezoutians have many applications in the theory of equations, system and control theory, etc., we refer the reader to the survey article of Helmke and Fuhrmann [10] and the book of Barnett [1] and the references therein. Recently the (classical) Bezoutian has been generalized to some other forms, in which the polynomial Bezoutian is an important direction of the research (e.g., see [3, 4, 7, 12, 13, 14, 18, 19]). At the same time we have observed that in the recent work of Helmke and Fuhrmann [10], Fuhrmann and Datta [6], Mani and Hartwig [13], and Yang [18], etc., some properties of Bezoutians and their relation to system theoretic problems were derived by using operator approach and viewing the Bezoutian as a matrix representation of a certain operator in the dual bases. While in the book of Heinig and Rost [9], a comprehensive discussion for the properties of classical Bezoutians was presented by using the methods of generating function and matrix algebra.

From definition (1.1), it is easy to see that polynomial Bezoutian preserves some elementary properties of classical Bezoutian, such as $B_Q(p,q)$ is symmetric, bilinear in p and q and satisfies $B_Q(p,q) = -B_Q(q,p)$. To present more properties of polynomial Bezoutians in this note we restrict ourselves to the methods of generating functions and matrix algebras. In the framework of unified algebraic methods, we can carry out an in-depth study for polynomial Bezoutians.

Let's first introduce some notation associated with polynomial Bezoutians. Note that (1.1) may be written simply in matrix form, i.e.,

$$R(x,y) = \pi(x)B(p,q)\pi(y)^t = Q(x)B_Q(p,q)Q(y)^t, \tag{1.2}$$

where superscript t denotes the transpose of a vector or a matrix throughout the paper. In particular, for $q(x) = 1$ and any polynomial $p(x)$ of degree n:

$$p(x) = \sum_{k=0}^{n} p_k x^k = \sum_{k=0}^{n} \theta_k Q_k(x), \tag{1.3}$$

we have the difference quotient form

$$D_p(x,y) = \frac{p(x)-p(y)}{x-y} = \pi(x)B(p,1)\pi(y)^t = Q(x)B_Q(p,1)Q(y)^t,$$

where

$$S(p) = B(p,1) = \begin{pmatrix} p_1 & p_2 & \cdots & \cdots & p_n \\ p_2 & p_3 & \dots & p_n & 0 \\ \vdots & \vdots & & & \vdots \\ p_n & 0 & \dots & \dots & 0 \end{pmatrix} \tag{1.4}$$

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and

$$S_Q(p) = B_Q(p, 1) \tag{1.5}$$

are called the *symmetrizer* [11] and the *generalized symmetrizer* [18] of $p(x)$ with respect to $\pi(x)$ and $Q(x)$, respectively. It can be seen that $S_Q(p)$ is (left) upper triangular and is congruent to $S(p)$ (see Lemma 2.2 below). In the case of $Q(x) = \pi(x)$, (1.5) degenerates to (1.4), thus, $S_Q(p)$ is a generalization of $S(p)$. The symmetrizer has some connections with the Barnett factorization and the triangular factorization of Bezoutians.

For a sequence of polynomials $Q_0(x)$, $Q_1(x)$, $\dots$, $Q_n(x)$ with deg $Q_k(x) = k$, we can assume that they satisfy the following relations

$$Q_0(x) = \alpha_0, Q_k(x) = \alpha_k x Q_{k-1}(x) - \sum_{i=1}^{k} a_{k-i,k} Q_{k-i}(x), \quad k = 1, 2, \dots, n, \tag{1.6}$$

where α_k, $a_{k-i,k}$ $(i = 1, \dots, k, k = 1, \dots, n)$ are uniquely determined by $Q_0(x), Q_1(x), \dots, Q_n(x)$ and α_k are not zeros. For polynomial $p(x)$, its (second) *companion matrix* $C(p)$ with respect to $\pi(x)$ and *confederate matrix* $C_Q(p)$ with respect to $Q(x)$ are defined as follows

$$C(p) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -p_0/p_n \\ 1 & 0 & \dots & 0 & -p_1/p_n \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -p_{n-1}/p_n \end{pmatrix}, \tag{1.7}$$

$$C_Q(p) = \begin{pmatrix} a_{01}/\alpha_1 & a_{02}/\alpha_2 & a_{03}/\alpha_3 & \cdots & \frac{1}{\alpha_n}(a_{0n} - \theta_0/\theta_n) \\ 1/\alpha_1 & a_{12}/\alpha_2 & a_{13}/\alpha_3 & \cdots & \frac{1}{\alpha_n}(a_{1n} - \theta_1/\theta_n) \\ 0 & 1/\alpha_2 & a_{23}/\alpha_3 & \cdots & \frac{1}{\alpha_n}(a_{2n} - \theta_2/\theta_n) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1/\alpha_{n-1} & \frac{1}{\alpha_n}(a_{n-1,n} - \theta_{n-1}/\theta_n) \end{pmatrix}. \tag{1.8}$$

The companion (confederate) matrix has intertwining relations with the classical (polynomial) Bezoutian and Hankel (generalized Hankel) matrix (see [5],[14]). We note that the matrix in (1.8) is in Hessenberg form.

Among the properties of classical Bezoutians there exist three well-known and important results. They are the Barnett factorization formula (see [2, 8]):

$$B(p, q) = S(p) q(C(p)^t), \tag{1.9}$$

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the intertwining relation with the companion matrix $C(p)$ (see [5]):

$$B(p,q)C(p)^t = C(p)B(p,q), \tag{1.10}$$

and the Bezoutian reduction via the confluent Vandermonde matrix (see [16]):

$$V(p)^t B(p,q)V(p) = \mathrm{diag}\Big[R_{n_i} p_i(J_{x_i}) q(J_{x_i})\Big]_{i=1}^{r}, \tag{1.11}$$

where $p_i(x) = p(x)/(x-x_i)^{n_i}$ $(1 \le i \le r)$ and

$$V(p)^t = \mathrm{col}[V(x_i)]_{i=1}^{r}, \quad V(x_i) = \mathrm{col}\Big[\frac{1}{j!}\pi^{(j)}(x_i)\Big]_{j=0}^{n_i-1}, \tag{1.12}$$

in which x_i are the zeros of $p(x)$ with multiplicities $n_i (i = 1,\dots,r, n_1+\cdots+n_r = n)$, and

$$R_{n_i} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}, \quad J_{x_i} = \begin{pmatrix} x_i & 1 & 0 & \cdots & 0 \\ 0 & x_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & x_i \end{pmatrix} \tag{1.13}$$

stand for the reflection matrix and the Jordan block of order $n_i \times n_i$ corresponding to x_i, respectively. In particular, if $p(x)$ has only simple zeros $x_1,\dots,x_n$, then (1.11) degenerates to

$$V(p)^t B(p,q)V(p) = \mathrm{diag}\Big[p'(x_i)q(x_i)\Big]_{i=1}^{n},$$

where $V(p) = (x_j^{i-1})_{i,j=1}^{n}$ is the classical Vandermonde matrix corresponding to $p(x)$. Equations (1.9) and (1.10) have theoretical meanings, while (1.11) is often used in root localization problems.

The outline of this note is as follows. In Section 2 we use the pure algebraic methods to re-derive three main results in [18] for polynomial Bezoutians; in contrast to operator approaches, are proofs are simpler and easier. Section 3 is devoted to investigation of some other properties of polynomial Bezoutians, they are mainly the generalizations of some results in [9]. In Section 4 we consider nonsymmetric Lyapunov-type equations of polynomial Bezoutians, which extend some results of Pták [15]. From the point of view of the theory of displacement structures, such equations can be seen as the displacement structures of the polynomial Bezoutians.

2. New proofs of three known results. Henceforth, let $T = [t_{ij}]_{i,j=1}^{n}$ be the transformation matrix from the standard power basis $\pi(x)$ to the general polynomial basis $Q(x)$, i.e.,

$$Q(x) = \pi(x)T. \tag{2.1}$$

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The matrix T in (2.1) is indeed a basis transformation matrix. Here we want to use it to give an in-depth characterization of polynomial Bezoutians by (2.1).

We begin by establishing a similarity relation between $C(p)$ and $C_Q(p)$ and a congruence relation between Bezoutians $B(p,q)$ and $B_Q(p,q)$.

LEMMA 2.1. *Let polynomial $p(x) = \sum_{k=0}^{n} p_k x^k = \sum_{k=0}^{n} \theta_k Q_k(x)$ be of degree n. Then the companion matrix $C(p)$ and the confederate matrix $C_Q(p)$ are related by the similarity equation*

$$C_Q(p) = T^{-1}C(p)T. \tag{2.2}$$

Proof. In terms of (1.6) and (1.8), it is easy to verify the equality

$$xQ(x) - Q(x)C_Q(p) = p(x)[0, \dots, 0, 1/(\alpha_n\theta_n)].$$

Thus

$$xQ(x) = Q(x)C_Q(p) \bmod p(x).$$

By means of $Q(x) = \pi(x)T$, we deduce $x\pi(x) = \pi(x)TC_Q(p)T^{-1} \bmod p(x)$. Thereby

$$\pi(x)C(p) = \pi(x)TC_Q(p)T^{-1} \bmod p(x).$$

The last equation is equivalent to $C(p) = TC_Q(p)T^{-1}$, or $C_Q(p) = T^{-1}C(p)T$. ◻

LEMMA 2.2. *The Bezoutians $B(p,q)$ and $B_Q(p,q)$ are related by the congruence relation*

$$TB_Q(p,q)T^t = B(p,q). \tag{2.3}$$

In particular,

$$TS_Q(p)T^t = S(p). \tag{2.4}$$

Proof. Substituting $Q(x) = \pi(x)T$ into (1.2), we have

$$\pi(x)B(p,q)\pi(y)^t = \pi(x)TB_Q(p,q)T^t\pi(y)^t.$$

Since $\pi(x)$ is a basis, thus (2.3) is deduced. If $q(x) = 1$, (2.4) is derived. ◻

REMARK 2.3. The congruence relationship in Lemma 2.2 implies that $B_Q(p,q)$ has the same inertia or signature as $B(p,q)$. Therefore the classical Hermite-Fujiwara and Routh-Hurwitz inertia and stability criteria can be generalized to the case of

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polynomial Bezoutians. For example, for the case of the polynomial Bezoutian of interpolatory type, we refer to [19].

It is well known that Bezoutian $B(p,q)$ is invertible if and only if p and q are coprime. On the other hand, from Lemma 2.2 we know that $B(p,q)$ is invertible if and only if $B_Q(p,q)$ is invertible. Therefore we deduce immediately the following result.

Corollary 2.4. *The polynomial Bezoutian $B_Q(p,q)$ is invertible if and only if p and q are coprime.*

To this end we pure use algebraic methods to derive afresh three main results in [18] obtained by the third author. These results are the generalized Barnett formula, the intertwining relation between polynomial Bezoutian and confederate matrix, and the generalized Bezoutian reduction via polynomial Vandermonde matrix. Comparing with operator approach used there the algebraic methods are easier.

Proposition 2.5. ([18]) *Assume that the matrices $B_Q(p,q)$, $S_Q(p)$, $C_Q(p)$ are defined as before. Then the generalized Barnett formula is satisfied:*

$$B_Q(p,q) = S_Q(p)q\Big(C_Q(p)^t\Big),$$

where $q(A)$ denotes the matrix polynomial q in matrix A.

Proof. In terms of Lemma 2.2, (1.8), the symmetry of Bezoutians $B(p,q)$ and $B_Q(p,q)$, and Lemma 2.1 successively, we have

$$\begin{aligned} B_Q(p,q) &= T^{-1}B(p,q)(T^{-1})^t = T^{-1}S(p)q(C(p)^t)(T^{-1})^t \\ &= T^{-1}S(p)(T^t)^{-1}T^t q(C(p)^t)(T^{-1})^t = S_Q(p)q\Big(T^tC(p)^t(T^{-1})^t\Big) \\ &= S_Q(p)q\Big(T^{-1}C(p)T\Big)^t = S_Q(p)q\Big(C_Q(p)^t\Big), \end{aligned}$$

which finishes the proof. ◻

Proposition 2.6. ([18]) *The polynomial Bezoutian matrix $B_Q(p,q)$ and the confederate matrix $C_Q(p)$ satisfy the following intertwining relation:*

$$B_Q(p,q)C_Q(p)^t = C_Q(p)B_Q(p,q).$$

Proof. In terms of Lemmas 2.2 and 2.1 and (1.10), we have

$$\begin{aligned} B_Q(p,q)C_Q(p)^t &= T^{-1}B(p,q)(T^{-1})^tT^tC(p)^t(T^{-1})^t \\ &= T^{-1}B(p,q)C(p)^t(T^{-1})^t = T^{-1}C(p)B(p,q)(T^{-1})^t \\ &= T^{-1}TC_Q(p)T^{-1}TB_Q(p,q)T^t(T^{-1})^t \\ &= C_Q(p)B_Q(p,q), \end{aligned}$$

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which completes the proof. □

Proposition 2.7. ([18]) *Let $V_Q(p)$ defined by*

$$V_Q(p)^t = \mathrm{col}[V_Q(x_i)]_{i=1}^r, \quad V_Q(x_i) = \mathrm{col}\Big[\frac{1}{j!}Q^{(j)}(x_i)\Big]_{j=0}^{n_i-1} \tag{2.5}$$

be the polynomial Vandermonde matrix corresponding to $p(x) = \prod_{i=1}^r (x - x_i)^{n_i}$ and the polynomial basis $Q(x)$. Then $B_Q(p, q)$ can be reduced by $V_Q(p)$:

$$V_Q(p)^t B_Q(p, q)V_Q(p) = \mathrm{diag}\Big[R_{n_i} p_i(J_{x_i})q(J_{x_i})\Big]_{i=1}^r, \tag{2.6}$$

where $p_i(x) = p(x)/(x - x_i)^{n_i}$ and R_{n_i} and J_{x_i} are defined as in (1.13).

Proof. By taking jth derivatives at x_i and dividing by $j!$ on both sides of (2.1) ($i = 1, \ldots, r, j = 0, 1, \ldots, n_i - 1$), and combining all together in matrix form, we obtain

$$V_Q(p)^t = V(p)^t T, \tag{2.7}$$

where $V_Q(p)$ and $V(p)$ are defined by (2.5) and (1.11), respectively. By substitution of (2.7) and (2.3) into the left side of (1.11), (2.6) is immediately deduced. □

With the help of (2.7), we note that the well known formula

$$C(p)^t V(p) = V(p)J,$$

where $J = \mathrm{diag}(J_{x_i})_{i=1}^r$ is the Jordan matrix, can be extended to the general polynomial case:

$$C_Q(p)^t V_Q(p) = V_Q(p)J. \tag{2.8}$$

Indeed, post-multiply both sides of the equation $V(p)^t C(p) = J^t V(p)^t$ by T, and rewrite it as the form

$$V(p)^t T T^{-1} C(p)T = J^t V(p)^t T.$$

In terms of (2.7) and (2.3), we conclude $V_Q(p)^t C_Q(p) = J^t V_Q(p)^t$, which is equivalent to (2.8).

3. Some new properties. In this section we mainly investigate some other properties of polynomial Bezoutians, which could be viewed as the generalizations of some results in [9] and [10]. It turns out that polynomial Bezoutians preserve most properties of classical Bezoutians.

First, the generalized Barnett's formula implies the following two results which are the generalizations of Propositions 2.10 and 2.11 in [9], respectively.

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PROPOSITION 3.1. *Suppose that polynomials* p, f, g *satisfy* $\deg fg \leq \deg p = n$. *Then*

$$B_Q(p, fg) = B_Q(p, f)S_Q(p)^{-1}B_Q(p, g).$$

Proof. In view of Proposition 2.5, we deduce that

$$B_Q(p, fg) = S_Q(p)f[C_Q(p)^t]S_Q(p)^{-1}S_Q(p)g[C_Q(p)^t] = B_Q(p, f)S_Q(p)^{-1}B_Q(p, g).$$

This completes the proof. □

For convenience, with the help of reflection matrix R_n, we introduce the so-called *generalized reflection matrix* R_Q with respect to $Q(x)$, which is defined by

$$R_Q := T^t R_n T. \tag{3.1}$$

Since R_n is a symmetric matrix, then $R_Q^t = R_Q$.

We have a further consequence of Proposition 2.5 as follows.

PROPOSITION 3.2. *Assume that* $p(x)$ *an* $q(x)$ *are polynomials of degree* n. *Then*

$$B_Q(p, q) = [C_Q(p)^n - C_Q(q)^n]N,$$

where $N = S_Q(p)R_QS_Q(q) = S_Q(q)R_QS_Q(p)$.

Proof. We check this directly instead of using the generalized Barnett's formula. In view of Prop. 2.11 in [9] we have

$$B(p, q) = [C(p)^n - C(q)^n]M,$$

where $M = S(p)R_nS(q) = S(q)R_nS(p)$. Using Lemmas 2.2 and 2.1, we get

$$TB_Q(p, q)T^t = T[C_Q(p)^n - C_Q(q)^n]T^{-1}TS_Q(p)T^tR_nTS_Q(q)T^t.$$

Eliminating T and T^t in both sides in the last equality and using (3.1), the assertion is deduced. □

Bezoutians have some interesting triangular factorizations, where are summed up by Helmke and Fuhrmann in [10]. Now we extend these factorizations for polynomial Bezoutians. In the sequel let

$$\widehat{a}(x) = x^n a(x^{-1})$$

denote the *reciprocal polynomial* of $a(x)$ and $S_{\widehat{Q}}(\widehat{a})$ stand for the generalized symmetrizer of $\widehat{a}$ with respect to the polynomial sequence $\widehat{Q}(x) := \pi(x)(T^t)^{-1}$. In terms of (2.4), we can write

$$S_{\widehat{Q}}(\widehat{a}) = [(T^t)^{-1}]^{-1}S(\widehat{a})(T^{-1})^{-1} = T^tS(\widehat{a})T. \tag{3.2}$$

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Now we give a generalization of Proposition 5.1 in [10], see also [11, Chap.13].

Proposition 3.3. *Assume that $p(x)$ and $q(x)$ are polynomials of degree n. Then $B_Q(p,q)$ has the following representations:*

$$\begin{aligned} B_Q(p,q) &= [S_Q(p)S_{\widehat{Q}}(\widehat{q}) - S_Q(q)S_{\widehat{Q}}(\widehat{p})]R_Q^{-1} \\ &= -R_Q^{-1}[S_{\widehat{Q}}(\widehat{p})S_Q(q) - S_{\widehat{Q}}(\widehat{q})S_Q(p)]. \end{aligned} \tag{3.3}$$

Proof. We only verify the first equality in (3.3), the second is similarly derived. From Theorem 5.1 in [10] and $R_n = R_n^{-1}$, we have

$$B(p,q) = S(p)S(\widehat{q})R_n^{-1} - S(q)S(\widehat{p})R_n. \tag{3.4}$$

Multiplying on the left side and the right side in (3.4) by T^{-1} and $(T^t)^{-1}$, respectively, and using Lemma 2.2, one can obtain

$$\begin{aligned} B_Q(p,q) = {} & T^{-1}S(p)(T^t)^{-1}T^tS(\widehat{q})TT^{-1}R_n^{-1}(T^t)^{-1} \\ & - T^{-1}S(q)(T^t)^{-1}T^tS(\widehat{p})TT^{-1}R_n(T^t)^{-1}. \end{aligned}$$

In view of Lemma 2.2 and equations (3.2) and (3.1), the last equality is equivalent to the first equality in (3.3). ∎

A class of matrices close to the Bezoutians is the class of resultant matrices. The resultant matrices have many applications in describing the kernel of Bezoutians and the connections between the inertias of Bezoutians and polynomials. Heinig and Rost described some connections between these two classes in [9]. To this end we will generalize such relations to the general polynomial case.

Assume that $a(x) = \sum_{k=0}^{n} a_k x^k$ and $b(x) = \sum_{k=0}^{m} b_k x^k$ are of degree n and m, respectively. The *resultant matrix* of $a(x)$ and $b(x)$ is of the form

$$\operatorname{Res}(a,b) = \begin{pmatrix} a_0 & a_1 & \cdots & a_n & & \\ & \ddots & \ddots & & \ddots & \\ & & a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_m & & \\ & \ddots & \ddots & & \ddots & \\ & & b_0 & b_1 & \cdots & b_m \end{pmatrix}_{m+n}. \tag{3.5}$$

Now we introduce the so-called *generalized resultant matrix*. For the sake of simplicity, in the sequel, we assume that polynomials $a(x)$ and $b(x)$ are all of degree n. Since $R_n = R_n^{-1}$, then the resultant matrix $\operatorname{Res}(a,b)$ can be written in the form

$$\operatorname{Res}(a,b) = \begin{pmatrix} S(\widehat{a})R_n^{-1} & R_nS(a) \\ S(\widehat{b})R_n^{-1} & R_nS(b) \end{pmatrix}_{2n}, \tag{3.6}$$

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where $S(a)$ and $S(\widehat{a})$ denote the symmtrizers of $a(x)$ and $\widehat{a}(x) = x^n a(x^{-1})$, respectively. We call the matrix defined by

$$\begin{aligned} \mathrm{Res}_Q(a,b) &:= \begin{pmatrix} T^t & 0 \\ 0 & T^t \end{pmatrix} \mathrm{Res}(a,b) \begin{pmatrix} (T^t)^{-1} & 0 \\ 0 & (T^t)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} T^t S(\widehat{a}) T T^{-1} R_n^{-1} (T^t)^{-1} & T^t R_n T T^{-1} S(a) (T^t)^{-1} \\ T^t S(\widehat{b}) T T^{-1} R_n^{-1} (T^t)^{-1} & T^t R_n T T^{-1} S(b) (T^t)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} S_{\widehat{Q}}(\widehat{a}) R_Q^{-1} & R_Q S_Q(a) \\ S_{\widehat{Q}}(\widehat{b}) R_Q^{-1} & R_Q S_Q(b) \end{pmatrix} \end{aligned} \tag{3.7}$$

the generalized resultant matrix of $a(x)$ and $b(x)$ with respect to $Q(x)$.

For the sake of such definition, we mainly have two considerations. One is, (3.7) has the same form as (3.5) and in the case of $Q(x) = \pi(x)$, $\mathrm{Res}_Q(a,b)$ degenerates to $\mathrm{Res}(a,b)$. The other, which we want to emphasize, is the representation (3.7) has many advantages and applications.

The following two results establish the connections between the generalized resultant matrix and the polynomial Bezoutian. They are the generalizations of Propositions 2.12 and 2.13 in [9] (see also [10, Th.5.2]).

Proposition 3.4. *Assume that $a(x) = \sum_{k=0}^n a_k x^k$ and $b(x) = \sum_{k=0}^n b_k x^k$ are all of degree n. Then the following holds:*

$$\mathrm{Res}_Q(a,b) = L_l \begin{bmatrix} B_Q(a,b) & 0 \\ 0 & I_n \end{bmatrix} L_r, \tag{3.8}$$

where

$$L_l = \begin{bmatrix} 0 & I_n \\ S_Q(a)^{-1} & R_Q S_Q(b) S_Q(a)^{-1} R_Q^{-1} \end{bmatrix},$$

$$L_r = \begin{bmatrix} I_n & 0 \\ -R_Q S_Q(a) C_{\widehat{Q}}(a)^n & R_Q S_Q(a) \end{bmatrix},$$

and $C_{\widehat{Q}}(a) := T^t C(a) (T^t)^{-1}$ is the confederate matrix of $a(x)$ with respect to $\widehat{Q}(x)$.

Proof. In terms of Prop. 2.12 in [9], we have

$$\mathrm{Res}(a,b) = P \begin{bmatrix} B(a,b) & 0 \\ 0 & I_n \end{bmatrix} Q, \tag{3.9}$$

where

$$P = \begin{bmatrix} 0 & I_n \\ S(a)^{-1} & R_n S(b) S(a)^{-1} R_n^{-1} \end{bmatrix}, \quad Q = \begin{bmatrix} I_n & 0 \\ -R_n S(a) C(a)^n & R_n S(a) \end{bmatrix}.$$

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Multiplying on the left side by $\mathrm{diag}(T^t, T^t)$ and the right side by $\mathrm{diag}((T^t)^{-1}, (T^t)^{-1})$ of (3.9), respectively, and writing the middle block matrix in (3.9) as

$$\begin{bmatrix} B(a,b) & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & (T^t)^{-1} \end{bmatrix} \begin{bmatrix} B_Q(a,b) & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} T^t & 0 \\ 0 & T^t \end{bmatrix},$$

after elementary computation, we get

$$\begin{aligned} \mathrm{Res}_Q(a,b) = &\begin{bmatrix} 0 & I_n \\ S_Q(a)^{-1} & R_Q S_Q(b) S_Q(a)^{-1} R_Q^{-1} \end{bmatrix} \times \\ &\begin{bmatrix} B_Q(a,b) & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -R_Q S_Q(a) C_{\widehat{Q}}(a)^n & R_Q S_Q(a) \end{bmatrix}, \end{aligned}$$

which is equal to (3.8). This completes the proof. □

Proposition 3.5. *With notation defined as above, we have*

$$\mathrm{Res}_Q(a,b)^t \begin{bmatrix} 0 & R_Q^{-1} \\ R_Q^{-1} & 0 \end{bmatrix} \mathrm{Res}_Q(a,b) = \begin{bmatrix} 0 & B_Q(a,b) \\ B_Q(a,b) & 0 \end{bmatrix}. \tag{3.10}$$

Proof. By (2.24) in Prop. 2.14 in [9], we have

$$\mathrm{Res}(a,b)^t \begin{bmatrix} 0 & R_n \\ R_n & 0 \end{bmatrix} \mathrm{Res}(a,b) = \begin{bmatrix} 0 & B(a,b) \\ B(a,b) & 0 \end{bmatrix}.$$

In view of $R_n = R_n^{-1}$ and (2.3), the last equality is equivalent to

$$\begin{aligned} &\mathrm{Res}(a,b)^t \begin{bmatrix} 0 & R_n^{-1} \\ R_n^{-1} & 0 \end{bmatrix} \mathrm{Res}(a,b) \\ &= \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} 0 & B_Q(a,b) \\ B_Q(a,b) & 0 \end{bmatrix} \begin{bmatrix} T^t & 0 \\ 0 & T^t \end{bmatrix}. \end{aligned} \tag{3.11}$$

By (3.7) this implies htat

$$\mathrm{Res}_Q(a,b)^t \begin{bmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix} \begin{bmatrix} 0 & R_n^{-1} \\ R_n^{-1} & 0 \end{bmatrix} \begin{bmatrix} (T^t)^{-1} & 0 \\ 0 & (T^t)^{-1} \end{bmatrix} \mathrm{Res}_Q(a,b)$$

equals

$$\begin{bmatrix} 0 & B_Q(a,b) \\ B_Q(a,b) & 0 \end{bmatrix}.$$

Using (3.1), we immediately deduce the assertion (3.10), and the proof is complete. □

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REMARK 3.6. Comparing Propositions 3.5 and 3.3 with Propositions 2.14 in [9] and 5.1 in [10], respectively, we think that the following another representations of similar results on classical Bezoutians might be more natural and suitable. That is,

$$\mathrm{Res}(a,b)^t \begin{bmatrix} 0 & R_n^{-1} \\ R_n^{-1} & 0 \end{bmatrix} \mathrm{Res}(a,b) = \begin{bmatrix} 0 & B(a,b) \\ B(a,b) & 0 \end{bmatrix}$$

and

$$\begin{aligned} B(p,q) = & \quad [S(p)S(\widehat{q}) - S(q)S(\widehat{p})]R_n^{-1} \\ = & \quad -R_n^{-1}[S(\widehat{p})S(q) - S(\widehat{q})S(p)]. \end{aligned}$$

The causation is just the equality $R_n = R_n^{-1}$.

For further discussions on the connections between polynomial Bezoutians and generalized resultant matrices, we introduce an $n \times (n+r)$ matrix operator

$$D_n(f) := \begin{pmatrix} f_0 & f_1 & \cdots & f_r & & \\ & \ddots & \ddots & & \ddots & \\ & & f_0 & f_1 & \cdots & f_r \end{pmatrix}_{n\times(n+r)}$$

for the polynomial $f(x) = \sum_{i=0}^{r} f_i x^i$ of degree $r (r \leq n-1)$.

First we need the following assertion which can be viewed as a extended property of the generating function (see [9, Prop.1.8]).

PROPOSITION 3.7. *Let $Q(x)$ be the general polynomial sequence defined as before, $C \in \mathbb{C}^{n\times n}$ and $\widetilde{C} \in \mathbb{C}^{(n-r)\times(n-s)}$. If there exist polynomials $a(x)$ and $b(x)$ with $\deg a = r, \deg b = s$ satisfying the condition*

$$C_Q(x,y) := Q(x)CQ(y)^t = a(x)\widetilde{Q}_r(x)\widetilde{C}\widetilde{Q}_s(y)^t b(y), \tag{3.12}$$

in which $\widetilde{Q}_r(x) = (Q_0(x), Q_1(x), \ldots, Q_{n-r-1}(x))$, then

$$C = W_{n-r}(a)^t \widetilde{C} W_{n-s}(b),$$

where

$$W_{n-r}(a) := T_{n-r}^t D_{n-r}(a)(T^t)^{-1} \tag{3.13}$$

and T_{n-r} is the $(n-r)$th leading submatrix of matrix T.

Proof. By the definition of $\widetilde{Q}_r(x)$, we have $\widetilde{Q}_r(x) = (1, x, \ldots, x^{n-r-1})T_{n-r}$. Thus

$$\begin{aligned} & a(x)\widetilde{Q}_r(x)\widetilde{C}\widetilde{Q}_s(y)^t b(y) \\ &= a(x)(1, x, \ldots, x^{n-r-1})T_{n-r}\widetilde{C}T_{n-s}^t(1, y, \ldots, y^{n-s-1})^t b(y) \\ &= \pi(x)D_{n-r}(a)^t T_{n-r}\widetilde{C}T_{n-s}^t D_{n-s}(b)\pi(y)^t \\ &= Q(x)T^{-1}D_{n-r}(a)^t T_{n-r}\widetilde{C}T_{n-s}^t D_{n-s}(b)(T^t)^{-1}Q(y)^t \\ &= Q(x)W_{n-r}(a)^t\widetilde{C}W_{n-r}(b)Q(y)^t, \end{aligned}$$

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where $W_{n-r}(a) = T_{n-r}^t D_{n-r}(a)(T^t)^{-1}$. Since $Q(x)$ is a basis sequence, (3.12) implies the assertion. ∎

From Lemma 3.7 we gain two interesting properties, which give the representations of Bezoutian $B_Q(p,q)$ as a product of generalized resultant matrices with polynomial Bezoutians of factors. They are the generalizations of Propositions 2.6 and 2.14 in [9], respectively.

Proposition 3.8. *If* $p(x) = \tilde{p}(x)d(x), q(x) = \tilde{q}(x)d(x)$ *with* $\deg q \leq \deg p = n$, $\deg d = r$ *and* $(\tilde{p}, \tilde{q}) = 1$. *Then*

$$B_Q(p,q) = W_{n-r}(d)^t B_Q(\tilde{p}, \tilde{q}) W_{n-r}(d).$$

Proof. Writing

$$\frac{p(x)q(y) - p(y)q(x)}{x-y} = d(x)\frac{\tilde{p}(x)\tilde{q}(y) - \tilde{p}(y)\tilde{q}(x)}{x-y}d(y),$$

one can deduce that

$$Q(x)B_Q(p,q)Q(y)^t = d(x)\widetilde{Q}_r(x)B_Q(\tilde{p},\tilde{q})\widetilde{Q}_r(y)^t d(y).$$

From Lemma 3.7, the conclusion is immediately deduced. ∎

Proposition 3.9. *Suppose that the polynomials* $a(x) = a_1(x)a_2(x), b(x) = b_1(x)b_2(x)$ *are all of degree* n *and satisfy* $\deg a_i = \deg b_i = n_i$, $i = 1, 2$. *Then*

$$B_Q(a,b) = \widetilde{\mathrm{Res}}_Q(a_2, b_1)^t \begin{bmatrix} B_{\widetilde{Q}}(a_1, b_1) & 0 \\ 0 & B_{\overline{Q}}(a_2, b_2) \end{bmatrix} \widetilde{\mathrm{Res}}_Q(b_2, a_1), \tag{3.14}$$

where $\widetilde{Q}(x) = (Q_0(x), \ldots, Q_{n_1-1}(x))$ *and* $\overline{Q}(x) = (Q_0(x), \ldots, Q_{n_2-1}(x))$, *and*

$$\widetilde{\mathrm{Res}}_Q(b_2, a_1) := \begin{bmatrix} T_{n_1}^t & 0 \\ 0 & T_{n_2}^t \end{bmatrix} \mathrm{Res}(b_2, a_1)(T^t)^{-1}.$$

Proof. In view of $a(x) = a_1(x)a_2(x), b(x) = b_1(x)b_2(x)$, we evaluate

$$\begin{aligned}\frac{a(x)b(y) - a(y)b(x)}{x-y} &= a_2(x)\frac{a_1(x)b_1(y) - a_1(y)b_1(x)}{x-y}b_2(y) \\ &\quad + b_1(x)\frac{a_2(x)b_2(y) - a_2(y)b_2(x)}{x-y}a_1(y).\end{aligned}$$

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Thereby

$$\begin{aligned} &Q(x)B_Q(a,b)Q(y)^t \\ &= a_2(x)\widetilde{Q}(x)B_{\widetilde{Q}}(a_1,b_1)\widetilde{Q}(y)^t b_2(y) + b_1(x)\overline{Q}(x)B_{\overline{Q}}(a_2,b_2)\overline{Q}(y)^t a_1(y). \end{aligned}$$

By Lemma 3.7, we have

$$\begin{aligned} B_Q(a,b) &= W_{n_1}(a_2)^t B_{\widetilde{Q}}(a_1,b_1)W_{n_1}(b_2) + W_{n_2}(b_1)^t B_{\overline{Q}}(a_2,b_2)W_{n_2}(a_1) \\ &= \begin{bmatrix} W_{n_1}(a_2) \\ W_{n_2}(b_1) \end{bmatrix}^t \begin{bmatrix} B_{\widetilde{Q}}(a_1,b_1) & 0 \\ 0 & B_{\overline{Q}}(a_2,b_2) \end{bmatrix} \begin{bmatrix} W_{n_1}(b_2) \\ W_{n_2}(a_1) \end{bmatrix}. \end{aligned}$$

In view of (3.10), the last equality is equivalent to (3.14). □

To this end we will study two properties of polynomial Bezoutians on the translation and scalar multiplication transformations of variables.

Proposition 3.10. *Suppose that $p(x), q(x)$ are two polynomials of degrees n and m, respectively. Denote $p_\alpha = p_\alpha(x) = p(x+\alpha), q_\alpha = q_\alpha(x) = q(x+\alpha)$. Then*

$$B_Q(p_\alpha, q_\alpha) = V_Q(\alpha)B_Q(p,q)V_Q(\alpha)^t,$$

where $V_Q(\alpha) = T^{-1}V(\alpha)T$ with $V(\alpha) = \left[\begin{pmatrix} j \\ i \end{pmatrix}\alpha^{j-i}\right]_{i,j=0}^{n-1}$

Proof. Introduce linear transformation σ in $\mathbb{C}_n[x]$ such that

$$\sigma(f) = f(x+\alpha), \quad f \in \mathbb{C}_n[x].$$

It is easy to check that

$$\sigma\pi(x) = (1, x+\alpha, \ldots, (x+\alpha)^{n-1}) = \pi(x)V(\alpha),$$

where $V(\alpha) = \left[\begin{pmatrix} j \\ i \end{pmatrix}\alpha^{j-i}\right]_{i,j=0}^{n-1}$, with convention $\begin{pmatrix} j \\ i \end{pmatrix} = 0$ for $j < i$. In view of $Q(x) = \pi(x)T$, we deduce that $\sigma Q(x) = Q(x)T^{-1}V(\alpha)T$, which is equivalent to

$$Q(x+\alpha) = Q(x)V_Q(\alpha),$$

where $V_Q(\alpha) = T^{-1}V(\alpha)T$. Considering the generating function of $B_Q(p_\alpha, q_\alpha)$ as:

$$\begin{aligned} R(x,y) &= Q(x)B_Q(p_\alpha,q_\alpha)Q(y)^t \\ &= \frac{p(x+\alpha)q(y+\alpha) - q(x+\alpha)p(y+\alpha)}{(x+\alpha)-(y+\alpha)} \\ &= Q(x+\alpha)B_Q(p,q)Q(y+\alpha)^t \\ &= Q(x)V_Q(\alpha)B_Q(p,q)V_Q(\alpha)^t Q(y), \end{aligned}$$

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we obtain $B_Q(p_\alpha, q_\alpha) = V_Q(\alpha) B_Q(p,q) V_Q(\alpha)^t$. $\square$

Using a method similar that in the proof of Proposition 3.10, we can deduce the following result.

PROPOSITION 3.11. *Suppose that $p(x)$ and $q(x)$ are defined as before. Denote $p^\alpha = p^\alpha(x) = p(\alpha x), q^\alpha = q^\alpha(x) = q(\alpha x)$. Then*

$$B_Q(p^\alpha, q^\alpha) = \alpha \Lambda_Q(\alpha) B_Q(p,q) \Lambda_Q(\alpha)^t,$$

where $\Lambda_Q(\alpha) = T^{-1}\Lambda(\alpha)T$ with $\Lambda(\alpha) = \mathrm{diag}[\alpha^i]_{i,=0}^{n-1}$.

4. Nonsymmetric Lyapunov-type equations of polynomial Bezoutians. In this section we will investigate some characteristic properties of polynomial Bezoutians as solutions of some nonsymmetric Lyapunov-type equations. For convenience, from now on we assume that the polynomial sequence $Q(x)$ satisfies the recurrence relation:

$$Q_0(x) = \delta_0, Q_k(x) = \delta_k Q_0(x) + x\sum_{i=1}^{k} w_{i,k+1} Q_{i-1}(x), \quad k = 1, 2, \ldots, n. \tag{4.1}$$

We introduce matrix W_Q associated with the relation (4.1) as

$$W_Q = \begin{pmatrix} 0 & w_{12} & \cdots & w_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & w_{n-1,n} \\ 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{4.2}$$

Let Z designate the forward shift matrix of order n:

$$Z = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then matrices W_Q and Z are similar. We formulate this as follows.

LEMMA 4.1. *The matrix W_Q and the forward shift matrix Z of order n satisfy the similarity relation*

$$W_Q = T^{-1} Z T.$$

Hereafter T stands for the transition matrix from $\pi(x)$ to the sequence $Q(x)$ in (4.1).

Proof. Introduce a linear function σ on the linear space $\mathbb{C}_n[x]$:

$$\sigma(f) = \frac{f - f_0}{x}, \quad f = \sum_{k=0}^{n-1} f_k x^k \in \mathbb{C}_n[x].$$

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It is easy to verify

$$\sigma(1, x, \dots, x^{n-1}) = (1, x, \dots, x^{n-1})Z,$$

$$\sigma(Q_0(x), Q_1(x), \dots, Q_{n-1}(x)) = (Q_0(x), Q_1(x), \dots, Q_{n-1}(x))W_Q.$$

Considering the relation $Q(x) = \pi(x)T$, one can immediately obtain $W_Q = T^{-1}ZT$. ∎

We also need the following two results.

Lemma 4.2. *Let $p(x)$ be a polynomial of degree n and $\widehat{p}(x) = x^n p(x^{-1})$ be its reciprocal polynomial. Then we have the following relation*

$$\widehat{p}(W_Q)R_Q^{-1} = S_Q(p).$$

Proof. Direct verification implies $\widehat{p}(Z)R_n = S(p)$. By the formula $R_Q = T^t R_n T$ and Lemmas 4.1 and 2.2 we have

$$\begin{aligned}\widehat{p}(W_Q)R_Q^{-1} &= \widehat{p}(W_Q)T^{-1}R_n(T^{-1})^t = T^{-1}\widehat{p}(Z)TT^{-1}R_n(T^{-1})^t \\ &= T^{-1}\widehat{p}(Z)R_n(T^{-1})^t = T^{-1}S(p)(T^{-1})^t = S_Q(p).\end{aligned}$$

This proof is complete. ∎

The next Lemma comes from an Exercise in [11, Chapter 12, Section 3].

Lemma 4.3. *Let $A, B, G \in \mathbb{C}^{n\times n}$. If $\lambda\mu \neq 1$ for all $\lambda \in \sigma(A)$ and $\mu \in \sigma(B)$, then the matrix equation*

$$X - AXB = G$$

has a unique solution, where $\sigma(A)$ represents the spectrum of matrix A.

Now we generalize nonsymmetric Lyapunov-type equations in [15] of the form

$$X - ZXC(p)^t = W$$

to the polynomial case, where shift matrix Z and companion matrix $C(p)^t$ will be replaced by W_Q and $C_Q(p)^t$, respectively. The results obtained extend part results of [15, Propositions 2.3 and 2.6].

Proposition 4.4. *Let $p(x) = \sum_{k=0}^{n} p_k x^k$. Then the Bezoutian $B_Q(p, x^{k-1})$ is the unique solution of matrix equation*

$$X - W_Q X C_Q(p)^t = \widetilde{u}\widetilde{e}_k^t \quad (1 \le k \le n), \tag{4.3}$$

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where $\widetilde{u} = T^{-1}(p_1, \ldots, p_n)^t, \widetilde{e}_k = T^{-1}e_k$ *and* e_k *is the* k*th unit column vector.*

Proof. Let $N = [e_n, 0, \cdots, 0]$. From Lemma 4.3 it is obvious to see that the matrix equation

$$X - W_Q X C_Q(p)^t = T^{-1}N(T^{-1})^t$$

has a unique solution, and the solution is $R_Q^{-1} = T^{-1}R_n(T^{-1})^t$ by Lemmas 4.1 and 2.1. Direct calculation gives

$$(p_1, \ldots, p_n)^t e_k^t = \widehat{p}(Z)N[C(p)^t]^{k-1}, \quad 1 \leq k \leq n.$$

Thus, in terms of Lemma 4.1, we have

$$\widetilde{u}\widetilde{e}_k^t = \widehat{p}(W_Q)T^{-1}N(T^{-1})^t[C_Q(p)^t]^{k-1}, \quad 1 \leq k \leq n.$$

Therefore, in view of Lemma 4.2 and Proposition 2.5, (4.3) has a unique solution

$$X = \widehat{p}(W_Q)R_Q^{-1}[C_Q(p)^t]^{k-1} = S_Q(p)[C_Q(p)^t]^{k-1} = B_Q(p, x^{k-1}).$$

This completes the proof. ◻

Proposition 4.5. *With the aforementioned notation and a polynomial* $p(x)$ *of degree* $n-1$*, there exists nonzero polynomial* $q(x)$ *such that the Bezoutian* $B = B_Q(p, q)$ *is the unique solution of the equation*

$$X - W_Q X C_Q(p)^t = \widetilde{u}w^t, \tag{4.4}$$

for a certain nonzero column vector $w \in \mathbb{C}^n$.

Proof. Suppose first that there exists a polynomial $q(x) = \sum_{k=1}^n q_{k-1}x^{k-1} \neq 0$, such that

$$B = B_Q(p, q) = \sum_{k=1}^n q_{k-1}B_Q(p, x^{k-1}).$$

By Proposition 4.4 the Bezoutian B satisifes the equation

$$B - W_Q B C_Q(p)^t = \widetilde{u}\sum_{k=1}^n q_{k-1}\widetilde{e}_k^t (1 \leq k \leq n).$$

Taking $w = \sum_{k=1}^n q_{k-1}\widetilde{e}_k = \sum_{k=1}^n q_{k-1}T^{-1}e_k$, the condition $q(x) \neq 0$ and nonsingularity of T implies $w \neq 0$. The uniqueness of the solution is guaranteed by Proposition 4.4. ◻

We note that nonsymmetric Lyapunov-type (4.3) and (4.4) exhibit the displacement structure of polynomial Bezoutians. The theory of displacement structures for

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structured matrices (such as Hankel, Toeplitz, Cauchy and Vandermonde matrices etc.) have been extensively studied in recent years.

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