Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 361-369, March 2024.

COMMUTING ADDITIVE MAPS ON UPPER TRIANGULAR AND STRICTLY UPPER TRIANGULAR INFINITE MATRICES*

DI-CHEN LAN† AND CHENG-KAI LIU†

Abstract. Let $\mathbb{F}$ be a field, let $N_\infty(\mathbb{F})$ be the ring of all $\mathbb{N}\times\mathbb{N}$ strictly upper triangular matrices over $\mathbb{F}$, and let $T_\infty(\mathbb{F})$ be the ring of all $\mathbb{N}\times\mathbb{N}$ upper triangular matrices over $\mathbb{F}$. In this paper, we completely characterize additive maps $f : N_\infty(\mathbb{F}) \to T_\infty(\mathbb{F})$ satisfying $[f(x), x] = 0$ for all $x \in N_\infty(\mathbb{F})$. As applications, we obtain the finite fields versions of the two main results recently obtained by Slowik and Ahmed [*Electron. J. Linear Algebra* 37:247–255, 2021].

Key words. Commuting map, Functional identity, Infinite (strictly) upper triangular matrix ring.

AMS subject classifications. 15A78, 15A27.

1. Introduction and results. Let R be an associative ring with center $Z(R)$. For $a, b \in R$, let $[a, b] = ab - ba$ be the commutator of a and b. A map $f : R \to R$ is called additive if $f(x + y) = f(x) + f(y)$ for all $x, y \in R$. A map $f : R \to R$ is called commuting if $[f(x), x] = 0$ for all $x \in R$. The usual goal when dealing with a commuting map is to characterize its form.

The study of commuting additive maps was initiated by Divinsky and Posner. In 1955, Divinsky [13] proved that if a simple artinian ring R admits a commuting automorphism σ, then either R is commutative or σ is the identity map. On the other hand, in 1957 Posner [19] proved that if a prime ring R admits a commuting derivation d, then either R is commutative or $d = 0$. In 1993, Brešar [3] extended above two results to general additive maps and proved that if R is a prime ring with extended centroid C and $f : R \to R$ is a commuting additive map, then f must be of the form $f(x) = \lambda x + \mu(x)$ for all $x \in R$, where $\lambda \in C$ and $\mu : R \to C$ is an additive map. This important result had been generalized to many different rings and operator algebras. We refer the reader to references [4, 5] for the developments and applications of the theory of commuting maps. Recently, commuting maps on subrings or subsets of matrix rings have been widely investigated in the literature (see [1, 2, 6–12, 14–18, 20–23] for instance). In 2000, Beidar, Brešar, and Chebotar [1] showed that if $\mathbb{F}$ is a field, $T_n(\mathbb{F})$ is the ring of all $n \times n$ upper triangular matrices over $\mathbb{F}$ for an integer $n \geq 2$, and $f : T_n(\mathbb{F}) \to T_n(\mathbb{F})$ is a commuting linear map, then f is of the form $f(x) = \lambda x + \mu(x)$ for all $x \in T_n(\mathbb{F})$, where $\lambda \in \mathbb{F}$ and $\mu : T_n(\mathbb{F}) \to Z(T_n(\mathbb{F}))$ is a linear map. This result was later extended to commuting additive maps on the ring of all upper triangular matrices over fields by Eremita in [14]. In 2016, Bounds [2] successfully characterized commuting linear maps on the ring of all strictly upper triangular matrices over fields of characteristic 0. Precisely, he proved the following:

Theorem B. ([2]) *Let $n \geq 4$ be an integer and let $N_n(\mathbb{F})$ be the ring of all $n \times n$ strictly upper triangular matrices over a field $\mathbb{F}$ of characteristic 0. Assume that $f : N_n(\mathbb{F}) \to N_n(\mathbb{F})$ is a linear map such that*

*Received by the editors on December 13, 2023. Accepted for publication on March 25, 2024. Handling Editor: Michael Tsatsomeros. Corresponding Author: Cheng-Kai Liu.

†Department of Mathematics, National Changhua University of Education, Changhua 500, Taiwan (tsubasacasim@gmail.com, ckliu@cc.ncue.edu.tw).

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 361-369, March 2024.

$[f(x),x]=0$ for all $x\in N_n(\mathbb{F})$. Then, there exist $\lambda\in\mathbb{F}$ and a linear map $\mu: N_n(\mathbb{F})\to\Omega$ such that $f(x)=\lambda x+\mu(x)$ for all $x\in N_n(\mathbb{F})$, where $\Omega=\{\alpha E_{1,n-1}+\beta E_{1,n}+\gamma E_{2,n}\mid \alpha,\beta,\gamma\in\mathbb{F}\}$.

Let $\mathbb{F}$ be a field. We denote by $T_\infty(\mathbb{F})$ the ring of all $\mathbb{N}\times\mathbb{N}$ upper triangular matrices over $\mathbb{F}$ and $N_\infty(\mathbb{F})$ the ring of all $\mathbb{N}\times\mathbb{N}$ strictly upper triangular matrices over $\mathbb{F}$. As usual, for $i,j\in\mathbb{N}$ with $i\le j$, let $E_{i,j}\in T_\infty(\mathbb{F})$ denote the matrix unit with 1 in the (i,j)-entry and 0 in any other entry. It is known that $E_{i,j}E_{k,\ell}=\delta_{jk}E_{i,\ell}$, where δ is the Kronecker delta. Note that the set $\{E_{i,j}\mid i,j\in\mathbb{N},\ i\le j\}$ does not form a basis of $T_\infty(\mathbb{F})$ over $\mathbb{F}$. However, for abbreviation, if $a=[a_{s,t}]\in T_\infty(\mathbb{F})$, we will formally write $a=\sum_{s,t=1,s\le t}^{\infty}a_{s,t}E_{s,t}$. Also, if $a=[a_{s,t}]\in N_\infty(\mathbb{F})$, we will write $a=\sum_{s,t=1,s<t}^{\infty}a_{s,t}E_{s,t}$. The symbol 0 may stand for the zero element of $\mathbb{F}$ as well as for the zero matrix of $T_\infty(\mathbb{F})$. Motivated by Theorem B, in 2021 Slowik and Ahmed [20] described commuting additive maps on the ring of all infinite strictly upper triangular matrices over infinite fields. Precisely, they proved the following:

Theorem SA1. ([20, Theorem 1.1]) *Let $\mathbb{F}$ be an infinite field and let $N_\infty(\mathbb{F})$ be the ring of all $\mathbb{N}\times\mathbb{N}$ strictly upper triangular matrices over $\mathbb{F}$. Suppose that $f: N_\infty(\mathbb{F})\to N_\infty(\mathbb{F})$ is an additive map such that $[f(x),x]=0$ for all $x\in N_\infty(\mathbb{F})$. Then, there exists $\lambda\in\mathbb{F}$ such that $f(x)=\lambda x$ for all $x\in N_\infty(\mathbb{F})$.*

Moreover, in [20] Slowik and Ahmed also characterized commuting additive maps on the ring of all infinite upper triangular matrices over infinite fields as follows:

Theorem SA2. (See [20, Theorem 1.2]) *Let $\mathbb{F}$ be an infinite field and let $T_\infty(\mathbb{F})$ be the ring of all $\mathbb{N}\times\mathbb{N}$ upper triangular matrices over $\mathbb{F}$. Suppose that $f: T_\infty(\mathbb{F})\to T_\infty(\mathbb{F})$ is an additive map such that $[f(x),x]=0$ for all $x\in T_\infty(\mathbb{F})$. Then, there exist $\lambda\in\mathbb{F}$ and an additive map $\mu: T_\infty(\mathbb{F})\to\mathbb{F}I_\infty$ such that $f(x)=\lambda x+\mu(x)$ for all $x\in T_\infty(\mathbb{F})$, where I_∞ is the identity matrix of $T_\infty(\mathbb{F})$.*

It is natural to ask the question whether Theorem SA1 and Theorem SA2 remain true when the scalar field $\mathbb{F}$ is assumed to be a finite field. The purpose of this paper is to give an affirmative answer to this question. Precisely, we will prove the following:

Theorem 1.1. *Let $\mathbb{F}$ be a field and let $N_\infty(\mathbb{F})$ be the ring of all $\mathbb{N}\times\mathbb{N}$ strictly upper triangular matrices over $\mathbb{F}$ and let $T_\infty(\mathbb{F})$ be the ring of all $\mathbb{N}\times\mathbb{N}$ upper triangular matrices over $\mathbb{F}$. Suppose that $f: N_\infty(\mathbb{F})\to T_\infty(\mathbb{F})$ is an additive map satisfying $[f(x),x]=0$ for all $x\in N_\infty(\mathbb{F})$. Then, there exist $\lambda\in\mathbb{F}$ and an additive map $\mu: N_\infty(\mathbb{F})\to\mathbb{F}I_\infty$ such that $f(x)=\lambda x+\mu(x)$ for all $x\in N_\infty(\mathbb{F})$, where I_∞ is the identity matrix of $T_\infty(\mathbb{F})$.*

As applications of Theorem 1.1, we generalize Theorem SA1 and Theorem SA2 as follows:

Corollary 1.2. *Let $\mathbb{F}$ be a field and let $N_\infty(\mathbb{F})$ be the ring of all $\mathbb{N}\times\mathbb{N}$ strictly upper triangular matrices over $\mathbb{F}$. Suppose that $f: N_\infty(\mathbb{F})\to N_\infty(\mathbb{F})$ is an additive map such that $[f(x),x]=0$ for all $x\in N_\infty(\mathbb{F})$. Then, there exists $\lambda\in\mathbb{F}$ such that $f(x)=\lambda x$ for all $x\in N_\infty(\mathbb{F})$.*

Corollary 1.3. *Let $\mathbb{F}$ be a field and let $T_\infty(\mathbb{F})$ be the ring of all $\mathbb{N}\times\mathbb{N}$ upper triangular matrices over $\mathbb{F}$. Suppose that $f: T_\infty(\mathbb{F})\to T_\infty(\mathbb{F})$ is an additive map such that $[f(x),x]=0$ for all $x\in T_\infty(\mathbb{F})$. Then, there exist $\lambda\in\mathbb{F}$ and an additive map $\mu: T_\infty(\mathbb{F})\to\mathbb{F}I_\infty$ such that $f(x)=\lambda x+\mu(x)$ for all $x\in T_\infty(\mathbb{F})$, where I_∞ is the identity matrix of $T_\infty(\mathbb{F})$.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 361-369, March 2024.

It is noteworthy to mention that our approaches to the proofs of this paper are quite different from those in [20] and are based on the detailed and systematic computations of the actions of commuting additive maps on matrix units.

2. Proof of Theorem 1.1.

The goal of this section is to prove Theorem 1.1. Let $\mathbb{F}$ be a field, let $N_\infty(\mathbb{F})$ be the ring of all $\mathbb{N}\times\mathbb{N}$ strictly upper triangular matrices over $\mathbb{F}$ and let $T_\infty(\mathbb{F})$ be the ring of all $\mathbb{N}\times\mathbb{N}$ upper triangular matrices over $\mathbb{F}$. Suppose that $f: N_\infty(\mathbb{F})\to T_\infty(\mathbb{F})$ is an additive map satisfying $[f(x),x]=0$ for all $x\in N_\infty(\mathbb{F})$, that is,

$$f(x)x = xf(x), \tag{2.1}$$

for all $x\in N_\infty(\mathbb{F})$. Replacing x with $x+y$ in (2.1), we obtain

$$f(x)y - yf(x) = xf(y) - f(y)x, \tag{2.2}$$

for all $x,y\in N_\infty(\mathbb{F})$. For two integers $i,j\in\mathbb{N}$ with $i<j$, we write

$$f(\alpha E_{i,j}) = \sum_{s,t\in\mathbb{N}, s\le t} a^{i,j}_{s,t}(\alpha)E_{s,t},$$

for all $\alpha\in\mathbb{F}$, where each $a^{i,j}_{s,t}:\mathbb{F}\to\mathbb{F}$ is a map for $s,t\in\mathbb{N}$. Since f is an additive map, we can see that each $a^{i,j}_{s,t}$ is also an additive map for $s,t\in\mathbb{N}$. In particular, from $a^{i,j}_{s,t}(0)=a^{i,j}_{s,t}(0+0)=a^{i,j}_{s,t}(0)+a^{i,j}_{s,t}(0)$ it follows that $a^{i,j}_{s,t}(0)=0$ for all $s,t\in\mathbb{N}$.

Lemma 2.1. *Let $i,j\in\mathbb{N}$ with $i<j$. Then, $a^{i,j}_{s,i}=0$ for every $s\in\mathbb{N}$ with $s<i$ and $a^{i,j}_{j,t}=0$ for every $t\in\mathbb{N}$ with $j<t$.*

Proof. Setting $x=\alpha E_{i,j}$ in (2.1), we have

$$f(\alpha E_{i,j})\alpha E_{i,j} = \alpha E_{i,j} f(\alpha E_{i,j}), \tag{2.3}$$

for all $\alpha\in\mathbb{F}$. Let $s\in\mathbb{N}$ with $s<i$. Multiplying (2.3) by $E_{s,s}$ from the left and by $E_{j,j}$ from the right, we obtain $E_{s,s}f(\alpha E_{i,j})\alpha E_{i,j}=0$. This implies that $a^{i,j}_{s,i}(\alpha)\alpha=0$ for all $\alpha\in\mathbb{F}$. Since $\mathbb{F}$ is a field, we have $a^{i,j}_{s,i}(\alpha)=0$ for all $\alpha\in\mathbb{F}$ with $\alpha\neq 0$. Recall that $a^{i,j}_{s,i}(0)=0$. Hence, $a^{i,j}_{s,i}(\alpha)=0$ for all $\alpha\in\mathbb{F}$. So $a^{i,j}_{s,i}=0$. Let $t\in\mathbb{N}$ with $j<t$. Multiplying (2.3) by $E_{i,i}$ from the left and by $E_{t,t}$ from the right, we obtain $0=\alpha E_{i,j}f(\alpha E_{i,j})E_{t,t}$. This implies that $\alpha a^{i,j}_{j,t}(\alpha)=0$ for all $\alpha\in\mathbb{F}$. Since $\mathbb{F}$ is a field, we have $a^{i,j}_{j,t}(\alpha)=0$ for all $\alpha\in\mathbb{F}$ with $\alpha\neq 0$. Recall that $a^{i,j}_{j,t}(0)=0$. Hence, $a^{i,j}_{j,t}(\alpha)=0$ for all $\alpha\in\mathbb{F}$. So $a^{i,j}_{j,t}=0$, as desired. ■

Lemma 2.2. *Let $i,j\in\mathbb{N}$ with $i<j$. Then, $a^{i,j}_{i,i}=a^{i,j}_{j,j}=a^{i,j}_{s,s}$ for every $s\in\mathbb{N}$.*

Proof. Multiplying (2.3) by $E_{i,i}$ from the left and by $E_{j,j}$ from the right, we obtain $E_{i,i}f(\alpha E_{i,j})\alpha E_{i,j} = \alpha E_{i,j}f(\alpha E_{i,j})E_{j,j}$. This implies that $a^{i,j}_{i,i}(\alpha)\alpha=\alpha a^{i,j}_{j,j}(\alpha)$ for all $\alpha\in\mathbb{F}$. Then, $(a^{i,j}_{i,i}(\alpha)-a^{i,j}_{j,j}(\alpha))\alpha=0$ for all $\alpha\in\mathbb{F}$. Since F is a field, we have $a^{i,j}_{i,i}(\alpha)=a^{i,j}_{j,j}(\alpha)$ for all $\alpha\in\mathbb{F}$ with $\alpha\neq 0$. Recall that $a^{i,j}_{i,i}(0)=a^{i,j}_{j,j}(0)=0$. Hence, $a^{i,j}_{i,i}(\alpha)=a^{i,j}_{j,j}(\alpha)$ for all $\alpha\in\mathbb{F}$. So $a^{i,j}_{i,i}=a^{i,j}_{j,j}$. Let $s\in\mathbb{N}$ with $s\neq i,j$. We divide the proof into three cases.

Case 1. $s<i<j$. Setting $x=\alpha E_{i,j}$ and $y=E_{s,i}$ in (2.2), we have

$$f(\alpha E_{i,j})E_{s,i} - E_{s,i}f(\alpha E_{i,j}) = \alpha E_{i,j}f(E_{s,i}) - f(E_{s,i})\alpha E_{i,j}, \tag{2.4}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 361-369, March 2024.

for all $\alpha \in \mathbb{F}$. Multiplying (2.4) by $E_{s,s}$ from the left and by $E_{i,i}$ from the right, we obtain $E_{s,s}f(\alpha E_{i,j})E_{s,i} - E_{s,i}f(\alpha E_{i,j})E_{i,i} = 0$. This implies that $a^{i,j}_{s,s}(\alpha) - a^{i,j}_{i,i}(\alpha) = 0$ for all $\alpha \in \mathbb{F}$. So $a^{i,j}_{s,s} = a^{i,j}_{i,i}$.

Case 2. $i < j < s$. Setting $x = \alpha E_{i,j}$ and $y = E_{j,s}$ in (2.2), we have

$$f(\alpha E_{i,j})E_{j,s} - E_{j,s}f(\alpha E_{i,j}) = \alpha E_{i,j}f(E_{j,s}) - f(E_{j,s})\alpha E_{i,j}, \tag{2.5}$$

for all $\alpha \in \mathbb{F}$. Multiplying (2.5) by $E_{j,j}$ from the left and by $E_{s,s}$ from the right, we obtain $E_{j,j}f(\alpha E_{i,j})E_{j,s} - E_{j,s}f(\alpha E_{i,j})E_{s,s} = 0$. This implies that $a^{i,j}_{j,j}(\alpha) - a^{i,j}_{s,s}(\alpha) = 0$ for all $\alpha \in \mathbb{F}$. So $a^{i,j}_{j,j} = a^{i,j}_{s,s}$.

Case 3. $i < s < j$. Setting $x = \alpha E_{i,j}$ and $y = E_{s,j}$ in (2.2), we have

$$f(\alpha E_{i,j})E_{s,j} - E_{s,j}f(\alpha E_{i,j}) = \alpha E_{i,j}f(E_{s,j}) - f(E_{s,j})\alpha E_{i,j}, \tag{2.6}$$

for all $\alpha \in \mathbb{F}$. Note that $E_{s,s}f(E_{s,j})\alpha E_{i,j} = 0$ as $s > i$ and $f(E_{s,j}) \in T_\infty(\mathbb{F})$. Multiplying (2.6) by $E_{s,s}$ from the left and by $E_{j,j}$ from the right and using $E_{s,s}f(E_{s,j})\alpha E_{i,j} = 0$, we obtain $E_{s,s}f(\alpha E_{i,j})E_{s,j} - E_{s,j}f(\alpha E_{i,j})E_{j,j} = 0$. This implies that $a^{i,j}_{s,s}(\alpha) - a^{i,j}_{j,j}(\alpha) = 0$ for all $\alpha \in \mathbb{F}$. So $a^{i,j}_{s,s} = a^{i,j}_{j,j}$.

Now by Cases 1,2,3, we see that $a^{i,j}_{i,i} = a^{i,j}_{j,j} = a^{i,j}_{s,s}$ for every $s \in \mathbb{N}$, proving the lemma. ■

Lemma 2.3. *Let $i, j \in \mathbb{N}$ with $i < j$. Then, $a^{i,j}_{i,t} = 0$ for every $t \in \mathbb{N}$ with $i < t < j$.*

Proof. Let $t \in \mathbb{N}$ with $i < t < j$. Setting $x = \alpha E_{i,j}$ and $y = E_{t,j}$ in (2.2), we have

$$f(\alpha E_{i,j})E_{t,j} - E_{t,j}f(\alpha E_{i,j}) = \alpha E_{i,j}f(E_{t,j}) - f(E_{t,j})\alpha E_{i,j}, \tag{2.7}$$

for all $\alpha \in \mathbb{F}$. By Lemma 2.2, $a^{t,j}_{j,j} = a^{t,j}_{i,i}$. With this, we see that

$$\alpha E_{i,j}f(E_{t,j})E_{j,j} - E_{i,i}f(E_{t,j})\alpha E_{i,j} = \alpha(a^{t,j}_{j,j}(1) - a^{t,j}_{i,i}(1))E_{i,j} = 0, \tag{2.8}$$

for all $\alpha \in \mathbb{F}$. Now multiplying (2.7) by $E_{i,i}$ from the left and by $E_{j,j}$ from the right and using (2.8), we obtain $E_{i,i}f(\alpha E_{i,j})E_{t,j} = 0$. This implies that $a^{i,j}_{i,t}(\alpha) = 0$ for all $\alpha \in \mathbb{F}$. So $a^{i,j}_{i,t} = 0$, as desired. ■

Lemma 2.4. *Let $i, j \in \mathbb{N}$ with $i < j$. Then, $a^{i,j}_{s,j} = 0$ for every $s \in \mathbb{N}$ with $s < i$.*

Proof. Let $s \in \mathbb{N}$ with $s < i$. Clearly, $s < i < j$. Setting $x = \alpha E_{i,j}$ and $y = E_{j,j+1}$ in (2.2), we have

$$f(\alpha E_{i,j})E_{j,j+1} - E_{j,j+1}f(\alpha E_{i,j}) = \alpha E_{i,j}f(E_{j,j+1}) - f(E_{j,j+1})\alpha E_{i,j}, \tag{2.9}$$

for all $\alpha \in \mathbb{F}$. Now multiplying (2.9) by $E_{s,s}$ from the left and by $E_{j+1,j+1}$ from the right, we obtain $E_{s,s}f(\alpha E_{i,j})E_{j,j+1} = 0$. This implies that $a^{i,j}_{s,j}(\alpha) = 0$ for all $\alpha \in \mathbb{F}$. So $a^{i,j}_{s,j} = 0$, as desired. ■

Lemma 2.5. *Let $i, j \in \mathbb{N}$ with $2 \leq i < j$. Then, $a^{i,j}_{i,t} = 0$ for every $t \in \mathbb{N}$ with $j < t$.*

Proof. Let $t \in \mathbb{N}$ with $j < t$. Clearly, $2 \leq i < j < t$. Setting $x = \alpha E_{i,j}$ and $y = E_{1,i}$ in (2.2), we have

$$f(\alpha E_{i,j})E_{1,i} - E_{1,i}f(\alpha E_{i,j}) = \alpha E_{i,j}f(E_{1,i}) - f(E_{1,i})\alpha E_{i,j}, \tag{2.10}$$

for all $\alpha \in \mathbb{F}$. Now multiplying (2.10) by $E_{1,1}$ from the left and by $E_{t,t}$ from the right, we obtain $-E_{1,i}f(\alpha E_{i,j})E_{t,t} = 0$. This implies that $a^{i,j}_{i,t}(\alpha) = 0$ for all $\alpha \in \mathbb{F}$. So $a^{i,j}_{i,t} = 0$, as desired. ■

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 361-369, March 2024.

Lemma 2.6. *Let $j \in \mathbb{N}$ with $2 \le j$. Then, $a_{1,t}^{1,j} = 0$ for every $t \in \mathbb{N}$ with $j < t$.*

Proof. Let $t \in \mathbb{N}$ with $2 \le j < t$. Setting $x = \alpha E_{1,j}$ and $y = E_{t,t+1}$ in (2.2), we have

$$f(\alpha E_{1,j})E_{t,t+1} - E_{t,t+1}f(\alpha E_{1,j}) = \alpha E_{1,j}f(E_{t,t+1}) - f(E_{t,t+1})\alpha E_{1,j}, \tag{2.11}$$

for all $\alpha \in \mathbb{F}$. Multiplying (2.11) by $E_{1,1}$ from the left and by $E_{t+1,t+1}$ from the right, we obtain

$$E_{1,1}f(\alpha E_{1,j})E_{t,t+1} = \alpha E_{1,j}f(E_{t,t+1})E_{t+1,t+1}, \tag{2.12}$$

for all $\alpha \in \mathbb{F}$. By Lemma 2.4, $a_{j,t+1}^{t,t+1} = 0$. Thus, $\alpha E_{1,j}f(E_{t,t+1})E_{t+1,t+1} = 0$. With this and (2.12), we have $E_{1,1}f(\alpha E_{1,j})E_{t,t+1} = 0$. This implies that $a_{1,t}^{1,j}(\alpha) = 0$ for all $\alpha \in \mathbb{F}$. So $a_{1,t}^{1,j} = 0$, as desired. ■

Lemma 2.7. *Let $i, j \in \mathbb{N}$ with $i < j$. Then, $a_{s,t}^{i,j} = 0$ for every $s, t \in \mathbb{N}$ with $j < s < t$.*

Proof. Let $s, t \in \mathbb{N}$ with $j < s < t$. Setting $x = \alpha E_{i,j}$ and $y = E_{j,s}$ in (2.2), we have

$$f(\alpha E_{i,j})E_{j,s} - E_{j,s}f(\alpha E_{i,j}) = \alpha E_{i,j}f(E_{j,s}) - f(E_{j,s})\alpha E_{i,j}, \tag{2.13}$$

for all $\alpha \in \mathbb{F}$. Now multiplying (2.13) by $E_{j,j}$ from the left and by $E_{t,t}$ from the right, we obtain $-E_{j,s}f(\alpha E_{i,j})E_{t,t} = 0$. This implies that $a_{s,t}^{i,j}(\alpha) = 0$ for all $\alpha \in \mathbb{F}$. So $a_{s,t}^{i,j} = 0$, as desired. ■

Lemma 2.8. *Let $i, j \in \mathbb{N}$ with $i < j$. Then, $a_{s,t}^{i,j} = 0$ for every $s, t \in \mathbb{N}$ with $i < s < j$ and $s < t$.*

Proof. Let $s, t \in \mathbb{N}$ with $i < s < j$ and $s < t$. Setting $x = \alpha E_{i,j}$ and $y = E_{i,s}$ in (2.2), we have

$$f(\alpha E_{i,j})E_{i,s} - E_{i,s}f(\alpha E_{i,j}) = \alpha E_{i,j}f(E_{i,s}) - f(E_{i,s})\alpha E_{i,j}, \tag{2.14}$$

for all $\alpha \in \mathbb{F}$. We divide the proof into two cases.

Case 1. $j \neq t$. Multiplying (2.14) by $E_{i,i}$ from the left and by $E_{t,t}$ from the right, we obtain

$$-E_{i,s}f(\alpha E_{i,j})E_{t,t} = \alpha E_{i,j}f(E_{i,s})E_{t,t}, \tag{2.15}$$

for all $\alpha \in \mathbb{F}$. Assume first that $j > t$. Then, $\alpha E_{i,j}f(E_{i,s})\alpha E_{t,t} = 0$ as $f(E_{i,s}) \in T_\infty(\mathbb{F})$. With this and (2.15), we obtain $-E_{i,s}f(\alpha E_{i,j})E_{t,t} = 0$. This implies that $a_{s,t}^{i,j}(\alpha) = 0$ for all $\alpha \in \mathbb{F}$. So $a_{s,t}^{i,j} = 0$, as desired. Assume next that $t > j$. In this case, $i < s < j < t$. Now by Lemma 2.7, $a_{j,t}^{i,s} = 0$. Thus, $\alpha E_{i,j}f(E_{i,s})E_{t,t} = 0$. With this and (2.15), we obtain $-E_{i,s}f(\alpha E_{i,j})E_{t,t} = 0$. This implies that $a_{s,t}^{i,j}(\alpha) = 0$ for all $\alpha \in \mathbb{F}$. So $a_{s,t}^{i,j} = 0$, as desired.

Case 2. $j = t$. Multiplying (2.14) by $E_{i,i}$ from the left and by $E_{j,j}$ from the right, we obtain

$$-E_{i,s}f(\alpha E_{i,j})E_{j,j} = \alpha E_{i,j}f(E_{i,s})E_{j,j} - E_{i,i}f(E_{i,s})\alpha E_{i,j}, \tag{2.16}$$

for all $\alpha \in \mathbb{F}$. By Lemma 2.2, $a_{j,j}^{i,s} = a_{i,i}^{i,s}$. With this, we see that

$$\alpha E_{i,j}f(E_{i,s})E_{j,j} - E_{i,i}f(E_{i,s})\alpha E_{i,j} = \alpha(a_{j,j}^{i,s}(1) - a_{i,i}^{i,s}(1))E_{i,j} = 0, \tag{2.17}$$

for all $\alpha \in \mathbb{F}$. Applying (2.17) to (2.16), we obtain $-E_{i,s}f(\alpha E_{i,j})E_{j,j} = 0$. This implies that $a_{s,j}^{i,j}(\alpha) = 0$ for all $\alpha \in \mathbb{F}$. So $a_{s,j}^{i,j} = 0$. From $j = t$, $a_{s,t}^{i,j} = 0$ follows, as desired. ■

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 361-369, March 2024.

Lemma 2.9. *Let $i, j \in \mathbb{N}$ with $i < j$. Then, $a^{i,j}_{s,t} = 0$ for every $s, t \in \mathbb{N}$ with $s < t < i$.*

Proof. Let $s, t \in \mathbb{N}$ with $s < t < i$. Clearly, $s < t < i < j$. Setting $x = \alpha E_{i,j}$ and $y = E_{t,i}$ in (2.2), we have

$$f(\alpha E_{i,j})E_{t,i} - E_{t,i}f(\alpha E_{i,j}) = \alpha E_{i,j}f(E_{t,i}) - f(E_{t,i})\alpha E_{i,j}, \tag{2.18}$$

for all $\alpha \in \mathbb{F}$. Multiplying (2.18) by $E_{s,s}$ from the left and by $E_{i,i}$ from the right, we obtain $E_{s,s}f(\alpha E_{i,j})E_{t,i} = 0$. This implies that $a^{i,j}_{s,t}(\alpha) = 0$ for all $\alpha \in \mathbb{F}$. So $a^{i,j}_{s,t} = 0$, as desired. ■

Lemma 2.10. *Let $i, j \in \mathbb{N}$ with $i < j$. Then, $a^{i,j}_{s,t} = 0$ for every $s, t \in \mathbb{N}$ with $s < i < t < j$.*

Proof. Let $s, t \in \mathbb{N}$ with $s < i < t < j$. Setting $x = \alpha E_{i,j}$ and $y = E_{t,j}$ in (2.2), we have

$$f(\alpha E_{i,j})E_{t,j} - E_{t,j}f(\alpha E_{i,j}) = \alpha E_{i,j}f(E_{t,j}) - f(E_{t,j})\alpha E_{i,j}, \tag{2.19}$$

for all $\alpha \in \mathbb{F}$. Multiplying (2.19) by $E_{s,s}$ from the left and by $E_{j,j}$ from the right, we obtain

$$E_{s,s}f(\alpha E_{i,j})E_{t,j} = -E_{s,s}f(E_{t,j})\alpha E_{i,j}, \tag{2.20}$$

for all $\alpha \in \mathbb{F}$. By Lemma 2.9, $a^{t,j}_{s,i} = 0$ as $s < i < t < j$. Thus, $-E_{s,s}f(E_{t,j})\alpha E_{i,j} = 0$. With this and (2.20), we obtain $E_{s,s}f(\alpha E_{i,j})E_{t,j} = 0$. This implies that $a^{i,j}_{s,t}(\alpha) = 0$ for all $\alpha \in \mathbb{F}$. So $a^{i,j}_{s,t} = 0$, as desired. ■

Lemma 2.11. *Let $i, j \in \mathbb{N}$ with $i < j$. Then, $a^{i,j}_{s,t} = 0$ for every $s, t \in \mathbb{N}$ with $s < i < j < t$.*

Proof. Let $s, t \in \mathbb{N}$ with $s < i < j < t$. Setting $x = \alpha E_{i,j}$ and $y = E_{t,t+1}$ in (2.2), we have

$$f(\alpha E_{i,j})E_{t,t+1} - E_{t,t+1}f(\alpha E_{i,j}) = \alpha E_{i,j}f(E_{t,t+1}) - f(E_{t,t+1})\alpha E_{i,j}, \tag{2.21}$$

for all $\alpha \in \mathbb{F}$. Now multiplying (2.21) by $E_{s,s}$ from the left and by $E_{t+1,t+1}$ from the right, we obtain $E_{s,s}f(\alpha E_{i,j})E_{t,t+1} = 0$. This implies that $a^{i,j}_{s,t}(\alpha) = 0$ for all $\alpha \in \mathbb{F}$. So $a^{i,j}_{s,t} = 0$, as desired. ■

Lemma 2.12. *Let $\mathbb{F}$ be a field. Suppose that $f : N_\infty(\mathbb{F}) \to T_\infty(\mathbb{F})$ is an additive map such that $[f(x), x] = 0$ for all $x \in N_\infty(\mathbb{F})$. Then for every $i, j \in \mathbb{N}$ with $i < j$, there exist an additive map $a^{i,j}_{i,j} : \mathbb{F} \to \mathbb{F}$ and an additive map $\mu_{i,j} : \mathbb{F} \to \mathbb{F}$ such that $f(\alpha E_{i,j}) = a^{i,j}_{i,j}(\alpha)E_{i,j} + \mu_{i,j}(\alpha)I_\infty$ for all $\alpha \in \mathbb{F}$.*

Proof. Let $i, j \in \mathbb{N}$ with $i < j$. Write $f(\alpha E_{i,j}) = \sum_{s,t=1,s\leq t}^{\infty} a^{i,j}_{s,t}(\alpha)E_{s,t}$ for all $\alpha \in \mathbb{F}$, where each $a^{i,j}_{s,t} : \mathbb{F} \to \mathbb{F}$ is an additive map. By Lemmas 2.4 and 2.8, $a^{i,j}_{s,j} = 0$ for all $s \in \mathbb{N}$ with $s < j$ and $s \neq i$ and by Lemma 2.1, $a^{i,j}_{j,t} = 0$ for all $t \in \mathbb{N}$ with $j < t$. Next by Lemma 2.1, $a^{i,j}_{s,i} = 0$ for all $s \in \mathbb{N}$ with $s < i$ and by Lemmas 2.3, 2.5 and 2.6, $a^{i,j}_{i,t} = 0$ for all $t \in \mathbb{N}$ with $i < t$ and $t \neq j$. Moreover, by Lemmas 2.7, 2.8, 2.9, 2.10 and 2.11, $a^{i,j}_{s,t} = 0$ for all $s, t \in \mathbb{N}$ with $s < t$ and $s, t \notin \{i, j\}$. Finally, by Lemma 2.2, $a^{i,j}_{1,1} = a^{i,j}_{s,s}$ for all $s \in \mathbb{N}$. With these, we obtain $f(\alpha E_{i,j}) = a^{i,j}_{i,j}(\alpha)E_{i,j} + \sum_{s=1}^{\infty} a^{i,j}_{s,s}(\alpha)E_{s,s} = a^{i,j}_{i,j}(\alpha)E_{i,j} + \mu_{i,j}(\alpha)I_\infty$ for all $\alpha \in \mathbb{F}$, where $\mu_{i,j} = a^{i,j}_{1,1}$. This proves the lemma. ■

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.12, for every $i, j \in \mathbb{N}$ with $i < j$, there exist an additive map $a^{i,j}_{i,j} : \mathbb{F} \to \mathbb{F}$ and an additive map $\mu_{i,j} : \mathbb{F} \to \mathbb{F}$ such that

$$f(\alpha E_{i,j}) = a^{i,j}_{i,j}(\alpha)E_{i,j} + \mu_{i,j}(\alpha)I_\infty, \tag{2.22}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 361-369, March 2024.

for all $\alpha \in \mathbb{F}$.

Let $i \in \mathbb{N}$. Setting $x = \alpha E_{i,i+1}$ and $y = \beta E_{i+1,i+2}$ in (2.2), we have

$$f(\alpha E_{i,i+1})\beta E_{i+1,i+2} - \beta E_{i+1,i+2} f(\alpha E_{i,i+1}) = \alpha E_{i,i+1} f(\beta E_{i+1,i+2}) - f(\beta E_{i+1,i+2})\alpha E_{i,i+1}, \tag{2.23}$$

for all $\alpha, \beta \in \mathbb{F}$. Multiplying (2.23) by $E_{i,i}$ from the left and by $E_{i+2,i+2}$ from the right, we obtain $E_{i,i} f(\alpha E_{i,i+1})\beta E_{i+1,i+2} = \alpha E_{i,i+1} f(\beta E_{i+1,i+2})E_{i+2,i+2}$. This implies that

$$a_{i,i+1}^{i,i+1}(\alpha)\beta = \alpha a_{i+1,i+2}^{i+1,i+2}(\beta), \tag{2.24}$$

for all $\alpha, \beta \in \mathbb{F}$ and $i \in \mathbb{N}$. By (2.24), we have

$$a_{1,2}^{1,2}(\alpha)\beta = \alpha a_{2,3}^{2,3}(\beta), \tag{2.25}$$

and

$$a_{2,3}^{2,3}(\alpha)\beta = \alpha a_{3,4}^{3,4}(\beta), \tag{2.26}$$

for all $\alpha, \beta \in \mathbb{F}$. Setting $\beta = 1$ in (2.25), we obtain $a_{1,2}^{1,2}(\alpha) = \lambda\alpha$ for all $\alpha \in \mathbb{F}$, where $\lambda = a_{2,3}^{2,3}(1) \in \mathbb{F}$. Next applying $a_{1,2}^{1,2}(\alpha) = \lambda\alpha$ to (2.25), we get $a_{2,3}^{2,3}(\beta) = \lambda\beta$ for all $\beta \in \mathbb{F}$. Consequently, $a_{1,2}^{1,2}(\alpha) = a_{2,3}^{2,3}(\alpha) = \lambda\alpha$ for all $\alpha \in \mathbb{F}$. Similarly, using (2.26) and $a_{2,3}^{2,3}(\alpha) = \lambda\alpha$, we obtain $a_{2,3}^{2,3}(\alpha) = a_{3,4}^{3,4}(\alpha) = \lambda\alpha$ for all $\alpha \in \mathbb{F}$. Now using (2.24) repeatedly, we conclude that

$$a_{i,i+1}^{i,i+1}(\alpha) = \lambda\alpha, \tag{2.27}$$

for all $\alpha \in \mathbb{F}$ and $i \in \mathbb{N}$. Let $i, j \in \mathbb{N}$ with $i < j$. Setting $x = \alpha E_{i,j}$ and $y = E_{j,j+1}$ in (2.2), we have

$$f(\alpha E_{i,j})E_{j,j+1} - E_{j,j+1} f(\alpha E_{i,j}) = \alpha E_{i,j} f(E_{j,j+1}) - f(E_{j,j+1})\alpha E_{i,j}, \tag{2.28}$$

for all $\alpha \in \mathbb{F}$. Multiplying (2.28) by $E_{i,i}$ from the left and by $E_{j+1,j+1}$ from the right, we obtain

$$E_{i,i} f(\alpha E_{i,j})E_{j,j+1} = \alpha E_{i,j} f(E_{j,j+1})E_{j+1,j+1}.$$

This implies that $a_{i,j}^{i,j}(\alpha) = \alpha a_{j,j+1}^{j,j+1}(1)$ for all $\alpha \in \mathbb{F}$. Recall that $a_{j,j+1}^{j,j+1}(1) = \lambda$ by (2.27). Hence, $a_{i,j}^{i,j}(\alpha) = \lambda\alpha$ for all $\alpha \in \mathbb{F}$. With this and (2.22), we see that for every $i, j \in \mathbb{N}$ with $i < j$,

$$f(\alpha E_{i,j}) = \lambda\alpha E_{i,j} + \mu_{i,j}(\alpha)I_\infty, \tag{2.29}$$

for all $\alpha \in \mathbb{F}$. Let $i, j \in \mathbb{N}$ with $i < j$. Clearly, by (2.29) $f(E_{i,j}) = \lambda E_{i,j} + \mu_{i,j}(1)I_\infty$. Setting $y = E_{i,j}$ in (2.2), we obtain

$$\begin{aligned} f(x)E_{i,j} - E_{i,j} f(x) &= x f(E_{i,j}) - f(E_{i,j})x \\ &= x(\lambda E_{i,j} + \mu_{i,j}(1)I_\infty) - (\lambda E_{i,j} + \mu_{i,j}(1)I_\infty)x \\ &= (\lambda x)E_{i,j} - E_{i,j}(\lambda x), \end{aligned}$$

for all $x \in N_\infty(\mathbb{F})$. This implies that

$$(f(x) - \lambda x)E_{i,j} = E_{i,j}(f(x) - \lambda x), \tag{2.30}$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 361-369, March 2024.

for all $x \in N_\infty(\mathbb{F})$ and $i, j \in \mathbb{N}$ with $i < j$. Clearly, from (2.30) it follows that $f(x) - \lambda x \in \mathbb{F}I_\infty$ for all $x \in N_\infty(\mathbb{F})$. Let $\mu : N_\infty(\mathbb{F}) \to \mathbb{F}I_\infty$ be the additive map defined by $\mu(x) = f(x) - \lambda x$ for all $x \in N_\infty(\mathbb{F})$. Then, $f(x) = \lambda x + \mu(x)$ for all $x \in N_\infty(\mathbb{F})$. This proves the theorem. ■

Proof of Corollary 1.2. By Theorem 1.1, there exist $\lambda \in \mathbb{F}$ and an additive map $\mu : N_\infty(\mathbb{F}) \to \mathbb{F}I_\infty$ such that $f(x) = \lambda x + \mu(x)$ for all $x \in N_\infty(\mathbb{F})$. By assumption, $f(x) \in N_\infty(\mathbb{F})$ for all $x \in N_\infty(\mathbb{F})$. Thus, $\mu(x) = f(x) - \lambda x \in N_\infty(\mathbb{F})$ for all $x \in N_\infty(\mathbb{F})$. From $\mathbb{F}I_\infty \cap N_\infty(\mathbb{F}) = \{0\}$, it follows that $\mu(x) = 0$ for all $x \in N_\infty(\mathbb{F})$. This implies that $f(x) = \lambda x$ for all $x \in N_\infty(\mathbb{F})$, as desired. ■

Proof of Corollary 1.3. By Theorem 1.1, there exist $\lambda \in \mathbb{F}$ and an additive map $\nu : N_\infty(\mathbb{F}) \to \mathbb{F}I_\infty$ such that

$$f(y) = \lambda y + \nu(y), \tag{2.31}$$

for all $y \in N_\infty(\mathbb{F})$. Similarly, in view of (2.2), we have

$$f(x)y - yf(x) = xf(y) - f(y)x, \tag{2.32}$$

for all $x, y \in T_\infty(\mathbb{F})$. Let $x \in T_\infty(\mathbb{F})$ and $y \in N_\infty(\mathbb{F})$. By (2.31) and (2.32), we have

$$f(x)y - yf(x) = xf(y) - f(y)x = x(\lambda y + \nu(y)) - (\lambda y + \nu(y))x = (\lambda x)y - y(\lambda x).$$

This implies that $(f(x) - \lambda x)y = y(f(x) - \lambda x)$ for all $x \in T_\infty(\mathbb{F})$ and $y \in N_\infty(\mathbb{F})$. In particular, $(f(x) - \lambda x)E_{i,j} = E_{i,j}(f(x) - \lambda x)$ for all $x \in T_\infty(\mathbb{F})$ and $i, j \in \mathbb{N}$ with $i < j$. Hence, $f(x) - \lambda x \in \mathbb{F}I_\infty$ for all $x \in T_\infty(\mathbb{F})$. Let $\mu : T_\infty(\mathbb{F}) \to \mathbb{F}I_\infty$ be the additive map defined by $\mu(x) = f(x) - \lambda x$ for all $x \in T_\infty(\mathbb{F})$. Then, $f(x) = \lambda x + \mu(x)$ for all $x \in T_\infty(\mathbb{F})$, as desired. ■

Acknowledgment. The authors would like to thank the referee for the very thorough reading of the paper and valuable comments.

REFERENCES

[1] K.I. Beidar, M. Brešar, and M.A. Chebotar. Functional identities on upper triangular matrix algebras. *J. Math. Sci.* (*New York*), 102:4557–4565, 2000.

[2] J. Bounds. Commuting maps over the ring of strictly upper triangular matrices. *Linear Algebra Appl.,* 507:132–136, 2016.

[3] M. Brešar. Centralizing mappings and derivations in prime rings. *J. Algebra,* 156:385–394, 1993.

[4] M. Brešar. Commuting maps: A survey. *Taiwanese J. Math.,* 8:361–397, 2004.

[5] M. Brešar, M.A. Chebotar, and W.S. Martindale III. *Functional Identities.* Frontiers in Mathematics. Birkhauser Verlag, Basel, 2007.

[6] M. Brešar and P. Šemrl. Continuous commuting functions on matrix algebras. *Linear Algebra Appl.,* 568:29–38, 2019.

[7] W.S. Cheung. Commuting maps of triangular algebras. *J. London Math. Soc.,* 63:117–127, 2001.

[8] W.L. Chooi, K.H. Kwa, and L.Y. Tan. Commuting maps on invertible triangular matrices over $\mathbb{F}_2$. *Linear Algebra Appl.,* 583:77–101, 2019.

[9] W.L. Chooi, K.H. Kwa, and L.Y. Tan. Commuting maps on rank k triangular matrices. *Linear Multilinear Algebra,* 68:1021–1030, 2020.

[10] W.L. Chooi, M.H.A. Mutalib, and L.Y. Tan. Commuting maps on rank one triangular matrices. *Linear Algebra Appl.,* 626:34–55, 2021.

[11] W.L. Chooi and Y.N. Tan. A note on commuting additive maps on rank k symmetric matrices. *Electron. J. Linear Algebra,* 37:734–746, 2021.

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 40, pp. 361-369, March 2024.

[12] P.-H. Chou and C.-K. Liu. Power commuting additive maps on rank-k linear transformations. *Linear Multilinear Algebra,* 69:403–427, 2021.

[13] N. Divinsky. On commuting automorphisms of rings. *Trans. Roy. Soc. Canada. Sect.* III, 49:19–22, 1955.

[14] D. Eremita. Functional identities of degree 2 in triangular rings. *Linear Algebra Appl.,* 438:584–597, 2013.

[15] W. Franca. Commuting maps on some subsets of matrices that are not closed under addition. *Linear Algebra Appl.*, 437:388–391, 2012.

[16] W. Franca and N. Louza. Generalized commuting maps on the set of singular matrices. *Electron. J. Linear Algebra,* 35:533–542, 2019.

[17] W. Franca and N. Louza. Commuting maps on idempotents and functional identities on finite subsets of fields. *Linear Multilinear Algebra*, 2023. Published online: 25 Oct 2023. https://doi.org/10.1080/03081087.2023.2271640.

[18] S.-W. Ko and C.-K. Liu. Commuting maps on strictly upper triangular matrix rings. *Oper. Matrices,* 17:1023–1036, 2023.

[19] E.C. Posner. Derivations in prime rings. *Proc. Amer. Math. Soc.,* 8:1093–1100, 1957.

[20] R. Slowik and D.A.H. Ahmed. M-commuting maps on triangular and strictly triangular infinite matrices. *Electron. J. Linear Algebra,* 37:247–255, 2021.

[21] Y. Wang. Functional identities in upper triangular matrix rings revisited. *Linear Multilinear Algebra,* 67:348–359, 2019.

[22] Z.-K. Xiao and F. Wei. Commuting mappings of generalized matrix algebras. *Linear Algebra Appl.,* 433:2178–2197: 2010.

[23] X. Xu and X. Yi. Commuting maps on rank-k matrices. *Electron. J. Linear Algebra,* 27:735–741, 2014.