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A NEW WEIGHTED SPECTRAL GEOMETRIC MEAN AND PROPERTIES*

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Abstract. In this paper, we introduce a new weighted spectral geometric mean:

$$F_t(A,B) = (A^{-1}\sharp_t B)^{1/2} A^{2-2t} (A^{-1}\sharp_t B)^{1/2}, \quad t \in [0,1],$$

where A and B are positive definite matrices. We study basic properties and inequalities for $F_t(A,B)$. We also establish the Lie–Trotter formula for $F_t(A,B)$. Finally, we extend some of the results on $F_t(A,B)$ to symmetric space of noncompact types.

Key words. Weighted F-mean, Positive definite matrices, The Lie–Trotter formulas, Noncompact semisimple Lie group, Kostant pre-order.

AMS subject classifications. 15A16, 15A45, 22E46.

1. Introduction. Let $\mathbb{P}_n$ be the set of all $n\times n$ positive definite matrices, $\mathbb{H}_n$ be the space of all $n\times n$ Hermitian matrices, and $\mathrm{U}(n)$ be the group of $n\times n$ unitary matrices. For $A, B \in \mathbb{P}_n$, the geometric mean $A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ was firstly defined by Pusz and Woronowicz [18] in 1975. They showed that it is the unique positive definite solution to the Riccati equation:

$$XA^{-1}X = B.$$

It is well known [3] that the geometric mean $A\sharp B$ is the midpoint of the geodesic

$$A\sharp_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}, \quad t \in [0,1],$$

joining A and B under the Riemannian metric $\delta_R(A,B) = \|\log(A^{-1/2}BA^{-1/2})\|_F$, where $\|\cdot\|_F$ denotes the Frobenius norm [4]. See [1, 5, 10, 11, 13] for more results on the geometric mean.

The spectral geometric mean of $A, B \in \mathbb{P}_n$ was introduced by Fiedler and Pták in 1997 [6], and one of its formulations is

$$A\natural B := (A^{-1}\sharp B)^{1/2} A (A^{-1}\sharp B)^{1/2}. \tag{1.1}$$

It is called *the spectral geometric mean* because $(A\natural B)^2$ is similar to AB and that the eigenvalues of their spectral mean are the positive square roots of the corresponding eigenvalues of AB [6, Theorem 3.2].

In 2007, Kim and Lim [13] established a matrix exponential formula for the geometric and spectral geometric means of positive definite matrices. In the same paper, they also defined the weighted spectral geometric mean as:

$$A\natural_t B := \left(A^{-1}\sharp B\right)^t A \left(A^{-1}\sharp B\right)^t, \quad t \in [0,1]. \tag{1.2}$$

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It is obvious that $A\natural_t B$ is a curve joining A and B. In 2015, Kim and Lee [12] studied the relative operator entropy by the spectral geometric mean. They also studied several properties similar to those of the Tsallis relative operator entropy by the usual geometric mean. Recently, Gan, Liu, and Tam [8] and Gan and Tam [9] studied $A\natural_t B$ and obtained some nice properties. A nice survey and new properties related to the weighted spectral geometric mean $A\natural_t B$ can be found in the recent paper of Gan and Kim [7].

Note that in (1.2) the geometric mean $A^{-1}\sharp B$ is a main component of the weighted spectral mean $A\natural_t B$, while the middle term is A, independent of t. We define a new weighted mean below.

Definition 1.1. *Let $A, B \in \mathbb{P}_n$. Define*

$$F_t(A,B) := (A^{-1}\sharp_t B)^{1/2} A^{2-2t} (A^{-1}\sharp_t B)^{1/2}, \quad t \in [0,1]. \tag{1.3}$$

It is obvious that $F_0(A,B) = A$ and $F_1(A,B) = B$, and hence $F_t(A,B)$ is a curve joining A and B. For $t = \frac{1}{2}$, $F_{\frac{1}{2}}(A,B)$ is the spectral geometric mean (1.1). We call $F_t(A,B)$ weighted F-mean and it is different from (1.2).

From the Riccati equation, it is obvious that $A\sharp X = B$ if and only if $X = BA^{-1}B$. Therefore, $F_t(A,B)$ is the unique positive definite solution X to

$$A^{2(t-1)}\sharp X = (A^{-1}\sharp_t B)^{1/2}.$$

Proposition 1.2 (See [2]). *For any differentiable curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{P}_n$ with $\gamma(0) = I$,*

$$e^{\gamma'(0)} = \lim_{t\to 0} \gamma^{1/t}(t) = \lim_{n\to\infty} \gamma^n(1/n).$$

Notice that for $X, Y \in \mathbb{H}_n$ and $\alpha \in [0,1]$, the following curves are smooth and pass through the identity matrix I at $t = 0$:

$$\begin{aligned}
\gamma_1(t) &= e^{t(1-\alpha)X/2} e^{t\alpha Y} e^{t(1-\alpha)X/2},\\
\gamma_2(t) &= (1-\alpha)e^{tX} + \alpha e^{tY},\\
\gamma_3(t) &= ((1-\alpha)e^{-tX} + \alpha e^{-tY})^{-1},\\
\gamma_4(t) &= e^{tX}\sharp_\alpha e^{tY},\\
\gamma_5(t) &= e^{tX}\natural_\alpha e^{tY}.
\end{aligned}$$

Applying Proposition 1.2 one obtains the following Lie–Trotter formulas:

$$\begin{aligned}
e^{(1-\alpha)X+\alpha Y} &= \lim_{n\to\infty} (e^{t(1-\alpha)X/2n} e^{t\alpha Y/n} e^{t(1-\alpha)X/2n})^n\\
&= \lim_{n\to\infty} ((1-\alpha)e^{tX/n} + \alpha e^{tY/n})^n\\
&= \lim_{n\to\infty} ((1-\alpha)e^{-tX/n} + \alpha e^{-tY/n})^{-n}\\
&= \lim_{n\to\infty} (e^{tX/n}\sharp_\alpha e^{tY/n})^n\\
&= \lim_{n\to\infty} (e^{tX/n}\natural_\alpha e^{tY/n})^n.
\end{aligned}$$

We will establish in Theorem 3.1 the Lie–Trotter formula for F_t, namely,

$$\lim_{p\to 0} F_t^{1/p}(e^{pA}, e^{pB}) = e^{(1-t)A+tB},$$

when $A, B \in \mathbb{H}_n$, and $t \in [0,1]$. We study basic properties and inequalities for $F_t(A,B)$ and extend some of the results on $F_t(A,B)$ to symmetric space of noncompact type.

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2. Properties and inequalities for $F_t(A,B)$. In this section, we establish some basic properties and inequalities for $F_t(A,B)$. Given $A, B \in \mathbb{H}_n$, we denote by $A \leq B$ the Löwner order, that is, $B - A \geq 0$, that is, positive semidefinite. First, we recall some known properties [16] of the weighted geometric mean.

Lemma 2.1. *Let $A, B, C, D \in \mathbb{P}_n$ and $t \in [0,1]$. We have*

1. $A\sharp_t B = A^{1-t}B^t$ *if A and B commute.*
2. $(aA)\sharp_t(bB) = a^{1-t}b^t(A\sharp_t B)$ *for $a, b > 0$.*
3. $A\sharp_t B = B\sharp_{1-t}A$.
4. $(A\sharp_t B)^{-1} = A^{-1}\sharp_t B^{-1}$.
5. $M^*(A\sharp_t B)M = (M^*AM)\sharp_t(M^*BM)$ *for any $M \in \mathrm{GL}_n(\mathbb{C})$ which denotes the group of all invertible matrices over $\mathbb{C}$.*
6. *(Löwner–Heinz)* $A\sharp_t B \leq C\sharp_t D$ *if $A \leq C$, $B \leq D$.*
7. $(\lambda A + (1-\lambda)B)\sharp_t(\lambda C + (1-\lambda)D) \geq \lambda(A\sharp_t C) + (1-\lambda)(B\sharp_t D)$, *for $\lambda \in [0,1]$.*
8. $((1-t)A^{-1} + tB^{-1})^{-1} \leq A\sharp_t B \leq (1-t)A + tB$.

The following proposition lists some basic properties of $F_t(A,B)$. Some properties are similar to those of weighted geometric mean [16] and are not hard to prove. Proofs are presented here for the sake of completeness.

Proposition 2.2. *Let $A, B \in \mathbb{P}_n$. The following properties hold for all $t \in [0,1]$.*

1. $F_t(A,B) = A^{1-t}B^t$ *if A and B commute.*
2. $F_t(aA, bB) = a^{1-t}b^t F_t(A,B)$ *for $a, b > 0$.*
3. $U^*F_t(A,B)U = F_t(U^*AU, U^*BU)$ *for $U \in \mathrm{U}(n)$.*
4. $F_t^{-1}(A,B) = F_t(A^{-1}, B^{-1})$.
5. $\det F_t(A,B) = (\det A)^{1-t}(\det B)^t$.
6. $2((1-t)A + tB^{-1})^{-1/2} - A^{2(t-1)} \leq F_t(A,B) \leq [2((1-t)A^{-1} + tB)^{-1/2} - A^{-2(t-1)}]^{-1}$, *where the second inequality holds when $2((1-t)A^{-1} + tB)^{-1/2} - A^{-2(t-1)}$ is invertible.*
7. $\operatorname{Tr} F_t(A,B) \leq (\operatorname{Tr} A)^{1-t}\,(\operatorname{Tr} B)^t \leq (1-t)\operatorname{Tr} A + t\operatorname{Tr} B$.

Proof. (1) Since A and B commute, so do A^{-1} and B. Thus, $A^{-1}\sharp_t B = (A^{-1})^{1-t}B^t$ and we have

$$F_t(A,B) = (A^{-1}\sharp_t B)^{1/2}A^{2-2t}(A^{-1}\sharp_t B)^{1/2} = (A^{-1+t}B^t)^{1/2}A^{2-2t}(A^{-1+t}B^t)^{1/2} = A^{1-t}B^t.$$

(2) For any $a, b > 0$, by Lemma 2.1 (2), we have

$$\begin{aligned} F_t(aA, bB) &= \Big((aA)^{-1}\sharp_t(bB)\Big)^{1/2}(aA)^{2-2t}\Big((aA)^{-1}\sharp_t(bB)\Big)^{1/2} \\ &= a^{1-t}b^t(A^{-1}\sharp_t B)^{1/2}A^{2-2t}(A^{-1}\sharp_t B)^{1/2} \\ &= a^{1-t}b^t F_t(A,B). \end{aligned}$$

(3) Note that $U^*(A\sharp_t B)^{1/2}U = (U^*(A\sharp_t B)U)^{1/2}$ and $U^*A^{2-2t}U = (U^*AU)^{2-2t}$ for any $U \in \mathrm{U}(n)$. Then

$$\begin{aligned} U^*F_t(A,B)U &= U^*(A^{-1}\sharp_t B)^{1/2}A^{2-2t}(A^{-1}\sharp_t B)^{1/2}U \\ &= U^*(A^{-1}\sharp_t B)^{1/2}UU^*A^{2-2t}UU^*(A^{-1}\sharp_t B)^{1/2}U \\ &= \left((U^*\left(A^{-1}\sharp_t B\right)U\right)^{1/2}(U^*AU)^{2-2t}\left(U^*\left(A^{-1}\sharp_t B\right)U\right)^{1/2} \\ &= F_t(U^*AU, U^*BU), \end{aligned}$$

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where the last equality follows from $U^*(A^{-1}\sharp_t B)U = (U^*A^{-1}U)\sharp_t(U^*BU)$.

(4) By Lemma 2.1 (4), we have

$$\begin{aligned} F_t(A,B)^{-1} &= \left[\left(A^{-1}\sharp_t B\right)^{1/2} A^{2-2t}\left(A^{-1}\sharp_t B\right)^{1/2}\right]^{-1} \\ &= \left(A^{-1}\sharp_t B\right)^{-1/2} A^{2t-2}\left(A^{-1}\sharp_t B\right)^{-1/2} \\ &= \left(A\sharp_t B^{-1}\right)^{1/2} A^{2t-2}\left(A\sharp_t B^{-1}\right)^{1/2} \\ &= F_t(A^{-1},B^{-1}). \end{aligned}$$

(5) Since $\det(AB) = \det A \det B$, we obtain

$$\det F_t(A,B) = \det\left(A^{-1}\sharp_t B\right)\det\left(A^{2-2t}\right) = (\det A)^{t-1}(\det B)^t(\det A)^{2-2t} = (\det A)^{1-t}(\det B)^t.$$

(6) Let $X = F_t(A,B)$. By the Arithmetic–Geometric–Harmonic mean inequalities and the operator monotonicity of the function $X \mapsto X^t$ when $t \in [0,1]$, we have

$$\left(\frac{A^{-2(t-1)} + X^{-1}}{2}\right)^{-1} \le A^{2(t-1)}\sharp X = (A^{-1}\sharp_t B)^{1/2} \le \Big((1-t)A^{-1} + tB\Big)^{1/2}. \tag{2.4}$$

Then, we have

$$\frac{A^{-2(t-1)} + X^{-1}}{2} \ge \Big((1-t)A^{-1} + tB\Big)^{-1/2}.$$

Hence,

$$X^{-1} \ge 2\Big((1-t)A^{-1} + tB\Big)^{-1/2} - A^{-2(t-1)}.$$

Consequently,

$$X \le \left[2\Big((1-t)A^{-1} + tB\Big)^{-1/2} - A^{-2(t-1)}\right]^{-1}.$$

Since $F_t(A,B) = (F_t(A^{-1},B^{-1}))^{-1}$, we obtain the first inequality.

Using the second inequality in (2.4) and similar arguments, one can prove the second inequality.

(7) Let ρ and σ be two density matrices. We show that

$$T = \{t \in [0,1] : \operatorname{Tr} F_t(\rho,\sigma) \le 1\} = [0,1].$$

Since $0,1 \in T$, according to the continuity of the trace function it is enough to verify $t = 1/2 \in T$. We have

$$\operatorname{Tr} F_{\frac{1}{2}}(\rho,\sigma) = \operatorname{Tr}(\rho\natural\sigma) = \operatorname{Tr}((\rho^{1/2}\sigma\rho^{1/2})^{1/2}) \le 1.$$

Now, let $\rho = \dfrac{A}{\operatorname{Tr} A}$ and $\sigma = \dfrac{B}{\operatorname{Tr} B}$. By the second property in this proposition, we have

$$\operatorname{Tr}(F_t(\rho,\sigma)) = (\operatorname{Tr} A)^{t-1}(\operatorname{Tr} B)^{-t}\operatorname{Tr}(F_t(A,B)) \le 1.$$

Consequently, $\operatorname{Tr} F_t(A,B) \le (\operatorname{Tr} A)^{1-t}(\operatorname{Tr} B)^t$. $\square$

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Remark 2.3. An analog of Lemma 2.1(3) for $F_t(A,B)$ is not true, that is, the equality $F_t(A,B) = F_{1-t}(B,A)$ does not hold. Indeed, from the last identity we have

$$(A^{-1}\sharp_t B)^{1/2}A^{2-2t}(A^{-1}\sharp_t B)^{1/2} = (A^{-1}\sharp_t B)^{-1/2}B^{2t}(A^{-1}\sharp_t B)^{-1/2},$$

or equivalently,

$$B^{2t} = (A^{-1}\sharp_t B)A^{2-2t}(A^{-1}\sharp_t B).$$

According to the Riccati equation, it implies that

$$A^{-1}\sharp_t B = B^{2t}\sharp A^{2t-2},$$

which is not true.

3. The Lie–Trotter formula for $\boldsymbol{F_t(A,B)}$. In this section, we establish the Lie–Trotter formulas for $F_t(A,B)$. Let us start with the version for two positive definite matrices.

Theorem 3.1. *Let $A,B \in \mathbb{H}_n$ and $t \in [0,1]$. Then*

$$\lim_{p\to 0} F_t^{1/p}(e^{pA}, e^{pB}) = e^{(1-t)A+tB}.$$

Proof. Since $F_t^{-1}(A,B) = F_t(A^{-1},B^{-1})$ we have

$$\lim_{p\to 0^-} F_t^{1/p}\left(e^{pA}, e^{pB}\right) = \lim_{p\to 0^-} F_t^{-1/p}\left(e^{-pA}, e^{-pB}\right) = \lim_{p\to 0^+} F_t^{1/p}\left(e^{pA}, e^{pB}\right).$$

So we only need to prove

$$\lim_{p\to 0^+} F_t(e^{pA}, e^{pB})^{1/p} = e^{(1-t)A+tB}.$$

For $p \in (0,1)$, we may express $p = \frac{1}{m+s}$, where $m \in \mathbb{N}$, and $s \in (0,1)$. Set

$$X(p) := F_t(e^{pA}, e^{pB}), \qquad Y(p) := e^{p[(1-t)A+tB]}.$$

We have

$$\begin{aligned}
&\|F_t(e^{pA}, e^{pB})^{1/p} - e^{(1-t)A+tB}\| \\
&= \|X(p)^{1/p} - Y(p)^{1/p}\| \\
&\le \|X(p)^{1/p} - X(p)^m\| + \|X(p)^m - Y(p)^m\| + \|Y(p)^m - Y(p)^{1/p}\|. \qquad (3.5)
\end{aligned}$$

By [17, Theorem 1.1],

$$e^{pA}\sharp_t e^{pB} \prec_{\log} e^{p[(1-t)A+tB]},$$

so we have

$$\|e^{pA}\sharp_t e^{pB}\| \le \|Y(p)\| \le e^{p[(1-t)\|A\|+t\|B\|]}.$$

Therefore,

$$\begin{aligned}
\|X(p)\| &= \|\left(e^{-pA}\sharp_t e^{pB}\right)^{\frac{1}{2}} e^{p(2-2t)A}\left(e^{-pA}\sharp_t e^{pB}\right)^{\frac{1}{2}}\| \\
&\le \|e^{-pA}\sharp_t e^{pB}\|^{\frac{1}{2}}\|e^{p(2-2t)A}\|\,\|e^{-pA}\sharp_t e^{pB}\|^{\frac{1}{2}} \\
&\le e^{\frac{p}{2}[(1-t)\|A\|+t\|B\|]}e^{p(2-2t)\|A\|}e^{\frac{p}{2}[(1-t)\|A\|+t\|B\|]} \\
&= e^{p[(3-3t)\|A\|+t\|B\|]}.
\end{aligned}$$

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As $pm \leq 1$, we have $\|X(p)\|^m \leq e^{pm[(3-3t)\|A\|+2t\|B\|]} \leq e^{[(3-3t)\|A\|+t\|B\|]} < \infty$. Consequently, the first term in (3.5)

$$\|X(p)^{1/p} - X(p)^m\| = \|X(p)^{m+s} - X(p)^m\| \leq \|X(p)\|^m \|X(p)^s - I\| \to 0 \quad \text{as} \quad p \to 0^+,$$

since $X(p) \to I$ as $p \to 0^+$ by (1.3) and $s \in (0,1)$. Similarly, the third term in (3.5)

$$\|Y(p)^m - Y(p)^{1/p}\| = \|Y(p)^m - Y(p)^{m+s}\| \leq \|Y(p)\|^m \|I - Y(p)^s\| \to 0 \quad \text{as} \quad p \to 0^+.$$

Now the second term in (3.5)

$$\|X(p)^m - Y(p)^m\| = \|\sum_{j=0}^{m-1} X(p)^{m-1-j}(X(p) - Y(p))Y(p)^j\| \leq mM^{m-1}\|X(p) - Y(p)\|,$$

where $M := \max\{\|X(p)\|, \|Y(p)\|\}$. As $p(m-1) \leq 1$, we have

$$\begin{aligned} M^{m-1} &\leq \max\left\{e^{p(m-1)[(3-3t)\|A\|+t\|B\|}, e^{p(m-1)[(1-t)\|A\|+t\|B\|]}\right\} \\ &\leq \max\left\{e^{(3-3t)\|A\|+t\|B\|}, e^{(1-t)\|A\|+t\|B\|}\right\} \\ &< \infty. \end{aligned}$$

Using the power series expansion of the matrix exponential $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$, we have

$$\begin{aligned} & e^{-pA} \sharp_t e^{pB} \\ &= e^{\frac{-pA}{2}} \left(e^{\frac{pA}{2}} e^{pB} e^{\frac{pA}{2}}\right)^t e^{\frac{-pA}{2}} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-pA}{2}\right)^k \left[\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{pA}{2}\right)^k \sum_{k=0}^{\infty} \frac{(pB)^k}{k!} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{pA}{2}\right)^k\right] \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{-pA}{2}\right)^k \\ &= \left(I - \frac{pA}{2} + o(p)\right)\left[\left(I + \frac{pA}{2} + o(p)\right)(I + pB + o(p))\left(I + \frac{pA}{2} + o(p)\right)\right]^t \left(I - \frac{pA}{2} + o(p)\right) \\ &= \left(I - \frac{pA}{2} + o(p)\right)[I + p(A+B) + o(p)]^t \left(I - \frac{pA}{2} + o(p)\right) \\ &= I + p[-(1-t)A + tB] + o(p), \end{aligned}$$

and

$$e^{p(2-2t)A} = \sum_{k=0}^{\infty} \frac{1}{k!}(p(2-2t)A)^k = I + p(2-2t)A + o(p).$$

Hence,

$$\begin{aligned} X(p) &= \left(e^{-pA} \sharp_t e^{pB}\right)^{\frac{1}{2}} e^{p(2-2t)A} \left(e^{-pA} \sharp_t e^{pB}\right)^{\frac{1}{2}} \\ &= [I + p(-(1-t)A + tB) + o(p)]^{\frac{1}{2}} [I + p(2-2t)A + o(p)] [I + p(-(1-t)A + tB)] \frac{1}{2} \\ &= \left[I + \frac{p}{2}(-(1-t)A + tB) + o(p)\right] [I + p(2-2t)A + o(p)] \left[I + \frac{p}{2}(-(1-t)A + tB) + o(p)\right] \\ &= I + p((1-t)A + tB) + o(p). \end{aligned}$$

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As $Y(p) := e^{p[(1-t)A+tB]} = I + p((1-t)A + tB) + o(p)$, we have $\|X(p) - Y(p)\| \le cp^2$ for some constant c. Then

$$\|X(p)^m - Y(p)^m\| \le mM^{m-1}cp^2 \le \frac{m}{(m+s)}M^{m-1}cp \to 0 \quad \text{as} \quad p \to 0^+,$$

since M^{m-1} is bounded. Thus, all three terms in (3.5) converge to 0 as $p \to 0^+$ and hence the proof is completed. ∎

THEOREM 3.2. *Let $(\alpha_1, ..., \alpha_{m-1}) \in \mathbb{R}^{m-1}$, and $X_1, X_2, ..., X_m \in \mathbb{H}_n$. The curve*

$$\gamma(t) := F_{\alpha_{m-1}}\left(e^{tX_m}, F_{\alpha_{m-2}}\left(e^{tX_{m-1}}, F_{\alpha_{m-3}}(...F_{\alpha_1}(e^{tX_2}, e^{tX_1})\right)\right),$$

is a differentiable curve with $\gamma(0) = I$ and

$$\gamma'(0) = \sum_{k=1}^{m}\prod_{i=k}^{m}\alpha_i\,(1-\alpha_{k-1})\,X_k,$$

where $\alpha_0 = 0$ and $\alpha_m = 1$. In particular, if $\alpha_k = \dfrac{k}{k+1}$, for $k = 1, 2, \dots, m-1$, then $\gamma'(0) = \dfrac{1}{m}\sum_{k=1}^{m} X_k$.

Proof. Let

$$\begin{aligned}\beta(t) := F_{\alpha_1}\left(e^{tX_2}, e^{tX_1}\right) &= \left(e^{-tX_2}\sharp_{\alpha_1}e^{tX_1}\right)^{\frac{1}{2}} e^{t(2-2\alpha_1)X_2}\left(e^{-tX_2}\sharp_{\alpha_1}e^{tX_1}\right)^{\frac{1}{2}} \\ &= \varphi(t)^{\frac{1}{2}}e^{t(2-2\alpha_1)X_2}\varphi(t)^{\frac{1}{2}},\end{aligned}$$

where $\varphi(t) = e^{-tX_2}\sharp_{\alpha_1}e^{tX_1} = e^{-\frac{tX_2}{2}}\left(e^{\frac{tX_2}{2}}e^{tX_1}e^{\frac{tX_2}{2}}\right)^{\alpha_1}e^{-\frac{tX_2}{2}}$. We have

$$\begin{aligned}&\frac{d}{dt}\varphi(t) \\ &= -\frac{X_2}{2}e^{-\frac{tX_2}{2}}\left(e^{\frac{tX_2}{2}}e^{tX_1}e^{\frac{tX_2}{2}}\right)^{\alpha_1}e^{-\frac{tX_2}{2}} - e^{-\frac{tX_2}{2}}\left(e^{\frac{tX_2}{2}}e^{tX_1}e^{\frac{tX_2}{2}}\right)^{\alpha_1}e^{-\frac{tX_2}{2}}\frac{X_2}{2} \\ &\quad +\alpha_1 e^{-\frac{tX_2}{2}}\left(e^{\frac{tX_2}{2}}e^{tX_1}e^{\frac{tX_2}{2}}\right)^{\alpha_1-1}\frac{d}{dt}\left(e^{\frac{tX_2}{2}}e^{tX_1}e^{\frac{tX_2}{2}}\right)e^{-\frac{tX_2}{2}} \\ &= -\frac{X_2}{2}e^{-\frac{tX_2}{2}}\left(e^{\frac{tX_2}{2}}e^{tX_1}e^{\frac{tX_2}{2}}\right)^{\alpha_1}e^{-\frac{tX_2}{2}} - e^{-\frac{tX_2}{2}}\left(e^{\frac{tX_2}{2}}e^{tX_1}e^{\frac{tX_2}{2}}\right)^{\alpha_1}e^{-\frac{tX_2}{2}}\frac{X_2}{2} \\ &\quad +\alpha_1 e^{-\frac{tX_2}{2}}\left(e^{\frac{tX_2}{2}}e^{tX_1}e^{\frac{tX_2}{2}}\right)^{\alpha_1-1}\left(\frac{X_2}{2}e^{\frac{tX_2}{2}}e^{tX_1}e^{\frac{tX_2}{2}} + e^{\frac{tX_2}{2}}e^{tX_1}e^{\frac{tX_2}{2}}\frac{X_2}{2} + e^{\frac{tX_2}{2}}e^{tX_1}X_1e^{\frac{tX_2}{2}}\right)e^{-\frac{tX_2}{2}}.\end{aligned}$$

Therefore,

$$\left.\frac{d}{dt}\varphi(t)\right|_{t=0} = -X_2 + \alpha_1(X_2 + X_1) = (\alpha_1 - 1)X_2 + \alpha_1X_1.$$

On the other hand,

$$\begin{aligned}\frac{d}{dt}\beta(t) = &\frac{1}{2}\varphi(t)^{-\frac{1}{2}}\frac{d}{dt}\varphi(t)e^{t(2-2\alpha_1)X_2}\varphi(t)^{\frac{1}{2}} + \frac{1}{2}\varphi(t)^{\frac{1}{2}}e^{t(2-2\alpha_1)X_2}\varphi(t)^{-\frac{1}{2}}\frac{d}{dt}\varphi(t) \\ &+(2-2\alpha_1)\varphi(t)^{\frac{1}{2}}e^{t(2-2\alpha_1)X_2}X_2\varphi(t)^{\frac{1}{2}}.\end{aligned}$$

Thus,

$$\left.\frac{d}{dt}\beta(t)\right|_{t=0} = (\alpha_1 - 1)X_2 + \alpha_1X_1 + (2-2\alpha_1)X_2 = (1-\alpha_1)X_2 + \alpha_1X_1.$$

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Set

$$\xi(t) := F_{\alpha_m}\left(e^{tX_{m+1}}, \gamma(t)\right) = L(t)^{\frac{1}{2}} e^{t(2-2\alpha_m)X_{m+1}} L(t)^{\frac{1}{2}},$$

where $L(t) = e^{-tX_{m+1}} \sharp_{\alpha_m} \gamma(t)$. Since

$$\begin{aligned}\frac{d}{dt}\xi(t) = \frac{1}{2}L(t)^{-\frac{1}{2}}\frac{d}{dt}L(t)e^{t(2-2\alpha_m)X_{m+1}}L(t)^{\frac{1}{2}} + \frac{1}{2}L(t)^{\frac{1}{2}}e^{t(2-2\alpha_m)X_{m+1}}L(t)^{-\frac{1}{2}}\frac{d}{dt}L(t)\\ +(2-2\alpha_m)L(t)^{\frac{1}{2}}e^{t(2-2\alpha_m)X_{m+1}}X_{m+1}L(t)^{\frac{1}{2}},\end{aligned}$$

by the previous argument, we have

$$\frac{d}{dt}L(t)\bigg|_{t=0} = -X_{m+1} + \alpha_m(X_{m+1} + \gamma'(0)).$$

Therefore,

$$\frac{d}{dt}\xi(t)\bigg|_{t=0} = (1-\alpha_m)X_{m+1} + \sum_{k=1}^{m}\prod_{i=k}^{m}\alpha_i\left(1-\alpha_{k-1}\right)X_k = \sum_{k=1}^{m+1}\prod_{i=k}^{m+1}\alpha_i\left(1-\alpha_{k-1}\right)X_k,$$

where $\alpha_0 = 0$ and $\alpha_{m+1} = 1$. □

4. Generalizations to semisimple Lie groups. The readers are referred to [8, 9] regarding Lie theoretic preliminaries and notations. Let G be a noncompact connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $\Theta : G \to G$ be a Cartan involution of G, and let K be the fixed point set of Θ, which is is an analytic subgroup of G. Denote by θ the differential map $d\Theta$ of Θ. Then $\theta : \mathfrak{g} \to \mathfrak{g}$ is a Cartan involution. and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a *Cartan decomposition*, where $\mathfrak{k}$ is the eigenspace of θ corresponding to the eigenvalue 1 and $\mathfrak{p}$ is the eigenspace of θ corresponding to the eigenvalue -1. For each $X \in \mathfrak{g}$, let $e^X = \exp X$ be the exponential of X. Let $P = \{e^X : X \in \mathfrak{p}\}$. The map $\mathfrak{p} \times K \to G$, defined by $(X, k) \mapsto e^X k$, is a diffeomorphism. So each $g \in G$ can be uniquely written as:

$$g = pk, \tag{4.6}$$

with $p = p(g) \in P$ and $k = k(g) \in K$. The decomposition $G = PK$ is called a *Cartan decomposition of* G. For example, when $G = \mathrm{SL}_n(\mathbb{C})$, $K = \mathrm{SU}(n)$, and $P \subset \mathbb{P}_n$ consisting of positive definite matrices of determinant 1.

The mapping $p \mapsto p^{1/2}K$ identifies P with G/K as a symmetric space of noncompact type. The t-geometric mean of $p, q \in P$ was defined in [17] as:

$$p\sharp_t q = p^{1/2}\left(p^{-1/2}qp^{-1/2}\right)^t p^{1/2}, \quad t \in [0,1].$$

It is the unique geodesic in P from p (at $t = 0$) to q (at $t = 1$). It is known that $p\sharp_t q = q\sharp_{1-t} p$ and $(p\sharp_t q)^{-1} = p^{-1}\sharp_t q^{-1}$. When $t = 1/2$, we abbreviate $p\sharp_{1/2} q$ as $p\sharp q$.

Denote by $\prec_G$ the Kostant pre-order on G [14, 17, 8, 9]. Given $f, g \in G$, by setting $f \prec_G g$ means (see [14, p. 426])

$$\mathcal{A}(f) \subset \mathcal{A}(g), \tag{4.7}$$

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where

$$\mathcal{A}(g) := \exp \operatorname{conv}(W(\log h(g))) \subset A,$$

in which conv $(\cdot)$ denotes the convex hull of the underlying set, W is the Weyl group of $(\mathfrak{g}, \mathfrak{a})$, $\mathfrak{a}$ is a fixed maximal abelian subalgebra in $\mathfrak{p}$, with A as the analytic group, and $h(g)$ is the hyperbolic component of g in its complete multiplicative Jordan decomposition. It is known from [14, Theorem 3.1] that this pre-order $\prec_G$ is independent of the choice of $\mathfrak{a}$.

Let $p, q \in P$. Define

$$F_t(p,q) = (p^{-1}\sharp_t q)^{1/2} p^{2-2t} (p^{-1}\sharp_t q)^{1/2}, \quad t \in [0,1].$$

Some properties in Proposition 2.2 can be extended to P.

PROPOSITION 4.1. *Let $p, q \in P$. The following properties hold for all $t \in [0,1]$.*

1. $F_t(p,q) = p^{1-t}q^t$ *if p and q commute.*
2. $kF_t(p,q)k^{-1} = F_t(kpk^{-1}, kqk^{-1})$ *for $k \in K$.*
3. $F_t^{-1}(p,q) = F_t(p^{-1}, q^{-1})$.

Proof. (1) When p and q commute, so do p^{-1} and q. Thus, $p^{-1}\sharp_t q = (p^{-1})^{1-t}q^t$ and hence

$$F_t(p,q) = (p^{-1}\sharp_t q)^{1/2} p^{2-2t} (p^{-1}\sharp_t q)^{1/2} = (p^{-1})^{1-t}q^t p^{2-2t}(p^{-1})^{1-t}q^t = p^{1-t}q^t.$$

The proof of (2) and (3) are similar to Proposition 2.2 (3) and (4), respectively. ∎

Similarly, Theorem 3.1 can be extended to P.

THEOREM 4.2. *For $X, Y \in \mathfrak{p}$ and $t \in [0,1]$,*

$$\lim_{s\to 0} \mathcal{A}(F_t^{1/s}(e^{sX}, e^{sY})) = \mathcal{A}(e^{(1-t)X+tY}). \tag{4.8}$$

Proof. We will first show that

$$\pi(F_t^{1/r}(p^r, q^r)) = F^{1/r}((\pi(p))^r, (\pi(q))^r). \tag{4.9}$$

To show this, note that for finite dimensional representation π, there exists an inner product on V such that $\pi(z)$ is positive definite for all $z \in P$ [14, p. 435]. As $\pi(p^{-1}\sharp_t q) = (\pi(p))^{-1}\sharp_t \pi(q)$, we have

$$\begin{aligned}
\pi(F_t^{1/r}(p^r, q^r)) &= (\pi(F_t(p^r, q^r)))^{1/r} \\
&= (\pi[(p^{-1}\sharp_t q)^{1/2} p^{2-2t} (p^{-1}\sharp_t q)^{1/2}])^{1/r} \\
&= ((\pi((p^{-1}\sharp_t q))^{1/2} (\pi(p))^{2-2t} (\pi((p^{-1}\sharp_t q))^{1/2})^{1/r} \\
&= (((\pi(p))^{-1}\sharp_t \pi(q))^{1/2} (\pi(p))^{2-2t} ((\pi(p))^{-1}\sharp_t \pi(q))^{1/2})^{1/r} \\
&= F^{1/r}((\pi(p))^r, (\pi(q))^r).
\end{aligned}$$

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By (4.9) and the fact that $\pi(e^X) = e^{d\pi(X)}$, where $X \in \mathfrak{p}$, for any finite dimensional representation π of G, where $d\pi$ denotes the differential of π at the identity, we have

$$\begin{aligned}
\pi(\lim_{s\to 0} F_t^{1/s}(e^{sX}, e^{sY})) &= \lim_{s\to 0} \pi(F_t^{1/s}(e^{sX}, e^{sY})) \\
&= \lim_{s\to 0} F_t^{1/s}(\pi(e^{sX}), \pi(e^{sY})) \\
&= \lim_{s\to 0} F_t^{1/s}(e^{s\, d\pi(X)}, e^{s\, d\pi(Y)}) \\
&= e^{(1-t)d\pi(X)+td\pi(Y)} \quad \text{(by Theorem 3.1)} \\
&= \pi(e^{(1-t)X+tY}).
\end{aligned}$$

By [14, Theorem 3.1], the compact convex sets $\mathcal{A}(F_t^{1/s}(e^{sX}, e^{sY})) \subset A$ converge to $\mathcal{A}(e^{(1-t)X+tY})$ with respect to the Hausdorff metric for convex sets, as $s \to 0$, that is, the limit (4.8) holds. ∎

We conjecture that $\lim_{s\to 0} F_t^{1/s}\left(e^{sX}, e^{sY}\right) = e^{(1-t)X+tY}$ for $X, Y \in \mathfrak{p}$.

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