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COMMUTATORS OF SKEW-INVOLUTIONS*

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Abstract. Let $\mathrm{SL}_n(\mathbb{F})$ be the group of all $n \times n$ matrices over a field $\mathbb{F}$ with determinant 1. Denote by I (I_n) the ($n \times n$) identity matrix. A matrix A is called skew-involution if $A^2 = -I$. It is proved that every matrix in $\mathrm{SL}_{2n}(\mathbb{F})$ is a product of at most three commutators of skew-involutions if $\mathbb{F} \neq \mathbb{Z}_3$ and $\mathrm{SL}_{2n}(\mathbb{F}) \neq \mathrm{SL}_2(\mathbb{Z}_2)$, and at most four commutators of skew-involutions if $\mathbb{F} = \mathbb{Z}_3$ and $n > 1$. Every complex symplectic matrix is a product of two commutators of complex symplectic skew-involutions, and every real symplectic matrix is a product of not more than four commutators of real symplectic skew-involutions.

Key words. Involution, Skew-involution, Commutator, Symplectic matrix.

AMS subject classifications. 15A23, 20H20.

1. Introduction. Let $M_n(\mathbb{F})$ be the set of all $n \times n$ matrices with over a field $\mathbb{F}$ and let $\mathrm{SL}_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) |\det A = 1\}$. An involution (a skew-involution) is a matrix A satisfying $A^2 = I$ ($A^2 = -I$). Let $\Omega_{2n} \in M_{2n}(\mathbb{F})$ be the skew-involution

$$\Omega_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

An $A \in M_{2n}(\mathbb{F})$ is called symplectic if it satisfies $A^{\mathrm{T}}\Omega_{2n}A = \Omega_{2n}$.

Expressing elements of some matrix groups as products of involutions is an interesting topic discussed by many scholars (see e.g. [1, 5, 6, 11, 14, 18]). Denote by $[X, Y] = XYX^{-1}Y^{-1}$ the commutator of matrices X and Y. Decomposing matrices into commutators of involutions has also drawn the attention of scholars (see e.g. [7, 8, 9, 10, 16, 19]). In [13], Joven and Paras discussed the decomposition into skew-involutions of the elements of $\mathrm{SL}_{2n}(\mathbb{F})$ for an arbitrary field $\mathbb{F}$ and the symplectic matrix groups over the real field $\mathbb{R}$ and the complex field $\mathbb{C}$. In this article, we consider the decompositions into commutators of skew-involutions.

If $p(x) = x^2 + 1$ has a root $a \in \mathbb{F}$, then A is a commutator of two involutions X and Y if and only if A is a commutator of two skew-involutions aX and aY. Denote by $\mathrm{char}\mathbb{F}$ the characteristic of $\mathbb{F}$. If $\mathrm{char}\mathbb{F} \neq 2$, then by Theorem 2.8 in [7], every element of the group $\mathrm{SL}_n(\mathbb{F})$ can be written as a product of at most two commutators of involutions, and therefore, it can also be written as a product of at most two commutators of skew-involutions. If $\mathrm{char}\mathbb{F} = 2$, then an involution is also a skew-involution, and every commutator of involutions is also a commutator of skew-involutions. Son *et al.* gave conclusions in Theorem 1 and Propositions 2 and 3 of [16] that every matrix in $\mathrm{SL}_n(\mathbb{F})(n > 1)$ is a product of at most two commutators of (skew-)involutions when $\mathbb{F}$ has at least three elements, a product of at most three commutators of (skew-)involutions when $\mathbb{F} = \mathbb{Z}_2$ and n is odd, and every matrix in $\mathrm{SL}_2(\mathbb{Z}_2)$ is a commutator of (skew-)involutions.

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If $p(x) = x^2 + 1$ has no root in $\mathbb{F}$, then the minimal polynomial of a skew-involution $P \in \mathrm{M}_n(\mathbb{F})$ is the irreducible polynomial $p(x)$. By the rational canonical form theorem, P is similar to

$$\bigoplus_k C(p(x)) = \bigoplus_k \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Thus, $n = 2k$ is even, and P is similar to Ω_{2k}. Hence, every skew-involution in $\mathrm{M}_{2k}(\mathbb{F})$ is in $\mathrm{SL}_{2k}(\mathbb{F})$ for some $k \in \mathbb{N}$, and it is similar to Ω_{2k}. In particular, if A is a product of commutators of skew-involutions, then $A \in \mathrm{SL}_{2k}(\mathbb{F})$.

In this paper, we consider products of commutators of skew-involutions in $\mathrm{SL}_{2n}(\mathbb{F})$. In Section 2, we give some elementary properties of commutators of skew-involutions. We show in Section 3 that every $A \in \mathrm{SL}_{2n}(\mathbb{F})$ is a product of commutators of skew-involutions if and only if $\mathbb{F} \neq \mathbb{Z}_2, \mathbb{Z}_3$ or $n > 1$. In Section 4, we prove that every real symplectic matrix is a product of four commutators of real symplectic skew-involutions. While in Section 5, we give a necessary and sufficient condition for a symplectic matrix over the complex number field $\mathbb{C}$ to be a commutator of symplectic skew-involutions and prove that every complex symplectic matrix is a product of two commutators of complex symplectic skew-involutions.

2. Preliminaries. Denote by $\mathrm{Sp}(2n, \mathbb{F})$ the subgroup of $\mathrm{SL}_{2n}(\mathbb{F})$ consisting of all the symplectic matrices over a field $\mathbb{F}$. The following gives a necessary and sufficient condition for a nonsingular matrix to be a product of two (symplectic) skew-involutions

LEMMA 2.1. *If $A \in M_n(\mathbb{F})$ is nonsingular, then A is a product of two (symplectic) skew-involutions if and only if there is a (symplectic) skew-involution P such that $P^{-1}AP = A^{-1}$.*

Proof. Let $A = ST$ be a product of two (symplectic) matrices with $S^2 = T^2 = -I_n$. If we take $P = S$, then $P^{-1}AP = -S(ST)S = TS = A^{-1}$. Conversely, if $P^{-1}AP = A^{-1}$ such that P is a (symplectic) skew-involution, then $A = (-P)(PA)$. Here, $-P$ is a (symplectic) skew-involution and so is PA, since $(-P)^2 = P^2 = -I_n$ and $(PA)^2 = -(P^{-1}AP)A = -A^{-1}A = -I_n$. ∎

Since the commutator of two (symplectic) skew-involutions S and T is $STS^{-1}T^{-1} = ST(-S)(-T) = (ST)^2$, we have the following by Lemma 2.1.

COROLLARY 2.2. *If $A \in M_n(\mathbb{F})$ is nonsingular, then A is a commutator of (symplectic) skew-involutions if and only if there is a $B \in M_n(\mathbb{F})$ such that $B^2 = A$ and B is a product of two (symplectic) skew-involutions, if and only if there is a $B \in M_n(\mathbb{F})$ such that $B^2 = A$ and B is similar to its inverse by a (symplectic) skew-involution.*

Denote by $A \oplus B$ the direct sum of the square matrices A and B. By Corollary 2.2, $A^2 \oplus A^{-2}$ is a commutator of skew-involutions since $A^2 \oplus A^{-2} = (A \oplus A^{-1})^2$ and $\Omega_{2n}^{-1}(A \oplus A^{-1})\Omega_{2n} = (A \oplus A^{-1})^{-1}$. Observe that

$$A^2 \oplus A^{-2} = \left[\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & -A^{-1} \\ A & 0 \end{pmatrix} \right] = \left[\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & A^{-1} \\ A & 0 \end{pmatrix} \right],$$

is a commutator of two skew-involutions and also a commutator of two involutions. If A is symmetric, then the two skew-involutions are also symplectic skew-involutions. Then, we have the following.

LEMMA 2.3. *If $A \in M_n(\mathbb{F})$ is nonsingular, then $A^2 \oplus A^{-2}$ is*

(a) *a commutator of involutions,*

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(b) *a commutator of skew-involutions, and*

(c) *a commutator of symplectic skew-involutions, when A is symmetric.*

Let $A=(A_{ij})\in M_{2k}(\mathbb{F})$ and $B=(B_{ij})\in M_{2l}(\mathbb{F})$, where each $A_{ij}\in M_k(\mathbb{F})$ and each $B_{ij}\in M_l(\mathbb{F})$. The *expanding sum* of the matrices A and B is defined to be

$$A\boxplus B=\begin{pmatrix} A_{11}\oplus B_{11} & A_{12}\oplus B_{12}\\ A_{21}\oplus B_{21} & A_{22}\oplus B_{22}\end{pmatrix}\in M_{2k+2l}(\mathbb{F}).$$

One can check that $A\boxplus B$ is permutation similar to $A\oplus B$. Moreover, $A\boxplus B$ is symplectic if and only if both A and B are symplectic. The preceding is also true if "symplectic" is replaced with "involution," or "skew-involution." In addition, if $C\in M_{2k}(\mathbb{F})$ and $D\in M_{2l}(\mathbb{F})$, then $(A\boxplus B)(C\boxplus D)=(AC)\boxplus(BD)$. The preceding facts give the following remark.

Remark 2.4. *Let $\mathcal{CSI}(m)$ be the set of all products of m commutators of (symplectic) skew-involutions over $\mathbb{F}$.*

(a) $\mathcal{CSI}(1)\subset\mathcal{CSI}(2)\subset\mathcal{CSI}(3)\subset\cdots$.

(b) *$\mathcal{CSI}(m)$ is closed under expanding summation, i.e., if $A\in\mathcal{CSI}(m)$, $B\in\mathcal{CSI}(m)$, then $A\boxplus B\in\mathcal{CSI}(m)$. When $\mathcal{CSI}(m)$ is just the set of all products of m commutators of skew-involutions, "expanding summation" can be replaced by "direct sum."*

(c) *$\mathcal{CSI}(m)$ is invariant under (symplectic) similarity, i.e., if $A\in\mathcal{CSI}(m)$ and P is a (symplectic) matrix of the same size as A, then $P^{-1}AP\in\mathcal{CSI}(m)$.*

(d) *If $A\in\mathcal{CSI}(m)$, then $A^{-1},A^{\top}\in\mathcal{CSI}(m)$.*

Remark 2.4 also holds for the set of all products of m (symplectic) involutions and the conclusions (b), (c) and (d) are true for the set of all products of m (symplectic) skew-involutions.

Lemma 2.5. *If the characteristic of the field $\mathbb{F}$ is prime, then the quadratic equation $x^2+y^2+1=0$ in x and y has solutions in $\mathbb{F}$.*

Proof. If $\mathrm{char}\mathbb{F}=2$, then $(0,1)$ is a solution of $x^2+y^2+1=0$. If $\mathrm{char}\mathbb{F}=p\neq 2$, then $\mathbb{Z}_p$ is a subfield of $\mathbb{F}$. Suppose that ξ is a generator of $\mathbb{Z}_p^*$. Since $\mathbb{Z}_p^2$ has $\frac{p+1}{2}$ square elements, it must intersect the set $p-1-\mathbb{Z}_p^2=\{p-1,p-1-\xi^2,p-1-\xi^4,\ldots,p-1-\xi^{2p-2}\}$. Thus, there exist two elements x and y in $\mathbb{Z}_p$ satisfying $x^2+y^2+1=0$. ∎

If the characteristic is zero, there are fields wherein $x^2+1=0$ has no root but $x^2+y^2+1=0$ has solutions in them. Here, we give an example. Let $\mathbb{Q}(\sqrt{-2})=\mathbb{Q}[t]/(t^2+2)$. Then, $x^2+1=0$ has no root in $\mathbb{Q}[t]/(t^2+2)$ but $(t,1)$ is a solution of $x^2+y^2+1=0$ in that field.

Let $h(x)=\sum_{i=0}^{l}c_ix^i$ be a monic polynomial over $\mathbb{F}$ with positive degree l.

$$C(h)=\begin{pmatrix}0 & \cdots & 0 & -c_0\\ 1 & & & -c_1\\ & \ddots & & \vdots\\ & & 1 & -c_{l-1}\end{pmatrix}\in M_l(\mathbb{F}),$$

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is called the Frobenius block or companion matrix of $h(x)$. The Frobenius (rational) canonical form theorem states that if $A \in M_n(\mathbb{F})$, then A is similar to a direct sum of Frobenius blocks $\oplus_{i=1}^k C(h_i)$, where each $h_i = p_i(x)^{l_i}$ for some positive integer l_i and irreducible polynomials $p_i(x)$.

Lemma 2.6. *Let $A \in M_n(\mathbb{F})$. The determinant of $A^2 + I_n$ is a sum of two squares.*

Proof. We first consider the companion matrix of $h(x)$. One can check that the determinant of $C(h)^2 + I_l$ equals $a^2 + b^2$, where

$$a = 1 + \sum_{i=1}^{\frac{l-1}{2}} (-1)^i c_{l-2i}, \ \ b = \sum_{i=1}^{\frac{l+1}{2}} (-1)^i c_{l+1-2i},$$

if l is odd, while

$$a = 1 + \sum_{i=1}^{\frac{l}{2}} (-1)^i c_{l-2i}, \ \ b = \sum_{i=1}^{\frac{l}{2}} (-1)^i c_{l+1-2i},$$

if l is even.

Now we finish the proof. By the rational canonical form theorem, A is similar to a direct sum of companion matrices $\oplus_{i=1}^k C(h_i)$. Suppose the degree of h_i is l_i. Then, $A^2 + I_n$ is similar to $\oplus_{i=1}^k (C(h_i)^2 + I_{l_i})$. The determinant of $C(h_i)^2 + I_{l_i}$ can be written as $a_i^2 + b_i^2$ for some a_i and b_i in $\mathbb{F}$. Then the determinant of $A^2 + I_n$ is $\det(A^2 + I_n) = \Pi_{i=1}^k \det(C(h_i)^2 + I_{l_i}) = \Pi_{i=1}^k (a_i^2 + b_i^2)$. Since the product $(a_i^2 + b_i^2)(a_j^2 + b_j^2) = (a_i a_j + b_i b_j)^2 + (a_i b_j - b_i a_j)^2$ is still a sum of squares of two elements in $\mathbb{F}$, by induction we know that the determinant of $A^2 + I_n$ can be written as $p^2 + q^2$ for some $p, q \in \mathbb{F}$. ∎

3. Commutators of skew-involutions in $\mathrm{SL}_{2n}(\mathbb{F})$**.** Let $A \in \mathrm{SL}_{2n}(\mathbb{F})$. From Section 3.2 of [13], we know that $\mathrm{SL}_2(\mathbb{Z}_3)$ cannot be generated by skew-involutions, and it is known that $\mathrm{SL}_2(\mathbb{Z}_3)$ properly contains its commutator subgroup (see [4] Section 107). So we divide our discussion into two cases: (i)$|\mathbb{F}| \geq 4$ and (ii) $n > 1$ and $\mathbb{F} = \mathbb{Z}_3$.

3.1. Case when $|\mathbb{F}| \geq 4$**.** We first consider nonscalar matrices with determinant 1, the case $A = -I_{2n}$, and finally $A = \alpha I_{2n}$ with $\alpha^{2n} = 1, \alpha \neq \pm 1$.

For the nonscalar matrices of determinant 1, we prove the following theorem.

Theorem 3.1. *If $\mathbb{F}$ is a field with at least four elements, then every nonscalar $A \in \mathrm{SL}_{2n}(\mathbb{F})$ is a product of at most two commutators of skew-involutions.*

Proof. If $\mathbb{F} = \mathbb{Z}_5$ or if the characteristic of the field is two, then the result follows from [Son *et al.* [16], Theorem 1] since every matrix in $\mathrm{SL}_{2n}(\mathbb{F})$ is a product of at most 2 commutators of involutions, but in such $\mathbb{F}$ the equation $x^2 + 1$ has a root, so commutators of involutions can be replaced by commutators of skew-involutions.

Suppose that $\mathbb{F} \neq \mathbb{Z}_5$ has at least 4 elements and the characteristic is different from 2. Then there exists $\alpha \in \mathbb{F}$ such that $\alpha^2 \neq \alpha^{-2}$ and since $\mathbb{F}$ has at least 4 elements, by Main Theorem of [2] one can write the non-scalar $A \in \mathrm{SL}_{2n}(\mathbb{F})$ as $A = BC$ where B and C are nonderogatory with eigenvalues α^2 (n times) and α^{-2} (n times). Both B and C are similar to the direct sum of the companion matrices $C(p(x)) \oplus C(q(x))$ where $p(x) = (x - \alpha^2)^n$ and $q(x) = (x - \alpha^{-2})^n$. The matrix $C(p(x))$ is similar to $(C((x-\alpha)^n))^2$ and $C(q(x))$ is similar to $(C((x-\alpha)^n))^{-2}$. Then the claim follows by Lemma 2.3(b) and Remark 2.4(c). ∎

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Now suppose A is scalar. If $B \in M_{2n}(\mathbb{F})$ is a skew-involution, then $I_{2n} = [B, B]$. If $A = -I_{2n}$, we have the following lemma.

Lemma 3.2. *If n is even or there exist $a, b \in \mathbb{F}$ such that $a^2 + b^2 + 1 = 0$, then $-I_{2n}$ is a commutator of skew-involutions in $\mathrm{SL}_{2n}(\mathbb{F})$. Otherwise, $-I_{2n}$ is not a commutator of skew-involutions in $\mathrm{SL}_{2n}(\mathbb{F})$ and $-I_2$ is a product of three commutators of skew-involutions in $\mathrm{SL}_2(\mathbb{F})$ and no fewer.*

Proof. If $n = 2k$ is even, then

$$-I_{2n} = -I_{4k} = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}^2 \oplus \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}^{-2},$$

is a commutator of skew-involutions in $\mathrm{SL}_{2n}(\mathbb{F})$ by Lemma 2.3(b). If there exist $a, b \in \mathbb{F}$ such that $a^2+b^2+1 = 0$, one checks that

$$-I_{2n} = \left[\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} aI_n & bI_n \\ bI_n & -aI_n \end{pmatrix} \right],$$

and the two matrices of the commutator are skew-involutions. Thus, $-I_{2n}$ is a commutator of skew-involutions in $\mathrm{SL}_{2n}(\mathbb{F})$.

If n is odd and the quadratic equation $a^2 + b^2 + 1 = 0$ in a and b has no solution in $\mathbb{F}$, it can be proved that $-I_{2n} = -I_{4k+2}$ is not a commutator of skew-involutions. Otherwise, there is a $B \in M_{2n}(\mathbb{F})$ such that $B^2 = -I_{4k+2}$ and B is similar to its inverse via some skew-involution S by Corollary 2.2, say $S^{-1}BS = B^{-1}$ and $S^2 = -I_{4k+2}$. Since $B^2 = -I_{2n}$, there exists an R such that $R^{-1}BR = \Omega_{2n}$. If we set $T = R^{-1}SR$, then T is a skew-involution and $T^{-1}\Omega_{2n}T = \Omega_{2n}^{-1} = -\Omega_{2n}$. Hence, T has the form $T = \begin{pmatrix} M & N \\ N & -M \end{pmatrix}$ for some $M, N \in M_n(\mathbb{F})$ such that $MN = NM$ and $M^2 + N^2 = -I_n$. From the Frobenius (rational) canonical form theorem, M is similar to a direct sum of Frobenius blocks by some matrix P, say $P^{-1}MP = \oplus_{i=1}^{k} C(h_i)$ and each h_i is a positive integer power of an irreducible polynomial. Without loss of generality, suppose $P^{-1}MP = M_1 \oplus M_2$, where $M_1 = \oplus_{i=1}^{m_1} C((x^2+1)^{k_i})$ is a matrix of order $2\sum_{i=1}^{m_1} k_i$, and $M_2 = \oplus_{i=1}^{m_2} C(h_i)$ is an odd order matrix with each h_i not divisible by x^2+1. Write $P^{-1}NP$ as $\begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}$, where N_1 is a matrix of order $2r$ with $r = \sum_{i=1}^{m_1} k_i$. From $MN = NM$, we get $(P^{-1}MP)(P^{-1}NP) = (P^{-1}NP)(P^{-1}MP)$. Then $M_1N_2 = N_2M_2$ and $M_2N_3 = N_3M_1$. Since a necessary and sufficient condition for $AX - XB = C$ to have a unique solution is that the spectra of A and B are disjoint, we obtain $N_2 = N_3 = 0$. From $M^2 + N^2 = -I_n$, we get $(P^{-1}MP)^2 + (P^{-1}NP)^2 = -I_n$, which implies $M_1^2 + N_1^2 = -I_{2r}$ and $M_2^2 + N_4^2 = -I_{n-2r}$. Since n is odd, by Lemma 2.6 we have $\det(M_2^2 + I_{n-2r}) = \det(-N_4^2) = -(\det N_4)^2 = p^2 + q^2$ for some $p, q \in \mathbb{F}$. Since h_i is not divisible by $x^2 + 1$ for each $C(h_i)$ in M_2, we have $\det N_4 \neq 0$ and so $\left(\frac{p}{\det N_4}\right)^2 + \left(\frac{q}{\det N_4}\right)^2 + 1 = 0$, which contradicts the hypothesis that $a^2 + b^2 + 1 = 0$ has no solution in $\mathbb{F}$.

We now consider $-I_2$. We know by the preceding that $-I_2$ is not a commutator of skew-involutions. We show that $-I_2$ is not a product of two commutators of skew-involutions in $\mathrm{SL}_2(\mathbb{F})$. Otherwise, there exists a D such that $-I_2 = D(-D^{-1})$, and both D and $-D^{-1}$ are commutators of skew-involutions in $\mathrm{SL}_2(\mathbb{F})$. Suppose $D = [X, Y]$ and $-D^{-1} = [\tilde{X}, \tilde{Y}]$ with $X^2 = Y^2 = \tilde{X}^2 = \tilde{Y}^2 = -I_2$. Since $a^2 + b^2 + 1 = 0$ has no solution in $\mathbb{F}$, every skew-involution in $\mathrm{SL}_2(\mathbb{F})$ is similar to Ω_2, say $P^{-1}XP = \Omega_2$, for some $P \in \mathrm{GL}_2(\mathbb{F})$. Observe that $S \in \mathrm{SL}_2(\mathbb{F})$ is a skew-involution if and only if

$$S = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \text{ where } bc = -1 - a^2 \text{ with } b, c \neq 0 \text{ (assuming that } a^2 + b^2 + 1 = 0 \text{ has no solutions in } \mathbb{F}).$$

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One can assume

$$P^{-1}YP = \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \text{ where } yz = -1 - x^2 \text{ with } y, z \neq 0.$$

Then, the trace of D is

$$\mathrm{tr}D = \mathrm{tr}(P^{-1}DP) = \mathrm{tr}[P^{-1}XP, P^{-1}YP] = \mathrm{tr}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} x & y \\ z & -x \end{pmatrix}\right] = (y-z)^2 - 2 = (y+z)^2 + 2 + 4x^2.$$

Similarly, it can be concluded that $\mathrm{tr}(-D^{-1}) = (\tilde{y} - \tilde{z})^2 - 2 = (\tilde{y} + \tilde{z})^2 + 2 + 4\tilde{x}^2$ for some $\tilde{x}, \tilde{y}, \tilde{z} \in \mathbb{F}$ with $-1 - \tilde{x}^2 = \tilde{y}\tilde{z} \neq 0$. From Corollary 2.2, $-D^{-1}$ and $-D$ are similar. Thus, $(\tilde{y} - \tilde{z})^2 - 2 = \mathrm{tr}(-D^{-1}) = \mathrm{tr}(-D) = -\mathrm{tr}D = -(y+z)^2 - 2 - 4x^2$, which means $(\tilde{y} - \tilde{z})^2 + (y+z)^2 + 4x^2 = 0$. Since $a^2 + b^2 + 1 = 0$ has no solution in $\mathbb{F}$, we have $\mathrm{char}\mathbb{F} \neq 2$. If $x \neq 0$, then $(\frac{\tilde{y}-\tilde{z}}{2x})^2 + (\frac{y+z}{2x})^2 + 1 = 0$ which contradicts the fact the quadratic equation $a^2 + b^2 + 1 = 0$ in a and b has no solution in $\mathbb{F}$. If $x = 0$ and $y + z \neq 0$, then $(\frac{\tilde{y}-\tilde{z}}{y+z})^2 + 1 = 0$ and the quadratic equation $a^2 + b^2 + 1 = 0$ has a solution $a = 0$ and $b = \frac{\tilde{y}-\tilde{z}}{y+z}$, a contradiction. If $x = 0$ and $y + z = 0$, then $y = -z = \pm 1$ since $yz = -1$. Hence, $\tilde{y} = \tilde{z}$, which implies $\tilde{y}^2 + \tilde{z}^2 + 1 = 0$, a contradiction. In conclusion, $-I_2$ is not a product of two or fewer commutators of skew-involutions in $\mathrm{SL}_2(\mathbb{F})$.

Write $-I_2 = \mathrm{diag}(y^2, y^{-2})\mathrm{diag}(-y^{-2}, -y^2)$ with $y \in \mathbb{F}$ and $y \neq 0, 1, -1$. Then, $y^2 \neq y^{-2}$, otherwise y would satisfy $(x-1)(x+1)(x^2+1) = 0$, and we are under the assumption that $y \neq \pm 1$ and $a^2+b^2+1 = 0$ has no solution in $\mathbb{F}$ - in fact, it is enough that $x^2+1 = 0$ has no root in $\mathbb{F}$. Since $\mathrm{diag}(y^2, y^{-2})$ is a commutator of skew-involutions from Lemma 2.3 and $\mathrm{diag}(-y^{-2}, -y^2)$ is a product of two commutators of skew-involutions from Theorem 3.1, $-I_2$ is a product of three commutators of skew-involutions in $\mathrm{SL}_2(\mathbb{F})$ and no fewer. □

Corollary 3.3. *If* $\mathrm{char}\mathbb{F} \neq 0$*, then* $-I_{2n}$ *is a commutator of skew-involutions in* $\mathrm{SL}_{2n}(\mathbb{F})$*.*

Proof. If $\mathrm{char}\mathbb{F} = p \neq 0$, by Lemma 2.5, there exist two elements a and b in $\mathbb{Z}_p$ satisfying $a^2 + b^2 + 1 = 0$. By Lemma 3.2, $-I_{2n}$ is a commutator of skew-involutions in $\mathrm{SL}_{2n}(\mathbb{F})$. □

If $A \neq \pm I_{2n}$ is a scalar matrix αI_{2n} with determinant 1, then the following lemma holds.

Lemma 3.4. *Assume that* $\alpha^{2n} = 1$ *and* $\alpha \neq \pm 1$*. If* n *is even or there exist two elements* a, b *in* $\mathbb{F}$ *such that* $a^2 + b^2 + 1 = 0$*, then* αI_{2n} *is a product of at most two commutators of skew-involutions in* $\mathrm{SL}_{2n}(\mathbb{F})$*. Otherwise,* αI_{2n} *is a product of at most three commutators of skew-involutions.*

Proof. Since $\alpha^{2n} = 1$, we have $\alpha I_{2n} \in \mathrm{SL}_{2n}(\mathbb{F})$. Write $\alpha I_{2n} = BC$ where

$$B = \bigoplus_{i=1}^{n} \alpha^{2i-1} I_2, \ C = \bigoplus_{i=1}^{n} \alpha^{2n+2-2i} I_2.$$

Note that C is permutation similar to $(\mathrm{diag}(1, \alpha, \alpha^2, \ldots, \alpha^{n-1}))^2 \oplus (\mathrm{diag}(1, \alpha, \alpha^2, \ldots, \alpha^{n-1}))^{-2}$, hence C is a commutator of skew-involutions in $\mathrm{SL}_{2n}(\mathbb{F})$ by Lemma 2.3(b) and Remark 2.4(c).

When $n = 2k$ is even, B is permutation similar to

$$\left(\bigoplus_{i=1}^{k} \begin{pmatrix} 0 & 1 \\ \alpha^{2i-1} & 0 \end{pmatrix}\right)^2 \oplus \left(\bigoplus_{i=1}^{k} \begin{pmatrix} 0 & 1 \\ \alpha^{2i-1} & 0 \end{pmatrix}\right)^{-2},$$

which is also a commutator of skew-involutions in $\mathrm{SL}_{2n}(\mathbb{F})$ by Lemma 2.3(b) and Remark 2.4(c).

When $n = 2k-1$ is odd, B is permutation similar to

$$\left(\bigoplus_{i=1}^{k-1} \begin{pmatrix} 0 & 1 \\ \alpha^{2i-1} & 0 \end{pmatrix}\right)^2 \oplus \left(\bigoplus_{i=1}^{k-1} \begin{pmatrix} 0 & 1 \\ \alpha^{2i-1} & 0 \end{pmatrix}\right)^{-2} \oplus \begin{pmatrix} \alpha^{2k-1} & 0 \\ 0 & \alpha^{2k-1} \end{pmatrix}.$$

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Since $\alpha^{2n} = \alpha^{4k-2} = 1$, we have $\alpha^{2k-1} = \pm 1$. If $\alpha^{2k-1} = 1$, then B is a commutator of skew-involutions in $\mathrm{SL}_{2n}(\mathbb{F})$ by Lemma 2.3(b) and Remark 2.4(b)(c). If $\alpha^{2k-1} = -1$ and there exist two elements a and b in $\mathbb{F}$ such that $a^2 + b^2 + 1 = 0$, then B is a commutator of skew-involutions in $\mathrm{SL}_{2n}(\mathbb{F})$ by Lemmas 3.2 and 2.3(b) and Remark 2.4(b)(c). Hence, αI_{2n} is a product of two commutators of skew-involutions in $\mathrm{SL}_{2n}(\mathbb{F})$.

When $n = 2k - 1$ is odd and the quadratic equation $a^2 + b^2 + 1 = 0$ in a and b has no solution in $\mathbb{F}$, write $\alpha I_{2n} = (\alpha^2 I_n \oplus \alpha^{-2} I_n)(\alpha^{-1} I_n \oplus \alpha^3 I_n)$. The first factor is a commutator of skew-involutions from Lemma 2.3(b). Since $n = 2k - 1$ is odd and $\alpha \neq \pm 1$, the second factor $\alpha^{-1} I_n \oplus \alpha^3 I_n$ is not a scalar matrix. By Theorem 3.1, it is a product of two commutators of skew-involutions. Then αI_{2n} can be written as a product of three commutators of skew-involutions in $\mathrm{SL}_{2n}(\mathbb{F})$. ∎

From Theorem 3.1 and Lemmas 3.2 and 3.4, we get the following.

THEOREM 3.5. *If $\mathbb{F}$ is a field with at least four elements, then every matrix $A \in \mathrm{SL}_{2n}(\mathbb{F})$ is a product of at most three commutators of skew-involutions. Moreover, if n is even or there exist two elements a and b in $\mathbb{F}$ such that $a^2 + b^2 + 1 = 0$, then every matrix $A \in \mathrm{SL}_{2n}(\mathbb{F})$ is a product of at most two commutators of skew-involutions in $\mathrm{SL}_{2n}(\mathbb{F})$.*

3.2. Case when $n > 1$ and $\mathbb{F} = \mathbb{Z}_3$. Denote by E_{ij} the matrix with a unique nonzero entry equal to 1 in the position (i, j).

LEMMA 3.6. *The matrix $A = I_{2n-1} + \sum_{i=1}^{2n-2} a_{i,i+1} E_{i,i+1}$ is similar to a product of two involutions A_1 and A_2 in $\mathrm{SL}_{2n}(\mathbb{F})$, where the multiplicities of -1 as an eigenvalue of A_1 and A_2 have the same parity.*

Proof. Let

$$A_1 = \begin{pmatrix} 1 & & & & & & & \\ & -1 & a_{23} & & & & & \\ & & 1 & & & & & \\ & & & -1 & a_{45} & & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & -1 & a_{2n-2,2n-1} \\ & & & & & & & 1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & a_{12} & & & & & & \\ & -1 & & & & & & \\ & & 1 & a_{34} & & & & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & a_{2n-3,2n-2} & \\ & & & & & & -1 & \\ & & & & & & & 1 \end{pmatrix}.$$

From the proof of [12] Theorem 2.3, we get A is similar to $A_1 A_2 = (-A_1)(-A_2)$ and both A_1 and A_2 are involutions. Note that either A_1 and A_2 or $-A_1$ and $-A_2$ have -1 as an eigenvalue with odd (even) multiplicity, so we get the conclusion. ∎

Now we give the main result for the case when $n > 1$ and $\mathbb{F} = \mathbb{Z}_3$.

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THEOREM 3.7. *If n is an integer greater than 1, then every $A \in \mathrm{SL}_{2n}(\mathbb{Z}_3)$ is a product of at most four commutators of skew-involutions.*

Proof. If A is scalar, then $A = I_{2n}$ or $A = -I_{2n}$. By Corollary 3.3, $-I_{2n}$ is a commutator of skew-involutions. In either case, A is a commutator of skew-involutions in $\mathrm{SL}_{2n}(\mathbb{Z}_3)$.

If A is nonscalar, then, by Theorem 1 of [17], we can write $A = BC$, where B is lower triangularizable with eigenvalues $\beta_1 = -1, \beta_i = 1 (i = 2, 3, \ldots, 2n)$ and C is simultaneously upper triangularizable with eigenvalues $\gamma_1 = -1, \gamma_i = 1 (i = 2, 3, \ldots, 2n)$. Both B and C are similar to a direct sum $[-1] \oplus \bigoplus_{i=1}^{k} J_{m_i}(1)$ for some positive integer m_i and k, where $J_{m_i}(1)$ is the $m_i \times m_i$ Jordan blocks associated to the eigenvalue 1. By Lemma 3.6, $\bigoplus_{i=1}^{k} J_{m_i}(1)$ is similar to a product of two involutions A_1 and A_2 and both of them have -1 as an eigenvalue with the same odd multiplicity $2l - 1$ for some integer l. Then A_1 and A_2 are similar to $-I_{2l-1} \oplus I_{2n-2l}$, and $[-1] \oplus A_1, [-1] \oplus A_2$ are both similar to $-I_{2l} \oplus I_{2n-2l}$. Since $-I_{2l} \oplus I_{2n-2l}$ is a commutator of skew-involutions from Corollary 3.3 and Remark 2.4(b), both B and C can be written as products of two commutators of skew-involutions in $\mathrm{SL}_{2k}(\mathbb{Z}_3)$ by Remark 2.4(c). Hence, if A is nonscalar, then A is a product of at most four commutators of skew-involutions. ∎

Since a commutator of skew-involutions is also a product of two skew-involutions, Theorem 3.8 of [13] can be deduced from Theorem 3.7.

It is known that $\mathrm{SL}_2(\mathbb{Z}_2)$ and $\mathrm{SL}_2(\mathbb{Z}_3)$ properly contain their commutator subgroups (see [4] Sections 106 and 107). Since a commutator of involutions is also a commutator of skew-involutions when $x^2 + 1$ has a root in $\mathbb{F}$, every $A \in \mathrm{SL}_{2n}(\mathbb{F}) \neq \mathrm{SL}_2(\mathbb{Z}_2), \mathrm{SL}_2(\mathbb{Z}_3)$ can be written as a product of commutators of skew-involutions by [7, Theorem 2.8] and [15, Lemma 1.2]. Then from Theorems 3.5 and 3.7, we can obtain the following.

THEOREM 3.8. *Every $A \in \mathrm{SL}_{2n}(\mathbb{F})$ is a product of commutators of skew-involutions if and only if $\mathrm{SL}_{2n}(\mathbb{F}) \neq \mathrm{SL}_2(\mathbb{Z}_2), \mathrm{SL}_2(\mathbb{Z}_3)$.*

4. Products of real symplectic skew-involutions. In [13], Joven and Paras proved that every real symplectic matrix in $\mathrm{Sp}_{2n}(\mathbb{R})$ is a product of six real symplectic skew-involutions. Via a similar matrix factorization, we prove the following.

THEOREM 4.1. *Every $A \in \mathrm{Sp}_{2n}(\mathbb{R})$ is a product of four commutators of real symplectic skew-involutions.*

Proof. From the proof of Theorem 4.3 in [13], we know that every $A \in \mathrm{Sp}_{2n}(\mathbb{R})$ has the decomposition $A = QR$ where Q is similar to $D \oplus D^{-1}$ by a real orthosymplectic matrix P and D is a positive diagonal matrix, and R is similar to

$$\Theta = \boxplus_{j=1}^{n} \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix}, \ \theta_1, \ldots, \theta_n \in \mathbb{R}$$

by a real orthosymplectic matrix S. Suppose $D = \mathrm{diag}(d_1, \ldots, d_n)$ for some positive $d_1, \ldots, d_n \in \mathbb{R}$. Then there exists a diagonal matrix $D_1 = \mathrm{diag}(\sqrt{d_1}, \ldots, \sqrt{d_n})$ and $D_1^2 = D$. By Lemma 2.3(c) and Remark 2.4(c), Q is a commutator of real symplectic skew-involutions. Since each summand in the expanding sum of Θ is in $\mathrm{SL}_2(\mathbb{R})$, it follows from Theorem 3.5 and Remark 2.4(b)(c) that R is a product of at most three commutators of real symplectic skew-involutions. Thus, A is a product of at most four commutators of real symplectic skew-involutions. ∎

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5. Products of complex symplectic skew-involutions. Let $A \in \mathrm{Sp}_{2n}(\mathbb{C})$. Since x^2+1 has a root in $\mathbb{C}$, by [8, Theorem 1.1] , we know that every symplectic complex matrix can be decomposed into a product of at most three commutators of symplectic skew-involutions. We can optimize this decomposition from "three" to "two". For this purpose, we need the following lemma which is called *symplectic Jordan form* given by de la Cruz in [3].

Lemma 5.1 (de la Cruz [3], Lemma 5). *Each symplectic complex matrix is symplectically similar to the expanding sum of matrices of the following forms:*

- $J_k(\lambda) \oplus J_k(\lambda)^{-\top}$ *for* $\lambda \neq 0, \pm 1$,
- $J_{2k-1}(\epsilon) \oplus J_{2k-1}(\epsilon)^{-\top}$ *for* $\epsilon = \pm 1$, *or*
- $\pm\varepsilon(k)$, *where*

$$\varepsilon(k) = \begin{pmatrix} J_k(1) & \begin{pmatrix} 0_{k-1,k} \\ u \end{pmatrix} \\ 0 & J_k(1)^{-\top} \end{pmatrix} \in M_{2k}(\mathbb{C}),$$

and $u = ((-1)^{k+1}, (-1)^k, \cdots, 1)$ *is the last row of* $J_k(1)^{-\top}$.

Suppose μ is a square root of $\lambda \neq 0$. Then $(J_k(\mu) \oplus J_k(\mu)^{-\top})^2$ is a commutator of complex symplectic skew-involutions by Corollary 2.2 and [13] Lemma 5.2. Since $J_k(\lambda) \oplus J_k(\lambda)^{-\top}$ is similar to $(J_k(\mu) \oplus J_k(\mu)^{-\top})^2$, the following lemma holds.

Lemma 5.2. *If k is a positive integer and $\lambda \neq 0$, then $J_k(\lambda) \oplus J_k(\lambda)^{-\top}$ is a commutator of complex symplectic skew-involutions.*

By taking $\lambda = \pm 1$ in Lemma 5.2, we obtain the following corollaries.

Corollary 5.3. *If k is a positive integer and $\epsilon = \pm 1$, then $J_{2k-1}(\epsilon) \oplus J_{2k-1}(\epsilon)^{-\top}$ is a commutator of complex symplectic skew-involutions.*

Corollary 5.4. *If k is a positive integer, then $\pm(\varepsilon(k) \oplus \varepsilon(k)^{-\top})$ is a commutator of complex symplectic skew-involutions.*

Since $\varepsilon(k)^2$ is similar to $\varepsilon(k)$, by Corollary 2.2 and [13] Lemma 5.5, we obtain the following.

Lemma 5.5. *If k is a positive integer, then $\varepsilon(k)$ is a commutator of complex symplectic skew-involutions.*

Since $-\varepsilon(k)$ has no square root in $\mathrm{Sp}_{2n}(\mathbb{C})$, by Corollary 2.2 $-\varepsilon(k)$ is not a commutator of complex symplectic skew-involutions. Note that $-\varepsilon(k) = (-I_{2k})\varepsilon(k)$, by Lemmas 5.2 and 5.5, we obtain the following.

Lemma 5.6. *If k is a positive integer, then $-\varepsilon(k)$ is a product of two commutators of complex symplectic skew-involutions and no fewer.*

By Corollaries 5.3, 5.4 and Lemmas 5.5, 5.6, we obtain the following.

Theorem 5.7. *Every $A \in \mathrm{Sp}_{2n}(\mathbb{C})$ is a product of two commutators of complex symplectic skew-involutions. Moreover, A is a commutator of complex symplectic skew-involutions if and only if for each k, the number of the expanding summands $-\varepsilon(k)$ of the symplectic Jordan form of A is even.*

Proof. The sufficiency follows from Corollaries 5.3, 5.4 and Lemmas 5.5, 5.6. It remains to show necessity. If the symplectic Jordan form of $A = B^2$ has an odd number of expanding summands $-\varepsilon(k)$, then the

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symplectic Jordan form of B has expanding summands $J_{2k}(i)$ and $J_{2k}(-i)$ and they differ in number. Since $J_{2k}(i)^{-1}$ is similar to $J_{2k}(-i)$, we have that B is not similar to B^{-1}. By Corollary 2.2, A is not a product of two commutators of complex symplectic skew-involutions, which is a contradiction. This proves necessity. ∎

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