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# POSITIVE DEFINITE SOLUTION OF THE MATRIX EQUATION $X = Q + A^H(I \otimes X - C)^{-\delta}A^*$

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**Abstract.** We consider the nonlinear matrix equation $X = Q+A^H(I\otimes X-C)^{-\delta}A$ $(0 < \delta \leq 1)$, where $Q$ is an $n\times n$ positive definite matrix, $C$ is an $mn \times mn$ positive semidefinite matrix, $I$ is the $m\times m$ identity matrix, and $A$ is an arbitrary $mn\times n$ matrix. We prove the existence and uniqueness of the solution which is contained in some subset of the positive definite matrices under the condition that $I \otimes Q > C$. Two bounds for the solution of the equation are derived. This equation is related to an interpolation problem when $\delta = 1$. Some known results in interpolation theory are improved and extended.

**Key words.** Nonlinear matrix equation, Positive definite solution, Interpolation theory.

**AMS subject classifications.** 15A24, 65H05.

**1. Introduction.** We consider the positive definite solution $X$ of the nonlinear matrix equation

$$X = Q + A^H(I \otimes X - C)^{-\delta}A, \quad 0 < \delta \leq 1, \tag{1.1}$$

where $Q$ is an $n \times n$ positive definite matrix, $A$ is an $mn \times n$ complex matrix, $C$ is an $mn \times mn$ positive semidefinite matrix, $I$ is the $m \times m$ identity matrix, $\otimes$ is the Kronecker product, and $A^H$ denotes the conjugate transpose of matrix $A$. When $\delta = 1$, (1.1) is connected to an interpolation problem (see [1]-[3]). The special cases of this equation have many applications in various areas, including control systems, ladder networks, dynamic programming, stochastic filtering, statistics (see [4]).

In recent years, many authors have been greatly interested in studying both the theory and numerical aspects of the positive definite solutions of the nonlinear matrix equations of the form (1.1) (see [1], [3]-[18]). Some special cases of (1.1) have been investigated. When $\delta = 1$, Ran et al [1] showed that (1.1) has a unique positive definite solution by using a reduction method and Sun [3] obtained the perturbation

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bounds and the residual bounds for an approximate solution of (1.1). In addition, Duan [17] proved that (1.1) always has a unique positive definite solution when $C = 0$. Hasanov [11] obtained that (1.1) has a unique positive definite solution under rigorous conditions when $C = 0$ and $m = 1$. In this article, we first claim that (1.1) always has a unique positive definite solution and then use a new approach that is different from [1] to prove our conclusions in Sections 2 and 3. We also obtain some bounds for the unique positive definite solution of (1.1).

Throughout this paper, $X > 0$ ($X \geq 0$) denotes that the matrix $X$ is positive definite (semidefinite). $B \otimes C$ denotes the Kronecker product of $B$ and $C$. If $B - C$ is positive definite (semidefinite), then we write $B > C$ ($B \geq C$). We use $\lambda_M(B)$ ($\sigma_M(B)$) and $\lambda_m(B)$ ($\sigma_m(B)$) to denote the maximal and minimal eigenvalues (singular values) of an $n \times n$ positive definite matrix $B$, respectively. Let $P(n)$ denote the set of $n \times n$ positive definite matrices, $\varphi(n)$ denote the matrix set defined by $\{X \in P(n) \mid I \otimes X > C\}$, $[B, C] = \{X \in P(n) \mid B \leq X \leq C\}$ and $(B, C) = \{X \in P(n) \mid B < X < C\}$. Unless otherwise stated, we suppose that $I \otimes Q > C$, the solutions of the matrix equations in this paper are positive definite and the solution of (1.1) is in $\varphi(n)$.

**2. The existence of a unique solution.** In this section, we prove that (1.1) always has a unique solution. We begin with some lemmas.

LEMMA 2.1. (1.1) *is equivalent to the following nonlinear matrix equation*

$$Y = \bar{Q} + \bar{A}^H(I \otimes Y)^{-\delta}\bar{A}, \tag{2.1}$$

*where* $Y = I \otimes X - C, \quad \bar{Q} = I \otimes Q - C, \quad \bar{A} = I \otimes A.$

*Proof.* Taking the Kronecker product of $I$ with both left sides of (1.1), we obtain

$$I \otimes X - I \otimes [A^H(I \otimes X - C)^{-\delta}A] = I \otimes Q.$$

Then

$$I \otimes X - C - (I \otimes A^H)[I \otimes (I \otimes X - C)^{-\delta}](I \otimes A) = I \otimes Q - C. \tag{2.2}$$

Noting that $I \otimes A^H = (I \otimes A)^H$ and $I \otimes Y^{-\delta} = (I \otimes Y)^{-\delta}$, we get (2.1) by substituting $Y$, $\bar{Q}$ and $\bar{A}$ for $I \otimes X - C$, $I \otimes Q - C$ and $I \otimes A$ in (2.2), respectively. Furthermore, (2.1) has a solution $\bar{Y} = I \otimes \bar{X} - C$ if $\bar{X}$ is a solution of (1.1). For the converse, it is easy to verify that (1.1) has a solution $\bar{X} = Q + A^H\bar{Y}^{-\delta}A$ if $\bar{Y}$ is a solution of (2.1). ∎

LEMMA 2.2. ([19, *p*.2]). *If* $A \geq B > 0$ *(or* $A > B > 0$*), then* $A^\alpha \geq B^\alpha > 0$ *(or* $A^\alpha > B^\alpha > 0$*) for all* $0 < \alpha \leq 1$*, and* $0 < A^\alpha \leq B^\alpha$ *(or* $0 < A^\alpha < B^\alpha$*) for all* $-1 \leq \alpha < 0$.

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To discuss the solution of (1.1), we define maps $f$ and $F$ as follows:

$$f(X) = Q + A^H(I \otimes X - C)^{-\delta}A, \quad X \in \varphi(n), \tag{2.3}$$

and

$$F(Y) = \bar{Q} + \bar{A}^H(I \otimes Y)^{-\delta}\bar{A}, \quad Y \in P(mn). \tag{2.4}$$

Observe that the solutions of (1.1) and (2.1) are fixed points of $f$ in $\varphi(n)$ and $F$ in $P(mn)$, respectively. Let

$$f^k(X) = f[f^{k-1}(X)], \quad F^k(Y) = F[F^{k-1}(Y)], \quad k = 2, 3, \ldots.$$

Lemma 2.3. *The map $F$ has the following properties:*

*(1) If $Y_1 \geq Y_2 \geq 0$, then $F(Y_2) \geq F(Y_1) \geq 0$ and $F^2(Y_1) \geq F^2(Y_2) \geq 0$.*
*(2) For any matrix $Y > 0$, $\bar{Q} \leq F^2(Y) \leq F(\bar{Q})$, and the set $\{Y \mid \bar{Q} \leq Y \leq F(\bar{Q})\}$ is mapped into itself by $F$.*
*(3) The sequence $\{F^{2k}(\bar{Q})\}_{k=0}^{\infty}$ is an increasing sequence of positive definite matrices converging to a positive definite matrix $Y^-$, which is a fixed point of $F^2$, i.e., $Y^- = F^2(Y^-)$, and the sequence $\{F^{2k+1}(\bar{Q})\}_{k=0}^{\infty}$ is a decreasing sequence of positive definite matrices converging to a positive definite matrix $Y^+$, which is also a fixed point of $F^2$, i.e., $Y^+ = F^2(Y^+)$.*
*(4) $F$ maps the set $\{Y \mid Y^- \leq Y \leq Y^+\}$ into itself. In particular, any solution of (2.1) is in between $Y^-$ and $Y^+$, and if $Y^- = Y^+$, then (2.1) has a unique solution.*

*Proof.* The proof is similar to that of Theorem 2.2 of [6] and is omitted here. ∎

From (4) of Lemma 2.3, we know that (2.1) has a unique solution if $Y^- = Y^+$. Next we will prove that $Y^- = Y^+$.

Lemma 2.4. *Let $\eta(t) = \frac{(1-t)\lambda_m(\bar{Q})}{t\lambda_M[F(\bar{Q})]}$. Then we have, for any $Y > 0$ and $t \in (0, 1)$,*

$$F^2(tY) \geq t[1 + \eta(t)]F^2(Y).$$

*Proof.* By (2) of Lemma 2.3, for any $Y > 0$, we have

$$F^2(Y) \leq F(\bar{Q}) \leq \lambda_M[F(\bar{Q})]I. \tag{2.5}$$

Hence

$$\begin{aligned} &F^2(tY) - t[1 + \eta(t)]F^2(Y) \\ &= \bar{Q} + \bar{A}^H[I \otimes F(tY)]^{-\delta}\bar{A} - t[1 + \eta(t)][\bar{Q} + \bar{A}^H(I \otimes F(Y))^{-\delta}\bar{A}] \end{aligned}$$

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$$
\begin{aligned}
&= (1-t)\bar{Q} + \bar{A}^H[I \otimes (\bar{Q} + t^{-\delta}\bar{A}^H(I \otimes Y)^{-\delta}\bar{A})]^{-\delta}\bar{A} \\
&\quad -t\bar{A}^H[I \otimes F(Y)]^{-\delta}\bar{A} - t\eta(t)F^2(Y) \\
&= (1-t)\bar{Q} + t\bar{A}^H[I \otimes (t^{\frac{1}{\delta}}\bar{Q} + t^{-\delta+\frac{1}{\delta}}\bar{A}^H(I \otimes Y)^{-\delta}\bar{A})]^{-\delta}\bar{A} \\
&\quad -t\bar{A}^H[I \otimes F(Y)]^{-\delta}\bar{A} - t\eta(t)F^2(Y).
\end{aligned}
$$

Since $0 < t^{\frac{1}{\delta}} < 1$ and $0 < t^{-\delta+\frac{1}{\delta}} < 1$, we have

$$t^{\frac{1}{\delta}}\bar{Q} < \bar{Q}, \quad t^{-\delta+\frac{1}{\delta}}[\bar{A}^H(I \otimes Y)^{-\delta}\bar{A}] < \bar{A}^H(I \otimes Y)^{-\delta}\bar{A}.$$

From Lemma 2.2 it follows that

$$[t^{\frac{1}{\delta}}\bar{Q} + t^{-\delta+\frac{1}{\delta}}\bar{A}^H(I \otimes Y)^{-\delta}\bar{A}]^{-\delta} \geq [\bar{Q} + \bar{A}^H(I \otimes Y)^{-\delta}\bar{A}]^{-\delta}, \tag{2.6}$$

which implies that

$$[I \otimes (t^{\frac{1}{\delta}}\bar{Q} + t^{-\delta+\frac{1}{\delta}}\bar{A}^H(I \otimes Y)^{-\delta}\bar{A})]^{-\delta} \geq [I \otimes (\bar{Q} + \bar{A}^H(I \otimes Y)^{-\delta}\bar{A})]^{-\delta}. \tag{2.7}$$

Hence, combining (2.5), (2.6) and (2.7), we have

$$
\begin{aligned}
F^2(tY) - t[1+\eta(t)]F^2(Y) &\geq (1-t)\bar{Q} - t\eta(t)F^2(Y) \\
&\geq (1-t)\lambda_m(\bar{Q})I - t\eta(t)\lambda_M[F(\bar{Q})]I \\
&= (1-t)\lambda_m(\bar{Q})I - t\frac{(1-t)\lambda_m(\bar{Q})}{t\lambda_M[F(\bar{Q})]}\lambda_M[F(\bar{Q})]I \\
&= 0,
\end{aligned}
$$

i.e., $F^2(tY) \geq t[1+\eta(t)]F^2(Y)$. ∎

LEMMA 2.5. *For any* $Y_1 \geq 0$ *and* $Y_2 \geq 0$, *we have* $\lambda_M(Y_1)Y_2 \geq \lambda_m(Y_2)Y_1$.

*Proof.* For any $Y_1 \geq 0$ and $Y_2 \geq 0$, it follows that

$$Y_2 \geq \lambda_m(Y_2)I, \quad \lambda_M(Y_1)I \geq Y_1.$$

Hence

$$\lambda_M(Y_1)Y_2 \geq \lambda_M(Y_1)\lambda_m(Y_2)I \geq \lambda_m(Y_2)Y_1. \quad ∎$$

THEOREM 2.6. (1.1) *always has a unique positive definite solution* $\bar{X}$ *and the sequence* $\{f^k(X_0)\}_{k=0}^{\infty}$ *converges to* $\bar{X}$ *for any* $X_0 \in \varphi(n)$, *where the map* $f$ *is defined by* (2.3).

*Proof.* We first consider (2.1) since (1.1) is equivalent to (2.1) according to Lemma 2.1. From Lemma 2.3, we know that there exist positive definite matrices $Y^- \in P(mn)$ and $Y^+ \in P(mn)$ such that $Y^+ \geq Y^-$ and

$$\lim_{k\to\infty} F^{2k}(\bar{Q}) = Y^-, \quad \lim_{k\to\infty} F^{2k+1}(\bar{Q}) = Y^+.$$

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We also know that the matrices $Y^-$ and $Y^+$ are fixed points of $F^2$, i.e.,

$$Y^- = F^2(Y^-)$$

and

$$Y^+ = F^2(Y^+).$$

By Lemma 2.5, $Y^- \geq \frac{\lambda_m(Y^-)}{\lambda_M(Y^+)} Y^+$, we define

$$t_0 = \sup\{t | Y^- \geq tY^+\}.$$

Evidently, $0 < t_0 < +\infty$, we now prove that $t_0 \geq 1$. Assume that $0 < t_0 < 1$, $\quad Y^- \geq t_0 Y^+$. By Lemmas 2.3 and 2.4, we have

$$Y^- = F^2(Y^-) \geq F^2(t_0 Y^+) \geq t_0[1 + \eta(t_0)]F^2(Y^+) = t_0[1 + \eta(t_0)]Y^+. \tag{2.8}$$

Since $t_0[1 + \eta(t_0)] > t_0$, (2.8) is contradictory to the definition of $t_0$, and therefore

$$t_0 \geq 1, \qquad Y^- = Y^+.$$

Let $\bar{Y} = Y^+$ (or $Y^-$). We know that $\lim\limits_{k\to\infty} F^k(\bar{Q}) = \bar{Y}$ is the unique solution of (2.1) by Lemma 2.3. Therefore, $\bar{X} = Q + A^H \bar{Y}^{-\delta} A$ is the unique solution of (1.1) by Lemma 2.1.

It remains to prove that the sequence $\{F^k(Y_0)\}_{k=0}^{\infty}$ converges to $\bar{Y}$ for any $Y_0$ in $P(mn)$. From Lemma 2.3, we have

$$\bar{Q} \leq F^2(Y_0) \leq F(\bar{Q}). \tag{2.9}$$

Taking $F$ in (2.9) yields

$$F^2(\bar{Q}) \leq F^3(Y_0) \leq F(\bar{Q}).$$

And taking $F$ in (2.9) repeatedly yields

$$F^{2k-2}(\bar{Q}) \leq F^{2k}(Y_0) \leq F^{2k-1}(\bar{Q}), \quad k = 1, 2, 3, \ldots,$$

$$F^{2k}(\bar{Q}) \leq F^{2k+1}(Y_0) \leq F^{2k-1}(\bar{Q}), \quad k = 1, 2, 3, \ldots.$$

It follows from the convergence of $\{F^k(\bar{Q})\}_{k=0}^{\infty}$ to the unique solution $\bar{Y}$ that the sequence $\{F^k(Y_0)\}_{k=0}^{\infty}$ converges to $\bar{Y}$ for any $Y_0 \in P(mn)$. From the maps $f$ and $F$ defined by (2.3) and (2.4), we have

$$F^k(Y_0) = I \otimes f^k(X_0) - C, \quad k = 1, 2, \ldots,$$

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where $X_0 \in \varphi(n)$, $\ Y_0 = I \otimes X_0 - C \in P(mn)$. Hence

$$\begin{aligned}
\lim_{k\to\infty}[I \otimes f^k(X_0)] &= \lim_{k\to\infty} F^k(Y_0) + C \\
&= \bar{Y} + C \\
&= \bar{Q} + \bar{A}^H(I \otimes \bar{Y})^{-\delta}\bar{A} + C \\
&= I \otimes (Q + A^H \bar{Y}^{-\delta} A) \\
&= I \otimes \bar{X},
\end{aligned}$$

i.e., $\lim_{k\to\infty} f^k(X_0) = \bar{X}$. □

**3. Some bounds for the unique solution.** In this section, we present two bounds for the unique solution (1.1).

THEOREM 3.1. *Let $\bar{X}$ be the unique positive definite solution of* (1.1). *Then* $\bar{X} \in [f^2(Q), f(Q)]$.

*Proof.* Let $\bar{Y} = I \otimes \bar{X} - C$. We know that $\bar{Y}$ is the unique positive definite solution of (2.1) by Lemma 2.1, i.e.,

$$\bar{Y} = \bar{Q} + \bar{A}^H(I \otimes \bar{Y})^{-\delta}\bar{A}.$$

Thus

$$\bar{Y} \geq \bar{Q},$$

which implies that

$$\bar{Y}^{-\delta} \leq \bar{Q}^{-\delta}.$$

Thus we have

$$\bar{Y} = \bar{Q} + \bar{A}^H(I_m \otimes \bar{Y})^{-\delta}\bar{A} \leq \bar{Q} + \bar{A}^H(I_m \otimes \bar{Q})^{-\delta}\bar{A}.$$

Hence

$$\bar{Q} \leq \bar{Y} \leq F(\bar{Q}). \tag{3.1}$$

From Lemma 2.2 it follows that

$$[F(\bar{Q})]^{-\delta} \leq \bar{Y}^{-\delta} \leq \bar{Q}^{-\delta},$$

which implies that

$$\bar{Q} + \bar{A}^H[I \otimes F(\bar{Q})]^{-\delta}\bar{A} \leq \bar{Q} + \bar{A}^H(I \otimes \bar{Y})^{-\delta}\bar{A} \leq \bar{Q} + \bar{A}^H(I \otimes \bar{Q})^{-\delta}\bar{A},$$

i.e.,

$$\bar{Y} \in [F^2(\bar{Q}), \quad F(\bar{Q})].$$

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Since

$$F^2(\bar{Q}) = I \otimes f^2(Q) - C, \quad F(\bar{Q}) = I \otimes f(Q) - C,$$

we have

$$I \otimes f^2(Q) - C \leq I \otimes \bar{X} - C \leq I \otimes f(Q) - C,$$

i.e., $\bar{X} \in [f^2(Q), \quad f(Q)]$. ☐

THEOREM 3.2. *Let $\bar{X}$ be the unique positive definite solution of* (1.1). *Then*

$$I \otimes \bar{X} \in [\alpha I + C, \beta I + C],$$

*where the pair $(\alpha, \beta)$ is a solution of the system*

$$\begin{cases} \alpha &= \lambda_m(\bar{Q}) + \sigma_m^2(A)\beta^{-\delta}, \\ \beta &= \lambda_M(\bar{Q}) + \sigma_M^2(A)\alpha^{-\delta}. \end{cases}$$

*Proof.* Let $\bar{Y} = I \otimes \bar{X} - C$. We know that $\bar{Y}$ is the unique solution of (2.1). From $\bar{A} = I \otimes A$, we have $\sigma_m(\bar{A}) = \sigma_m(A), \quad \sigma_M(\bar{A}) = \sigma_M(A)$. Define the sequences $\{\alpha_s\}$ and $\{\beta_s\}$ as follows:

$$\begin{aligned} \alpha_0 &= \lambda_m(\bar{Q}), \\ \beta_0 &= \lambda_M(\bar{Q}) + \sigma_M^2(\bar{A})\lambda_m^{-\delta}(\bar{Q}), \\ \alpha_s &= \lambda_m(\bar{Q}) + \sigma_m^2(\bar{A})\beta_{s-1}^{-\delta}, \\ \beta_s &= \lambda_M(\bar{Q}) + \sigma_M^2(\bar{A})\alpha_{s-1}^{-\delta}, \quad s = 1, 2, \ldots. \end{aligned}$$

We will prove that the sequences $\{\alpha_s\}$ and $\{\beta_s\}$ are monotonically increasing and monotonically decreasing, respectively. Obviously,

$$0 < \alpha_0 < \beta_0.$$

Hence

$$\begin{aligned} \alpha_1 &= \lambda_m(\bar{Q}) + \sigma_m^2(\bar{A})\beta_0^{-\delta} \geq \alpha_0, \\ \beta_1 &= \lambda_M(\bar{Q}) + \sigma_M^2(\bar{A})\alpha_0^{-\delta} = \beta_0. \end{aligned}$$

Suppose that $\alpha_k \geq \alpha_{k-1}, \quad \beta_k \leq \beta_{k-1}$. Then

$$\begin{aligned} \alpha_{k+1} &= \lambda_m(\bar{Q}) + \sigma_m^2(\bar{A})\beta_k^{-\delta} \geq \lambda_m(\bar{Q}) + \sigma_m^2(\bar{A})\beta_{k-1}^{-\delta} = \alpha_k, \\ \beta_{k+1} &= \lambda_M(\bar{Q}) + \sigma_M^2(\bar{A})\alpha_k^{-\delta} \leq \lambda_M(\bar{Q}) + \sigma_M^2(\bar{A})\alpha_{k-1}^{-\delta} = \beta_k. \end{aligned}$$

Therefore, we can get $\alpha_s \geq \alpha_{s-1}, \ \beta_{s-1} \geq \beta_s$ for $s = 1, 2, \ldots$ by induction.

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Next, we will prove that the unique solution $\bar{Y}$ of (2.1) lies in $[\alpha_s I, \beta_s I]$, for $s = 0, 1, 2, \ldots$. It is easy to see that

$$\bar{Q} \leq \bar{Y} \leq \bar{Q} + \bar{A}^H(I \otimes \bar{Q})^{-\delta}\bar{A}. \tag{3.2}$$

By $\bar{Q} \geq \sigma_m(\bar{Q})I = \alpha_0 I$, we have $\bar{Y} \geq \alpha_0 I$ and

$$\begin{aligned} \bar{Q} + \bar{A}^H(I \otimes \bar{Q})^{-\delta}\bar{A} &\leq \bar{Q} + \lambda_m^{-\delta}(\bar{Q})\bar{A}^H\bar{A} \\ &\leq [\lambda_M(\bar{Q}) + \lambda_m^{-\delta}(\bar{Q})\sigma_M^2(\bar{A})]I \\ &= \beta_0 I. \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we have $\bar{Y} \in [\alpha_0 I,\ \beta_0 I]$. Suppose that $\bar{Y} \in [\alpha_k I,\ \beta_k I]$. Then

$$\begin{aligned} \bar{Y} &= \bar{Q} + \bar{A}^H(I \otimes \bar{Y})^{-\delta}\bar{A} \\ &\geq \bar{Q} + \beta_k^{-\delta}\bar{A}^H\bar{A} \\ &\geq [\lambda_m(\bar{Q}) + \beta_k^{-\delta}\sigma_m^2(\bar{A})]I \\ &= \alpha_{k+1} I \end{aligned}$$

and

$$\begin{aligned} \bar{Y} &= \bar{Q} + \bar{A}^H(I \otimes \bar{Y})^{-\delta}\bar{A} \\ &\leq \bar{Q} + \alpha_k^{-\delta}\bar{A}^H\bar{A} \\ &\leq [\lambda_M(\bar{Q}) + \alpha_k^{-\delta}\sigma_M^2(\bar{A})]I \\ &= \beta_{k+1} I. \end{aligned}$$

Consequently, the sequences $\{\alpha_s\}$ and $\{\beta_s\}$ are convergent. Let $\alpha = \lim_{s\to\infty} \alpha_s$, $\beta = \lim_{s\to\infty} \beta_s$. Then $\bar{Y} \in [\alpha I, \beta I]$. Since $\bar{Y} = I \otimes \bar{X} - C$, we have that $I_m \otimes \bar{X}$ belongs to $[\alpha I + C, \beta I + C]$. $\square$

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