

GROUP INVERSE FOR THE BLOCK MATRIX WITH TWO IDENTICAL SUBBLOCKS OVER SKEW FIELDS*

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Abstract. Let K be a skew field and $K^{n \times n}$ be the set of all $n \times n$ matrices over K . The purpose of this paper is to give some necessary and sufficient conditions for the existence and the representations of the group inverse of the block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ under some conditions.

Key words. Skew fields, Block matrix, Group inverse.

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1. Introduction. Let K be a skew field, \mathbb{C} be the complex number field, $K^{m \times n}$ be the set of all $m \times n$ matrices over K , and I_n be the $n \times n$ identity matrix over K . For a matrix $A \in K^{n \times n}$, the matrix $X \in K^{n \times n}$ satisfying

$$A^k X A = A^k, \quad X A X = X, \quad A X = X A$$

is called the Drazin inverse of A and is denoted by $X = A^D$, where k is the index of A , i.e., the smallest non-negative integer such that $\text{rank}(A^k) = \text{rank}(A^{k+1})$. We denote such a k by $\text{Ind}(A)$. It is well-known that A^D exists and is unique (see [2]). If $\text{Ind}(A) = 1$, A^D is also called the group inverse of A and is denoted by $A^\#$. Then $A^\#$ exists if and only if $\text{rank}(A) = \text{rank}(A^2)$ (see [1, 3, 11-14, 24, 25, 29]). We denote $I - A A^\#$ by A^π .

The group inverse of block matrices has numerous applications in matrix theory, such as singular differential and difference equations, Markov chains, iterative methods and so on (see [12]-[14]). For instance, Y. Wei et al. studied the representation of the group inverse of a real singular Toeplitz matrix which arises in scientific computing and engineering (see [29]); in [25], S. Kirkland et al. investigated the representation of the group inverse of the Laplacian matrix of an undirected weighted graph G on n vertices; and in [24], G. Heinig studied the group inverse of the Sylvester transformation $\varphi(X) = AX - XB$, where $A \in \mathbb{C}^{m \times m}$ and $B \in \mathbb{C}^{n \times n}$; utilizing the theory of Drazin inverse, the differential equation $Ax' + Bx = f$ is studied in linear systems,

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where A, B are both $n \times n$ singular matrices and f is a vector-valued function (see [13]); in [3], R. Bru et al. gave the general explicit solution which is obtained when the symmetric singular linear control system satisfies the regularity condition.

In 1979, S. Campbell and C. Meyer proposed an open problem to find an explicit representation for the Drazin inverse of a 2×2 block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where the blocks A and D are supposed to be square matrices but their sizes need not be the same (see [12]). A simplified problem to find an explicit representation of the Drazin (group) inverse for block matrix $\begin{pmatrix} A & B \\ C & O \end{pmatrix}$ (A is square, O is square null matrix) was proposed by S. Campbell in 1983. This open problem was motivated in hoping to find general expressions for the solutions of the second-order system of the differential equations

$$Ex''(t) + Fx'(t) + Gx(t) = 0 \quad (t \geq 0),$$

where E is a singular matrix. Detailed discussions of the importance of the problem can be found in [11]. Until now, both problems have not been resolved. However, there have been some studies about the representations of the Drazin(group) inverse under certain conditions (see [4-10, 15-23, 26, 27]). For example, the formulae of Drazin(group) inverse of an operator matrix under some conditions are investigated in [18] and [19]. The representations for the Drazin(group) inverse of the sum of several matrices are given in [15] and [17]. In particular, J. Chen et al. [15] presented the representation of the group inverse of $M = P + Q + S$, where $PQ = QP = O$, $PS = SQ = O$, $P^\#$ and $Q^\#$ exist. Some conditions for the representations of group inverse for block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ are listed below:

- (i) $A^\#$ and $(D - CA^\#B)^\#$ exist [18].
- (ii) $D = O$, $A = B = I_d$, I_d , $C \in \mathbb{C}^{d \times d}$ [10].
- (iii) $D = O$, $A, B, C \in \{P, P^*, PP^*\}$, $P \in \mathbb{C}^{n \times n}$, $P^2 = P$, P^* is the conjugate transpose of P [9].
- (iv) $C = O$, A and D are square matrices over K [8].
- (v) $D = O$, $A = B = A^2$, $A, C \in K^{n \times n}$ [4].
- (vi) $D = O$, $A = B$, $\text{rank}(C) \geq \text{rank}(A)$, $A, C \in K^{n \times n}$ [5].
- (vii) A is invertible and $(D - CA^{-1}B)^\#$ exists; D is invertible and $(A - BD^{-1}C)^\#$ exists; B or C is invertible, where $A, B, C \in K^{n \times n}$ [6].
- (viii) $D = O$, $A = c_1B + c_2C$, where non-zero elements c_1 and c_2 are in the center of K ; $D = O$, $A = B^kC^l$, where k and l are positive integers [7].

In order to further solve the problem proposed in [11], in this paper, we give the sufficient conditions or the necessary and sufficient conditions for the existence and the representations of the group inverse for block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ ($A, B \in K^{n \times n}$) when A and B satisfy one of the following conditions:

- (1) $B^\#$ and $(B^\pi A)^\#$ exist;

- (2) B^\sharp and $(AB^\pi)^\sharp$ exist;
- (3) B^\sharp exists and $BAB^\pi = O$;
- (4) B^\sharp exists and $B^\pi AB = O$.

2. Some Lemmas.

LEMMA 2.1. *Let $A \in K^{n \times n}$. Then A has a group inverse if and only if there exist nonsingular matrices $P \in K^{n \times n}$ and $A_1 \in K^{r \times r}$ such that $A = P \begin{pmatrix} A_1 & O \\ O & O \end{pmatrix} P^{-1}$ and $A^\sharp = P \begin{pmatrix} A_1^\sharp & O \\ O & O \end{pmatrix} P^{-1}$, where $\text{rank}(A) = r$.*

Proof. Theorem 2.2.2 of [28] holds over the complex number field and also over skew fields. \square

LEMMA 2.2. [7] *Let $A, G \in K^{n \times n}$, $\text{Ind}(A) = k$. Then $G = A^D$ if and only if*

$$A^k GA = A^k, \quad AG = GA, \quad \text{rank}(G) \leq \text{rank}(A^k).$$

LEMMA 2.3. *Let $M = \begin{pmatrix} A & B \\ B & O \end{pmatrix}$, $S = B^\pi AB^\pi$, $A, B \in K^{n \times n}$.*

- (i) *If B^\sharp and $(B^\pi A)^\sharp$ exist, then S^\sharp and M^\sharp exist.*
- (ii) *If B^\sharp exists and $BAB^\pi = O$, then M^\sharp exists if and only if $(AB^\pi)^\sharp$ exists.*

Proof. Suppose $\text{rank}(B) = r$. Applying Lemma 2.1, there exist invertible matrices $P \in K^{n \times n}$ and $B_1 \in K^{r \times r}$ such that

$$B = P \begin{pmatrix} B_1 & O \\ O & O \end{pmatrix} P^{-1} \quad \text{and} \quad B^\sharp = P \begin{pmatrix} B_1^{-1} & O \\ O & O \end{pmatrix} P^{-1}.$$

Let $A = P \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} P^{-1}$, where $A_1 \in K^{r \times r}$, $A_2 \in K^{r \times (n-r)}$, $A_3 \in K^{(n-r) \times r}$ and $A_4 \in K^{(n-r) \times (n-r)}$.

- (i) Because $(B^\pi A)^\sharp$ exists, we have $\text{rank}(B^\pi A) = \text{rank}(B^\pi A)^2$, that is,

$$\text{rank}(A_3 \ A_4) = \text{rank}(A_4 A_3 \ A_4^2).$$

Since $\text{rank}(A_3 \ A_4) = \text{rank}(A_4(A_3 \ A_4)) \leq \text{rank}(A_4)$ and $\text{rank}(A_3 \ A_4) \geq \text{rank}(A_4)$, we have $\text{rank}(A_3 \ A_4) = \text{rank}(A_4)$.

So there exists a matrix $X \in K^{(n-r) \times r}$ such that $A_3 = A_4 X$.

Hence, $\text{rank}(A_4) = \text{rank}(A_4^2)$, i.e., $A_4^\#$ exists. Noting that

$$\begin{aligned} S &= B^\pi AB^\pi \\ &= P \begin{pmatrix} O & O \\ O & I_{n-r} \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} O & O \\ O & I_{n-r} \end{pmatrix} P^{-1} \\ &= P \begin{pmatrix} O & O \\ O & A_4 \end{pmatrix} P^{-1}, \end{aligned}$$

we see that $S^\#$ exists. Since

$$\begin{aligned} \text{rank}(M) &= \text{rank} \begin{pmatrix} A_1 & A_2 & B_1 & O \\ A_3 & A_4 & O & O \\ B_1 & O & O & O \\ O & O & O & O \end{pmatrix} = \text{rank} \begin{pmatrix} O & O & B_1 & O \\ O & A_4 & O & O \\ B_1 & O & O & O \\ O & O & O & O \end{pmatrix} \\ &= 2r + \text{rank}(A_4) \end{aligned}$$

and

$$\begin{aligned} \text{rank}(M^2) &= \text{rank} \begin{pmatrix} A^2 + B^2 & AB \\ BA & B^2 \end{pmatrix} = \text{rank} \begin{pmatrix} A^2 - ABB^\#A + B^2 & O \\ O & B^2 \end{pmatrix} \\ &= \text{rank} \begin{pmatrix} B_1^2 + A_2A_3 & A_2A_4 & O & O \\ A_4A_3 & A_4^2 & O & O \\ O & O & B_1^2 & O \\ O & O & O & O \end{pmatrix}, \end{aligned}$$

by $A_3 = A_4X$, we get

$$\text{rank}(M^2) = \text{rank} \begin{pmatrix} B_1^2 & O & O & O \\ O & A_4^2 & O & O \\ O & O & B_1^2 & O \\ O & O & O & O \end{pmatrix} = 2r + \text{rank}(A_4^2),$$

and

$$\text{rank}(M) = \text{rank}(M^2),$$

i.e., $M^\#$ exists.

(ii) If $BAB^\pi = O$, then $A_2 = O$, thus $AB^\pi = P \begin{pmatrix} O & O \\ O & A_4 \end{pmatrix} P^{-1}$,

$$\text{rank}(M) = \text{rank} \begin{pmatrix} A_1 & O & B_1 & O \\ A_3 & A_4 & O & O \\ B_1 & O & O & O \\ O & O & O & O \end{pmatrix} = \text{rank} \begin{pmatrix} O & O & B_1 & O \\ O & A_4 & O & O \\ B_1 & O & O & O \\ O & O & O & O \end{pmatrix}$$

$$= 2r + \text{rank}(A_4)$$

and

$$\begin{aligned} \text{rank}(M^2) &= \text{rank} \begin{pmatrix} A^2 + B^2 & AB \\ BA & B^2 \end{pmatrix} = \text{rank} \begin{pmatrix} A^2 - ABB^\sharp A + B^2 & O \\ O & B \end{pmatrix} \\ &= \text{rank} \begin{pmatrix} B_1^2 & O & O & O \\ A_4 A_3 & A_4^2 & O & O \\ O & O & B_1^2 & O \\ O & O & O & O \end{pmatrix} = \text{rank} \begin{pmatrix} B_1^2 & O & O & O \\ O & A_4^2 & O & O \\ O & O & B_1^2 & O \\ O & O & O & O \end{pmatrix} \\ &= 2r + \text{rank}(A_4^2). \end{aligned}$$

As $(AB^\pi)^\sharp$ exists, we get $\text{rank}(A_4) = \text{rank}(A_4^2)$. Thus $\text{rank}(M) = \text{rank}(M^2)$; that is, M^\sharp exists, so the “if” part holds. Now we prove the “only if” part.

Since M^\sharp exists if and only if $\text{rank}(M) = \text{rank}(M^2)$, $\text{rank}(A_4) = \text{rank}(A_4^2)$, i.e., $(AB^\pi)^\sharp$ exists. \square

LEMMA 2.4. *Let $A, B \in K^{n \times n}$, $S = B^\pi A B^\pi$. Suppose B^\sharp and $(B^\pi A)^\sharp$ exist. Then S^\sharp exists and the following conclusions hold:*

- (i) $B^\pi A S^\sharp A = B^\pi A$;
- (ii) $B^\pi A S^\sharp = S^\sharp A B^\pi$;
- (iii) $B S^\sharp = S^\sharp B = B^\sharp S^\sharp = S^\sharp B^\sharp = O$.

Proof. Suppose $\text{rank}(B) = r$. Applying Lemma 2.1, there exist invertible matrices $P \in K^{n \times n}$ and $B_1 \in K^{r \times r}$ such that $B = P \begin{pmatrix} B_1 & O \\ O & O \end{pmatrix} P^{-1}$ and

$$B^\sharp = P \begin{pmatrix} B_1^{-1} & O \\ O & O \end{pmatrix} P^{-1}.$$

Let $A = P \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} P^{-1}$, where $A_1 \in K^{r \times r}$, $A_2 \in K^{r \times (n-r)}$, $A_3 \in K^{(n-r) \times r}$ and $A_4 \in K^{(n-r) \times (n-r)}$.

From Lemma 2.3 (i) and the proof of Lemma 2.3 (i), we get S^\sharp exists and

$$S^\sharp = P \begin{pmatrix} O & O \\ O & A_4^\sharp \end{pmatrix} P^{-1}.$$

$$(i) \quad B^\pi A S^\sharp A = P \begin{pmatrix} O & O \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} O & O \\ O & A_4^\sharp \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} P^{-1}$$

$$= P \begin{pmatrix} O & O \\ A_4 A_4^\# A_3 & A_4 \end{pmatrix} P^{-1}.$$

By $A_3 = A_4 X$, $X \in K^{(n-r) \times r}$, we have

$$B^\pi A S^\# A = P \begin{pmatrix} O & O \\ A_3 & A_4 \end{pmatrix} P^{-1} = B^\pi A.$$

$$(ii) \quad B^\pi A S^\# = P \begin{pmatrix} O & O \\ O & A_4 A_4^\# \end{pmatrix} P^{-1} = S^\# A B^\pi.$$

$$(iii) \quad B S^\# = P \begin{pmatrix} B_1 & O \\ O & O \end{pmatrix} \begin{pmatrix} O & O \\ O & A_4^\# \end{pmatrix} P^{-1} = O.$$

Similarly, we obtain $S^\# B = B^\# S^\# = S^\# B^\# = O$. \square

LEMMA 2.5. Let $A, B \in K^{n \times n}$. If $BAB^\pi = O$, $B^\#$ and $(AB^\pi)^\#$ exist, then

- (a) $A(AB^\pi)^\# AB^\pi = AB^\pi$;
- (b) $A(AB^\pi)^\# = (AB^\pi)^\# AB^\pi$;
- (c) $B^\# AB^\pi = O, (AB^\pi)^\# B = O, B(AB^\pi)^\# = O$.

Proof. Suppose $\text{rank}(B) = r$. Applying Lemma 2.1, there exist invertible matrices $P \in K^{n \times n}$ and $B_1 \in K^{r \times r}$ such that

$$B = P \begin{pmatrix} B_1 & O \\ O & O \end{pmatrix} P^{-1} \quad \text{and} \quad B^\# = P \begin{pmatrix} B_1^{-1} & O \\ O & O \end{pmatrix} P^{-1}.$$

Let $A = P \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} P^{-1}$, where $A_1 \in K^{r \times r}$, $A_2 \in K^{r \times (n-r)}$, $A_3 \in K^{(n-r) \times r}$

and $A_4 \in K^{(n-r) \times (n-r)}$. By $BAB^\pi = O$, we get $A = P \begin{pmatrix} A_1 & O \\ A_3 & A_4 \end{pmatrix} P^{-1}$. Then

$$(a) \quad A(AB^\pi)^\# AB^\pi = P \begin{pmatrix} A_1 & O \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} O & O \\ O & A_4^\# \end{pmatrix} \begin{pmatrix} O & O \\ O & A_4 \end{pmatrix} P^{-1} \\ = P \begin{pmatrix} O & O \\ O & A_4 \end{pmatrix} P^{-1} = AB^\pi.$$

$$(b) \quad A(AB^\pi)^\# = P \begin{pmatrix} A_1 & O \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} O & O \\ O & A_4^\# \end{pmatrix} P^{-1} = P \begin{pmatrix} O & O \\ O & A_4^\# A_4 \end{pmatrix} P^{-1} \\ = (AB^\pi)^\# AB^\pi.$$

$$(c) \quad B^\# AB^\pi = P \begin{pmatrix} B_1^{-1} & O \\ O & O \end{pmatrix} \begin{pmatrix} O & O \\ O & A_4 \end{pmatrix} P^{-1} = O.$$

Similarly, we have $(AB^\pi)^\# B = O, B(AB^\pi)^\# = O$. \square

3. Conclusions.

THEOREM 3.1. Let $M = \begin{pmatrix} A & B \\ B & O \end{pmatrix}$, where $A, B \in K^{n \times n}$. Suppose B^\sharp and $(B^\pi A)^\sharp$ exist. Then M^\sharp exists and

$$M^\sharp = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},$$

where

$$\begin{aligned} U_{11} &= S^\sharp + (S^\sharp A - I)BB^\sharp AB^\pi AS^\sharp - (S^\sharp A - I)BB^\sharp AB^\pi; \\ U_{12} &= B^\sharp - S^\sharp AB^\sharp; \\ U_{21} &= B^\sharp - B^\sharp AS^\sharp + B^\sharp A(S^\sharp A - I)BB^\sharp AB^\pi - B^\sharp A(S^\sharp A - I)BB^\sharp AB^\pi AS^\sharp; \\ U_{22} &= B^\sharp AS^\sharp AB^\sharp - B^\sharp AB^\sharp; \\ S &= B^\pi AB^\pi. \end{aligned}$$

Proof. The existence of M^\sharp and S^\sharp have been given in Lemma 2.3 (i).

Let $X = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$. Then

$$MX = \begin{pmatrix} AU_{11} + BU_{21} & AU_{12} + BU_{22} \\ BU_{11} & BU_{12} \end{pmatrix} \text{ and } XM = \begin{pmatrix} U_{11}A + U_{12}B & U_{11}B \\ U_{21}A + U_{22}B & U_{21}B \end{pmatrix}.$$

We prove $X = M^\sharp$. Applying (i)–(iii) of Lemma 2.4, we get the following identities:

$$\begin{aligned} AU_{11} + BU_{21} &= AS^\sharp + A(S^\sharp A - I)BB^\sharp AB^\pi AS^\sharp - A(S^\sharp A - I)BB^\sharp AB^\pi + BB^\sharp \\ &\quad - BB^\sharp AS^\sharp + BB^\sharp A(S^\sharp A - I)BB^\sharp AB^\pi - BB^\sharp A(S^\sharp A - I)BB^\sharp AB^\pi AS^\sharp \\ &= BB^\sharp + B^\pi AS^\sharp + B^\pi A(S^\sharp A - I)BB^\sharp AB^\pi AS^\sharp - B^\pi A(S^\sharp A - I)BB^\sharp AB^\pi \\ &\stackrel{(i)}{=} B^\pi AS^\sharp + BB^\sharp, \end{aligned}$$

and

$$\begin{aligned} U_{11}A + U_{12}B &= S^\sharp A + (S^\sharp A - I)BB^\sharp AB^\pi AS^\sharp A - (S^\sharp A - I)BB^\sharp AB^\pi A + BB^\sharp \\ &\quad - S^\sharp ABB^\sharp \\ &\stackrel{(i)}{=} S^\sharp AB^\pi + BB^\sharp \\ &\stackrel{(ii)}{=} B^\pi AS^\sharp + BB^\sharp. \end{aligned}$$

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Hence,

$$\begin{aligned}
AU_{11} + BU_{21} &= U_{11}A + U_{12}B. \\
AU_{12} + BU_{22} &= AB^\sharp - B^\pi AS^\sharp AB^\sharp - BB^\sharp AB^\sharp \\
&\stackrel{(i)}{=} AB^\sharp - B^\pi AB^\sharp - BB^\sharp AB^\sharp \\
&= O, \\
U_{11}B &= S^\sharp B + (S^\sharp A - I)BB^\sharp AB^\pi AS^\sharp B - (S^\sharp A - I)BB^\sharp AB^\pi B \\
&= S^\sharp B + (S^\sharp A - I)BB^\sharp AB^\pi AS^\sharp B \\
&\stackrel{(iii)}{=} O.
\end{aligned}$$

Therefore, $AU_{12} + BU_{22} = U_{11}B$. Similarly, we can get

$$\begin{aligned}
BU_{11} &= U_{21}A + U_{22}B = BB^\sharp AB^\pi (I - AS^\sharp), \\
BU_{12} &= U_{21}B = BB^\sharp.
\end{aligned}$$

Consequently, $MX = XM = \begin{pmatrix} BB^\sharp + B^\pi AS^\sharp & O \\ BB^\sharp AB^\pi (I - AS^\sharp) & BB^\sharp \end{pmatrix}.$

It is easy to compute

$$MXM = \begin{pmatrix} BB^\sharp + B^\pi AS^\sharp & O \\ BB^\sharp AB^\pi (I - AS^\sharp) & BB^\sharp \end{pmatrix} \begin{pmatrix} A & B \\ B & O \end{pmatrix} = \begin{pmatrix} A & B \\ B & O \end{pmatrix} = M.$$

Suppose $\text{rank}(B) = r$. By Lemma 2.1, there exist invertible matrices $P \in K^{n \times n}$ and $B_1 \in K^{r \times r}$ such that $B = P \begin{pmatrix} B_1 & O \\ O & O \end{pmatrix} P^{-1}$ and $B^\sharp = P \begin{pmatrix} B_1^{-1} & O \\ O & O \end{pmatrix} P^{-1}$.

Let $A = P \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} P^{-1}$, where $A_1 \in K^{r \times r}$, $A_2 \in K^{r \times (n-r)}$, $A_3 \in K^{(n-r) \times r}$ and $A_4 \in K^{(n-r) \times (n-r)}$.

$$\begin{aligned}
\text{rank}(X) &= \text{rank} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \\
&= \text{rank} \begin{pmatrix} S^\sharp + (S^\sharp A - I)BB^\sharp AB^\pi AS^\sharp - (S^\sharp A - I)BB^\sharp AB^\pi & (I - S^\sharp A)B^\sharp \\ B^\sharp & O \end{pmatrix} \\
&= \text{rank} \begin{pmatrix} S^\sharp & B^\sharp - S^\sharp AB^\sharp \\ B^\sharp & O \end{pmatrix} = \text{rank} \begin{pmatrix} O & O & B_1^{-1} & O \\ O & A_4^\sharp & O & O \\ B_1^{-1} & O & O & O \\ O & O & O & O \end{pmatrix} \\
&= 2r + \text{rank}(A_4^\sharp) = 2r + \text{rank}(A_4) = \text{rank}(M).
\end{aligned}$$

From Lemma 2.2, we get $X = M^\sharp$. \square

Now we present an example to show the representation of the group inverse for a block matrix over the real quaternion skew field.

EXAMPLE 3.2. Let the real quaternion skew field $K = \{a + bi + cj + dk\}$, where a, b, c and d are real numbers. Let $M = \begin{pmatrix} A & B \\ B & O \end{pmatrix}$, where $A = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$ and $B = \begin{pmatrix} i & j \\ 0 & 0 \end{pmatrix}$.

By computation, B^\sharp and $(B^\pi A)^\sharp$ exist, and $B^\sharp = \begin{pmatrix} -i & -j \\ 0 & 0 \end{pmatrix}$, $S^\sharp = \begin{pmatrix} 0 & 1 \\ 0 & -k \end{pmatrix}$.

By Theorem 3.1, M^\sharp exists and $U_{11} = \begin{pmatrix} 0 & 1 \\ 0 & -k \end{pmatrix}$, $U_{12} = \begin{pmatrix} -i & -j \\ 0 & 0 \end{pmatrix}$, $U_{21} = \begin{pmatrix} -i & 0 \\ 0 & 0 \end{pmatrix}$, $U_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, so $M^\sharp = \begin{pmatrix} 0 & 1 & -i & -j \\ 0 & -k & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

THEOREM 3.3. Let $M = \begin{pmatrix} A & B \\ B & O \end{pmatrix}$, where $A, B \in K^{n \times n}$. Suppose B^\sharp and $(AB^\pi)^\sharp$ exist. Then M^\sharp exists and

$$M^\sharp = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix},$$

where

$$\begin{aligned} U_{11} &= S^\sharp + S^\sharp AB^\pi ABB^\sharp(AS^\sharp - I) - B^\pi ABB^\sharp(AS^\sharp - I); \\ U_{12} &= B^\sharp - S^\sharp AB^\sharp - S^\sharp AB^\pi ABB^\sharp(AS^\sharp - I)AB^\sharp + B^\pi ABB^\sharp(I - AS^\sharp)AB^\sharp; \\ U_{21} &= B^\sharp - B^\sharp AS^\sharp; \\ U_{22} &= B^\sharp AS^\sharp AB^\sharp - B^\sharp AB^\sharp; \\ S &= B^\pi AB^\pi. \end{aligned}$$

Proof. The proof is similar to that of Theorem 3.1. \square

THEOREM 3.4. Let $M = \begin{pmatrix} A & B \\ B & O \end{pmatrix}$, where $A, B \in K^{n \times n}$. Suppose B^\sharp exists and $BAB^\pi = O$. Then

(i) M^\sharp exists if and only if $(AB^\pi)^\sharp$ exists.

(ii) If M^\sharp exists, then

$$M^\sharp = \begin{pmatrix} U & V \\ B^\sharp & -B^\sharp AB^\sharp \end{pmatrix},$$

where

$$\begin{aligned} U &= B^\pi A(B^\sharp)^2 - (AB^\pi)^\sharp AB^\pi A(B^\sharp)^2 + (AB^\pi)^\sharp; \\ V &= -B^\pi A(B^\sharp)^2 AB^\sharp + (AB^\pi)^\sharp AB^\pi A(B^\sharp)^2 AB^\sharp - (AB^\pi)^\sharp AB^\sharp + B^\sharp. \end{aligned}$$

Proof. (i) The existence of M^\sharp has been given in Lemma 2.3 (ii).

(ii) Let $X = \begin{pmatrix} U & V \\ B^\sharp & -B^\sharp AB^\sharp \end{pmatrix}$. Then

$$MX = \begin{pmatrix} AU + BB^\sharp & AV - BB^\sharp AB^\sharp \\ BU & BV \end{pmatrix} \text{ and } XM = \begin{pmatrix} UA + VB & UB \\ B^\sharp AB^\pi & B^\sharp B \end{pmatrix}.$$

We prove $MX = XM$ by Lemma 2.5.

$$\begin{aligned} AU + BB^\sharp &= AB^\pi A(B^\sharp)^2 - A(AB^\pi)^\sharp AB^\pi A(B^\sharp)^2 + A(AB^\pi)^\sharp + BB^\sharp \\ &\stackrel{(a)}{=} AB^\pi A(B^\sharp)^2 - AB^\pi A(B^\sharp)^2 + A(AB^\pi)^\sharp + BB^\sharp \\ &= A(AB^\pi)^\sharp + BB^\sharp. \\ UA + VB &= B^\pi A(B^\sharp)^2 AB^\pi - (AB^\pi)^\sharp AB^\pi A(B^\sharp)^2 AB^\pi + (AB^\pi)^\sharp AB^\pi + B^\sharp B \\ &\stackrel{(c)}{=} (AB^\pi)^\sharp AB^\pi + B^\sharp B \\ &\stackrel{(b)}{=} A(AB^\pi)^\sharp + BB^\sharp. \end{aligned}$$

Thus,

$$\begin{aligned} AU + BB^\sharp &= UA + VB. \\ AV - BB^\sharp AB^\sharp &= -AB^\pi A(B^\sharp)^2 AB^\sharp + A(AB^\pi)^\sharp AB^\pi A(B^\sharp)^2 AB^\sharp - A(AB^\pi)^\sharp AB^\sharp \\ &\quad + AB^\sharp - BB^\sharp AB^\sharp \\ &\stackrel{(a)}{=} -A(AB^\pi)^\sharp AB^\sharp + B^\pi AB^\sharp \\ UB &= B^\pi AB^\sharp - (AB^\pi)^\sharp AB^\pi AB^\sharp + (AB^\pi)^\sharp B \\ &\stackrel{(b)}{=} B^\pi AB^\sharp - A(AB^\pi)^\sharp AB^\sharp + (AB^\pi)^\sharp B \\ &\stackrel{(c)}{=} -A(AB^\pi)^\sharp AB^\sharp + B^\pi AB^\sharp. \end{aligned}$$

Hence, $AV - BB^\sharp AB^\sharp = UB$. Similarly, we can obtain

$$BU = B^\sharp AB^\pi = O, BV = B^\sharp B.$$

Consequently,

$$MX = XM = \begin{pmatrix} A(AB^\pi)^\sharp + BB^\sharp & -A(AB^\pi)^\sharp AB^\sharp + B^\pi AB^\sharp \\ O & BB^\sharp \end{pmatrix}.$$

Applying Lemma 2.5, it is easy to compute

$$\begin{aligned} MXM &= \begin{pmatrix} A(AB^\pi)^\sharp + BB^\sharp & -A(AB^\pi)^\sharp AB^\sharp + B^\pi AB^\sharp \\ O & BB^\sharp \end{pmatrix} \begin{pmatrix} A & B \\ B & O \end{pmatrix} \\ &= \begin{pmatrix} A & B \\ B & O \end{pmatrix} = M. \end{aligned}$$

Suppose $\text{rank}(B) = r$. Applying Lemma 2.1, there exist invertible matrices $P \in K^{n \times n}$ and $B_1 \in K^{r \times r}$ such that

$$B = P \begin{pmatrix} B_1 & O \\ O & O \end{pmatrix} P^{-1} \text{ and } B^\sharp = P \begin{pmatrix} B_1^{-1} & O \\ O & O \end{pmatrix} P^{-1}.$$

Let $A = P \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} P^{-1}$, where $A_1 \in K^{r \times r}$, $A_2 \in K^{r \times (n-r)}$, $A_3 \in K^{(n-r) \times r}$, and $A_4 \in K^{(n-r) \times (n-r)}$. Then

$$\begin{aligned} \text{rank}(X) &= \text{rank} \begin{pmatrix} U & V \\ B^\sharp & -B^\sharp AB^\sharp \end{pmatrix} = \text{rank} \begin{pmatrix} U & B^\sharp \\ B^\sharp & O \end{pmatrix} \\ &= \text{rank} \begin{pmatrix} (AB^\pi)^\sharp & B^\sharp \\ B^\sharp & O \end{pmatrix} = \text{rank} \begin{pmatrix} O & O & B_1^{-1} & O \\ O & A_4^\sharp & O & O \\ B_1^{-1} & O & O & O \\ O & O & O & O \end{pmatrix} \\ &= 2r + \text{rank}(A_4^\sharp). \end{aligned}$$

Since $\text{rank}(M) = 2r + \text{rank}(A_4)$, $\text{rank}(X) = \text{rank}(M)$, thus $X = M^\sharp$ by Lemma 2.2. \square

EXAMPLE 3.5. Let $M = \begin{pmatrix} A & B \\ B & O \end{pmatrix}$, where $A = \begin{pmatrix} 1 & 1+i \\ i & 0 \end{pmatrix}$ and $B = \begin{pmatrix} i & i \\ 0 & 0 \end{pmatrix}$, $i = \sqrt{-1}$.

By computation, B^\sharp exists, $BAB^\pi = O$ and $B^\sharp = \begin{pmatrix} -i & -i \\ 0 & 0 \end{pmatrix}$.

From Theorem 3.4, M^\sharp exists, $U = \begin{pmatrix} 0 & -i \\ 0 & i \end{pmatrix}$, $V = \begin{pmatrix} 0 & 0 \\ -i & -i \end{pmatrix}$ and $-B^\sharp AB^\sharp$

$$= \begin{pmatrix} 1+i & 1+i \\ 0 & 0 \end{pmatrix}, \text{ then } M^\# = \begin{pmatrix} 0 & -i & 0 & 0 \\ 0 & i & -i & -i \\ -i & -i & 1+i & 1+i \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

THEOREM 3.6. Let $M = \begin{pmatrix} A & B \\ B & O \end{pmatrix}$, where $A, B \in K^{n \times n}$. Suppose that $B^\#$ exists and $B^\pi AB = O$. Then

(i) $M^\#$ exists if and only if $(B^\pi A)^\#$ exists.

(ii) If $M^\#$ exists, then

$$M^\# = \begin{pmatrix} U & B^\# \\ V & -B^\#AB^\# \end{pmatrix},$$

where

$$\begin{aligned} U &= (B^\#)^2 AB^\pi - (B^\#)^2 AB^\pi A(B^\pi A)^\# + (B^\pi A)^\#; \\ V &= -B^\#A(B^\#)^2 AB^\pi + B^\#A(B^\#)^2 AB^\pi A(B^\pi A)^\# - B^\#A(B^\pi A)^\# + B^\#. \end{aligned}$$

Proof. The proof is similar to that of Theorem 3.4. \square

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