



## INVERSE OF THE SQUARED DISTANCE MATRIX OF A COMPLETE MULTIPARTITE GRAPH\*

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**Abstract.** Let  $G$  be a connected graph on  $n$  vertices and  $d_{ij}$  be the length of the shortest path between vertices  $i$  and  $j$  in  $G$ . We set  $d_{ii} = 0$  for every vertex  $i$  in  $G$ . The squared distance matrix  $\Delta(G)$  of  $G$  is the  $n \times n$  matrix with  $(i, j)^{th}$  entry equal to 0 if  $i = j$  and equal to  $d_{ij}^2$  if  $i \neq j$ . For a given complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices, under some condition we find the inverse  $\Delta(K_{n_1, n_2, \dots, n_t})^{-1}$  as a rank-one perturbation of a symmetric Laplacian-like matrix  $\mathcal{L}$  with  $\text{rank}(\mathcal{L}) = n - 1$ . We also investigate the inertia of  $\mathcal{L}$ .

**Key words.** Squared distance matrix, Complete  $t$ -partite graphs, Laplacian-like matrix, Inverse, Inertia.

**AMS subject classifications.** 05C12, 05C50.

**1. Introduction and motivation.** Let  $G$  be a connected graph on vertices  $1, 2, \dots, n$  and  $d(i, j)$  be the length of the shortest path between vertices  $i$  and  $j$ . We set  $d(i, i) = 0$  for  $i = 1, 2, \dots, n$ . The distance matrix of a graph  $G$  on  $n$  vertices is an  $n \times n$  matrix  $D(G) = [d_{ij}]$ , where  $d_{ij} = d(i, j)$ . In the literature, the Hadamard product  $D(G) \circ D(G)$  is called the squared distance matrix of  $G$  and is defined as  $\Delta(G) = [d_{ij}^2]$ . Thus,  $\Delta(G)$  is a real symmetric matrix, and the eigenvalues of  $\Delta(G)$  are real.

Before proceeding further, we now introduce a few notations. Let  $I_n$  and  $\mathbf{1}_n$  denote the identity matrix and the column vector of all ones, respectively. We use  $\mathbf{0}_{m \times n}$  to represent zero matrix of order  $m \times n$ . We simply write  $\mathbb{1}$ ,  $I$ , and  $\mathbf{0}$  if there is no scope of confusion with respect to the order. Further,  $J_{m \times n}$  denotes the  $m \times n$  matrix of all ones, and if  $m = n$ , we use the notation  $J_m$ . Given a matrix  $A$ , we write  $A^t$  to denote the transpose of the matrix  $A$ . For a symmetric matrix  $A$ , the inertia of  $A$ , denoted by  $\text{In}(A)$ , is the triplet  $(\mathbf{n}_+(A), \mathbf{n}_0(A), \mathbf{n}_-(A))$ , where  $\mathbf{n}_+(A)$ ,  $\mathbf{n}_0(A)$ , and  $\mathbf{n}_-(A)$  denote the number of positive eigenvalues of  $A$ , the number of zero eigenvalue of  $A$ , and the number of negative eigenvalues of  $A$ , respectively.

A well-known result is due to Graham and Pollak [9]; if  $T$  is a tree with  $n$  vertices, then the determinant of the distance matrix  $D(T)$  is given by  $\det D(T) = (-1)^{n-1}(n-1)2^{n-2}$ . Thus, the determinant does not depend on the structure of the tree but only on the number of vertices. Later in [10], the inverse of  $D(T)$  is obtained by Graham and Lovász as a rank-one perturbation of the Laplacian matrix of  $T$ . These two results invoke significant interest, and several extensions and generalizations have been obtained (e.g., see [2, 6, 15, 18]). Given a graph  $G$ , the primary objective of these results is to define a matrix  $\mathcal{L}$  called Laplacian-like matrix satisfying  $\mathcal{L}\mathbf{1} = \mathbf{0}$  and  $\mathbf{1}^t\mathcal{L} = \mathbf{0}$  and find the inverse of the  $D(G)$  as a rank-one perturbation of  $\mathcal{L}$ .

\*Received by the editors on November 2, 2023. Accepted for publication on June 24, 2024. Handling Editor: K.C. Sivakumar. Corresponding Author: Sumit Mohanty.

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In [3, 4], Bapat and Sivasubramanian first studied the squared distance matrix of a tree. To be specific, Bapat and Sivasubramanian proved a formula for the determinant of the squared distance matrix  $\Delta(T)$  of a tree  $T$  in [3]. Later, in [4], they obtained the inverse  $\Delta(T)^{-1}$  (whenever it exists) as a rank-one perturbation of a matrix and also computed the inertia of  $\Delta(T)$ . Bapat, in [4], studied the determinant and the inverse of the squared distance matrix of a weighted tree. In [7], Das and Mohanty studied the squared distance matrix of a complete multipartite graph  $K_{n_1, n_2, \dots, n_t}$ . To be precise, in [7], authors first computed the inertia, energy of  $\Delta(K_{n_1, n_2, \dots, n_t})$ . Next, for fixed values of  $n$  and  $t$ , they discussed the existence and the uniqueness of graphs for which the spectral radius and the energy of  $\Delta(K_{n_1, n_2, \dots, n_t})$  attained its maximum and minimum value. In [13], Mahato and Kannan obtained the determinant and the inverse of the squared distance matrix of a tree with matrix weights. In [12], Howell, Kempton, Sandall, and Sinkovic gave an alternative proof to obtain the inertia of the squared distance matrix of a tree due to Bapat and Sivasubramanian in [4] and also studied the inertia for a unicyclic graph. In the literature, the spectral properties and the inverse of complete multipartite graphs have been studied with respect to the distance matrix and the adjacency matrix (e.g., see [6, 8, 14, 16]).

In this article, we obtain a Laplacian-like matrix  $\mathcal{L}$  with  $\text{rank}(\mathcal{L}) = n - 1$  for a given complete multipartite graph  $K_{n_1, n_2, \dots, n_t}$  and prove that the inverse of  $\Delta(K_{n_1, n_2, \dots, n_t})$  (whenever it exists) is a rank-one perturbation of  $\mathcal{L}$ . In this process, we find a few interesting recurrence-type relations involving  $n_1, n_2, \dots, n_t$ . We also observe a few properties of  $\mathcal{L}$ , compute the  $\text{In}(\mathcal{L})$  if  $\det \Delta(K_{n_1, n_2, \dots, n_t}) \neq 0$ , and give a conjecture about the  $\text{In}(\mathcal{L})$  if  $\det \Delta(K_{n_1, n_2, \dots, n_t}) = 0$ .

This article is organized as follows. In Section 2, we prove a few preliminary results that are necessary for the subsequent sections. In Section 3, we compute the inverse of the squared distance matrix  $\Delta(K_{n_1, n_2, \dots, n_t})$  (whenever it exists). Finally, in Section 4, we obtain a few properties of  $\mathcal{L}$  and investigate the inertia  $\text{In}(\mathcal{L})$ .

**2. Some preliminary results.** Let  $A$  be an  $n \times n$  matrix. Let  $A(i | j)$  be the submatrix obtained by deleting the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column and for  $1 \leq i, j \leq n$ , the cofactor  $c(i, j)$  is defined as  $(-1)^{i+j} \det A(i | j)$ . We use the notation  $\text{cof } A$  to denote the sum of all cofactors of  $A$  and state the following result.

LEMMA 2.1. [1] *Let  $A$  be an  $n \times n$  matrix. Let  $M$  be the matrix obtained from  $A$  by subtracting the first row from all other rows and then subtracting the first column from all other columns. Then*

$$\text{cof } A = \det M(1|1).$$

The following is a standard result on computing the determinant of block matrices.

PROPOSITION 2.2. [17] *Let  $A_{11}$  and  $A_{22}$  be square matrices. Then*

$$\det \left[ \begin{array}{c|c} A_{11} & \mathbf{0} \\ \hline A_{21} & A_{22} \end{array} \right] = \det A_{11} \det A_{22}.$$

We will prove a lemma, which will help us to compute the  $\text{cof } \Delta(K_{n_1, n_2, \dots, n_t})$ .

LEMMA 2.3. *Let  $C_m$  be an  $m \times m$  matrix of the following form:*

$$C_m = \begin{bmatrix} n_1 & -4(n_1 - 1) & -4(n_1 - 1) & \cdots & -4(n_1 - 1) \\ n_2 & 2(n_2 - 2) & -n_2 & \cdots & -n_2 \\ n_3 & -n_3 & 2(n_3 - 2) & \cdots & -n_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_m & -n_m & -n_m & \cdots & 2(n_m - 2) \end{bmatrix}.$$

The determinant of the above matrix is given by:

$$\det C_m = \sum_{i=1}^m \left( n_i \prod_{\substack{j=1 \\ j \neq i}}^m (3n_j - 4) \right).$$

*Proof.* Adding the first column to all the remaining columns of  $C_m$  yields the following matrix:

$$\begin{bmatrix} n_1 & -3n_1 + 4 & -3n_1 + 4 & \cdots & -3n_1 + 4 \\ n_2 & 3n_2 - 4 & 0 & \cdots & 0 \\ n_3 & 0 & 3n_3 - 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_m & 0 & 0 & \cdots & 3n_m - 4 \end{bmatrix}.$$

Now, expanding along the first row, we get

$$\det C_m = n_1 \prod_{j=2}^m (3n_j - 4) + (3n_1 - 4) \sum_{i=2}^m \left( n_i \prod_{\substack{j=2 \\ j \neq i}}^m (3n_j - 4) \right),$$

and the desired result follows. □

Let  $\Delta(K_{n_1, n_2, \dots, n_t})$  be the squared distance matrix of the complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$ . Then  $\Delta(K_{n_1, n_2, \dots, n_t})$  can be expressed in the following block form:

$$(2.1) \quad \Delta(K_{n_1, n_2, \dots, n_t}) = \begin{bmatrix} 4(J_{n_1} - I_{n_1}) & J_{n_1 \times n_2} & \cdots & J_{n_1 \times n_t} \\ J_{n_2 \times n_1} & 4(J_{n_2} - I_{n_2}) & \cdots & J_{n_2 \times n_t} \\ \vdots & \cdots & \ddots & \vdots \\ J_{n_t \times n_1} & J_{n_t \times n_2} & \cdots & 4(J_{n_t} - I_{n_t}) \end{bmatrix}.$$

Throughout this article, we assume that the vertices of  $\Delta(K_{n_1, n_2, \dots, n_t})$  are indexed as in equation (2.1).

**THEOREM 2.4.** *Let  $\Delta(K_{n_1, n_2, \dots, n_t})$  be the squared distance matrix of the complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices. Then, the sum of the cofactors of the squared distance matrix is given by:*

$$\text{cof } \Delta(K_{n_1, n_2, \dots, n_t}) = (-4)^{n-t} \left[ \sum_{i=1}^t \left( n_i \prod_{\substack{j=1 \\ j \neq i}}^t (3n_j - 4) \right) \right].$$

*Proof.* For complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  with  $n_i = 1$  for all  $1 \leq i \leq t$ , we have  $K_{n_1, n_2, \dots, n_t} = K_t$ . Then,  $\Delta(K_t) = D(K_t) = J_n - I_n$  and the result is true as  $\text{cof } \Delta(K_t) = \text{cof } D(K_t) = (-1)^{t-1} t$ . For other cases, without loss of generality, let  $n_1 > 1$  and  $M$  be the matrix obtained from  $\Delta(K_{n_1, n_2, \dots, n_t})$  by subtracting the first row from all other rows and then subtracting the first column from all other columns. Then the block form of the matrix  $M(1|1)$  is given by:

$$\begin{bmatrix} -4(J_{n_1-1} + I_{n_1-1}) & -4J_{(n_1-1) \times n_2} & \cdots & -4J_{(n_1-1) \times n_t} \\ -4J_{n_2 \times (n_1-1)} & 2J_{n_2} - 4I_{n_2} & \cdots & -J_{n_2 \times n_t} \\ \vdots & \cdots & \ddots & \vdots \\ -4J_{n_t \times (n_1-1)} & -J_{n_t \times n_2} & \cdots & 2J_{n_t} - 4I_{n_t} \end{bmatrix}.$$

First, for each partition of  $M(1|1)$ , we subtract the first column from all other columns and then add all the rows to the first row. Further, we shift the first column of each of the  $t$ -partitions to the first  $t$  columns and repeat the same operation for the rows. Then, the resulting matrix is of the following block form:

$$\left[ \begin{array}{c|c} \tilde{C}_t & \mathbf{0} \\ \hline * & -4I_{n-(t+1)} \end{array} \right],$$

where

$$\tilde{C}_t = \begin{bmatrix} -4n_1 & -4(n_1 - 1) & -4(n_1 - 1) & \cdots & -4(n_1 - 1) \\ -4n_2 & 2(n_2 - 2) & -n_2 & \cdots & -n_2 \\ -4n_3 & -n_3 & 2(n_3 - 2) & \cdots & -n_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -4n_t & -n_t & -n_t & \cdots & 2(n_t - 2) \end{bmatrix}.$$

Using the Proposition 2.2 and Lemma 2.3, the result follows.  $\square$

We now recall a result that gives a formula to obtain the determinant of  $\Delta(K_{n_1, n_2, \dots, n_t})$ .

**THEOREM 2.5.** [7, Corollary 3.5] *Let  $\Delta(K_{n_1, n_2, \dots, n_t})$  be the squared distance matrix of complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices. Then, the determinant of the squared distance matrix is given by:*

$$\det \Delta(K_{n_1, n_2, \dots, n_t}) = (-4)^{n-t} \left[ \sum_{i=1}^t \left( n_i \prod_{\substack{j=1 \\ j \neq i}}^t (3n_j - 4) \right) + \prod_{i=1}^t (3n_i - 4) \right].$$

The following result discusses the cases in which the determinant and the sum of the cofactors of the squared distance matrix of  $K_{n_1, n_2, \dots, n_t}$  are zero.

**THEOREM 2.6.** *Let  $G$  be a complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices and  $\Delta(G)$  be the squared distance matrix of  $G$ . If  $h = |\{i : n_i = 1\}|$ , then the following results hold:*

(i)  $\det \Delta(G) = 0$  only if  $\frac{t}{4} + \frac{3}{4} < h \leq \frac{t}{2} + \frac{1}{2}$ . Furthermore, let  $t = s + h$  such that  $n_i \geq 2$  for  $1 \leq i \leq s$  and  $n_i = 1$  for  $s + 1 \leq i \leq t = s + h$ . Then,  $\det \Delta(G) = 0$  if and only if  $h - 1 = \sum_{i=1}^s \frac{n_i}{3n_i - 4}$ .

(ii)  $\text{cof} \Delta(G) = 0$  only if  $\frac{t}{4} < h \leq \frac{t}{2}$ . Furthermore, let  $t = s + h$  such that  $n_i \geq 2$  for  $1 \leq i \leq s$  and  $n_i = 1$  for  $s + 1 \leq i \leq t = s + h$ . Then,  $\text{cof} \Delta(G) = 0$  if and only if  $h = \sum_{i=1}^s \frac{n_i}{3n_i - 4}$ .

*Proof.* By Theorems 2.4 and 2.5, it is easy to see that  $\det \Delta(G)$  and  $\text{cof} \Delta(G)$  are nonzero if  $n_i \geq 2$  for  $1 \leq i \leq t$ . Thus,  $\det \Delta(G)$  and  $\text{cof} \Delta(G)$  are 0 only if some of the  $n_i$ 's are 1. Let  $t = s + h$  and  $n_1, n_2, \dots, n_t$  be positive integers such that  $n_i \geq 2$  for  $1 \leq i \leq s$  and  $n_i = 1$  for  $s + 1 \leq i \leq t = s + h$ .

Now substituting  $n_i = 1$  for  $s + 1 \leq i \leq t = s + h$  in Theorem 2.5, the determinant of the squared distance matrix  $\Delta(G)$  is given by:

$$\begin{aligned} \det \Delta(G) &= (-4)^{n-t} \left[ \sum_{i=1}^t \left( n_i \prod_{\substack{j=1 \\ j \neq i}}^t (3n_j - 4) \right) + \prod_{i=1}^t (3n_i - 4) \right] \\ &= (-4)^{n-t} \left[ (-1)^h \sum_{i=1}^s \left( n_i \prod_{\substack{j=1 \\ j \neq i}}^s (3n_j - 4) \right) + (-1)^{h-1} h \prod_{i=1}^s (3n_i - 4) + (-1)^h \prod_{i=1}^s (3n_i - 4) \right] \\ &= (-4)^{n-t} (-1)^{h-1} \left[ (h-1) \prod_{i=1}^s (3n_i - 4) - \sum_{i=1}^s \left( n_i \prod_{\substack{j=1 \\ j \neq i}}^s (3n_j - 4) \right) \right] \\ &= (-4)^{n-t} (-1)^{h-1} \left[ (h-1) - \sum_{i=1}^s \frac{n_i}{3n_i - 4} \right] \prod_{i=1}^s (3n_i - 4). \end{aligned}$$

Since  $n_i \geq 2$  for  $1 \leq i \leq s$ , so  $\prod_{i=1}^s (3n_i - 4) > 0$ . Thus,  $\det \Delta(G) = 0$  if and only if  $h - 1 = \sum_{i=1}^s \frac{n_i}{3n_i - 4}$ .

Further,  $h - 1 = \sum_{i=1}^s \frac{n_i}{3n_i - 4} = \frac{s}{3} + \frac{4}{3} \sum_{i=1}^s \frac{1}{3n_i - 4} = \frac{t-h}{3} + \frac{4}{3} \sum_{i=1}^s \frac{1}{3n_i - 4}$ , which implies that  $\sum_{i=1}^s \frac{1}{3n_i - 4} = h - \frac{t+3}{4}$ . Using  $n_i \geq 2$  for  $1 \leq i \leq s$ , we have  $0 < \sum_{i=1}^s \frac{1}{3n_i - 4} \leq \frac{s}{2} = \frac{t-h}{2}$ . Therefore,  $0 < h - \frac{t+3}{4} \leq \frac{t-h}{2}$  and hence  $\frac{t}{4} + \frac{3}{4} < h \leq \frac{t}{2} + \frac{1}{2}$ . This proves part (i).

Next, substituting  $n_i = 1$  for  $s + 1 \leq i \leq t = s + h$  in Theorem 2.4, the sum of the cofactors of the squared distance matrix  $\Delta(G)$  is given by:

$$\begin{aligned} \text{cof } \Delta(G) &= (-4)^{n-t} \left[ (-1)^h \sum_{i=1}^s \left( n_i \prod_{\substack{j=1 \\ j \neq i}}^s (3n_j - 4) \right) + (-1)^{h-1} h \prod_{i=1}^s (3n_i - 4) \right] \\ &= (-4)^{n-t} (-1)^{h-1} \left[ h - \sum_{i=1}^s \frac{n_i}{3n_i - 4} \right] \prod_{i=1}^s (3n_i - 4). \end{aligned}$$

Since  $n_i \geq 2$  for  $1 \leq i \leq s$ ,  $\text{cof } \Delta(G) = 0$  if and only if  $h = \sum_{i=1}^s \frac{n_i}{3n_i - 4}$ . Moreover, arguing similar

to part (i),  $h = \sum_{i=1}^s \frac{n_i}{3n_i - 4}$  yields that  $\sum_{i=1}^s \frac{1}{3n_i - 4} = h - \frac{t}{4}$ . Using  $n_i \geq 2$  for  $1 \leq i \leq s$ , we have  $0 < \sum_{i=1}^s \frac{1}{3n_i - 4} \leq \frac{s}{2} = \frac{t-h}{2}$  implies that  $0 < h - \frac{t}{4} \leq \frac{t-h}{2}$  and hence  $\frac{t}{4} < h \leq \frac{t}{2}$ . This completes the proof.  $\square$

**3. Inverse of  $\Delta(K_{n_1, n_2, \dots, n_t})$ .** In this section, we first find the inverse of the squared distance matrix  $\Delta(K_{n_1, n_2, \dots, n_t})$  as a rank-one perturbation of a Laplacian-like matrix subject to the condition that  $\text{cof } \Delta(K_{n_1, n_2, \dots, n_t}) \neq 0$ . Notice that by [6, Lemma 4.13], it is known that  $\text{cof } \Delta(K_{n_1, n_2, \dots, n_t}) \neq 0$  is a necessary condition to present the inverse of  $\Delta(K_{n_1, n_2, \dots, n_t})$  (whenever it exists) as a rank-one perturbation of a matrix. Next, whenever it exists, we provide a formula for the inverse of  $\Delta(K_{n_1, n_2, \dots, n_t})$ . Before proceeding

further, we introduce a few notations useful for the subsequent results. Let  $n_i \in \mathbb{N}$  for  $1 \leq i \leq t$  and  $t \geq 2$ , and we denote

$$(3.2) \quad \begin{cases} \Phi_{n_1, n_2, \dots, n_t} = \prod_{i=1}^t (3n_i - 4), \\ \Phi_{\widehat{n_i}} = \Phi_{n_1, n_2, \dots, n_{i-1}, n_{i+1}, \dots, n_t} = \prod_{\substack{k=1 \\ k \neq i}}^t (3n_k - 4), \\ \Phi_{\widehat{n_i, n_j}} = \Phi_{n_1, n_2, \dots, n_{i-1}, n_{i+1}, \dots, n_{j-1}, n_{j+1}, \dots, n_t} = \prod_{\substack{k=1 \\ k \neq i, j}}^t (3n_k - 4), \end{cases}$$

$$(3.3) \quad \begin{cases} \Psi_{n_1, n_2, \dots, n_t} = \sum_{i=1}^t \left( n_i \prod_{\substack{j=1 \\ j \neq i}}^t (3n_j - 4) \right), \\ \Psi_{\widehat{n_i}} = \Psi_{n_1, n_2, \dots, n_{i-1}, n_{i+1}, \dots, n_t} = \sum_{\substack{k=1 \\ k \neq i}}^t \left( n_k \prod_{\substack{l=1 \\ l \neq i}}^t (3n_l - 4) \right), \\ \Psi_{\widehat{n_i, n_j}} = \Psi_{n_1, n_2, \dots, n_{i-1}, n_{i+1}, \dots, n_{j-1}, n_{j+1}, \dots, n_t} = \sum_{\substack{k=1 \\ k \neq i, j}}^t \left( n_k \prod_{\substack{l=1 \\ l \neq i, j}}^t (3n_l - 4) \right), \end{cases}$$

and

$$(3.4) \quad \begin{cases} \Theta_{n_1, n_2, \dots, n_t} = \Phi_{n_1, n_2, \dots, n_t} + \Psi_{n_1, n_2, \dots, n_t}, \\ \Theta_{\widehat{n_i}} = \Phi_{\widehat{n_i}} + \Psi_{\widehat{n_i}}. \end{cases}$$

Next, we state a few identities based on the notations from equations (3.2)–(3.4) and are useful for our subsequent calculations. The proofs of these identities are mostly computational and hence omitted (for details of the proofs, see Appendix A).

LEMMA 3.1. *Let  $n_i \in \mathbb{N}$  for  $1 \leq i \leq t$  and  $t \geq 2$ . Then, we have the following identities:*

$$\begin{aligned} (a) \quad & \Theta_{n_1, n_2, \dots, n_t} = (3n_i - 4)\Phi_{\widehat{n_i}} + \sum_{k=1}^t n_k \Phi_{\widehat{n_k}}. \\ (b) \quad & \Theta_{n_1, n_2, \dots, n_t} = (3n_i - 4)\Theta_{\widehat{n_i}} + n_i \Phi_{\widehat{n_i}}. \\ (c) \quad & \Psi_{n_1, n_2, \dots, n_t} = (3n_i - 4)\Psi_{\widehat{n_i}} + n_i \Phi_{\widehat{n_i}}. \\ (d) \quad & \Psi_{\widehat{n_i}} = \sum_{\substack{k=1 \\ k \neq i}}^t n_k \Phi_{\widehat{n_i, n_k}} = (3n_j - 4)\Psi_{\widehat{n_i, n_j}} + n_j \Phi_{\widehat{n_i, n_j}}. \end{aligned}$$

Let  $G = (V, E)$  be a complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices such that the vertex  $V$  is partitioned into  $t$  subsets  $V_i$  for  $1 \leq i \leq t$  and  $|V_i| = n_i$ . Let  $\Delta(G)$  be the squared distance matrix of  $G$ . Using notations in equations (3.2)–(3.4), by Theorem 2.5, we have  $\det \Delta(G) = 0$  if and only if  $\Theta_{n_1, n_2, \dots, n_t} = 0$  and by Theorem 2.4, we have  $\text{cof } \Delta(G) = 0$  if and only if  $\Psi_{n_1, n_2, \dots, n_t} = 0$ . With the above notations and observations, we define a few parameters useful to find the inverse of  $\Delta(G)$ . If  $\text{cof } \Delta(G) \neq 0$ , then we define

the constant and an  $n$ -dimensional column vector  $\nu$  as follows:

$$(3.5) \quad \lambda = \frac{\det \Delta(G)}{\text{cof } \Delta(G)} = \frac{\Theta_{n_1, n_2, \dots, n_t}}{\Psi_{n_1, n_2, \dots, n_t}},$$

and

$$(3.6) \quad \nu(v) = \frac{1}{\Psi_{n_1, n_2, \dots, n_t}} \left( \sum_{i=1}^t \sum_{v \in V_i} \Phi_{\widehat{n_i}} \right), \text{ where } v \in V.$$

Next, we define the Laplacian-like matrix  $\mathcal{L}$  satisfying  $\mathcal{L}\mathbf{1} = \mathbf{0}$  and  $\mathbf{1}^t \mathcal{L} = \mathbf{0}$  with respect to the squared distance matrix  $\Delta(G)$  with  $\text{cof } \Delta(G) \neq 0$  as follows: Let  $\mathcal{L} = [\mathcal{L}_{uv}]_{u, v \in V}$ , where

$$(3.7) \quad \mathcal{L}_{uv} = \begin{cases} \frac{1}{2\Psi_{n_1, n_2, \dots, n_t}} \left[ \frac{n_i - 1}{2} \Theta_{\widehat{n_i}} + (n_i - 3) \Psi_{\widehat{n_i}} \right] = a_i & \text{if } u = v \text{ and } u, v \in V_i, \\ -\frac{1}{2\Psi_{n_1, n_2, \dots, n_t}} \left[ \frac{1}{2} \Theta_{\widehat{n_i}} + \Psi_{\widehat{n_i}} \right] = b_i & \text{if } u \neq v \text{ and } u, v \in V_i, \\ \frac{1}{\Psi_{n_1, n_2, \dots, n_t}} \Phi_{\widehat{n_i, n_j}} = c_{ij} & \text{if } u \neq v, u \in V_i \text{ and } v \in V_j. \end{cases}$$

In view of equations (2.1) and (3.7), the product of matrices  $\mathcal{L}$  and  $\Delta(G)$  is given by  $\mathcal{L}\Delta(G) = [(\mathcal{L}\Delta(G))_{uv}]_{u, v \in V}$ , where

$$(3.8) \quad (\mathcal{L}\Delta(G))_{uv} = \begin{cases} 4(n_i - 1)b_i + \sum_{\substack{k=1 \\ k \neq i}}^t n_k c_{ik} & \text{if } u = v \text{ and } u, v \in V_i, \\ 4a_i + 4(n_i - 2)b_i + \sum_{\substack{k=1 \\ k \neq i}}^t n_k c_{ik} & \text{if } u \neq v \text{ and } u, v \in V_i, \\ a_i + (n_i - 1)b_i + 4(n_j - 1)c_{ij} + \sum_{\substack{k=1 \\ k \neq i, j}}^t n_k c_{ik} & \text{if } u \neq v, u \in V_i \text{ and } v \in V_j. \end{cases}$$

LEMMA 3.2. Let  $G$  be a complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$ . Let  $\Delta(G)$  be the squared distance matrix of  $G$  and  $\text{cof } \Delta(G) \neq 0$ . Then  $\Delta(G)\nu = \lambda\mathbf{1}$ , where  $\lambda$  and  $\nu$  are defined in equations (3.5) and (3.6), respectively.

*Proof.* Let  $G = (V, E)$  be the complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices such that the vertex  $V$  is partitioned into  $t$  subsets  $V_i$  for  $1 \leq i \leq t$  and  $|V_i| = n_i$ . Let  $\eta = \Delta(G)\nu$ . We will show  $\eta(v) = \lambda$  for all  $v \in V$ . For  $v \in V_i$ , we have

$$\begin{aligned} \eta(v) &= 4(n_i - 1) \frac{\Phi_{\widehat{n_i}}}{\Psi_{n_1, n_2, \dots, n_t}} + \sum_{\substack{k=1 \\ k \neq i}}^t n_k \frac{\Phi_{\widehat{n_k}}}{\Psi_{n_1, n_2, \dots, n_t}} \\ &= \frac{1}{\Psi_{n_1, n_2, \dots, n_t}} \left[ (3n_i - 4) \Phi_{\widehat{n_i}} + \sum_{k=1}^t n_k \Phi_{\widehat{n_k}} \right] \\ &= \frac{\Theta_{n_1, n_2, \dots, n_t}}{\Psi_{n_1, n_2, \dots, n_t}} = \lambda. \end{aligned}$$

This completes the proof. □

LEMMA 3.3. Let  $G$  be a complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$ . Let  $\Delta(G)$  be the squared distance matrix of  $G$  and  $\text{cof } \Delta(G) \neq 0$ . If  $\mathcal{L}$  is the Laplacian-like matrix defined in equation (3.7), then  $\mathcal{L}\Delta(G) + I = \nu \mathbf{1}^t$ , where  $\nu$  is as defined in equation (3.6).

*Proof.* Let  $G = (V, E)$  be the complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices such that the vertex set  $V$  is partitioned into  $t$  subsets  $V_i$  for  $1 \leq i \leq t$  and  $|V_i| = n_i$ . We will use equations (3.2)–(3.4), identities of Lemma 3.1, equation (3.8), and consider the following cases to complete the proof.

**Case 1:** Let  $u = v$ , where  $u, v \in V_i$  for  $1 \leq i \leq t$ .

$$\begin{aligned} (\mathcal{L}\Delta(G) + I)_{uv} &= 1 + 4(n_i - 1)b_i + \sum_{\substack{k=1 \\ k \neq i}}^t n_k c_{ik} \\ &= 1 - \frac{n_i - 1}{\Psi_{n_1, n_2, \dots, n_t}} [\Theta_{\widehat{n_i}} + 2\Phi_{\widehat{n_i}}] + \frac{\Psi_{\widehat{n_i}}}{\Psi_{n_1, n_2, \dots, n_t}} \\ &= 1 - \frac{(n_i - 1)[3\Psi_{\widehat{n_i}} + \Phi_{\widehat{n_i}}] + \Psi_{\widehat{n_i}}}{\Psi_{n_1, n_2, \dots, n_t}} \\ &= 1 - \frac{[(3n_i - 4)\Psi_{\widehat{n_i}} + n_i\Phi_{\widehat{n_i}}] - \Phi_{\widehat{n_i}}}{\Psi_{n_1, n_2, \dots, n_t}} \\ &= 1 - \frac{\Psi_{n_1, n_2, \dots, n_t} - \Phi_{\widehat{n_i}}}{\Psi_{n_1, n_2, \dots, n_t}} = \frac{\Phi_{\widehat{n_i}}}{\Psi_{n_1, n_2, \dots, n_t}} = \nu(u). \end{aligned}$$

**Case 2:** Let  $u \neq v$ , where  $u, v \in V_i$  for  $1 \leq i \leq t$ .

$$\begin{aligned} (\mathcal{L}\Delta(G) + I)_{uv} &= 4a_i + 4(n_i - 2)b_i + \sum_{\substack{k=1 \\ k \neq i}}^t n_k c_{ik} \\ &= \frac{(n_i - 1)\Theta_{\widehat{n_i}} + 2(n_i - 3)\Psi_{\widehat{n_i}}}{\Psi_{n_1, n_2, \dots, n_t}} - \frac{(n_i - 2)\Theta_{\widehat{n_i}} + 2(n_i - 2)\Psi_{\widehat{n_i}}}{\Psi_{n_1, n_2, \dots, n_t}} + \frac{\Psi_{\widehat{n_i}}}{\Psi_{n_1, n_2, \dots, n_t}} \\ &= \frac{\Theta_{\widehat{n_i}} + [2(n_i - 3) - 2(n_i - 2) + 1]\Psi_{\widehat{n_i}}}{\Psi_{n_1, n_2, \dots, n_t}} \\ &= \frac{\Theta_{\widehat{n_i}} - \Psi_{\widehat{n_i}}}{\Psi_{n_1, n_2, \dots, n_t}} = \frac{\Phi_{\widehat{n_i}}}{\Psi_{n_1, n_2, \dots, n_t}} = \nu(u). \end{aligned}$$

**Case 3:** Let  $u \in V_i$  and  $v \in V_j$  for  $1 \leq i, j \leq t$  with  $i \neq j$ .

$$\begin{aligned} (\mathcal{L}\Delta(G) + I)_{uv} &= a_i + (n_i - 1)b_i + 4(n_j - 1)c_{ij} + \sum_{\substack{k=1 \\ k \neq i, j}}^t n_k c_{ik} \\ &= \frac{(n_i - 1)\Theta_{\widehat{n_i}} + 2(n_i - 3)\Psi_{\widehat{n_i}}}{4\Psi_{n_1, n_2, \dots, n_t}} - \frac{(n_i - 1)\Theta_{\widehat{n_i}} + 2(n_i - 1)\Psi_{\widehat{n_i}}}{4\Psi_{n_1, n_2, \dots, n_t}} \\ &\quad + \frac{4(n_j - 1)\Phi_{\widehat{n_i, n_j}}}{\Psi_{n_1, n_2, \dots, n_t}} + \frac{(3n_j - 4)\Psi_{\widehat{n_i, n_j}}}{\Psi_{n_1, n_2, \dots, n_t}} \\ &= -\frac{\Psi_{\widehat{n_i}}}{\Psi_{n_1, n_2, \dots, n_t}} + \frac{4(n_j - 1)\Phi_{\widehat{n_i, n_j}}}{\Psi_{n_1, n_2, \dots, n_t}} + \frac{(3n_j - 4)\Psi_{\widehat{n_i, n_j}}}{\Psi_{n_1, n_2, \dots, n_t}} \end{aligned}$$



$$\begin{aligned} &= -\frac{\Psi_{\widehat{n}_i}}{\Psi_{n_1, n_2, \dots, n_t}} + \frac{\left[ (3n_j - 4)\Psi_{\widehat{n}_i, n_j} + n_j\Phi_{\widehat{n}_i, n_j} \right] + (3n_j - 4)\Phi_{\widehat{n}_i, n_j}}{\Psi_{n_1, n_2, \dots, n_t}} \\ &= -\frac{\Psi_{\widehat{n}_i}}{\Psi_{n_1, n_2, \dots, n_t}} + \frac{\Psi_{\widehat{n}_i} + \Phi_{\widehat{n}_i}}{\Psi_{n_1, n_2, \dots, n_t}} = \frac{\Phi_{\widehat{n}_i}}{\Psi_{n_1, n_2, \dots, n_t}} = \nu(u). \end{aligned}$$

This completes the proof. □

We now show that if  $\Delta(K_{n_1, n_2, \dots, n_t})$  is invertible, then we find the inverse of the squared distance matrix  $\Delta(K_{n_1, n_2, \dots, n_t})$  as a rank-one perturbation of a Laplacian-like matrix  $\mathcal{L}$  subject to the condition that  $\text{cof } \Delta(K_{n_1, n_2, \dots, n_t}) \neq 0$ .

**THEOREM 3.4.** *Let  $G$  be the complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  and  $\Delta(G)$  be the squared distance matrix of  $G$ . If  $\det \Delta(G) \neq 0$  and  $\text{cof } \Delta(G) \neq 0$ , then*

$$\Delta(G)^{-1} = -\mathcal{L} + \frac{1}{\lambda}\nu\nu^t,$$

where  $\mathcal{L}$  is the Laplacian-like matrix defined in equation (3.7),  $\lambda$  and  $\nu$  are defined in equations (3.5) and (3.6), respectively.

*Proof.* Using Lemma 3.2, we have  $\nu^t\Delta(G) = \lambda\mathbf{1}^t$ , which implies that  $\nu\nu^t\Delta(G) = \lambda\nu\mathbf{1}^t$ . By equation (3.5),  $\det \Delta(G) \neq 0$  if and only if  $\lambda \neq 0$ , and hence using Lemma 3.3, we get  $\mathcal{L}\Delta(G) + I = \nu\mathbf{1}^t = \frac{1}{\lambda}\nu\nu^t\Delta(G)$ . Therefore,  $\Delta(G)^{-1} = -\mathcal{L} + \frac{1}{\lambda}\nu\nu^t$ . □

We conclude this section with the result that gives a block matrix form for the inverse of  $\Delta(K_{n_1, n_2, \dots, n_t})$ , whenever it exists.

**THEOREM 3.5.** *Let  $G$  be a complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  and  $\Delta(G)$  be the squared distance matrix of  $G$ . If  $\det \Delta(G) \neq 0$ , then the inverse in  $t \times t$  block form is given by  $\Delta(G)^{-1} = [X_{ij}]$ , where*

$$(3.9) \quad X_{ij} = \begin{cases} \left( \frac{3\Theta_{\widehat{n}_i} + \Phi_{\widehat{n}_i}}{4\Theta_{n_1, n_2, \dots, n_t}} \right) J_{n_i} - \frac{1}{4}I_{n_i} & \text{if } i = j, \\ -\frac{\Phi_{\widehat{n}_i, n_j}}{\Theta_{n_1, n_2, \dots, n_t}} J_{n_i \times n_j} & \text{if } i \neq j. \end{cases}$$

*Proof.* Let  $\Delta(G) = [\Delta_{ij}]$  be the  $t \times t$  block form of the squared distance matrix of  $G$  in equation (2.1) and  $X = [X_{ij}]$  be a  $t \times t$  block matrix, where  $X_{ij}$  is defined in equation (3.9). Consider the block matrix  $Y = \Delta(G)X = [Y_{ij}]$ , where  $Y_{ij} = \sum_{k=1}^t \Delta_{ik}X_{kj}$  for  $1 \leq i, j \leq t$ . We will use equations (3.2)–(3.4) and identities of Lemma 3.1 to show that  $Y = I$  to complete the proof.

**Case 1:** For  $i = j$ ;  $1 \leq i \leq t$ .

$$\begin{aligned} Y_{ii} &= (J_{n_i} - I_{n_i}) \left[ \left( \frac{3\Theta_{\widehat{n}_i} + \Phi_{\widehat{n}_i}}{\Theta_{n_1, n_2, \dots, n_t}} \right) J_{n_i} - I_{n_i} \right] - \sum_{\substack{k=1 \\ k \neq i}}^t \frac{\Phi_{\widehat{n}_k, n_i}}{\Theta_{n_1, n_2, \dots, n_t}} J_{n_i \times n_k} J_{n_k \times n_i} \\ &= \frac{3\Theta_{\widehat{n}_i} + \Phi_{\widehat{n}_i}}{\Theta_{n_1, n_2, \dots, n_t}} \times n_i J_{n_i} - J_{n_i} - \frac{3\Theta_{\widehat{n}_i} + \Phi_{\widehat{n}_i}}{\Theta_{n_1, n_2, \dots, n_t}} \times J_{n_i} + I_{n_i} - \sum_{\substack{k=1 \\ k \neq i}}^t \frac{\Phi_{\widehat{n}_k, n_i}}{\Theta_{n_1, n_2, \dots, n_t}} \times n_k J_{n_i} \end{aligned}$$

$$\begin{aligned}
 &= \left[ \frac{3n_i \Theta_{\widehat{n}_i} + n_i \Phi_{\widehat{n}_i}}{\Theta_{n_1, n_2, \dots, n_t}} - \frac{3\Theta_{\widehat{n}_i} + \Phi_{\widehat{n}_i}}{\Theta_{n_1, n_2, \dots, n_t}} - 1 \right] J_{n_i} + I_{n_i} - \sum_{\substack{k=1 \\ k \neq i}}^t \frac{n_k \Phi_{\widehat{n}_k \widehat{n}_i}}{\Theta_{n_1, n_2, \dots, n_t}} J_{n_i} \\
 &= \left[ \frac{3n_i \Theta_{\widehat{n}_i} + n_i \Phi_{\widehat{n}_i}}{\Theta_{n_1, n_2, \dots, n_t}} - \frac{3\Theta_{\widehat{n}_i} + \Phi_{\widehat{n}_i}}{\Theta_{n_1, n_2, \dots, n_t}} - 1 \right] J_{n_i} + I_{n_i} - \frac{\Psi_{\widehat{n}_i}}{\Theta_{n_1, n_2, \dots, n_t}} J_{n_i} \\
 &= \left[ \frac{3n_i \Theta_{\widehat{n}_i} + n_i \Phi_{\widehat{n}_i}}{\Theta_{n_1, n_2, \dots, n_t}} - \frac{3\Theta_{\widehat{n}_i} + \Phi_{\widehat{n}_i}}{\Theta_{n_1, n_2, \dots, n_t}} - 1 \right] J_{n_i} + I_{n_i} - \frac{\Theta_{\widehat{n}_i} - \Phi_{\widehat{n}_i}}{\Theta_{n_1, n_2, \dots, n_t}} J_{n_i} \\
 &= \left[ \frac{3n_i \Theta_{\widehat{n}_i} + n_i \Phi_{\widehat{n}_i}}{\Theta_{n_1, n_2, \dots, n_t}} - \frac{3\Theta_{\widehat{n}_i} + \Phi_{\widehat{n}_i}}{\Theta_{n_1, n_2, \dots, n_t}} - \frac{\Theta_{\widehat{n}_i} - \Phi_{\widehat{n}_i}}{\Theta_{n_1, n_2, \dots, n_t}} - 1 \right] J_{n_i} + I_{n_i} \\
 &= \left[ \frac{(3n_i - 4)\Theta_{\widehat{n}_i} + n_i \Phi_{\widehat{n}_i} - \Theta_{n_1, n_2, \dots, n_t}}{\Theta_{n_1, n_2, \dots, n_t}} \right] J_{n_i} + I_{n_i} = I_{n_i}.
 \end{aligned}$$

**Case 2:** For  $i \neq j$ ;  $1 \leq i, j \leq t$ .

$$\begin{aligned}
 Y_{ij} &= 4(J_{n_i} - I_{n_i}) \left[ -\frac{\Phi_{\widehat{n}_i, \widehat{n}_j}}{\Theta_{n_1, n_2, \dots, n_t}} J_{n_i \times n_j} \right] + J_{n_i \times n_j} \left[ \left( \frac{3\Theta_{\widehat{n}_j} + \Phi_{\widehat{n}_j}}{4\Theta_{n_1, n_2, \dots, n_t}} \right) J_{n_j} - \frac{1}{4} I_{n_j} \right] \\
 &\quad - \sum_{\substack{k=1 \\ k \neq i, j}}^t \frac{\Phi_{\widehat{n}_k, \widehat{n}_j}}{\Theta_{n_1, n_2, \dots, n_t}} J_{n_i \times n_k} J_{n_k \times n_j} \\
 &= \left[ -\frac{4n_i \Phi_{\widehat{n}_i, \widehat{n}_j}}{\Theta_{n_1, n_2, \dots, n_t}} + \frac{4\Phi_{\widehat{n}_i, \widehat{n}_j}}{\Theta_{n_1, n_2, \dots, n_t}} + \frac{n_j(3\Theta_{\widehat{n}_j} + \Phi_{\widehat{n}_j})}{4\Theta_{n_1, n_2, \dots, n_t}} - \frac{1}{4} \right] J_{n_i \times n_j} - \sum_{\substack{k=1 \\ k \neq i, j}}^t \frac{n_k \Phi_{\widehat{n}_k, \widehat{n}_j}}{\Theta_{n_1, n_2, \dots, n_t}} J_{n_i \times n_j} \\
 &= \left[ -\frac{3n_i \Phi_{\widehat{n}_i, \widehat{n}_j}}{\Theta_{n_1, n_2, \dots, n_t}} + \frac{4\Phi_{\widehat{n}_i, \widehat{n}_j}}{\Theta_{n_1, n_2, \dots, n_t}} + \frac{3n_j \Theta_{\widehat{n}_j} + n_j \Phi_{\widehat{n}_j}}{4\Theta_{n_1, n_2, \dots, n_t}} - \frac{1}{4} \right] J_{n_i \times n_j} - \sum_{\substack{k=1 \\ k \neq j}}^t \frac{n_k \Phi_{\widehat{n}_k, \widehat{n}_j}}{\Theta_{n_1, n_2, \dots, n_t}} J_{n_i \times n_j} \\
 &= \left[ -\frac{(3n_i - 4)\Phi_{\widehat{n}_i, \widehat{n}_j}}{\Theta_{n_1, n_2, \dots, n_t}} + \frac{3n_j \Theta_{\widehat{n}_j} + n_j \Phi_{\widehat{n}_j}}{4\Theta_{n_1, n_2, \dots, n_t}} - \frac{1}{4} \right] J_{n_i \times n_j} - \frac{\Psi_{\widehat{n}_j}}{\Theta_{n_1, n_2, \dots, n_t}} J_{n_i \times n_j} \\
 &= \left[ -\frac{\Phi_{\widehat{n}_j}}{\Theta_{n_1, n_2, \dots, n_t}} + \frac{3n_j \Theta_{\widehat{n}_j} + n_j \Phi_{\widehat{n}_j} - \Theta_{n_1, n_2, \dots, n_t}}{4\Theta_{n_1, n_2, \dots, n_t}} - \frac{\Psi_{\widehat{n}_j}}{\Theta_{n_1, n_2, \dots, n_t}} \right] J_{n_i \times n_j} \\
 &= \left[ -\frac{\Phi_{\widehat{n}_j}}{\Theta_{n_1, n_2, \dots, n_t}} + \frac{3n_j \Theta_{\widehat{n}_j} + n_j \Phi_{\widehat{n}_j} - \Theta_{n_1, n_2, \dots, n_t}}{4\Theta_{n_1, n_2, \dots, n_t}} - \frac{\Theta_{\widehat{n}_j} - \Phi_{\widehat{n}_j}}{\Theta_{n_1, n_2, \dots, n_t}} \right] J_{n_i \times n_j} \\
 &= \left[ \frac{(3n_j - 4)\Theta_{\widehat{n}_j} + n_j \Phi_{\widehat{n}_j} - \Theta_{n_1, n_2, \dots, n_t}}{4\Theta_{n_1, n_2, \dots, n_t}} \right] J_{n_i \times n_j} = \mathbf{0}_{n_i \times n_j}. \quad \square
 \end{aligned}$$

In the next section, we will discuss a few properties of the Laplacian-like matrix  $\mathcal{L}$  defined in equation (3.7) and also investigate the inertia  $\text{In}(\mathcal{L})$ .

**4. A few properties of the Laplacian-like matrix.** For a complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices, it is known that 0 is an eigenvalue of the Laplacian-like matrix  $\mathcal{L}$  defined in equation (3.7). In the next result, we prove that 0 is a simple eigenvalue and also compute the cofactor of any two elements of  $\mathcal{L}$  if  $\det \Delta(G) \neq 0$ .

**THEOREM 4.1.** *Let  $G = (V, E)$  be a complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices with  $\text{cof } \Delta(G) \neq 0$ . If  $\mathcal{L}$  is the Laplacian-like matrix defined in equation (3.7), then  $\text{rank}(\mathcal{L}) = n - 1$ . Furthermore, if  $\det \Delta(G) \neq 0$ , then the cofactors of any two elements of  $\mathcal{L}$  are equal to  $\frac{(-1)^{n-1}}{\text{cof } \Delta(G)}$ .*

*Proof.* Let  $G = (V, E)$  be a complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices with  $\text{cof } \Delta(G) \neq 0$ . By equation (3.7), we have that Laplacian-like matrix  $\mathcal{L}$  is a symmetric matrix and  $\mathcal{L}\mathbf{1} = \mathbf{0}$  and  $\mathbf{1}^t \mathcal{L} = \mathbf{0}$ . Thus, 0 is an eigenvalue of  $\mathcal{L}$  and  $\mathbf{1}$  is a corresponding eigenvector. Furthermore,  $\mathcal{L}$  and  $\Delta(G)$  are symmetric matrices, and  $\mathbf{1}$  is an eigenvector of  $\mathcal{L}\Delta(G)$  corresponding to the eigenvalue 0.

Let  $\mathbf{x}$  be an eigenvector of  $\mathcal{L}\Delta(G)$  corresponding to the eigenvalue 0, that is,  $\mathbf{x}^t \mathcal{L}\Delta(G) = 0$ . Assume that  $\mathbf{x}$  is not in the span of  $\{\mathbf{1}\}$ . By Lemma 3.3, we have  $\mathbf{x}^t(\mathcal{L}\Delta(G) + I) = \mathbf{x}^t \nu \mathbf{1}^t$  and hence using  $\mathbf{x}^t \mathcal{L}\Delta(G) = 0$ , we get  $\mathbf{x}^t = \mathbf{x}^t \nu \mathbf{1}^t$ . Which is a contradiction to our assumption. Therefore,  $\text{rank}(\mathcal{L}) = n - 1$ .

Next, let  $\det \Delta(G) \neq 0$ . Then, by Theorem 3.4, we get  $\Delta(G)^{-1} = -\mathcal{L} + \frac{1}{\lambda} \nu \nu^t$ . Thus, using the determinant property  $\det(A + uv^t) = \det(A) + v^t \text{adj}(A)u$ , we have

$$\det(\Delta(G)^{-1}) = \det(-\mathcal{L}) + \frac{1}{\lambda} \nu^t \text{adj}(-\mathcal{L})\nu.$$

Using  $\text{rank}(\mathcal{L}) = n - 1$ , we have  $\det(\Delta(G)^{-1}) = \frac{1}{\lambda} \nu^t \text{adj}(-\mathcal{L})\nu = \frac{(-1)^{n-1}}{\lambda} \nu^t \text{adj}(\mathcal{L})\nu$ . Since  $\mathcal{L}$  is a symmetric matrix and  $\mathcal{L}\mathbf{1} = \mathbf{0}$ , using [1, Lemma 4.2] the cofactors of any two elements of  $\mathcal{L}$  are equal, say,  $c$ . Then,

$$(4.10) \quad \det(\Delta(G)^{-1}) = \frac{(-1)^{n-1}}{\lambda} \nu^t (cJ)\nu = \frac{(-1)^{n-1}c}{\lambda} \nu^t J\nu = \frac{(-1)^{n-1}c}{\lambda} \left( \sum_{v \in V} \nu(v) \right)^2.$$

Using equation (3.6), we have

$$(4.11) \quad \begin{aligned} \sum_{v \in V} \nu(v) &= \sum_{v \in V} \left[ \frac{1}{\Psi_{n_1, n_2, \dots, n_t}} \left( \sum_{i=1}^t \sum_{v \in V_i} \Phi_{\widehat{n}_i} \right) \right] \\ &= \frac{1}{\Psi_{n_1, n_2, \dots, n_t}} \sum_{i=1}^t n_i \Phi_{\widehat{n}_i} \\ &= \frac{1}{\Psi_{n_1, n_2, \dots, n_t}} \sum_{i=1}^t n_i \prod_{\substack{j=1 \\ j \neq i}}^t (3n_j - 4) = 1. \end{aligned}$$

Substituting equations (3.5) and (4.11) in equation (4.10), we get  $c = \frac{(-1)^{n-1}}{\text{cof } \Delta(G)}$ . This completes the proof.  $\square$

We now calculate a few eigenvalues of the Laplacian-like matrix  $\mathcal{L}$ .

**PROPOSITION 4.2.** *Let  $G$  be a complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices with  $\text{cof } \Delta(G) \neq 0$  and  $\mathcal{L}$  be the Laplacian-like matrix defined in equation (3.7). Then, the following holds:*

- (i)  $\frac{1}{4}$  is an eigenvalue of  $\mathcal{L}$  with multiplicity at least  $n - t$ .
- (ii) If  $h = |\{i : n_i = 1\}|$  and  $h \geq 2$ , then 1 is an eigenvalue of  $\mathcal{L}$  with multiplicity at least  $h - 1$ .

*Proof.* Let  $G = (V, E)$  be a complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices such that the vertex  $V$  is partitioned into  $t$  subsets  $V_i$  for  $1 \leq i \leq t$  and  $|V_i| = n_i$ . Suppose  $h = |\{i : n_i = 1\}|$  and  $t = s + h$  such that  $n_i \geq 2$  for  $1 \leq i \leq s$  and  $n_i = 1$  for  $s + 1 \leq i \leq t = s + h$ .

Let  $\mathbf{e}(p, q)$  be an  $n$ -dimensional column vector whose  $p^{\text{th}}$  entry is 1,  $q^{\text{th}}$  entry is  $-1$  and 0 otherwise. Suppose the vertices of  $G$  are indexed as in equation (2.1). Consider the set of column vectors

$$\mathcal{E} = \{\mathbf{e}(1, j) \mid j = 2, \dots, n_1\} \cup \{\mathbf{e}(n_1 + 1, n_1 + j) \mid j = 2, \dots, n_2\} \\ \cup \dots \cup \left\{ \mathbf{e} \left( \sum_{k=1}^{s-1} n_k + 1, \sum_{k=1}^{s-1} n_k + j \right) \mid j = 2, \dots, n_s \right\}.$$

Let  $\mathcal{E}_1 = \{\mathbf{e}(1, j) \mid j = 2, \dots, n_1\}$  and  $\mathcal{E}_i = \left\{ \mathbf{e} \left( \sum_{k=1}^{i-1} n_k + 1, \sum_{k=1}^{i-1} n_k + j \right) \mid j = 2, \dots, n_i \right\}$  for  $i = 2, 3, \dots, s$ .

Then  $\mathcal{E} = \cup_{i=1}^s \mathcal{E}_i$ . Using the definition of  $\mathcal{L}$  as in equation (3.7), it is easy to see

$$(4.12) \quad \mathcal{L}\mathbf{x} = (a_i - b_i)\mathbf{x} \text{ for all } \mathbf{x} \in \mathcal{E}_i; 1 \leq i \leq s.$$

For  $1 \leq i \leq s$ , we have

$$\begin{aligned} a_i - b_i &= \frac{1}{2\Psi_{n_1, n_2, \dots, n_t}} \left[ \frac{n_i - 1}{2} \Theta_{\widehat{n_i}} + (n_i - 3) \Psi_{\widehat{n_i}} \right] + \frac{1}{2\Psi_{n_1, n_2, \dots, n_t}} \left[ \frac{1}{2} \Theta_{\widehat{n_i}} + \Psi_{\widehat{n_i}} \right] \\ &= \frac{1}{2\Psi_{n_1, n_2, \dots, n_t}} \left[ \frac{n_i}{2} \Theta_{\widehat{n_i}} + (n_i - 2) \Psi_{\widehat{n_i}} \right] \\ &= \frac{1}{2\Psi_{n_1, n_2, \dots, n_t}} \left[ \frac{n_i}{2} \Phi_{\widehat{n_i}} + \frac{n_i}{2} \Psi_{\widehat{n_i}} + (n_i - 2) \Psi_{\widehat{n_i}} \right] \\ &= \frac{1}{4\Psi_{n_1, n_2, \dots, n_t}} [n_i \Phi_{\widehat{n_i}} + (3n_i - 4) \Psi_{\widehat{n_i}}] = \frac{1}{4}. \end{aligned}$$

Thus, from equation (4.12), we get  $\mathcal{L}\mathbf{x} = \frac{1}{4}\mathbf{x}$  for all  $\mathbf{x} \in \mathcal{E}$ , and hence  $\frac{1}{4}$  is an eigenvalue of  $\mathcal{L}$  with multiplicity at least  $|\mathcal{E}| = \sum_{i=1}^s n_i - s$ . Since  $n_i = 1$  for  $s + 1 \leq i \leq t = s + h$ ,  $n - t = \sum_{i=1}^t n_i - t = \sum_{i=1}^s n_i - s$ . This proves part (i).

To prove part (ii), let us assume  $h = |\{i : n_i = 1\}|$  and  $h \geq 2$ . Let  $\widetilde{\mathcal{E}} = \{\mathbf{e}(s + i, t) : i = 1, 2, \dots, h - 1\}$ . For  $s + 1 \leq i < t = s + h$ , using the definition of  $\mathcal{L}$  as in equation (3.7), we have

$$\mathcal{L}\mathbf{x} = (a_i - c_{it})\mathbf{x} \text{ for all } \mathbf{x} \in \widetilde{\mathcal{E}}.$$

Since  $n_t = 1$ ,  $\Phi_{\widehat{n_i}} = (3n_t - 4)\Phi_{\widehat{n_i, n_t}} = -\Phi_{\widehat{n_i, n_t}}$ . Thus, for  $s + 1 \leq i < t = s + h$ , we have

$$\begin{aligned} a_i - c_{it} &= \frac{1}{2\Psi_{n_1, n_2, \dots, n_t}} \left[ \frac{n_i - 1}{2} \Theta_{\widehat{n_i}} + (n_i - 3) \Psi_{\widehat{n_i}} \right] - \frac{1}{\Psi_{n_1, n_2, \dots, n_t}} \Phi_{\widehat{n_i, n_t}} \\ &= \frac{1}{2\Psi_{n_1, n_2, \dots, n_t}} [-2\Psi_{\widehat{n_i}} - 2\Phi_{\widehat{n_i, n_t}}] \\ &= \frac{1}{\Psi_{n_1, n_2, \dots, n_t}} [-\Psi_{\widehat{n_i}} - \Phi_{\widehat{n_i, n_t}}] \\ &= \frac{1}{\Psi_{n_1, n_2, \dots, n_t}} [-\Psi_{\widehat{n_i}} + \Phi_{\widehat{n_i}}] \\ &= \frac{1}{\Psi_{n_1, n_2, \dots, n_t}} [(3n_i - 4)\Psi_{\widehat{n_i}} + n_i \Phi_{\widehat{n_i}}] = 1. \end{aligned}$$

Hence,  $\mathcal{L}\mathbf{x} = \mathbf{x}$  for all  $\mathbf{x} \in \widetilde{\mathcal{E}}$ , and this completes the proof.  $\square$

Given a real symmetric matrix  $M$  of order  $n \times n$ , we use the following convention where the eigenvalues of  $M$  are in decreasing order:

$$(4.13) \quad \lambda_1(M) \geq \lambda_2(M) \geq \dots \geq \lambda_{n-1}(M) \geq \lambda_n(M).$$

We now state the Weyls inequality that gives interlacing inequalities of a rank-one perturbation to a real symmetric matrix.

**THEOREM 4.3.** [11, Corollary 4.3.9] *Let  $A$  and  $B$  be real symmetric matrices of order  $n \times n$  with eigenvalues ordered as in equation (4.13) such that  $B = A + \alpha\alpha^t$ , where  $\alpha$  is a column vector. Then,*

$$\lambda_1(B) \geq \lambda_1(A) \geq \lambda_2(B) \geq \lambda_2(A) \geq \dots \geq \lambda_n(B) \geq \lambda_n(A).$$

In the following theorem, we compute the inertia of the Laplacian-like matrix  $\mathcal{L}$  subject to the condition  $\det \Delta(K_{n_1, n_2, \dots, n_t}) \neq 0$ .

**THEOREM 4.4.** *Let  $G$  be a complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices with  $\text{cof } \Delta(G) \neq 0$  and  $\mathcal{L}$  be the Laplacian-like matrix defined in equation (3.7). Then, the following holds:*

- (i) *If  $n_i \geq 2$  for  $1 \leq i \leq t$ , then  $\text{In}(\mathcal{L}) = (n - t, 1, t - 1)$ .*
- (ii) *Let  $t = s + h$  and  $h = |\{i : n_i = 1\}|$ . If  $n_i \geq 2$  for  $1 \leq i \leq s$  and  $n_i = 1$  for  $s + 1 \leq i \leq s + h$ , then*

$$(4.14) \quad \text{In}(\mathcal{L}) = \begin{cases} (n - s - 1, 1, s) & \text{if } h - 1 > \sum_{i=1}^s \frac{n_i}{3n_i - 4}, \\ (n - s, 1, s - 1) & \text{if } h - 1 < \sum_{i=1}^s \frac{n_i}{3n_i - 4}. \end{cases}$$

*Proof.* Let  $G$  be a complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices and  $n_i \geq 2$  for  $1 \leq i \leq t$ . From the proof of [7, Theorem 4.1], it is known that  $-4$  is the only negative eigenvalue of the squared distance matrix  $\Delta(G)$  with multiplicity of  $n - t$  and  $\text{In}(\Delta(G)) = (t, 0, n - t)$ . Thus, if  $\det \Delta(G) \neq 0$ ,  $-\frac{1}{4}$  is the only negative eigenvalue of  $\Delta(G)^{-1}$  with multiplicity of  $n - t$ , and  $\text{In}(\Delta(G)^{-1}) = (t, 0, n - t)$ . Furthermore, under the assumption  $\det \Delta(G) \neq 0$  and  $\text{cof } \Delta(G) \neq 0$ , using Theorem 3.4, the inverse of  $\Delta(G)$  can be written as:

$$\Delta(G)^{-1} = -\mathcal{L} + \frac{1}{\lambda} \nu \nu^t.$$

Therefore, using the fact that  $0$  is a simple eigenvalue of  $\mathcal{L}$  due to Theorems 4.1 and 4.3, we get

$$\begin{array}{ccccccccc} \lambda_{t-1}(-\mathcal{L}) & \geq & \lambda_t(\Delta(G)^{-1}) & \geq & \lambda_t(-\mathcal{L}) & \geq & \lambda_{t+1}(\Delta(G)^{-1}) & \geq & \lambda_{t+1}(-\mathcal{L}), \\ (+ve) & & (+ve) & & (0) & & (-ve) & & (-ve) \end{array}$$

that is,  $\lambda_t(\Delta(G)^{-1}) > 0$  and  $\lambda_{t+1}(\Delta(G)^{-1}) = -\frac{1}{4} < 0$  implies that  $\lambda_i(-\mathcal{L}) > 0$  for  $1 \leq i \leq t - 1$ ,  $\lambda_t(-\mathcal{L}) = 0$  and  $\lambda_i(-\mathcal{L}) = -\frac{1}{4}$  for  $t + 1 \leq i \leq n$ . Hence,  $\text{In}(-\mathcal{L}) = (t - 1, 1, n - t)$ , and this proves part (i).

Let  $t = s + h$  and  $h = |\{i : n_i = 1\}|$  with  $n_i \geq 2$  for  $1 \leq i \leq s$  and  $n_i = 1$  for  $s + 1 \leq i \leq s + h$ . From part (i) of Theorem 2.6,  $\det \Delta(G) \neq 0$  if and only if  $h - 1 \neq \sum_{i=1}^s \frac{n_i}{3n_i - 4}$ . Similar to part (i), we prove part (ii) using  $0$  is a simple eigenvalue of  $\mathcal{L}$  and the form  $\Delta(G)^{-1} = -\mathcal{L} + \frac{1}{\lambda} \nu \nu^t$ .

Let  $h - 1 > \sum_{i=1}^s \frac{n_i}{3n_i - 4}$ . From [7, Lemma 4.4 and Theorem 4.5],  $\lambda_i(\Delta(G)) > 0$  for  $1 \leq i \leq s + 1$ ,  $\lambda_i(\Delta(G)) < 0$  for  $s + 2 \leq i \leq n$ , and  $\text{In}(\Delta(G)) = (s + 1, 0, n - s - 1)$ . Using  $0$  is a simple eigenvalue of  $\mathcal{L}$  and Theorem 4.3, we get

$$\lambda_s(-\mathcal{L}) \geq \lambda_{s+1}(\Delta(G)^{-1}) \geq \lambda_{s+1}(-\mathcal{L}) \geq \lambda_{s+2}(\Delta(G)^{-1}) \geq \lambda_{s+2}(-\mathcal{L}),$$

(+ve)                      (+ve)                      (0)                      (-ve)                      (-ve)

which implies that  $\text{In}(-\mathcal{L}) = (s, 1, n - s - 1)$ . Hence,  $\text{In}(\mathcal{L}) = (n - s - 1, 1, s)$ .

Next, let  $h - 1 < \sum_{i=1}^s \frac{n_i}{3n_i - 4}$ . From [7, Lemma 4.4 and Theorem 4.5],  $\lambda_i(\Delta(G)) > 0$  for  $1 \leq i \leq s$ ,  $\lambda_i(\Delta(G)) < 0$  for  $s + 1 \leq i \leq n$ , and  $\text{In}(\Delta(G)) = (s, 0, n - s)$ . Using Theorem 4.3, we get

$$\lambda_{s-1}(-\mathcal{L}) \geq \lambda_s(\Delta(G)^{-1}) \geq \lambda_s(-\mathcal{L}) \geq \lambda_{s+1}(\Delta(G)^{-1}) \geq \lambda_{s+1}(-\mathcal{L}).$$

(+ve)                      (+ve)                      (0)                      (-ve)                      (-ve)

Hence,  $\text{In}(\mathcal{L}) = (n - s, 1, s - 1)$ . This completes the proof.  $\square$

**COROLLARY 4.5.** *Let  $G$  be a complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices with  $\text{cof } \Delta(G) \neq 0$  and  $\mathcal{L}$  be the Laplacian-like matrix defined in equation (3.7). If  $\det \Delta(G) \neq 0$ , then the following holds:*

- (i)  $\frac{1}{4}$  is an eigenvalue of  $\mathcal{L}$  with multiplicity  $n - t$ .
- (ii) If  $h = |\{i : n_i = 1\}|$  and  $h \geq 2$ , then 1 is an eigenvalue of  $\mathcal{L}$  with multiplicity  $h - 1$ .

*Proof.* Let  $G$  be a complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$  on  $n = \sum_{i=1}^t n_i$  vertices. We complete the proof by considering the following cases.

Let  $n_i \geq 2$  for  $1 \leq i \leq t$ . From the proof of part (i) of Theorem 4.4, it follows that  $\frac{1}{4}$  is an eigenvalue of  $\mathcal{L}$  with multiplicity at most  $n - t$ . Thus, the result follows from Proposition 4.2.

Next, let  $t = s + h$  with  $n_i \geq 2$  for  $1 \leq i \leq s$  and  $n_i = 1$  for  $s + 1 \leq i \leq t = s + h$ . From part (iii) of [7, Lemma 4.4],  $-4$  is an eigenvalue of  $\Delta(G)$  with multiplicity  $n - t$  and  $-1$  is an eigenvalue of  $\Delta(G)$  with multiplicity  $h - 1$ . Therefore, the result follows from the interlacing of eigenvalues of  $\Delta(G)^{-1}$  and  $-\mathcal{L}$  arguments used in the proof of Theorem 4.4.  $\square$

We conclude the article with a conjecture. Given a complete  $t$ -partite graph  $K_{n_1, n_2, \dots, n_t}$ , by part (i) of the Theorem 2.6  $\det \Delta(K_{n_1, n_2, \dots, n_t}) = 0$  if and only if  $h - 1 = \sum_{i=1}^s \frac{n_i}{3n_i - 4}$ , where  $t = s + h$  with  $n_i \geq 2$  for  $1 \leq i \leq s$  and  $n_i = 1$  for  $s + 1 \leq i \leq t = s + h$ . Based on the examples encountered during the preparation of this manuscript, we believe the following holds true. If  $\det \Delta(K_{n_1, n_2, \dots, n_t}) = 0$  and  $\mathcal{L}$  be the Laplacian-like matrix defined in equation (3.7), then

- (i)  $\frac{1}{4}$  and 1 are the only positive eigenvalues of  $\mathcal{L}$  with multiplicity  $n - t$  and  $h - 1$ , respectively.
- (ii) The inertia  $\text{In}(\mathcal{L}) = (n - s - 1, 1, s)$ .

**Acknowledgment.** The authors thank the anonymous referees for a careful reading of the manuscript and for various suggestions. The first author Joyentanuj Das is partially supported by the National Science and Technology Council in Taiwan (Grant ID: NSTC-111-2628-M-110-002) and EPSRC Early Career Fellowship (EP/T00729X/1) by U.K. Research and Innovation in the U.K.

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Appendices.

**A. Proof of the Lemma 3.1.** Let  $n_i \in \mathbb{N}$  for  $1 \leq i \leq t$  and  $t \geq 2$ . We complete the proof with repeated application of equations (3.2)–(3.4). Thus,

$$\begin{aligned} \Theta_{n_1, n_2, \dots, n_t} &= \Phi_{n_1, n_2, \dots, n_t} + \Psi_{n_1, n_2, \dots, n_t} \\ &= (3n_i - 4)\Phi_{\widehat{n_i}} + \sum_{k=1}^t \left( n_k \prod_{\substack{j=1 \\ j \neq k}}^t (3n_j - 4) \right) \\ &= (3n_i - 4)\Phi_{\widehat{n_i}} + \sum_{k=1}^t n_k \Phi_{\widehat{n_k}}. \end{aligned}$$

This proves part (a) of the lemma. Similarly, to prove part (b),

$$\begin{aligned} \Theta_{n_1, n_2, \dots, n_t} &= \Phi_{n_1, n_2, \dots, n_t} + \Psi_{n_1, n_2, \dots, n_t} \\ &= (3n_i - 4)\Phi_{\widehat{n_i}} + \sum_{k=1}^t \left( n_k \prod_{\substack{j=1 \\ j \neq k}}^t (3n_j - 4) \right) \end{aligned}$$

$$\begin{aligned}
 &= (3n_i - 4)\Phi_{\widehat{n_i}} + (3n_i - 4) \sum_{\substack{k=1 \\ k \neq i}}^t \left( n_k \prod_{\substack{j=1 \\ j \neq i, k}}^t (3n_j - 4) \right) + n_i \prod_{\substack{j=1 \\ j \neq i}}^t (3n_j - 4) \\
 &= (3n_i - 4)\Phi_{\widehat{n_i}} + (3n_i - 4)\Psi_{\widehat{n_i}} + n_i \prod_{\substack{j=1 \\ j \neq i}}^t (3n_j - 4) \\
 &= (3n_i - 4) (\Phi_{\widehat{n_i}} + \Psi_{\widehat{n_i}}) + n_i \Phi_{\widehat{n_i}} \\
 &= (3n_i - 4)\Theta_{\widehat{n_i}} + n_i \Phi_{\widehat{n_i}}.
 \end{aligned}$$

Next,

$$\begin{aligned}
 \Psi_{n_1, n_2, \dots, n_t} &= \sum_{k=1}^t \left( n_k \prod_{\substack{j=1 \\ j \neq k}}^t (3n_j - 4) \right) \\
 &= (3n_i - 4) \sum_{\substack{k=1 \\ k \neq i}}^t \left( n_k \prod_{\substack{j=1 \\ j \neq i, k}}^t (3n_j - 4) \right) + n_i \prod_{\substack{j=1 \\ j \neq i}}^t (3n_j - 4) \\
 &= (3n_i - 4)\Psi_{\widehat{n_i}} + n_i \Phi_{\widehat{n_i}}.
 \end{aligned}$$

This establish the part (c). Finally, to prove part (d),

$$\begin{aligned}
 \Psi_{\widehat{n_i}} &= \sum_{\substack{k=1 \\ k \neq i}}^t n_k \Phi_{\widehat{n_i, n_k}} \\
 &= (3n_j - 4) \sum_{\substack{k=1 \\ k \neq i, j}}^t \left( n_k \prod_{\substack{l=1 \\ l \neq i, j, k}}^t (3n_l - 4) \right) + n_i \prod_{\substack{l=1 \\ l \neq i, j}}^t (3n_l - 4) \\
 &= (3n_j - 4)\Psi_{\widehat{n_i, n_j}} + n_j \Phi_{\widehat{n_i, n_j}}.
 \end{aligned}$$

This completes the proof. □