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SOME SYMMETRIC SIGN PATTERNS REQUIRING UNIQUE INERTIA*

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Abstract. A sign pattern is a matrix whose entries are from the set $\{+,-,0\}$. A sign pattern requires unique inertia if every matrix in its qualitative class has the same inertia. For symmetric tree sign patterns, several necessary and sufficient conditions to require unique inertia are known. In this paper, sufficient conditions for symmetric tree sign patterns to require unique inertia based on the sign and position of the loops in the underlying graph are given. Further, some sufficient conditions for a symmetric sign pattern to require unique inertia if the underlying graph contains cycles are determined.

Key words. Sign pattern, Inertia, Inertia set.

AMS subject classifications. 15B35, 15A18, 05C50.

1. Introduction. An $n \times n$ matrix $\mathcal{P} = [p_{ij}]$ whose entries belong to the set $\{+,-,0\}$ is called a *sign pattern matrix* or a *sign pattern*. The set of all $n \times n$ sign patterns is denoted by $\mathbf{Q}_n$. The set of all real matrices

$$\mathcal{Q}(\mathcal{P}) = \{\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n\times n}|\ sign(a_{ij}) = p_{ij}\ for\ all\ i,j = 1,2,...,n\},$$

is called the *qualitative class* of the sign pattern $\mathcal{P} \in \mathbf{Q}_n$.

$(+) + (-)$ is called an *ambiguous entry* and is denoted by $\#$. Let $\mathcal{P}_1 = [p^1_{ij}]$, $\mathcal{P}_2 = [p^2_{ij}]$ be two sign patterns of order n. Then, $\mathcal{P}_1 + \mathcal{P}_2$ is defined unambiguously if $p^1_{ij}p^2_{ij} \neq -$ for all $i,\ j$ and the product $\mathcal{P}_1\mathcal{P}_2$ is defined unambiguously if, for all i,j; $\sum_{k=1}^{n} p^1_{ik}p^2_{kj}$ does not contain opposite signed terms.

A sign pattern $\mathcal{P} \in \mathbf{Q_n}$ is said to be *sign nonsingular* if every $A \in \mathcal{Q}(\mathcal{P})$ is nonsingular. Thus, $\mathcal{P}$ is sign nonsingular if and only if $det(\mathcal{P}) = +$ or $det(\mathcal{P}) = -$, that is, in the standard expansion of $det(\mathcal{P})$ into $n!$ terms, there is at least one nonzero term and all the nonzero terms have the same sign.

The *directed graph* D of an $n \times n$ sign pattern matrix $\mathcal{P} = [p_{ij}]$ is the directed graph having n vertices $\{1,2,...,n\}$, such that there is a directed edge in D from i to j, denoted by (i,j), if and only if $p_{ij} \neq 0$. The arc (i,j) is associated with the sign $+$ *or* $-$ if and only if $p_{ij} = +$ *or* $-$. If $i = j$, the arc (i,j) is called a loop. The degree $deg(v)$ of a vertex v in D is the number of edges of D incident with v, each loop counting as two edges. The *underlying graph* of $\mathcal{P}$ is a simple graph denoted by G with n vertices, and the edge (i,j) is defined if and only if $p_{ij} \neq 0$. If $\mathcal{P}$ is symmetric and the underlying graph G of $\mathcal{P}$ is a tree, with possible loops, then $\mathcal{P}$ is referred to as a *symmetric tree* sign pattern.

A product of the form $\gamma = p_{i_1i_2}p_{i_2i_3}\cdots p_{i_ki_{k+1}}$, where all the elements are nonzero and the index set $\{i_1,i_2,...,i_k,i_{k+1}\}$ consists of distinct indices, is called a *path* of length k from i_1 to i_{k+1}. We say that the position of i_l corresponding to the path $\gamma = p_{i_1i_2}p_{i_2i_3}\cdots p_{i_ki_{k+1}}$ is odd (even) when $l \leq k+1$ and the cardinality of the set $\{i_1,i_2,...,i_l\}$ is odd (even). Further, i_{k_1}, i_{k_2} are said to be in ascending position corresponding to the path $\gamma = p_{i_1i_2}p_{i_2i_3}\cdots p_{i_ki_{k+1}}$ if $k_1,k_2 \in \{1,2,...,k+1\}$ and $i_{k_1} < i_{k_2}$. Also i_{k_1}, i_{k_2} and

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i_{k_1}, i_{k_2}, i_{k_3} are in odd-even and odd-even-odd ascending positions, respectively, if $k_1, k_2, k_3 \in \{1, 2, ..., k+1\}$ and $i_{k_1} < i_{k_2} < i_{k_3}$, i_{k_1} is odd, i_{k_2} is even, and i_{k_3} is odd. Similarly, we can define for all other combinations of sequence of odd, even ascending positions. For example, in the following graph G given in Fig. 1, indices 2, 3, and 5 are in even-odd-even ascending position corresponding to the path $\gamma : p_{12}p_{23}p_{35}$. Also, indices 2, 3, and 5 are in odd-even-odd ascending position corresponding to the path $\gamma_1 : p_{23}p_{35}$.

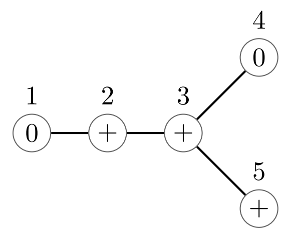

FIGURE 1. G.

If $i_1 = i_{k+1}$ and the index set $\{i_1, i_2, ..., i_k\}$ consists of distinct indices, then γ is called a *simple cycle* of length k. Suppose that γ_i is a simple cycle of length k_i, $i = 1, 2, .., l$, then the product of simple cycles, $\gamma = \gamma_1\gamma_2 \cdots \gamma_l$ is called a *composite cycle* of length $\sum_{i=1}^{l} k_i$. Throughout this paper, we assume all the cycles to be simple unless otherwise mentioned.

The *inertia* of a real symmetric matrix A is the triple of nonnegative numbers $(i_+(A), i_-(A), i_0(A))$ and could be denoted as $In(A)$, where $i_+(A),\ i_-(A),\ i_0(A)$ denotes the number of positive, negative, and zero eigenvalues of A, respectively, counted with their algebraic multiplicity.

An $n \times n$ sign pattern $\mathcal{P} = [p_{ij}]$ is called *symmetric* if $p_{ij} = p_{ji}$ for all i, j. For a symmetric sign pattern $\mathcal{P} \in \mathbf{Q}_n$, the *inertia set* is denoted by $In(\mathcal{P})$ and defined as $In(\mathcal{P}) = \{In(A) :\ A = A^T \in \mathcal{Q}(\mathcal{P})\}$. The symmetric sign pattern $\mathcal{P}$ is said to *require unique inertia* if $In(A_1) = In(A_2)$ for all real symmetric matrices $A_1,\ A_2 \in \mathcal{Q}(\mathcal{P})$.

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two sign patterns, then the matrices are called equivalent if one can be produced from the other through a series of permutation similarity, signature similarity, negation, and transposition. Being equivalent, $\mathcal{P}_1$ requires unique inertia if and only if $\mathcal{P}_2$ requires unique inertia.

In 2001, Hall et al. [1], proved that a symmetric sign pattern $\mathcal{P}$ requires unique inertia if and only if $smr(\mathcal{P}) = SMR(\mathcal{P})$, where $smr(\mathcal{P}) = min\{rank(A) : A = A^T,\ A \in \mathcal{Q}(\mathcal{P})\}$ and $SMR(\mathcal{P}) = max\{rank(A) : A = A^T,\ A \in \mathcal{Q}(\mathcal{P})\}$. They further characterized symmetric sign patterns requiring unique inertia in terms of the sum of all the maximum cycle lengths. Hall et al. in [1] characterized symmetric tree sign patterns with no loops and symmetric star sign patterns with loops that require unique inertia in terms of the position and sign of the loops. For symmetric sign patterns such that the underlying graph is a simple cycle with no loops, Hall et al. in [1] obtained the following result,

THEOREM 1.1. *[1, Theorem 4.6] Let $A \in \boldsymbol{Q}_n$ be a symmetric sign pattern with all diagonal entries equal to 0, and suppose $G(A)$ is a simple cycle of length n. If n is odd, then A is sign nonsingular, and hence, A requires unique inertia. If n is even, then A requires unique inertia if and only if A is sign nonsingular. More specifically, for even n, A requires unique inertia if and only if $\frac{1}{2}n$ is odd (respectively, even) and the number of $-$ entries on a simple n-cycle in A is even (respectively, odd).*

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In Section 2, we consider symmetric tree sign patterns with loops in the underlying graph. Hall and Li [2] showed that symmetric tri-diagonal sign patterns with no loops require unique inertia. They also characterized the inertia of symmetric tri-diagonal sign patterns with nonnegative diagonal entries in terms of the position of the loops. Also in [3], Lin et al. showed that under certain conditions $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}'$ require unique inertia, where $\mathcal{P}'$ is a principal subpattern of $\mathcal{P}$. In Section 2, Theorem 2.13 gives a sufficient condition for a symmetric tree sign pattern to require unique inertia in terms of the position of the loops. In Theorems 2.15 and 2.16, we characterize all symmetric tri-diagonal sign patterns that require unique inertia.

In Section 3, we consider symmetric sign patterns with cycles in the underlying graphs. We obtain some sufficient conditions for such sign patterns to require unique inertia. In Theorems 3.10 and 3.13, we consider nonnegative sign patterns with cycles of order $4k+2$, $k \in \mathbb{N}$, whereas in Theorem 3.17 we consider nonnegative sign patterns with cycles of order $4k+1$, $k \in \mathbb{N}$, and give sufficient conditions for such sign patterns to require unique inertia.

2. Inertia of a symmetric tree sign pattern. This section looks into symmetric tree sign patterns that require unique inertia. Assume that $\mathcal{P} \in \mathbf{Q}_n$ is a symmetric sign pattern such that the maximum cycle length is m in the underlying graph G. Let $B = B^T \in \mathcal{Q}(\mathcal{P})$ and the characteristic polynomial of B is given by

$$f(B) = \lambda^n - E_1(B)\lambda^{n-1} + E_2(B)\lambda^{n-2} - \cdots + (-1)^m E_m(B)\lambda^{n-m},$$

where $E_k(B)$ for $1 \leq k \leq m$ is the sum of all cycles (simple or composite) of length k in B properly signed.

Hall et al. [1] obtained the following two results:

Theorem 2.1. *[1, Theorem 3.5] Let $A \in \mathbf{Q}_n$ be a symmetric sign pattern, with the maximum length of the (composite) cycles in A equal to $m \geq 1$. Then, A requires unique inertia if and only if $E_m(B)$ has the same sign for all $B = B^T \in \mathcal{Q}(A)$. In particular, if all the terms in $E_m(B)$ have the same sign for any $B \in \mathcal{Q}(A)$, then A requires unique inertia.*

Theorem 2.2. *[1, Theorem 4.5] Let A be a symmetric tree sign pattern, with the maximum length of the cycles in A equal to $m \geq 1$. Then, A requires unique inertia if and only if all the terms in $E_m(B)$ have the same sign for any $B \in \mathcal{Q}(A)$. In this case, A requires rank m.*

The following results are by Hall and Li [2] for a nonnegative symmetric tri-diagonal sign pattern that requires unique inertia.

Proposition 2.3. *[2, Proposition 3.3] For the $n \times n$ symmetric tri-diagonal pattern*

$$A = \begin{pmatrix} * & + & 0 & \dots & \dots & 0 \\ + & * & + & 0 & \dots & 0 \\ 0 & + & * & + & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & + & * & + \\ 0 & 0 & \dots & 0 & + & * \end{pmatrix},$$

where each diagonal entry is either 0 or $+$, we have the following.

(a) For even n, A is sign nonsingular if and only if there are no two $+$ diagonal entries in A in odd-even ascending positions, respectively. In this case, $In(A) = (\frac{n}{2}, \frac{n}{2}, 0)$.

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(b) *For odd n, A is sign nonsingular if and only if there is at least one $+$ diagonal entry in an odd position, and there are not three $+$ diagonal entries in odd-even-odd ascending positions, respectively. In this case, $In(A) = (\frac{n+1}{2}, \frac{n-1}{2}, 0)$.*

Corollary 2.4. *[2, Corollary 3.4] For the $n \times n$ symmetric tri-diagonal pattern A as in Proposition 2.4, where each diagonal entry is 0 or $+$, we have the following.*

(a) *For even n, A requires unique inertia if and only if there are no two $+$ diagonal entries in A in odd-even ascending positions, respectively. In this case, $In(A) = (\frac{n}{2}, \frac{n}{2}, 0)$.*
(b) *For odd n where there is at least one $+$ diagonal entry in an odd position, A requires unique inertia if and only if there are not three $+$ diagonal entries in odd-even-odd ascending positions, respectively. In this case, $In(A) = (\frac{n+1}{2}, \frac{n-1}{2}, 0)$.*
(c) *For odd n where there are no $+$ diagonal entries in odd positions, A requires the unique inertia $(\frac{n-1}{2}, \frac{n-1}{2}, 1)$.*

Since for any symmetric sign pattern $\mathcal{P}$, $In(\mathcal{P}) = \{In(A) : A = A^T \in \mathcal{Q}(\mathcal{P})\}$. Therefore, we have the following lemmas.

Lemma 2.5. *Let $\mathcal{P}$ be an $n \times n$ symmetric sign pattern whose underlying graph is G. If G has a leaf u with $p_{u,u} = 0$ and v is its unique neighbour, then*

$$In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is the $(n-2) \times (n-2)$ principal subpattern with rows and columns u and v deleted.

Proof. If the underlying graph G is a tree, then the result follows from Lemma 4.3 in [3]. If G is not a tree, then since $\mathcal{P}$ is a symmetric sign pattern and the underlying graph G has a leaf with no loop, we can similarly conclude by following the proof of Lemma 4.3 in [3] that

$$In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is the $(n-2) \times (n-2)$ principal subpattern with rows and columns u and v deleted. ∎

Lemma 2.6. *Let $\mathcal{P}$ be a symmetric sign pattern whose underlying graph is G. If G has a leaf u with $p_{u,u} = +$ and v is its unique neighbour with $p_{v,v} \in \{0, -\}$, then*

$$In(\mathcal{P}) = (1, 0, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is obtained from $\mathcal{P}$ by removing the row and column corresponding to u and setting $p_{v,v} = -$. If G has a leaf u with $p_{u,u} = -$ and v is its unique neighbour with $p_{v,v} \in \{0, +\}$, then

$$In(\mathcal{P}) = (0, 1, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is obtained from $\mathcal{P}$ by removing the row and column corresponding to u and setting $p_{v,v} = +$.

Proof. If the underlying graph G is a tree, then the result follows from Lemma 4.4 in [3]. If G is not a tree, then since $\mathcal{P}$ is a symmetric sign pattern and the underlying graph G has a leaf with a loop and the unique neighbour of the leaf either has no loop or has a loop with opposite sign, we can similarly obtain the above result by following the proof of Lemma 4.4 in [3]. ∎

Suppose that $\mathcal{P}$ is a symmetric tree sign pattern containing loops. We now specify certain sufficient conditions on the underlying graph G of $\mathcal{P}$ such that $\mathcal{P}$ requires unique inertia.

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LEMMA 2.7. *Let $\mathcal{P}$ be an $n \times n$ symmetric tree sign pattern, whose underlying graph is G. Suppose that the number of loops in G is strictly less than 2. Then $\mathcal{P}$ requires unique inertia.*

Proof. We prove this theorem by using induction on n where n is the order of $\mathcal{P}$. For $n = 1$, $\mathcal{P}$ is a 1×1 sign pattern; therefore, inertia of $\mathcal{P}$ is unique. For $n = 2$, the only possible forms of $\mathcal{P}$ up to equivalence are $\begin{bmatrix} + & + \\ + & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$. In both cases, $\mathcal{P}$ has unique inertia, which is $(1, 1, 0)$. Therefore, $\mathcal{P}$ requires unique inertia.

Suppose that the statement holds for all $n \leq k-1$, $k \geq 3$. For $n = k$, since the number of loops in G is strictly less than 2, G has a leaf u such that $p_{u,u} = 0$. Let v be the unique neighbour of u. By applying Lemma 2.5 to u, v, we have

$$In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is the $(k-2) \times (k-2)$ principal subpattern of $\mathcal{P}$ with rows and columns corresponding to u and v deleted. Therefore, the underlying graph of $\mathcal{P}'$ is a forest with connected components $G_1, G_2, ..., G_s$ for some s. Thus, $\mathcal{P}' = \mathcal{P}_1 \bigoplus \mathcal{P}_2 \bigoplus \cdots \bigoplus \mathcal{P}_s$, where the underlying graph corresponding to $\mathcal{P}_i$ is G_i, $i = 1, 2, ..., s$. Since G_i is a tree with the number of vertices strictly less than k and with at most one loop, by the induction hypothesis $\mathcal{P}_i$ requires unique inertia for all $i = 1, 2, ..., s$. Therefore, $\mathcal{P}'$ and $\mathcal{P}$ require unique inertia. ∎

The following examples show that if $\mathcal{P}$ is an $n \times n$ symmetric sign pattern whose underlying graph G is a tree and if the number of loops in G is equal to 2, then $\mathcal{P}$ may not require unique inertia.

EXAMPLE 2.8. *Consider the symmetric sign pattern $\mathcal{P} = \begin{bmatrix} + & + \\ + & + \end{bmatrix}$ whose underlying graph is G given in Fig. 2, then $B_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \in \mathcal{Q}(\mathcal{P})$. Since $In(B_1) = (1, 0, 1)$ and $In(B_2) = (2, 0, 0)$, therefore $\mathcal{P}$ does not require unique inertia.*

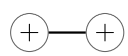

FIGURE 2. G.

LEMMA 2.9. *Let $\mathcal{P}$ be an $n \times n$ symmetric tree sign pattern, whose underlying graph is G with exactly two positive loops and no negative loops. If no odd length path from a leaf to another leaf of G has loops in odd-even ascending positions, respectively, then $\mathcal{P}$ requires unique inertia.*

Proof. We prove this by using induction on n, the order of $\mathcal{P}$. Since G has two loops and no odd length path from one leaf to another leaf of G has loops in odd-even ascending positions, respectively, $n > 2$. For $n = 3$, by Corollary 2.4, $\mathcal{P}$ requires unique inertia.

Now suppose that the statement holds for all $n \leq k-1$, $k \geq 4$. For $n = k$, if $\mathcal{P}$ is a symmetric tri-diagonal sign pattern then by Corollary 2.4, $\mathcal{P}$ requires unique inertia. If $\mathcal{P}$ is not a symmetric tri-diagonal sign pattern then G has leaf u such that $p_{u,u} = 0$. Let v be the unique neighbour of u. By applying Lemma 2.5 to u, v, we get

$$In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is the $(k-2) \times (k-2)$ principal subpattern of $\mathcal{P}$ with rows and columns corresponding to u and v deleted. Let G' be the underlying graph of $\mathcal{P}'$ with the connected components $G_1, G_2, ..., G_s$ for some s. Thus, $\mathcal{P}' = \mathcal{P}_1 \bigoplus \mathcal{P}_2 \bigoplus \cdots \bigoplus \mathcal{P}_s$, where the underlying graph corresponding to $\mathcal{P}_i$ is G_i, $i = 1, 2, ..., s$. If any

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component G_i of G' for $i \in \{1, 2, ..., s\}$ contains less than 2 loops, then by Lemma 2.7, $\mathcal{P}_i$ requires unique inertia. If G_i for some $i \in \{1, 2, ..., s\}$ contains two loops, consider if possible an odd length path $\bar{P}$ in G_i from a leaf x to a leaf y containing both the loops.

Case-1: Both x and y are leaves of G. Then, $\bar{P}$ has odd length in G, and by assumption, the loops are not in odd-even ascending positions, respectively.

Case-2: Either x or y is not a leaf of G. Suppose that x is not a leaf of G, then $\{(u, v), (v, x)\} \cup \bar{P}$ is a path in G from u to y. Since $\{(u, v), (v, x)\} \cup \bar{P}$ has odd length, by assumption, the loops are not in odd-even ascending positions, respectively, which is true for $\bar{P}$ as well.

Thus in both cases, each G_i satisfies the stated conditions and the order of each G_i is strictly less than k; therefore, by the induction hypothesis and Lemma 2.7, $\mathcal{P}_i$ requires unique inertia for all $i = 1, 2, ..., s$. Therefore, $\mathcal{P}'$ and $\mathcal{P}$ require unique inertia. □

The following example shows that the converse of the previous result is not true.

EXAMPLE 2.10. *Let $\mathcal{P}$ be the symmetric nonnegative sign pattern. From left to right, let $\mathcal{P}$, $\mathcal{P}'$ be the sign patterns corresponding to the graphs G, G', respectively, given in Fig. 3.*

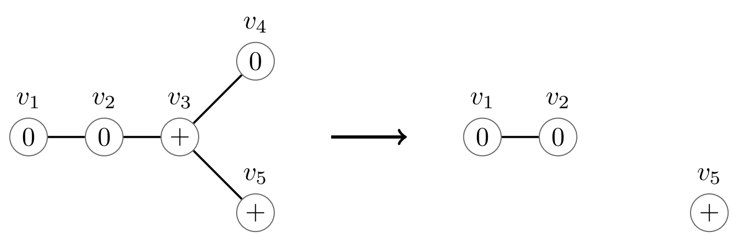

FIGURE 3. *Using Lemma 2.5 to determine the inertia.*

By using Lemma 2.5 to the edge (v_4, v_3), we have $In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}')$. Clearly, $\mathcal{P}'$ and $\mathcal{P}$ require unique inertia. But G contains the odd length path $\gamma : p_{v_1 v_2} p_{v_2 v_3} p_{v_3 v_5}$ from a leaf to another leaf which contains loops in odd-even ascending positions, respectively.

Also, the conclusion of Lemma 2.9 is not true if the underlying graph G of $\mathcal{P}$ has more than two loops.

EXAMPLE 2.11. *Let $\mathcal{P}$ be the symmetric nonnegative sign pattern associated with the graph is G given in Fig. 4.*

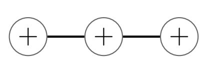

FIGURE 4. *G.*

By Corollary 2.4, $\mathcal{P}$ does not require unique inertia.

Hall and Li [2] obtained necessary and sufficient conditions for path sign patterns with nonnegative diagonal entries to require unique inertia. In the following results, we derived similar conditions for symmetric tree sign patterns with nonnegative diagonal entries to require unique inertia.

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LEMMA 2.12. *Let $\mathcal{P}$ be an $n \times n$ symmetric tree sign pattern, whose underlying graph is G with positive loops and no negative loops. If every path from one leaf to another leaf of G has an even length and no such path contains loops in odd-even-odd ascending positions, respectively, then $\mathcal{P}$ requires unique inertia.*

Proof. If the number of loops is less than or equal to, 2 then the result follows from Lemmas 2.7 and 2.9. Let us assume that the number of loops in G is greater than or equal to 3. We prove the theorem by using induction on n, the order of $\mathcal{P}$. Since G has at least 3 loops and every path from one leaf to another leaf of G has an even length which does not contain loops in odd-even-odd ascending positions, respectively, hence $n \geq 4$.

For $n = 4$, then $\mathcal{P}$ is either a symmetric tri-diagonal pattern or a symmetric star sign pattern. If $\mathcal{P}$ is a symmetric tri-diagonal sign pattern then by Corollary 2.4, $\mathcal{P}$ requires unique inertia. If $\mathcal{P}$ is a symmetric star sign pattern, then without loss of generality let,

$$\mathcal{P} = \begin{bmatrix} 0 & + & + & + \\ + & + & 0 & 0 \\ + & 0 & + & 0 \\ + & 0 & 0 & + \end{bmatrix}.$$

By Theorem (4.3) [1], $\mathcal{P}$ requires unique inertia, which is $(3, 1, 0)$.

Now suppose that the statement holds for all $n \leq k - 1$, where $k \geq 5$. For $n = k$, we have the following cases.

Case-1: G has a leaf without a loop. Let u be the leaf of G, such that $p_{u,u} = 0$. Let v be the unique neighbour of u. By applying Lemma 2.5 to u, v, we get

$$In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is the $(k-2) \times (k-2)$ principal subpattern of $\mathcal{P}$ with rows and columns corresponding to u and v deleted. Let G' be the underlying graph of $\mathcal{P}'$ with the connected components $G_1, G_2, ..., G_s$ for some s. Thus $\mathcal{P}' = \mathcal{P}_1 \bigoplus \mathcal{P}_2 \bigoplus \cdots \bigoplus \mathcal{P}_s$, where the underlying graph corresponding to $\mathcal{P}_i$ is G_i, for all $i = 1, 2, ..., s$.

If there is no path in G_i with more than 2 loops then since any path from one leaf to another leaf of G_i has an even length, by Lemmas 2.7 and 2.9, $\mathcal{P}_i$ requires unique inertia for all $i = 1, 2, ..., s$. If a component G_i of G', $i \in \{1, 2, ..., s\}$ contains three or more loops but there is no path from a leaf to another leaf in G_i with more than 2 loops then the result follows by the induction hypothesis. Otherwise, let $\bar{P}$ be a path from a leaf x to a leaf y of G_i which contains at least three loops. Then, we have the following cases.

Case-1a: Both x and y are leaves of G. If $\bar{P}$ has an even length then by assumption, the loops are not in odd-even-odd ascending positions, respectively.

Case-1b: Either x or y is not a leaf of G. Suppose that x is not a leaf of G, then $\{(u, v), (v, x)\} \cup \bar{P}$ is a path in G from u to y. Therefore, $\{(u, v), (v, x)\} \cup \bar{P}$ has an even length and the loops are not in odd-even-odd ascending positions, respectively, which is true for $\bar{P}$ as well. Thus, G_i satisfies the conditions stated and the order of G_i is strictly less than k; therefore by the induction hypothesis, $\mathcal{P}_i$ requires unique inertia. Therefore $\mathcal{P}'$ and $\mathcal{P}$ require unique inertia.

Case 2: Every leaf of G has a loop. Suppose that $R : uvr_1r_2...r_{2s-1}$ is a path from a leaf u to another leaf r_{2s-1} of G, such that $2s$ is the maximum length of any path in G. Since $p_{u,u} = +$, $p_{r_{2s-1},r_{2s-1}} = +$ and G does not contain loops in odd-even-odd ascending positions, respectively, $p_{v,v} = 0$. Since R is a maximum

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length path, all vertices adjacent to v other than r_1 (if there is one) are leaves. Let $u, u_1, ..., u_m$ be the leaves adjacent to v. Now, $p_{u,u} = +$, $p_{u_i,u_i} = +$ for $i = 1, 2, ..., m$ and $p_{v,v} = 0$. By recursively applying Lemma 2.6 to u, v and u_i, v, we get

$$In(\mathcal{P}) = (m + 1, 0, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is obtained from $\mathcal{P}$ by removing the rows and columns corresponding to $u, u_1, ..., u_m$ from $\mathcal{P}$ and setting $p_{v,v} = -$. Again in $\mathcal{P}'$, we have $p'_{v,v} = -$ and $p'_{r_1,r_1} \in \{0, +\}$, where r_1 is the unique neighbour of v in $\mathcal{P}'$. By applying Lemma 2.6, to v, r_1,

$$In(\mathcal{P}') = (0, 1, 0) + In(\mathcal{P}''),$$

where $\mathcal{P}''$ is obtained from $\mathcal{P}'$ by removing the row and column corresponding to v and setting $p'_{r_1,r_1} = +$. Since $\mathcal{P}''$ has order strictly less than k, similarly as in case-1, it can be shown that $\mathcal{P}''$ requires unique inertia. Since

$$In(\mathcal{P}) = (m + 1, 1, 0) + In(\mathcal{P}''),$$

$\mathcal{P}$ requires unique inertia. ∎

THEOREM 2.13. *Let $\mathcal{P}$ be an $n \times n$ symmetric tree sign pattern, whose underlying graph G has positive loops and no negative loops. If no odd length path from a leaf to another leaf of G contains loops in odd-even ascending positions, respectively, and no even length path from a leaf to another leaf of G contains loops in odd-even-odd ascending positions, respectively, then $\mathcal{P}$ requires unique inertia.*

Proof. If G has an odd length path from a leaf to another leaf then by assumption the loops are not in odd-even ascending positions, respectively, G must have a leaf which does not have a loop. Then, similarly as in the proof of Lemma 2.9, by using Lemma 2.5 and induction it can be shown that $\mathcal{P}$ requires unique inertia. Otherwise, there is no path from any leaf to another leaf of odd length, then the results follow from Lemmas 2.7 and 2.12. ∎

However, the converse of the above theorem is not true as the following example shows.

EXAMPLE 2.14. *Let* $\mathcal{P} = \begin{bmatrix} + & + & + & + \\ + & + & 0 & 0 \\ + & 0 & + & 0 \\ + & 0 & 0 & 0 \end{bmatrix}$ *be a nonnegative symmetric sign pattern whose underlying graph G given in Fig. 5.*

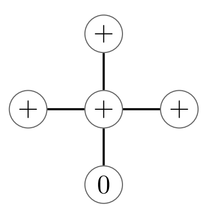

FIGURE 5. *G.*

By Theorem 4.3 [1], $\mathcal{P}$ requires unique inertia but the underlying graph of $\mathcal{P}$ does not satisfy the conditions stated in the previous theorem.

The following theorems generalize the results in Proposition 2.3 by Hall and Li [2], for a symmetric tree sign pattern where the loops in the underlying graph are not necessarily positively signed.

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THEOREM 2.15. *If $\mathcal{P}$ is a symmetric sign pattern whose underlying graph G is a path of odd length, then $\mathcal{P}$ requires unique inertia if and only if one of the following conditions holds,*

- *there exist no loops with odd-even ascending positions, respectively,*
- *if there are loops in odd-even ascending positions, respectively, then all such pairs have either the sign $+,-$ in this order, or the sign $-,+$ in this order.*

Proof. Let G have n vertices. Since G is a path of odd length, n is even. Suppose that there are no loops in the odd-even ascending positions, respectively, then the result follows from Proposition 2.3. Now suppose that G has loops in odd-even ascending positions, respectively, and without loss of generality let the sign of all such pairs be $+,-$, in this order. Then, all possible composite cycles of length n are one of the following:

i. $\frac{n}{2}$ distinct 2-cycles,
ii. $2k$ loops $(i_1,i_1),(i_2,i_2),...,(i_{2k},i_{2k})$, $k\in\mathbb{N}$, where $i_1,i_3,...,i_{2k-1}$ are odd, $i_2,i_4,...,i_{2k}$ are even and $i_1<i_2<\cdots<i_{2k}$ and $\frac{n-2k}{2}$ distinct 2-cycles which do not contain $i_1,i_2,...,i_{2k}$.

The sign of any nonzero term in $det(\mathcal{P})$ is $(-)^{\frac{n}{2}}$, so by Theorem 2.1, $\mathcal{P}$ requires unique inertia.

For the converse part, suppose there is a path in G which does not satisfy the above two conditions. Then, G must have two loops in odd-even ascending positions, respectively, with the same sign and by Corollary 2.4, $\mathcal{P}$ does not require unique inertia. $\square$

THEOREM 2.16. *If $\mathcal{P}$ is a symmetric sign pattern whose underlying graph G is a path of even length, then $\mathcal{P}$ requires unique inertia if and only if one of the following conditions holds,*

1. *there exist no loops with odd-even-odd ascending positions, respectively, and any two loops in odd positions have the same sign,*
2. *all loops in odd positions have the same sign and if there are loops in odd-even-odd ascending positions, respectively, then all such triplets have either the sign $+,-,+$, in this order, or the sign $-,+,-$, in this order.*

Proof. Let G have n vertices. Since the length of G is even, n is odd. First, suppose there are no loops in odd-even-odd ascending positions, respectively. If there is no loop in the odd position, the maximum length of the composite cycles in G is $n-1$, formed by $\frac{n-1}{2}$ distinct 2-cycles. Hence by Theorem 2.2, $\mathcal{P}$ requires unique inertia. If G has a loop in an odd position and no loops in odd-even-odd ascending positions, respectively, then the only composite cycle of length n is formed by a loop in an odd position and $\frac{n-1}{2}$ distinct 2-cycles. Since all loops in odd positions have the same sign, $\mathcal{P}$ is sign nonsingular and requires unique inertia.

Now suppose that G has loops in odd-even-odd ascending positions, respectively, and without loss of generality assume that any such combination of loops has the sign $+,-,+$, in this order. Then, all possible composite cycles of length n are one of the following:

i. $2k+1$ loops $(i_1,i_1),(i_2,i_2),...,(i_{2k+1},i_{2k+1})$, for some $k\in\mathbb{N}\cup\{0\}$ where $i_1,i_3,...,i_{2k+1}$ are odd, $i_2,i_4,...,i_{2k}$ are even and $i_1<i_2<\cdots<i_{2k+1}$ together with $\frac{n-(2k+1)}{2}$ distinct 2-cycles which does not contain $i_1,i_2,...,i_{2k_1+1}$.

Since the sign of all the nonzero terms in $det(\mathcal{P})$ is $(-)^{\frac{n-1}{2}}$, $\mathcal{P}$ requires unique inertia.

For the converse part, suppose that there is a path G which does not satisfy the above conditions. Then,

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either G has two loops in odd positions with opposite signs or if not then G has loops in odd-even-odd ascending positions, respectively, with the same sign. If G has two loops in odd positions with opposite signs, then G has two composite cycles of length n which are of opposite signs formed by the loops in odd positions and $\frac{n-1}{2}$ distinct 2-cycles. Therefore, $\mathcal{P}$ does not require unique inertia. If G has loops in odd-even-odd ascending positions, respectively with the same sign, then by Corollary 2.4, $\mathcal{P}$ does not require unique inertia. ∎

3. Inertia of a symmetric sign pattern with cycles in the underlying graph. In this section, we consider symmetric sign patterns such that the underlying graphs have cycles and derive sufficient conditions for such sign patterns to require unique inertia.

THEOREM 3.1. *Let $\mathcal{P}$ be an $n \times n$ nonnegative symmetric sign pattern whose underlying graph G is a cycle. Suppose that n is even, then the following is true:*

(a) If $n = 4k$ for some $k \in \mathbb{N}$ and G has no loops, then $\mathcal{P}$ does not require unique inertia.
(b) If $n = 4k + 2$ for some $k \in \mathbb{N}$ and the number of loops in G is at most 1, then $\mathcal{P}$ requires unique inertia.
(c) If $n = 4k + 2$ for some $k \in \mathbb{N}$ and the number of loops in G is more than 1, then $\mathcal{P}$ requires unique inertia if and only if the loops are not in odd-even ascending positions, respectively in any path in G.

Proof. Since the underlying graph of $\mathcal{P}$ is a simple cycle and $\mathcal{P}$ is nonnegative, without loss of generality let

$$\mathcal{P} = \begin{pmatrix} * & + & 0 & \dots & \dots & + \\ + & * & + & 0 & \dots & 0 \\ 0 & + & * & + & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & + & * & + \\ + & 0 & \dots & 0 & + & * \end{pmatrix},$$

where the nonzero diagonal elements are $+$.

(a) For all $A = A^T \in \mathcal{Q}(\mathcal{P})$,

$$\begin{aligned} det(A) &= (-1)^{n/2} a_{12}^2 a_{34}^2 \cdots a_{n-1n}^2 + (-1)^{n/2} a_{23}^2 a_{45}^2 \cdots a_{n1}^2 - 2a_{12}a_{23} \cdots a_{n1} \\ &= (a_{12}a_{34} \cdots a_{n-1n} - a_{23}a_{45} \cdots a_{n1})^2. \end{aligned} \tag{3.1}$$

By symmetrically emphasizing the entries $a_{i,i+1}$ and $a_{i+1,i}$ for odd i, we get $det(A) > 0$. Again if we take A such that $a_{12}a_{34} \cdots a_{n-1n} = a_{23}a_{45} \cdots a_{n1,}$ then $det(A) = 0$. So by Theorem 2.1, $\mathcal{P}$ does not require unique inertia.

(b) If the number of loops in G is at most 1, then for all $A = A^T \in \mathcal{Q}(\mathcal{P})$,

$$\begin{aligned} det(A) &= (-1)^{n/2} a_{12}^2 a_{34}^2 \cdots a_{n-1n}^2 + (-1)^{n/2} a_{23}^2 a_{45}^2 \cdots a_{n1}^2 - 2a_{12}a_{23} \cdots a_{n1} \\ &= -(a_{12}a_{34} \cdots a_{n-1n} + a_{23}a_{45} \cdots a_{n1})^2. \end{aligned} \tag{3.2}$$

Therefore, $sign(det(A)) = -$ for all $A = A^T \in \mathcal{Q}(\mathcal{P})$. By Theorem 2.1, $\mathcal{P}$ requires unique inertia.

(c) Suppose there is a path in G which contains two loops in odd-even ascending positions, respectively. Let $p_{ii} = +$ and $p_{kk} = +$, i is odd and k is even. Now by symmetrically emphasizing the entries

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$p_{s,s+1}$, $p_{s+1,s}$ for $s = 1, 3, ..., i-2, i+1, i+3..., k-2, k+1, k+3..., n-1$, and p_{ii}, p_{kk}, we get $A = A^T \in \mathcal{Q}(\mathcal{P})$ such that $sign(det(A)) = (-)^{\frac{n}{2}-1} = +$. Again by symmetrically emphasizing the entries $p_{s,s+1}$ and $p_{s+1,s}$ for odd s, we get $B = B^T \in \mathcal{Q}(\mathcal{P})$ such that $sign(det(B)) = (-)^{\frac{n}{2}} = -$. So by Theorem 2.1, $\mathcal{P}$ does not require unique inertia.

Conversely, suppose that there are no paths in G with loops in odd-even ascending positions, respectively, then for all $A = A^T \in \mathcal{Q}(\mathcal{P})$

$$\begin{aligned} det(A) &= (-1)^{n/2} a_{12}^2 a_{34}^2 \cdots a_{n-1n}^2 + (-1)^{n/2} a_{23}^2 a_{45}^2 \cdots a_{n1}^2 - 2a_{12}a_{23}\cdots a_{n1} \\ &= -(a_{12}a_{34}\cdots a_{n-1n} + a_{23}a_{45}\cdots a_{n1})^2. \end{aligned} \tag{3.3}$$

Therefore, $sign(det(A)) = -$ for all $A = A^T \in \mathcal{Q}(\mathcal{P})$. By Theorem 2.1, $\mathcal{P}$ requires unique inertia. ☐

THEOREM 3.2. *Let $\mathcal{P}$ be an $n \times n$ nonnegative symmetric sign pattern whose underlying graph G is a cycle. Suppose that n is odd, then the following is true:*

(a) If G has no loops, then $\mathcal{P}$ requires unique inertia.
(b) If $n = 4k + 3$ for some $k \in \mathbb{N}$ and G have at least one loop, then $\mathcal{P}$ does not require unique inertia.
(c) If $n = 4k + 1$ for some $k \in \mathbb{N}$ and the number of loops in G is at most 2, then $\mathcal{P}$ requires unique inertia.
(d) If $n = 4k + 1$ for some $k \in \mathbb{N}$ and the number of loops in G is more than 2, then $\mathcal{P}$ requires unique inertia if and only if the loops are not in odd-even-odd ascending positions, respectively, in any path in G.

Proof. Since the underlying graph of $\mathcal{P}$ is a simple cycle with loops and $\mathcal{P}$ is nonnegative, without loss of generality let

$$\mathcal{P} = \begin{pmatrix} * & + & 0 & \dots & \dots & + \\ + & * & + & 0 & \dots & 0 \\ 0 & + & * & + & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & + & * & + \\ + & 0 & \dots & 0 & + & * \end{pmatrix},$$

where the nonzero diagonals are $+$.

(a) For $A = A^T \in \mathcal{Q}(\mathcal{P})$, we have

$$det(A) = 2a_{12}a_{23}\cdots a_{n1}. \tag{3.4}$$

Therefore, $sign(det(A)) = +$ for all $A = A^T \in \mathcal{Q}(\mathcal{P})$. By Theorem 2.1, $\mathcal{P}$ requires unique inertia.

(b) Without loss of generality, let $p_{11} = +$. By symmetrically emphasizing the entries $p_{s,s+1}$, $p_{s+1,s}$ for even s and p_{11}, we get $A = A^T \in \mathcal{Q}(\mathcal{P})$ such that $sign(det(A)) = (-)^{\frac{n-1}{2}} = -$. Again by symmetrically emphasizing the entries $p_{s,s+1}$ and $p_{s+1,s}$ for all s, we get $B = B^T \in \mathcal{Q}(\mathcal{P})$ such that $sign(det(B)) = +$. So by Theorem 2.1, $\mathcal{P}$ does not require unique inertia.

(c) Since the number of loops in G is at most 2 and n is odd so the composite cycles of length n in G are one of the following:
   i. a loop and $\frac{n-1}{2}$, 2-cycles,
   ii. a simple cycle of length n.

   Thus, for all $A = A^T \in \mathcal{Q}(\mathcal{P})$, $sign(det(A)) = +$. By Theorem 2.1, $\mathcal{P}$ requires unique inertia.

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(d) Suppose there is a path in G which has three loops in odd-even-odd ascending positions, respectively. Without loss of generality, let p_{11}, p_{kk} and p_{ll} be in odd-even-odd ascending positions, respectively, such that $1 < k < l$. Now by symmetrically emphasizing the entries $p_{s,s+1}$, $p_{s+1,s}$ for $s = 2, 4, ..., k-2, k+1, k+3, ..., l-2, l+1, l+3, ..., n-1$, p_{11}, p_{kk} and p_{ll}, we get $A = A^T \in \mathcal{Q}(\mathcal{P})$ such that $sign(det(A)) = (-)^{\frac{n-3}{2}} = -$. Again by symmetrically emphasizing the entries p_{11}, $p_{s,s+1}$ and $p_{s+1,s}$ for $s = 2, 4, ..., n-1$, we get $B = B^T \in \mathcal{Q}(\mathcal{P})$ such that $sign(det(B)) = (-)^{\frac{n-1}{2}} = +$, so $\mathcal{P}$ does not require unique inertia.
Conversely, suppose that there are no paths in G such that three loops are in odd-even-odd ascending positions, respectively. So the possible composite cycles of length n are one of the following:
  i. a loop and $\frac{n-1}{2}$, 2-cycles,
  ii. a simple cycle of length n.
Thus for all $A = A^T \in \mathcal{Q}(\mathcal{P})$, $sign(det(A)) = +$. By Theorem 2.1, $\mathcal{P}$ requires unique inertia. ∎

Note that for any sign pattern $\mathcal{P}$, the inertia is the sum of the inertia of its irreducible components. So it is enough to consider the inertia of irreducible sign patterns. Consider irreducible symmetric sign patterns $\mathcal{P}$, such that the underlying graph G is connected but is not necessarily a tree with at least one leaf. We obtain sufficient conditions for such sign patterns $\mathcal{P}$ to require unique inertia. The following result follows from Lemma 2.5.

Theorem 3.3. *Let $\mathcal{P} = [p_{ij}]$ be a nonnegative symmetric sign pattern such that the underlying graph G is connected and has exactly one cycle $\mathcal{C}$ and no loops. If the order of $\mathcal{C}$ is either $4k+2$, $4k+1$ or $4k+3$ for some $k \in \mathbb{N}$, then $\mathcal{P}$ requires unique inertia.*

Proof. We prove this theorem by using induction on m where m is the number of leaves in G. If $m = 0$, then by Theorems 3.1 and 3.2, $\mathcal{P}$ requires unique inertia. If $m = 1$, suppose that u is the leaf of G and $R : uu_1u_2 \cdots u_tu_{t+1}$ be the path from u to a vertex u_{t+1} of $\mathcal{C}$ where $u_1, u_2, ..., u_t \notin \mathcal{C}$. Since u is a leaf of G with $p_{u,u} = 0$ and u_1 its unique neighbour, by applying Lemma 2.5 to $u_1, u_2, ..., u_t$ recursively, we get that $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}'$ requires unique inertia where $\mathcal{P}'$ is a principal subpattern of $\mathcal{P}$ and the underlying graph of $\mathcal{P}'$ is either a cycle or a tree with no loops. So by Lemma 2.7 and Theorems 3.1 and 3.2, $\mathcal{P}'$ and $\mathcal{P}$ require unique inertia.

Suppose that the statement is true for all $m \leq l-1$ where $l \geq 2$. For $m = l$, let w be a leaf of G and $R' : ww_1w_2 \cdots w_ru_{r+1}$ be the path from w to a vertex u_{r+1} of $\mathcal{C}$, where $w_1, w_2, ..., w_r \notin \mathcal{C}$. Since w is a leaf of G with $p_{w,w} = 0$ and w_1 its unique neighbour, by applying Lemma 2.5 to w, w_1, we have

$$In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}^1),$$

where $\mathcal{P}^1$ is the $(n-2) \times (n-2)$ principal subpattern of $\mathcal{P}$ with rows and columns corresponding to w and w_1 deleted. Let G^1 be the underlying graph of $\mathcal{P}^1$ with the connected components $G_1, G_2, ..., G_s$ for some s. Thus, $\mathcal{P}^1 = \mathcal{P}_1 \bigoplus \mathcal{P}_2 \bigoplus \cdots \bigoplus \mathcal{P}_s$, where the underlying graph corresponding to $\mathcal{P}_i$ is G_i, for all $i = 1, 2, ..., s$. If no G_i for $i \in \{1, 2, ..., s\}$ contain a cycle $\mathcal{C}$, then G_i is a tree for all $i = 1, 2, ..., s$. Therefore, $\mathcal{P}_i$, $\mathcal{P}^1$ requires unique inertia, which implies $\mathcal{P}$ requires unique inertia.

Otherwise without loss of generality, let G_1 be the component of G^1, which contains the cycle $\mathcal{C}$. Then, $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}^1$ requires unique inertia. If w_2 is not a leaf of G^1, then G^1 has less than l leaves. So by the induction hypothesis $\mathcal{P}^1$, $\mathcal{P}$ requires unique inertia. If w_2 is a leaf of G^1, then w_3 is its unique neighbour in G_1. Then, repeat the above processes in $\mathcal{P}_1$ in place of $\mathcal{P}$. Continuing in this way, we finally get a principal subpattern $\mathcal{P}^k$ of $\mathcal{P}$ such that the underlying graph $\mathcal{P}^k$ has less than equal to

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$l-1$ leaves. Then by the induction hypothesis, $\mathcal{P}^k$ requires unique inertia which implies $\mathcal{P}$ requires unique inertia. ∎

The following result is a generalization of a similar result Lemma 2.5.

THEOREM 3.4. *Let $\mathcal{P} = [p_{ij}]$ be an $n \times n$ symmetric sign pattern whose underlying graph is G. Suppose that G has a positive edge (u, v) with $deg(u) = deg(v) = 2$ where u is adjacent with u_1, $u_1 \neq v$, v is adjacent with v_1, $v_1 \neq u, u_1$.*

If $p_{u,u_1} = p_{v,v_1}$ and $p_{u_1,v_1} \in \{0, -\}$, then

$$In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is obtained from $\mathcal{P}$ by deleting the rows and columns corresponding to u, v and setting $p_{u_1,v_1} = p_{v_1,u_1} = -$.

If $p_{u,u_1} \neq p_{v,v_1}$ and $p_{u_1,v_1} \in \{0, +\}$, then

$$In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is obtained from $\mathcal{P}$ by deleting the rows and columns corresponding to u, v and setting $p_{u_1,v_1} = p_{v_1,u_1} = +$.

Proof. Without loss of generality, assume $u = 1$, $v = 2$, $u_1 = 3$ and $v_1 = 4$. Here we prove the case with $p_{u,u_1} = p_{v,v_1}$. Let $A = A^T \in \mathcal{Q}(\mathcal{P})$, then A can be written as

$$A = \left[\begin{array}{cc|cccc} 0 & a_{12} & a_{13} & 0 & \cdots & 0 \\ a_{12} & 0 & 0 & a_{24} & \cdots & 0 \\ \hline a_{13} & 0 & & & & \\ 0 & a_{24} & & & & \\ \vdots & \vdots & & B & & \\ 0 & 0 & & & & \end{array}\right] = \left[\begin{array}{c|c} A_{11} & C^T \\ \hline C & B \end{array}\right] \ (say).$$

Then, A is congruent to

$$\left[\begin{array}{c|c} A_{11} & \mathbf{0}^T \\ \hline \mathbf{0} & B - CA_{11}^{-1}C^T \end{array}\right] = \left[\begin{array}{c|c} I_2 & \mathbf{0}^T \\ \hline -CA_{11}^{-1} & I_{n-2} \end{array}\right] \left[\begin{array}{c|c} A_{11} & C^T \\ \hline C & B \end{array}\right] \left[\begin{array}{c|c} I_2 & -(CA_{11}^{-1})^T \\ \hline \mathbf{0} & I_{n-2} \end{array}\right].$$

Since $A_{11}^{-1} = \begin{bmatrix} 0 & \frac{1}{a_{12}} \\ \frac{1}{a_{12}} & 0 \end{bmatrix}$, $B' = B - CA_{11}^{-1}C^T$ is obtained from B by adding $-\frac{a_{13}a_{24}}{a_{12}}$ to its $(1, 2)$ and $(2, 1)$ entries, all other entries of B' are same as that of B. Also since $-\frac{a_{13}a_{24}}{a_{12}} < 0$ and $p_{u_1v_1} \in \{0, -\}$, $B' \in \mathcal{Q}(\mathcal{P}')$. Since the inertia of A_{11} is $(1, 1, 0)$ for any nonzero a_{12}, it follows that

$$In(\mathcal{P}) \subseteq (1, 1, 0) + In(\mathcal{P}').$$

Since $A = A^T \in \mathcal{Q}(\mathcal{P})$ can be arbitrary, and any $B' \in \mathcal{Q}(\mathcal{P}')$ can be realized by some A, equality follows. ∎

From Theorem 3.4, we get the following corollary.

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Corollary 3.5. *Let $\mathcal{P} = [p_{ij}]$ be an $n \times n$ symmetric sign pattern whose underlying graph is G. Suppose that G has a negative edge (u, v) with $deg(u) = deg(v) = 2$ and u adjacent with u_1, $u_1 \neq v$, v adjacent with v_1, $v_1 \neq u, u_1$.*

If $p_{u,u_1} = p_{v,v_1}$ and $p_{u_1,v_1} \in \{0, +\}$, then

$$In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is obtained from $\mathcal{P}$ by deleting the rows and columns corresponding to u, v and setting $p_{u_1,v_1} = p_{v_1,u_1} = +$.

If $p_{u,u_1} \neq p_{v,v_1}$ and $p_{u_1,v_1} \in \{0, -\}$, then

$$In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is obtained from $\mathcal{P}$ by deleting the rows and columns corresponding to u, v and setting $p_{u_1,v_1} = p_{v_1,u_1} = -$.

Corollary 3.6. *Let $\mathcal{P} = [p_{ij}]$ be an $n \times n$ symmetric sign pattern whose underlying graph is G. Suppose that G has a positive edge (u, v) with $deg(u) = deg(v) = 2$ and u, v are adjacent with w.*

If $p_{u,w} = p_{v,w}$ and $p_{w,w} \in \{0, -\}$, then

$$In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is obtained from $\mathcal{P}$ by deleting the rows and columns corresponding to u, v and setting $p_{w,w} = -$.

If $p_{u,w} \neq p_{v,w}$ and $p_{w,w} \in \{0, +\}$, then

$$In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is obtained from $\mathcal{P}$ by deleting the rows and columns corresponding to u, v and setting $p_{w,w} = +$.

Proof. Similar to the proof of Theorem 3.4. ∎

The above theorem can be used to give an alternate proof of part (a) of Theorem 3.2.

Example 3.7. *Consider the symmetric sign pattern $\mathcal{P} = [p_{ij}]$, $p_{ij} \in \{0, +\}$.*

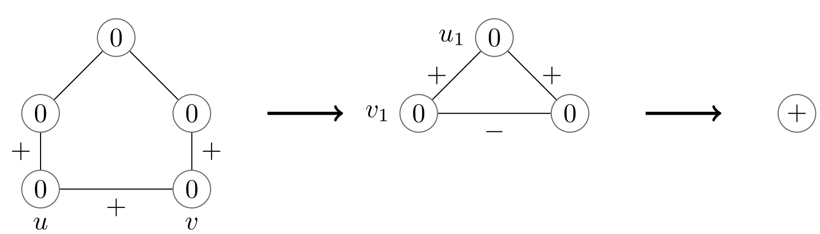

Figure 6. *Using Theorem 3.4 and Corollary 3.6 to determine the inertia.*

From left to right, let $\mathcal{P}$, $\mathcal{P}^1$ and $\mathcal{P}^2$ be the sign patterns corresponding to each of the graphs given in Fig. 6. By using Theorem 3.4 to the edge (u, v) and using Corollary 3.6 to the edge (u_1, v_1), it follows that

$$In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}^1) = (1, 1, 0) + (1, 1, 0) + In(\mathcal{P}^2).$$

Since $In(\mathcal{P}^2) = \{(1, 0, 0)\}$, therefore, the sign pattern $\mathcal{P}$ requires the unique inertia $(3, 2, 0)$.

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We use the following results to obtain sufficient conditions for symmetric sign patterns whose underlying graph contains cycles, to require unique inertia.

Lemma 3.8. *Let G be a connected graph such that none of the cycles in G share an edge. Suppose γ_1, γ_2, γ_3 are three distinct cycles in G such that $P_1 : u_1u_2\cdots u_{m_1}$ is a path from γ_1 to γ_2 where u_2, u_3,...,u_{m_1-1} is neither in γ_1 nor in γ_2 and $P_2 : v_1v_2\cdots v_{m_2}$ is a path from γ_1 to γ_3 where v_2, v_3,...,v_{m_2-1} is neither in γ_1 nor in γ_3. If $u_1 \neq v_1$, then every path from γ_2 to γ_3 must intersect γ_1.*

Proof. Suppose that P_1, P_2 intersect and $k_1 = min\{s \in \{1, 2, ..., m_1\} : u_s \textit{ lie in both } P_1,\ P_2\}$. Then, there exists $k_2 \in \{1, 2, ..., m_2\}$ such that $u_{k_1} = v_{k_2}$. Consider the paths $P_1' : u_1u_2\cdots u_{k_1}$ and $P_2' = v_{k_2}v_{k_2-1}\cdots v_1$ and let C_1 be a path in γ_1 from u_1 to v_1. The cycle in G formed by P_1', P_2' and C_1 share an edge with γ_1, which is a contradiction. Thus, P_1, P_2 do not intersect.

If possible, let P_3 be a path from γ_2 to γ_3 which does not intersect γ_1. Let $P_4 : w_1w_2\cdots w_{m_3}$ for some $m_3 \in \mathbb{N}$ be a path from γ_2 to γ_3 obtained from P_3 such that $w_2, w_3, ..., w_{m_3-1}$ is neither in γ_2 nor in γ_3. If $u_{m_1} \neq w_1$ let C_2 be a path from u_{m_1} to w_1 along the cycle γ_2 and if $v_{m_2} \neq w_{m_3}$ let C_3 be a path from w_{m_3} to v_{m_2} along the cycle γ_3, then $P_5 : C_2P_4C_3$ is a path from u_{m_1} to v_{m_2}. Note that $P_5 = P_4$ if $u_{m_1} = w_1$, $v_{m_2} = w_{m_3}$ and $P_5 = C_2P_4$ if $v_{m_2} = w_{m_3}$, $u_{m_1} \neq w_1$, etc. Suppose,

$$k_1 = min\{s \in \{1, 2, ..., m_1\} :\ u_s \textit{ is in both } P_1,\ P_5\},$$

$$k_2 = min\{s \in \{1, 2, ..., m_2\} :\ v_s \textit{ is in both } P_2,\ P_5\}.$$

Since P_1, P_2 do not intersect, $u_{k_1} \neq v_{k_2}$. Consider $\bar{P}_1$ the path segment of P_1 from u_1 to u_{k_1}, $\bar{P}_2$ the path segment of P_2 from v_1 to v_{k_2} and $\bar{P}_5$ the path segment of P_5 from u_{k_1} to v_{k_2}. Then, C_1, $\bar{P}_1$, $\bar{P}_5$, $\bar{P}_2$ forms a cycle in G distinct from γ_1 and sharing an edge with γ_1, which is a contradiction. Therefore, any path from γ_2 to γ_3 must intersect γ_1. ∎

Theorem 3.9. *Let G be a connected graph such that none of the cycles in G share an edge. Then, there exists a cycle γ and a vertex u of γ such that every path from γ to any other cycle of G contains u.*

Proof. If the number of cycles in G is strictly less than 3, then the theorem is true. Suppose that G has m cycles, where $m \geq 3$. Let γ_1 be a cycle in G. Since G is connected, so there exists another cycle γ_2 and a path $P_1 : u_1v_1^1v_2^1\cdots v_{k_1}^1u_2$ from γ_1 to γ_2 for some $k_1 \in \mathbb{N}$, where $v_i^1 \notin \gamma_1, \gamma_2$ for all $i = 1, 2, ..., k_1$.

If every path from γ_1 (*or* γ_2) to any other cycle of G contain u_1 (*or* u_2), then the result holds. If not there exists a vertex $u_2' \neq u_2$ in γ_2 and a cycle $\gamma_3 \neq \gamma_2$ such that $P_2 : u_2'v_1^2v_2^2\cdots v_{k_2}^2u_3$ is a path from γ_2 to γ_3 for some $k_2 \in \mathbb{N}$, where $v_i^2 \notin \gamma_2, \gamma_3$ for all $i = 1, 2, ..., k_2$. Since $u_2 \neq u_2'$ and $v_i^1, v_j^2 \notin \gamma_2$ for all $i = 1, 2, ..., k_1$, $j = 1, 2, ..., k_2$, so $P_1 \neq P_2$. Also, the cycles of G do not share any edge; thus, $\gamma_3 \neq \gamma_1$. Otherwise, γ_2 shares an edge either with the cycle formed by P_1, P_2 and a path segment of γ_1 from u_1 to u_3 and a path segment of γ_2 from u_2 to u_2' (if P_1, P_2 do not intersect) or with the cycle formed by certain path segments of P_1, P_2 and γ_2 (if P_1, P_2 intersects), which is a contradiction.

Since P_1 is a path from γ_2 to γ_1 and P_2 is a path from γ_2 to γ_3 and $u_2 \neq u_2'$, so by Lemma 3.8, every path from γ_3 to γ_1 must intersect γ_2. **(*)**

If every path from γ_3 to any other cycle of G contains u_3 then the result holds. Otherwise, there exists another vertex $u_3' \neq u_3$ of γ_3 and a cycle $\gamma_4 \neq \gamma_3$ such that $P_3 : u_3'v_1^3\cdots v_{k_3}^3u_4$ is a path from γ_3 to γ_4 for some $k_3 \in \mathbb{N}$, where $v_i^3 \notin \gamma_3, \gamma_4$ for all $i = 1, 2, ..., k_3$. Since $u_3 \neq u_3'$ and $v_i^2, v_j^3 \notin \gamma_3$ for all $i = 1, 2, ..., k_2$, $j = 1, 2, ..., k_3$, so $P_2 \neq P_3$. Again since the cycles of G do not share an edge, by repeating the argument used to show $\gamma_3 \neq \gamma_1$ we get, $\gamma_2 \neq \gamma_4$. Since P_2 is a path from γ_3 to γ_2 and P_3 is a path from γ_3 to γ_4 and $u_3 \neq u_3'$, so by Lemma 3.8, every path from γ_4 to γ_2 must intersect γ_3. If $\gamma_1 = \gamma_4$, then two cases arise.

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Case-1: P_3 intersects γ_2 at some vertex w. Then, similarly we get a contradiction to the assumption that no two cycles in G share an edge.

Case-2: P_3 does not intersect γ_2. Then, there exists a path from γ_3 to γ_1 which does not intersect γ_2, which is a contradiction to (*).

Therefore, the cycles γ_1, γ_2, γ_3, and γ_4 are all distinct. Since G has m cycles after a finite number of steps, the above process will terminate with a cycle γ_k and a vertex u_k in γ_k such that every path from γ_k to any other cycle in G contain u_k. Therefore, the result holds with $\gamma = \gamma_k$ and $u = u_k$. □

THEOREM 3.10. *Let $\mathcal{P} = [p_{ij}]$ be a nonnegative symmetric sign pattern of order n such that the underlying graph G of $\mathcal{P}$ contains no loops. Suppose that all the cycles of G have an order of the form $4k+2$, $k \in \mathbb{N}$, and no two cycles in G share an edge, then $\mathcal{P}$ requires unique inertia.*

Proof. If G does not have any cycle, then G is a tree and hence G requires unique inertia by Lemma 2.7. Suppose that G has at least one cycle. We prove this theorem by using induction on m, where m is the number of cycles in G. If $m = 1$ then by Theorem 3.3, the theorem is true. Suppose that the theorem holds for all $m,\ m \leq r-1$, $r \geq 2$.

For $m = r$, since no two cycles of G share an edge, by Theorem 3.9 there exists a cycle $\gamma : u_1u_2\cdots u_{4k+2}$ of length $4k+2$, and a vertex u_1 such that every path from γ to any other cycle of G contains u_1.

Case-1: Each of the vertices $u_2, ..., u_{4k+2}$ has degree equal to 2. Since (u_2, u_3) is positively signed with $deg(u_2) = deg(u_3) = 2$, u_2 adjacent with u_1, u_3 adjacent with u_4, $p_{u_1u_2} = p_{u_3u_4} = +$ and $p_{u_1u_4} = 0$, by Theorem 3.4,

$$In(\mathcal{P}) = (1,1,0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is the principal subpattern of $\mathcal{P}$ obtained by deleting the rows and columns corresponding to u_2, u_3 from $\mathcal{P}$ and setting $p_{u_1u_4} = -$. Therefore, the underlying graph of $\mathcal{P}'$ denoted by G' is obtained from G by replacing γ with the cycle $\gamma' : u_1u_4u_5\cdots u_{4k+2}$ of length $4k$, the edge (u_1, u_4) being negative (everything else in G' is same as in G). Now γ' has a positive edge (u_4, u_5) with $deg(u_4) = deg(u_5) = 2$, u_4 adjacent with u_1 and u_5 adjacent with u_6, $p_{u_1u_4} \neq p_{u_5u_6}$ and $p_{u_1u_6} \in \{0, +\}$. By Theorem 3.4,

$$In(\mathcal{P}') = (1,1,0) + In(\mathcal{P}''),\ or\ In(\mathcal{P}) = (2,2,0) + In(\mathcal{P}''),$$

where $\mathcal{P}''$ is the principal subpattern of $\mathcal{P}'$ obtained by deleting the rows and columns corresponding to u_4, u_5 from $\mathcal{P}'$ and setting $p_{u_1u_6} = +$. Therefore, the underlying graph of $\mathcal{P}''$ is obtained from G by replacing γ with the cycle $\gamma'' : u_1u_6u_7\cdots u_{4k+2}$ with all the edges positive.

Since γ has $4k+2$ vertices, continuing in this way, after $2k$ steps we get

$$In(\mathcal{P}) = (2k, 2k, 0) + In(\mathcal{P}^{2k}),$$

where $\mathcal{P}^{2k}$ is the principal subpattern of $\mathcal{P}$ obtained by deleting the rows and columns corresponding to $u_2, ..., u_{4k+1}$ and setting $p_{u_1u_{4k+2}} = +$. Therefore, the underlying graph of $\mathcal{P}^{2k}$ denoted by G^{2k} is obtained from G by replacing γ with the positive edge (u_1, u_{4k+2}). Since G^{2k} has strictly less than r cycles, so by the induction hypothesis $\mathcal{P}^{2k}$, $\mathcal{P}$ requires unique inertia.

Case-2: At least one of $u_2, u_3, ..., u_{4k+2}$ has degree greater than or equal to 3. Then as in the proof of Theorem 3.3, $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}'$ requires unique inertia where $\mathcal{P}'$ is a principal subpattern of $\mathcal{P}$ such that the underlying graph of $\mathcal{P}'$ has either strictly less than r cycles or otherwise

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contains the cycle γ where the degree of each of $u_2, u_3, ..., u_{4k+2}$ is equal to 2. Then either by applying the induction hypothesis or the proof of Case-1, $\mathcal{P}'$ and $\mathcal{P}$ require unique inertia. ∎

THEOREM 3.11. *Let $\mathcal{P} = [p_{ij}]$ be an $n \times n$ symmetric sign pattern whose underlying graph is G. Suppose that G has a positive edge (u, v) and a positive loop (u, u) with $deg(u) = 4$, $deg(v) = 2$ and u adjacent with u_1, $u_1 \neq v$, v adjacent with v_1, $v_1 \neq u, u_1$.*

If $p_{u,u_1} = p_{v,v_1}$, $p_{u_1,v_1} \in \{0, -\}$ and $p_{v_1,v_1} \in \{0, +\}$, then

$$In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is obtained from $\mathcal{P}$ by deleting the rows and columns corresponding to u, v and setting $p_{u_1,v_1} = p_{v_1,u_1} = -$, $p_{v_1,v_1} = +$.

If $p_{u,u_1} \neq p_{v,v_1}$, $p_{u_1,v_1} \in \{0, +\}$ and $p_{v_1,v_1} \in \{0, +\}$, then

$$In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}'),$$

where $\mathcal{P}'$ is obtained from $\mathcal{P}$ by deleting the rows and columns corresponding to u, v and setting $p_{u_1,v_1} = p_{v_1,u_1} = +$, $p_{v_1,v_1} = +$.

Proof. Without loss of generality, assume $u = 1$, $v = 2$, $u_1 = 3$ and $v_1 = 4$. Here, we prove the case with $p_{u,u_1} = p_{v,v_1}$. Let $A = A^T \in \mathcal{Q}(\mathcal{P})$, then A can be written as

$$A = \left[\begin{array}{c|c} A_{11} & C^T \\ \hline C & B \end{array}\right],$$

where $A_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & 0 \end{bmatrix}$ and B, C are the same as in Theorem 3.4. Then, A congruent to

$$\left[\begin{array}{c|c} A_{11} & \mathbf{0}^T \\ \hline \mathbf{0} & B' \end{array}\right], \text{ where } B' = B - CA_{11}^{-1}C^T.$$

Thus, B' is obtained from B by adding $-\frac{a_{13}a_{24}}{a_{12}}$ to its $(1, 2)$ and $(2, 1)$ entries and $\frac{a_{11}a_{24}^2}{a_{12}^2}$ to its $(2, 2)$ entry. Since $-\frac{a_{13}a_{24}}{a_{12}} < 0$, $p_{34} \in \{0, -\}$ and $p_{44} \in \{0, +\}$, it follows that $B' \in \mathcal{Q}(\mathcal{P}')$. Since the inertia of A_{11} is $(1, 1, 0)$ for any nonzero a_{12} and a_{11}, it follows that

$$In(\mathcal{P}) \subseteq (1, 1, 0) + In(\mathcal{P}').$$

Since $A = A^T \in \mathcal{Q}(\mathcal{P})$ can be arbitrary, and any $B' \in \mathcal{Q}(\mathcal{P}')$ can be realized by some A, equality follows. ∎

Note that the above theorem can be used to give an alternate proof of part (b) of Theorem 3.1.

EXAMPLE 3.12. *Consider a symmetric sign pattern $\mathcal{P} = [p_{ij}]$, $p_{ij} \in \{0, +\}$. From left to right, let $\mathcal{P}$, $\mathcal{P}^1$ and $\mathcal{P}^2$ be the sign patterns corresponding to each of the graphs G, G^1 and G^2, respectively, given in Fig. 7.*

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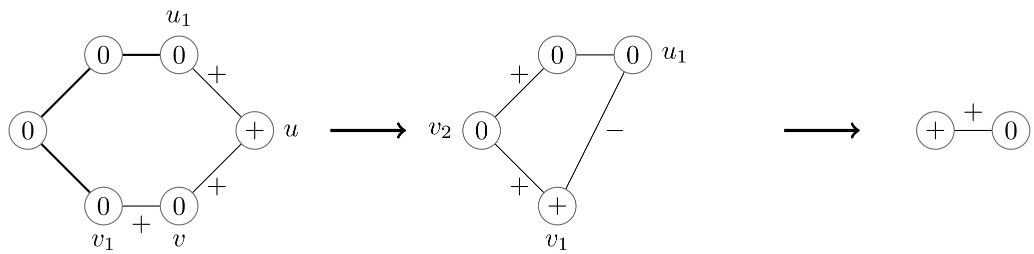

FIGURE 7. *Using Theorem 3.11 to determine the inertia.*

By using Theorem 3.11 to the edge (u, v) of G, we have $In(\mathcal{P}) = (1, 1, 0) + In(\mathcal{P}^1)$. Again, applying Theorem 3.11 to the edge (v_1, v_2) of G^1, we have $In(\mathcal{P}^1) = (1, 1, 0) + In(\mathcal{P}^2)$. Since $In(\mathcal{P}^2) = \{(1, 1, 0)\}$, therefore the sign pattern $\mathcal{P}$ requires unique inertia $(3, 3, 0)$.

THEOREM 3.13. *Let $\mathcal{P} = [p_{ij}]$, be a nonnegative symmetric sign pattern of order n such that the underlying graph G of $\mathcal{P}$ is connected with exactly one loop. Suppose that all the cycles in G have order $4k + 2$ for some $k \in \mathbb{N}$ and no two cycles share an edge. Then, $\mathcal{P}$ requires unique inertia.*

Proof. If G does not contain any cycle, then G is a tree with one loop hence, by Lemma 2.7, G requires unique inertia. Suppose that G has at least one cycle. We prove this theorem by using induction on m, where m is the number of cycles in G. If $m = 1$ suppose that $\gamma : u_1u_2 \cdots u_{4k+2}$ is the cycle of G and u is the vertex of G for which $p_{uu} \neq 0$.

Case-1: u is a vertex of γ. If the underlying graph of G is a cycle with one loop then by Theorem 3.1(b), $\mathcal{P}$ requires unique inertia. If the underlying graph G is not a cycle then by applying Lemma 2.5 recursively as in Theorem 3.3, $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}'$ requires unique inertia where the underlying graph of $\mathcal{P}'$ is either a tree with at most one loop or a cycle with exactly one loop. By Lemma 2.7 and Theorem 3.1, $\mathcal{P}'$ and $\mathcal{P}$ require unique inertia.

Case-2: u is not a vertex of γ. Then, there exists a path $P : u_tw_1w_2...w_s$ from a vertex u_t of γ to a leaf w_s for some $s \in \mathbb{N}$ such that $u = w_i$ for some $i \in \{1, 2, ..., s\}$ where $w_1, w_2, ..., w_s \notin \gamma$. If $u = w_i$ for some even i, then by applying Lemmas 2.5 and 2.6 recursively, $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}'$ requires unique inertia, where $\mathcal{P}'$ is a principal subpattern of $\mathcal{P}$ obtained either by deleting the rows and columns corresponding to $w_1, w_2, ..., w_s$ in $\mathcal{P}$ and setting $p_{u_tu_t} = +$ or by deleting the rows and columns corresponding to $u_t, w_1, w_2, ..., w_s$ in $\mathcal{P}$. Then similarly as in Case-1, $\mathcal{P}'$ and $\mathcal{P}$ require unique inertia.

If $u = w_i$ for some odd i, then by applying Lemmas 2.5 and 2.6 recursively, $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}'$ requires unique inertia, where $\mathcal{P}'$ is a principal subpattern of $\mathcal{P}$ obtained either by deleting the rows and columns corresponding to $w_1, w_2, ..., w_s$ and setting $p_{u_tu_t} = -$ or by deleting the rows and columns corresponding to $u_t, w_1, w_2, ..., w_s$. By applying Lemma 2.5 recursively to $\mathcal{P}'$, we get $\mathcal{P}'$ requires unique inertia if and only if $\mathcal{P}''$ requires unique inertia, where the underlying graph of $\mathcal{P}''$ is either a cycle or a tree with exactly one loop. Then, the result follows by Theorem 3.1(b).

Suppose that the theorem holds for all $m \leq r - 1$, $r \geq 2$. For $m = r$, since no two cycles of G share an edge, by Theorem 3.9 there exists a cycle $\gamma : u_1u_2 \cdots u_{4k+2}$ where u_1 is such that every path from γ to any other cycle of G contains u_1. Suppose that u is the vertex of G such that $p_{u,u} = +$. Then, we have the following cases.

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Case-1: u is a vertex of γ. Then, there exists s, $s \in \{1, 2, ..., 4k+2\}$ such that $u = u_s$. If all paths from a vertex of γ to a leaf of G intersect some cycle other than γ of G, then by recursively applying Theorems 3.4 and 3.11, we have

$$In(\mathcal{P}) = (2k, 2k, 0) + In(\mathcal{P}^{2k}),$$

where $\mathcal{P}^{2k}$ is a principal subpattern of $\mathcal{P}$ obtained by deleting the rows and columns of $\mathcal{P}$ corresponding to $u_2, ..., u_{4k+1}$ and setting $p_{u_1 u_{4k+2}} = +$ and $p_{u_1 u_1} = +$ if s is odd or $p_{u_{4k+2} u_{4k+2}} = +$ if s is even. Therefore, the underlying graph of $\mathcal{P}^{2k}$ is G^{2k} obtained from G by replacing γ with the positive edge (u_1, u_{4k+2}) and a loop (u_1, u_1) if s is odd or a loop (u_{4k+2}, u_{4k+2}) if s is even. Therefore, G^{2k} has strictly less than r cycles, so by the induction hypothesis $\mathcal{P}^{2k}$, $\mathcal{P}$ requires unique inertia.

If there exists a path P_i from u_i to a leaf w_i of G such that P_i does not intersect any other cycle of G. By applying Lemma 2.5 recursively, we get $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}'$ requires unique inertia, where the underlying graph of $\mathcal{P}'$ either has strictly less than r cycles or otherwise the underlying graph of $\mathcal{P}'$ has r cycles such that every path from a vertex of γ to a leaf of G intersect some other cycle of G. So $\mathcal{P}'$ and $\mathcal{P}$ require unique inertia followed either by the induction hypothesis or similarly as Case-1 in the previous paragraph.

Case-2: γ does not contain u.

Case-2a: There exists a path $P : u_t w_1 w_2 ... w_s$ from a vertex u_t to a leaf w_s for some $s \in \mathbb{N}$ such that $u = w_i$ for some $i = 1, 2, ..., s$, where $w_1, w_2, ..., w_s \notin \gamma$. Similarly as in Case-2 for the number of cycles $m = 1$, by applying Lemmas 2.5, 2.6 and Theorem 3.11 we get $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}^1$ requires unique inertia where the underlying graph of $\mathcal{P}^1$ is obtained from G by replacing the cycle γ with a positive edge with at most one loop. Since G^1 the underlying graph of $\mathcal{P}^1$ has a cycle γ_1 and a vertex $v_1 \in \gamma_1$ such that every path from γ_1 to any other cycle of G^1 contains v_1, continuing the same argument with $\mathcal{P}$ replaced by $\mathcal{P}^1$ we finally get a principal subpattern $\mathcal{P}^k$ of $\mathcal{P}$ such that $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}^k$ requires unique inertia where the underlying graph of $\mathcal{P}^k$ is a tree with at most one loop. Hence, $\mathcal{P}^k$, $\mathcal{P}$ require unique inertia by Lemma 2.7.

Case-2b: There exists no such path. Similarly, as in Theorem 3.10, we get $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}^1$ requires unique inertia where $\mathcal{P}^1$ is a principal subpattern of $\mathcal{P}$ whose underlying graph is obtained from $\mathcal{P}$ by replacing γ with a positive edge. By the induction hypothesis $\mathcal{P}^1$, $\mathcal{P}$ requires unique inertia. $\square$

In the above theorem if the number of loops in G is more than one, then $\mathcal{P}$ may not require unique inertia.

Example 3.14. *Let $\mathcal{P}$ be a symmetric sign pattern of order 6, whose underlying graph is G given in Fig. 8.*

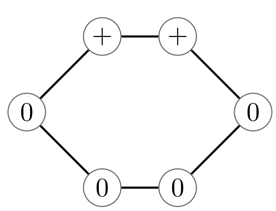

Figure 8. G.

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By Theorem 3.1(c), $\mathcal{P}$ does not require unique inertia.

LEMMA 3.15. *Let $\mathcal{P} = [p_{ij}]$ be a nonnegative symmetric sign pattern of order n such that underlying graph G of $\mathcal{P}$ is connected with exactly one loop and exactly one cycle of order $4k+1$ for some $k \in \mathbb{N}$. Suppose that the loop either belongs to the cycle or if not then the distance between the loop and the cycle is even. Then, $\mathcal{P}$ requires unique inertia.*

Proof. By applying Lemma 2.5 recursively to $\mathcal{P}$, we get that $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}'$ requires unique inertia where the underlying graph of $\mathcal{P}'$ is either a tree with at most one loop or a cycle with at most one positive loop. Then from Lemma 2.7 or part (c) of Theorem 3.2, it follows that $\mathcal{P}$, $\mathcal{P}'$ requires unique inertia. ∎

If in the above theorem, the distance between the loop and the cycle is odd then $\mathcal{P}$ may not require unique inertia.

EXAMPLE 3.16. *Let $\mathcal{P}$ be a symmetric sign pattern of order 6. From left to right, let $\mathcal{P}$, $\mathcal{P}'$ be the sign patterns corresponding to each of the graphs given in Fig. 9.*

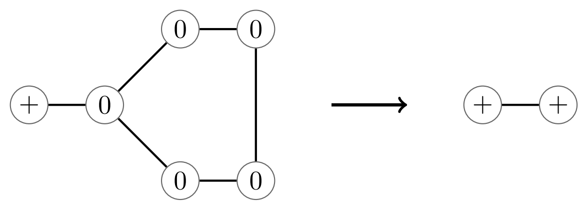

FIGURE 9. *Using Theorem 3.4 and Corollary 3.6 to determine the inertia.*

Then by Theorem 3.4 and Corollary 3.6, we have $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}'$ requires unique inertia. Clearly, $\mathcal{P}'$ and $\mathcal{P}$ does not require unique inertia.

THEOREM 3.17. *Let $\mathcal{P} = [p_{ij}]$ be a nonnegative symmetric sign pattern of order n such that underlying graph G of $\mathcal{P}$ is connected with no loops and at most two cycles with orders of the form, $4k+1$ for some $k \in \mathbb{N}$. Assume that the distance between the cycles is even and no two cycles share an edge. Then, $\mathcal{P}$ requires unique inertia.*

Proof. By Theorem 3.3, P requires unique inertia if G has fewer than two cycles. Assume that G has exactly two cycles. By applying Lemma 2.5 recursively, we get $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}'$ requires unique inertia, where the underlying graph G' of $\mathcal{P}'$ has either fewer than 2 cycles or exactly two cycles with no leaves.

If the underlying graph of $\mathcal{P}'$ has fewer than 2 cycles, then the result follows from Theorem 3.3. If the underlying graph of $\mathcal{P}'$ has two cycles γ and γ_1 (say), then by applying Theorems 3.4 and 3.11 recursively on the vertices of γ, we get $\mathcal{P}'$ requires unique inertia if and only if $\mathcal{P}''$ requires unique inertia, where the underlying graph of $\mathcal{P}''$ is obtained from G' by replacing γ with a positive loop such that the distance of the loop from γ_1 is even. By applying Lemma 3.15, we get $\mathcal{P}''$, $\mathcal{P}$ requires unique inertia. ∎

If the underlying graph G of $\mathcal{P}$ contain two cycles of orders of the form $4k+1$, $k \in \mathbb{N}$, and a loop with an even distance between the cycles, then $\mathcal{P}$ may not require unique inertia.

EXAMPLE 3.18. *Consider the symmetric sign pattern $\mathcal{P} = [p_{ij}]$, $p_{ij} \in \{0, +\}$. From left to right, let $\mathcal{P}$, $\mathcal{P}'$ be the sign patterns corresponding to the graphs G, G', respectively, given in Fig. 10.*

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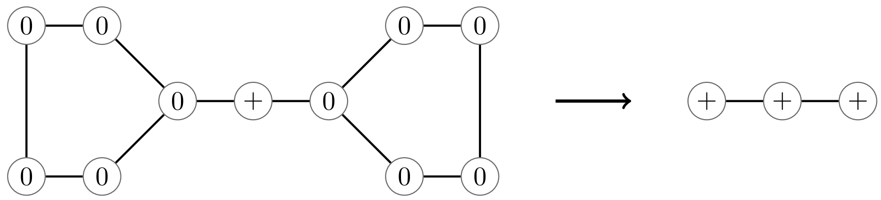

FIGURE 10. *Using Theorem 3.4 and Corollary 3.6 to determine the inertia.*

Then by recursively applying Theorem 3.4 and Corollary 3.6, we get that $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}'$ requires unique inertia. Since G' is a path of even length and the loops lie in odd-even-odd ascending positions, respectively, so by Proposition 2.3, $\mathcal{P}'$ and $\mathcal{P}$ does not require unique inertia.

If the underlying graph G of $\mathcal{P}$ contain two cycles with order of the form $4k+1$, $k \in \mathbb{N}$ such that the distance between the cycles is odd, then $\mathcal{P}$ may not require unique inertia.

EXAMPLE 3.19. *Consider the symmetric sign pattern $\mathcal{P} = [p_{ij}]$, $p_{ij} \in \{0, +\}$. From left to right, let $\mathcal{P}$, $\mathcal{P}'$ be the sign patterns corresponding to the graphs G, G', respectively, given in Fig. 11.*

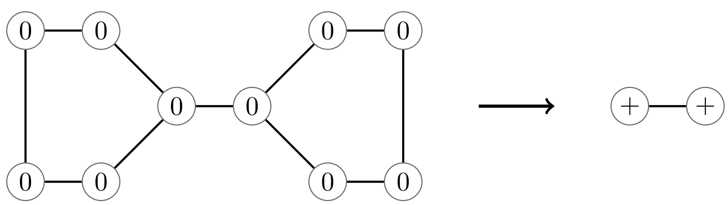

FIGURE 11. *Using Theorem 3.4 and Corollary 3.6 to determine the inertia.*

Then by recursively applying Theorem 3.4 and Corollary 3.6, we get that $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}'$ requires unique inertia, and G' is an edge with 2 positive loops. Clearly $\mathcal{P}'$ and $\mathcal{P}$ does not require unique inertia.

If the underlying graph G of $\mathcal{P}$ contain two cycles one with order of the form $4k+1$, $k \in \mathbb{N}$ and the other with order of the form $4k+3$, $k \in \mathbb{N}$, then $\mathcal{P}$ may not require unique inertia.

EXAMPLE 3.20. *Consider the symmetric sign pattern $\mathcal{P} = [p_{ij}]$, $p_{ij} \in \{0, +\}$. From left to right, let $\mathcal{P}$, $\mathcal{P}'$ be the sign patterns corresponding to the graphs G, G', respectively, given in Fig. 12.*

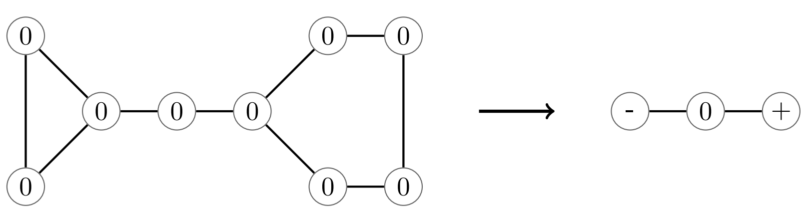

FIGURE 12. *Using Theorem 3.4 and Corollary 3.6 to determine the inertia.*

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Then by recursively applying Theorem 3.4 and Corollary 3.6, we get that $\mathcal{P}$ requires unique inertia if and only if $\mathcal{P}'$ requires unique inertia. Clearly, $(2,1,0),(1,2,0) \in In(\mathcal{P}')$; *hence, $\mathcal{P}'$ and $\mathcal{P}$ do not require unique inertia.*

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