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TOKEN GRAPHS OF CAYLEY GRAPHS AS LIFTS*

CRISTINA DALFÓ†, MIQUEL ÀNGEL FIOL‡, SOŇA PAVLÍKOVÁ§, AND JOZEF ŠIRÁŇ¶

Abstract. This paper describes a general method for representing k-token graphs of Cayley graphs as lifts of voltage graphs. This allows us to construct line graphs of circulant graphs and Johnson graphs as lift graphs on cyclic groups. As an application of the method, we derive the spectra of the considered token graphs. The method can also be applied to dealing with other matrices, such as the Laplacian or the signless Laplacian, and to construct token digraphs of Cayley digraphs.

Key words. Token graph, Cayley graph, Voltage graph, Lift.

AMS subject classifications. 05C35, 05C50.

1. Introduction. A considerable amount of research in algebraic graph theory has been devoted to constructions of larger graphs from smaller ones, enabling the determination of properties of larger graphs in an algebraically controllable way from properties of the smaller graphs. Although examples of such constructions abound, we will focus only on two of them, namely, the token graph construction and the lifting construction. They turn out to be related interestingly, and before an explanation, we offer an informal description of the two constructions.

Given a graph $G=(V,E)$ on n vertices, its k-token graph $F_k(G)$ has a vertex set consisting of all the $\binom{n}{k}$ configurations of k indistinguishable tokens placed in different vertices of G, and two vertices are adjacent if one is obtained from the other by moving one token along an edge of G. These graphs were also called *symmetric kth power of a graph* by Audenaert, Godsil, Royle, and Rudolph [2], and *k-tuple vertex graphs* in Alavi, Behzad, Erdős, and Lick [1]. Token graphs have applications in physics (quantum mechanics) and in the graph isomorphism problem (because invariants of the k-token $F_k(G)$ are also invariants of G). For more details, see again [2]. Some properties of token graphs were studied by Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia, and Wood [11], including the connectivity, diameter, clique and chromatic number, and Hamiltonian paths. In particular, the connectivity of token graphs of trees was studied by Fabila-Monroy, Leaños, and Trujillo-Negrete in [12].

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†Departament. de Matemàtica, Universitat de Lleida, Igualada (Barcelona), Catalonia (cristina.dalfo@udl.cat). Her research has been supported by AGAUR from the Catalan Government under project 2021SGR00434 and MICINN from the Spanish Government under project PID2020-115442RB-I00.

‡Departament de Matemàtiques, Universitat Politècnica de Catalunya, Barcelona, Catalonia, Barcelona Graduate School of Mathematics, Institut de Matemàtiques de la UPC-BarcelonaTech (IMTech) (miguel.angel.fiol@upc.edu). His research has been supported by AGAUR from the Catalan Government under project 2021SGR00434, MICINN from the Spanish Government under project PID2020-115442RB-I00, and a grant from the Universitat Politècnica de Catalunya with references AGRUPS-2022 and AGRUPS-2023.

§Institute of Information Engineering, Automation, and Mathematics, FCFT, Slovak Technical University, Bratislava, Slovakia (sona.pavlikova@stuba.sk). She gratefully acknowledges support for this research from APVV Research Grants 19-0308 and 22-0005, and VEGA Research Grants 1/0567/22 and 1/0069/23.

¶Department of Mathematics and Descriptive Geometry, Slovak University of Technology, Bratislava, Slovakia (jozef.siran@stuba.sk). He gratefully acknowledges support for this research from APVV Research Grants 19-0308 and 22-0005, and VEGA Research Grants 1/0567/22 and 1/0069/23.

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For an intuitive description of the lifting construction, think of a graph G (a *base graph*) endowed by an assignment α of elements of a group Γ on arcs of G (a *voltage assignment*). The pair (G, α) gives rise to a *lift* G^α, a larger graph that may be thought of as obtained by 'compounding' $|\Gamma|$ copies of G, joined in-between in a way dictated by the voltage assignment. A general necessary and sufficient condition for a graph H to arise as a lift of a (smaller) graph is the existence of a subgroup of the automorphism group of H with a free action on vertices of H; see Gross and Tucker [14]. Lifts have found numerous applications in areas of graph theory that are as versatile as the degree/diameter problem on the one hand and the Map Color Theorem on the other hand. In the first case, the diameter of the lift can be conveniently expressed in terms of voltages on the edges of the base graph. Besides its theoretical importance, this fact can be used to design efficient diameter-checking algorithms, see Baskoro, Branković, Miller, Plesník, Ryan, and Širáň [3]. In the context of the degree-diameter problem, we address the interested reader to the comprehensive survey by Miller and Širáň [16]. Other prominent examples of applications also include the use of lift graphs in the study of the automorphisms of G^α and in Cayley graphs, which are lifts of one-vertex graphs (with loops and semi-edges attached).

The surprising way the two constructions turn out to be related is based on the observation that *automorphism groups of k-token graphs of Cayley graphs admit subgroups acting freely on vertices* and, hence, they can be described as lifts of smaller graphs by voltage assignments. This enables one to link the study of k-token graphs, Cayley graphs, and lifts in a novel way by introducing a general method to represent k-token graphs of Cayley graphs as lifts of voltage graphs. In particular, these result in representations of line graphs of circulant graphs and Johnson graphs as lift of smaller graphs, with voltages in cyclic groups. As another application of this method, we derive the spectra of the token graphs considered and generate infinite families of graphs with given maximum degree, and eigenvalues contained in a certain interval.

This paper is structured as follows. In Section 2, there are some notation, the formal definitions, and the known results. In Section 3, we construct line graphs of circulant graphs and Johnson graphs as lift graphs on the cyclic group. Applying this method, we derive the spectra of the considered token graphs in Section 4. Finally, in Section 5, we explain how to extend our results in the following directions: computing eigenspaces, working with the universal matrix instead of the adjacency matrix, and dealing with digraphs.

2. Preliminaries.

Let G be a graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. If necessary, we consider every edge $\{u, v\}$ as a 'digon' formed by the arcs (or directed edges) $a^+ = (u, v)$ and $a^- = (v, u)$. For a given integer k such that $1 \le k \le n$, the *k-token graph* $F_k(G)$ of G is the graph in which the vertices of $F_k(G)$ correspond to k-subsets of $V(G)$. Two vertices are adjacent when the corresponding subsets' symmetric difference are the edge's end-vertices in $E(G)$. In particular, notice that $F_1(G) = G$ and, by symmetry, $F_k(G) = F_{n-k}(G)$.

Moreover, $F_k(K_n)$ is the Johnson graph $J(n, k)$. For example, Fig. 1 shows the Johnson graph $J(5, 2) \cong F_2(K_5)$, with vertex labels $\{i, j\} = ij$.

In general, the Johnson graph $J(n, k)$, with $k \le n - k$, is a distance-regular graph with degree $k(n - k)$, diameter $d = k$, and for $j = 0, 1, \ldots, d$, intersection parameters

$$b_j = (k - j)(n - k - j), \qquad c_j = j^2,$$

with eigenvalues

$$\lambda_j = (k - j)(n - k - j) - j,$$

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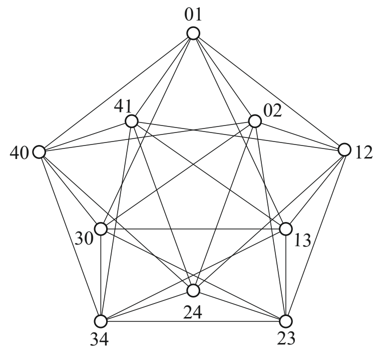

FIGURE 1. *The Johnson graph* $J(5,2)$.

and multiplicities

$$m_j = \binom{n}{j} - \binom{n}{j-1}.$$

For more information about Johnson graphs, see Godsil [13].

In our study, some infinite families of line graphs appear. Concerning their spectra, recall that if a k-regular graph G with n vertices and m edges has the spectrum

$$\operatorname{sp}(G) = \{k, \lambda_1^{[m_1]}, \ldots, \lambda_d^{[m_d]}\},$$

then its line graph $L(G)$ has spectrum

$$\operatorname{sp}(L(G)) = \{2k-2, (k-2+\lambda_1)^{[m_1]}, \ldots, (k-2+\lambda_d)^{[m_d]}, -2^{[m-n]}\}.$$

The graphs with the least eigenvalue not smaller than -2 are completely identified in three categories: the line graphs of bipartite graphs, the generalized line graphs, and a finite class of graphs arising from root systems, see Biggs [4], and Cameron, Goethals, Seide, and Shult [5].

Let $G = (V, E)$ be an undirected graph with loops and multiple edges allowed. Every edge $e = \{u, v\} \in E$ is made up of two opposite arcs, say $a^+ = (u, v)$ and $a^- = (v, u)$. Let Γ be a group. An (*ordinary*) *voltage assignment* on the graph $G = (V, E)$ is a mapping $\alpha : E \to \Gamma$ with the property that $\alpha(a^-) = (\alpha(a^+))^{-1}$ for every pair of opposite arcs a^+ and a^-. Then, the graph G together with the voltage assignment α (that is commonly referred to as 'base graph') determine a new graph G^α, called the *lift* of G, which is defined as follows. The vertex and arc sets of the lift are simply the Cartesian products $V^\alpha = V \times \Gamma$ and $E^\alpha = E \times \Gamma$, respectively. Moreover, for every arc $a \in E$ from a vertex u to a vertex v for any $u, v \in V$ (possibly, $u = v$) in G, and for every element $g \in \Gamma$, there is an arc $(a, g) \in E^\alpha$ from the vertex $(u, g) \in V^\alpha$ to the vertex $(v, g\alpha(a)) \in V^\alpha$. Notice that if two parallel arcs or loops of G receive the same voltage, then G^α has also parallel arcs or loops. For example, a base graph for the Johnson graph is also shown in Fig. 2.

The interest of this construction is that, from the base graph G and the voltages, we can deduce some properties of the lift graph G^α. This is done through an important matrix associated with G. When the group Γ is Abelian of rank r, say $\Gamma = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$, every entry of such a matrix is a polynomial in r variables $z_1, \ldots, z_r$ with integer coefficients. So it is called the associated 'polynomial matrix' $\boldsymbol{B}(z) = \boldsymbol{B}(z_1, \ldots, z_r)$. Thus, if in G there is an arc from u to v with voltage $(j_1, \ldots, j_r)$, for $j_i \in \mathbb{Z}_{n_i}$ and $i = 1, \ldots, r$, the entry

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uv of $\boldsymbol{B}(z)$ has a term of the form $z_1^{j_1}\cdots z_r^{j_r}$ (see examples with $r=1$ and $r=2$ in the following sections). Then, the whole (adjacency or Laplacian) spectrum and eigenspaces of G^α can be retrieved from $\boldsymbol{B}(z)$. More precisely, Dalfó, Fiol, Miller, Ryan, and Širáň [7] proved the following result.

Theorem 2.1 ([7]). *Let $R(n)$ be the set of nth roots of unity, and consider the base graph $G=(V,E)$ with voltage assignment α on the Abelian group $\Gamma=\mathbb{Z}_{n_1}\times\cdots\times\mathbb{Z}_{n_r}$. If $\boldsymbol{x}=(x_u)_{u\in V}$ is an eigenvector of $\boldsymbol{B}(z)$ and $z_i\in R(n_i)$ with eigenvalue λ, then the vector $\boldsymbol{\phi}=(\phi_{(u,\boldsymbol{j})})_{(u,\boldsymbol{j})\in V^\alpha}$ with $\boldsymbol{j}=(j_1,\ldots,j_r)\in\Gamma$ and components $\phi_{(u,\boldsymbol{j})}=z_1^{j_1}\cdots z_r^{j_r}x_u$ is an eigenvector of the lift graph G^α corresponding to the eigenvalue λ. Moreover, all the eigenvalues (including multiplicities) of G^α are obtained:*

$$\operatorname{sp} G^\alpha = \bigcup_{z_1\in R(n_1),\ldots,z_r\in R(n_r)} \operatorname{sp}(\boldsymbol{B}(z)).$$

For more information on lift graphs and digraphs, see Dalfó, Fiol, Pavlíková, and Širán [8], and Dalfó, Fiol, and Širáň [10].

3. Johnson graphs as lifts of voltage graphs. In this section, we show that the Johnson graphs with some given parameters can be obtained as lifts of base graphs on cyclic groups.

3.1. Johnson graphs $J(n,2)$ as lifts. We begin our study by considering the 2-token graphs of the complete graph K_n, where n is an odd number. These graphs correspond to the Johnson graphs $J(n,2)$ or line graphs $L(K_n)$.

Theorem 3.1. *Let n be an odd number, $n=2\nu+1$. Then, the Johnson graph $J(n,2)$ is a lift of a base graph G with ν vertices with voltages on the group $\mathbb{Z}_n$.*

Proof. As representatives of the vertices of $J(n,2)$ (2-subsets of $\mathbb{Z}_n$), that is, vertices of the base graph G, we choose $\{0,i\}$ with $i=1,2,\ldots,\nu$. To determine the voltages, we consider the vertices adjacent to $\{0,i\}$ in $J(n,2)$ in terms of the representatives. Note that $\{0,\nu+1\}=\{0,\nu\}-\nu$ means that, when ν is subtracted (mod n) from both elements of $\{0,\nu\}$, we get $\{-\nu,0\}=\{0,\nu+1\}$. Then, we obtain the following adjacencies:

$$\begin{aligned}
\{0,i\}\sim\ & \{0,1\},\{0,2\},\ldots,\{0,i-1\},\{0,i+1\},\{0,i+2\},\ldots,\{0,\nu\},\\
& \{0,\nu+1+j\}=\{0,\nu-j\}-(\nu-j),\ \text{for } j=0,1,\ldots,\nu-1,\\
& \{i,1+k\}=\{0,i-(1+k)\}+(1+k),\ \text{for } k=0,1,\ldots,i-2,\\
& \{i,i+h\}=\{0,h\}+i,\ \text{for } h=1,\ldots,\nu, \qquad (3.1)\\
& \{i,i+\nu+r\}=\{0,\nu-r+1\}-(\nu-i-r+1),\ \text{for } r=1,\ldots,\nu-i. \qquad (3.2)
\end{aligned}$$

Now, let us see that the constructed base graph $G=(V,E)$ with voltages $\alpha: E\to\mathbb{Z}_n$ forms a lift G^α that is isomorphic to $F_2(K_n)$, under the isomorphism $(\{0,i\},g)\mapsto\{g,g+i\}$. To this end, assume that, in G, the vertex $\{0,i\}$ is adjacent to $\{0,j\}$ through an arc with voltage $t\in\mathbb{Z}_n$. This means that, for some r, one of the following equalities hold:

(*i*) $\{r,i\}=\{0,j\}+t=\{t,t+j\}$;
(*ii*) $\{0,i+r\}=\{t,t+j\}$.

Then, in G^α, the vertex $(\{0,i\},g)\mapsto\{g,g+i\}$ is adjacent to the vertex $(\{0,j\},g+t)\mapsto\{g+t,g+t+j\}$. To prove that, in $F_2(K_n)$, a vertex adjacent to $\{g,g+i\}$ is $\{g+t,g+t+j\}$, we distinguish the possible

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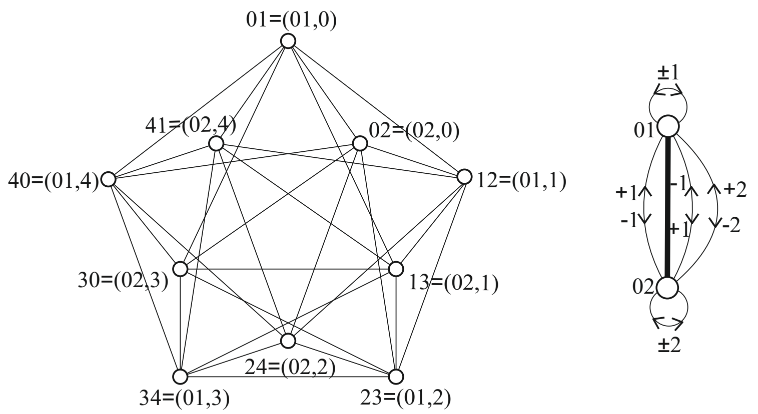

FIGURE 2. *The Johnson graph $J(5,2)$ and its base graph on $\mathbb{Z}_5$. The thick line represents the edge with voltage 0.*

cases: If in (i), $r = t$ and $i = t + j$, then $\{g+t, g+t+j\} = \{g+t, g+i\}$; on the contrary, if $r = t + j$ and $i = t$, then $\{g+i, g+t+j\} = \{g+t, g+i\}$. Similarly if, in (ii), $t = 0$ and $i + r = t + j$, then $\{g+t, g+t+j\} = \{g, g+i+r\}$; and, finally, if $t + j = 0$ and $i + r = t$, then $\{g+t, g+t+j\} = \{g+i+r, g\}$. Thus, in all cases $\{g+t, g+i\}$ or $\{g+i+r, g\}$ is a vertex adjacent to $\{g, g+i\}$, as claimed. ◻

COROLLARY 3.2. *The entries of the row $\{0, i\}$ of the polynomial matrix $\boldsymbol{B}(z)$, associated with the base graph G of Theorem 3.1, are given by the sums of the elements of the columns of the following array:*

$$\begin{array}{cccccccccc}
(1) & (2) & \cdots & (i-2) & (i-1) & \boldsymbol{(i)} & (i+1) & \ldots & (\nu-1) & (\nu) \\
1 & 1 & \cdots & 1 & 1 & \mathbf{0} & 1 & \cdots & 1 & 1 \\
\frac{1}{z} & \frac{1}{z^2} & \cdots & \frac{1}{z^{i-2}} & \frac{1}{z^{i-1}} & \boldsymbol{\frac{1}{z^i}} & \frac{1}{z^{i+1}} & \cdots & \frac{1}{z^{\nu-1}} & \frac{1}{z^\nu} \\
z^{i-1} & z^{i-2} & \cdots & z^2 & z & \mathbf{0} & \frac{1}{z} & \cdots & \frac{1}{z^{\nu-i-1}} & \frac{1}{z^{\nu-i}} \\
z^i & z^i & \cdots & z^i & z^i & \boldsymbol{z^i} & z^i & \cdots & z^i & z^i
\end{array}$$

Proof. This is a consequence of the adjacencies of the vertices $\{0, i\}$ in $J(n, 2)$, in terms of the representatives, given in the proof of Theorem 3.1. For example, notice that in such adjacencies, we have

$$\{0, i\} \sim \{i, 2i\} = \{0, i\} + 2i$$

between some of the items in (3.1) if $2i \le \nu$, or those in (3.2) if $2i \ge \nu + 1$. Then, the $(\{0, i\}, \{0, i\})$-entries of the diagonal of $\boldsymbol{B}(z)$ are $z^i + \frac{1}{z^i}$ for $i = 1, \ldots, \nu$, that is, the sum of the elements in column $\boldsymbol{(i)}$ of the array displayed in boldface.

EXAMPLE 3.3. *For $n = 5$ and 7, the corresponding polynomial matrices are, respectively:*

$$\boldsymbol{B}(z) = \begin{pmatrix} z + \frac{1}{z} & 1 + z + \frac{1}{z} + \frac{1}{z^2} \\ 1 + z + \frac{1}{z} + z^2 & z^2 + \frac{1}{z^2} \end{pmatrix}, \tag{3.3}$$

and

$$\boldsymbol{B}(z) = \begin{pmatrix} z + \frac{1}{z} & 1 + z + \frac{1}{z} + \frac{1}{z^2} & 1 + z + \frac{1}{z^2} + \frac{1}{z^3} \\ 1 + z + \frac{1}{z} + z^2 & z^2 + \frac{1}{z^2} & 1 + \frac{1}{z} + z^2 + \frac{1}{z^3} \\ 1 + \frac{1}{z} + z^2 + z^3 & 1 + z + \frac{1}{z^2} + z^3 & z^3 + \frac{1}{z^3} \end{pmatrix}. \tag{3.4}$$

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Then, for $n = 5$ and $n = 7$, the Johnson graphs $J(n, 2) \cong F_2(K_n)$ are a lift on the cyclic group $\mathbb{Z}_n$ of the base graphs shown in Figs. 2 and 3. In the case of $J(5, 2)$, the vertices are labeled as a 2-token of K_5, with vertices $0, 1, \ldots, 4$, and as the lift of the corresponding base graph with voltages on $\mathbb{Z}_5$ (thick lines correspond to voltage 0).

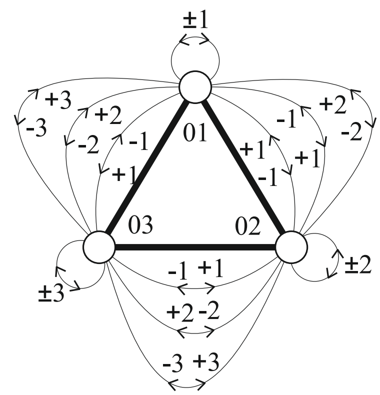

Figure 3. *The base graph of $J(7, 2)$. The thick lines represent the edges with voltage 0.*

Moreover, from the matrices (3.3) and (3.4), we obtain the spectra of $J(n, 2)$ for $n = 5$ and $n = 7$ shown in Tables 1 and 2.

Table 1
All the eigenvalues of the matrices $\boldsymbol{B}(\omega^r)$, which yield the eigenvalues of the Johnson graph $J(5, 2) = F_2(K_5)$.

$\omega = e^{i\frac{2\pi}{5}}$, $z = \omega^r$	$\lambda_{r,1}$	$\lambda_{r,2}$
$\mathrm{sp}(\boldsymbol{B}(\omega^0))$	6	-2
$\mathrm{sp}(\boldsymbol{B}(\omega^1)) = \mathrm{sp}(\boldsymbol{B}(\omega^4))$	1	-2
$\mathrm{sp}(\boldsymbol{B}(\omega^2)) = \mathrm{sp}(\boldsymbol{B}(\omega^3))$	1	-2

Table 2
All the eigenvalues of the matrices $\boldsymbol{B}(\omega^r)$, which yield the eigenvalues of the Johnson graph $J(7, 2) = F_2(K_7)$.

$\omega = e^{i\frac{2\pi}{7}}$, $z = \omega^r$	$\lambda_{r,1}$	$\lambda_{r,2}$	$\lambda_{r,3}$
$\mathrm{sp}(\boldsymbol{B}(\omega^0))$	10	-2	-2
$\mathrm{sp}(\boldsymbol{B}(\omega^1)) = \mathrm{sp}(\boldsymbol{B}(\omega^6))$	3	-2	-2
$\mathrm{sp}(\boldsymbol{B}(\omega^2)) = \mathrm{sp}(\boldsymbol{B}(\omega^5))$	3	-2	-2
$\mathrm{sp}(\boldsymbol{B}(\omega^3)) = \mathrm{sp}(\boldsymbol{B}(\omega^4))$	3	-2	-2

3.2. Line graphs of circulant graphs as lifts. In the following result, we show that, in fact, the polynomial matrix $\boldsymbol{B}(z)$ (associated with $J(n, 2)$ with odd n) contains information about the polynomial matrices of the line graphs of circulant graphs, which are Cayley graphs of the cyclic group $\mathbb{Z}_m$ with generators within the interval $[-(m-1), m-1]$.

Theorem 3.4. *Let n be an odd number, $n = 2\nu + 1$. Let $\boldsymbol{B}(z)$ be the polynomial matrix of the Johnson graph $J(n, 2) \cong F_2(K_n) \cong L(K_n)$, with rows and columns indexed by $\{0, 1\}, \{0, 2\}, \ldots, \{0, \nu\}$. Let $\boldsymbol{B}'(z)$ be the principal submatrix of $\boldsymbol{B}(z)$ with rows and columns indexed by $\{0, a_1\}, \{0, a_2\}, \ldots, \{0, a_s\}$ with $1 \leq$*

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$a_1 < a_2 < \cdots < a_s \le \nu$. Consider $\boldsymbol{B}'(z)$ as the polynomial matrix of a base graph on $\mathbb{Z}_m$ with $m \ge n$. Then, the lift graph defined by $\boldsymbol{B}'(z)$ is isomorphic to the line graph $L(H)$, where H is the circulant graph $\mathrm{Cay}(\mathbb{Z}_m; \pm a_1, \ldots, \pm a_s)$.

Proof. Reasoning as at the beginning of this section, notice that the representatives of the vertices of $L(H)$, as vertices of the base graph, can be chosen as $0a_1, 0a_2, \ldots, 0a_s$. Then, the reasoning is completely similar if we 'replace' the elements $1, 2, \ldots$ by $a_1, a_2, \ldots$ Indeed, the vertices of $L(H)$ adjacent to $\{0, a_i\}$, written in terms of the representatives, are

$$\begin{aligned}
\{0,a_i\} \sim\ & \{0,a_1\}, \{0,a_2\}, \ldots, \{0,a_{i-1}\}, \{0,a_{i+1}\}, \ldots, \{0,a_s\},\\
& \{0,-a_s\} = \{0,a_s\} - a_s, \ldots, \{0,-a_1\} = \{0,a_1\} - a_1,\\
& \{a_i, a_i - a_1\} = \{0,a_1\} + (a_i - a_1), \{a_i, a_i - a_2\} = \{0,a_2\} + (a_i - a_2), \ldots\\
& \{a_i, a_i - a_{i-1}\} = \{0,a_{i-1}\} + (a_i - a_{i-1}), \{a_i, a_i - a_{i+1}\} = \{0,a_{i+1}\} + (a_i - a_{i+1}), \ldots\\
& \{a_i, a_i - a_s\} = \{0,a_r\} + (a_i - a_r),\\
& \{a_i, a_i + a_1\} = \{0,a_1\} + a_i, \ldots, \{a_i, a_i + a_s\} = \{0,a_s\} + a_i.
\end{aligned}$$

Therefore, the entries of the row $\{0, a_i\}$ of the polynomial matrix $\boldsymbol{B}'(z)$ are given by the sums of the elements of the columns of the following array:

$$\begin{array}{ccccccccc}
(a_1) & (a_2) & \cdots & (a_{i-1}) & (\boldsymbol{a_i}) & (a_{i+1}) & \ldots & (a_{s-1}) & (a_s)\\
1 & 1 & \cdots & 1 & \mathbf{0} & 1 & \cdots & 1 & 1\\
\frac{1}{z^{a_1}} & \frac{1}{z^{a_2}} & \cdots & \frac{1}{z^{a_{i-1}}} & \boldsymbol{\frac{1}{z^{a_i}}} & \frac{1}{z^{a_{i+1}}} & \cdots & \frac{1}{z^{a_{s-1}}} & \frac{1}{z^{a_s}}\\
z^{a_i - a_1} & z^{a_i - a_2} & \cdots & z^{a_i - a_{i-1}} & \mathbf{0} & \frac{1}{z^{a_{i+1} - a_i}} & \cdots & \frac{1}{z^{a_s - a_i}} & \frac{1}{z^{a_s - a_i}}\\
z^{a_i} & z^{a_i} & \cdots & z^{a_i} & \boldsymbol{z^{a_i}} & z^{a_i} & \cdots & z^{a_i} & z^{a_i}
\end{array}$$

These correspond to the entries of the row indexed by $\{0, a_i\}$ of the principal submatrix $\boldsymbol{B}'(z)$ of $\boldsymbol{B}(z)$, as claimed. □

For instance, if $\boldsymbol{B}(z)$ is the polynomial matrix of the base graph of Fig. 3 and $\boldsymbol{B}'(z) = \boldsymbol{B}(z)$, the spectra of the graphs obtained by different values of m are

- $m = 7$: $\{10, 3^{[6]}, -2^{[14]}\}$ (strongly regular graph $J(7,2) \cong F_2(K_7) \cong L(K_7)$).
- $m = 8$: $\{10, 4^{[4]}, 2^{[3]}, -2^{[16]}\}$ (walk-regular graph isomorphic to the line graph of $\mathrm{Cay}(\mathbb{Z}_8; \pm 1, \pm 2, \pm 3)$).
- $m = 9$: $\{10, 4.880^{[2]}, 4^{[2]}, 2.652^{[2]}, 1.468^{[2]}, -2^{[18]}\}$ (graph isomorphic to the line graph of $\mathrm{Cay}(\mathbb{Z}_9; \pm 1, \pm 2, \pm 3)$).

Moreover, the $(2,3)$-submatrix $\boldsymbol{B}'(z)$ of the same matrix $\boldsymbol{B}(z)$, that is,

$$\boldsymbol{B}'(z) = \begin{pmatrix} z^2 + \frac{1}{z^2} & 1 + \frac{1}{z} + z^2 + \frac{1}{z^3} \\ 1 + z + \frac{1}{z^2} + z^3 & z^3 + \frac{1}{z^3} \end{pmatrix}, \tag{3.5}$$

corresponds to the base graph of $L(H)$ with $H = \mathrm{Cay}(\mathbb{Z}_m; \pm 2, \pm 3)$. Then if, for example, we take $m = 12$, the eigenvalues of $\boldsymbol{B}'(z)$ with $z = e^{i\frac{r2\pi}{12}}$, for $r = 0, \ldots, 11$, yield

$$\mathrm{sp}(L(H)) = \{6^{[1]}, 3^{[6]}, 2^{[1]}, 0^{[2]}, -2^{[12]}\}.$$

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3.3. Johnson graphs $J(n,k)$ as lifts. The following result is a generalization of Theorem 3.1 for $k \geq 2$.

THEOREM 3.5. *If n and $k(\leq n-k)$ are relatively prime integers, then the Johnson graph $J(n,k)$ is a lift of a base graph G with $\nu = \binom{n}{k}/n$ vertices with voltages on the group $\mathbb{Z}_n$.*

Proof. As before, the proof consists of identifying the representatives of the orbits (that is, the vertices of the base graph) and defining their 'weighted adjacencies' (the voltages). Since the procedure to find the latter is the same, we only need to show the former. Now, the representatives of the orbits correspond to distinct necklaces with k black beads (representing the k chosen elements of $\mathbb{Z}_n$) and $n-k$ white beads (corresponding to the remaining elements of $\mathbb{Z}_n$). Thus, two necklaces are equivalent if a given rotation of the other can obtain one. Moreover, if $\gcd(n,k) = 1$, all the orbits have n elements, as required. These correspond to 'aperiodic' necklaces (not consisting of a repeated sequence), also called Lyndon words, see Ruskey and Sawada [19] for an efficient algorithm to generate them. ◻

For example, to deal with $J(7,3) \cong F_3(K_7)$, and using the simplified notation ijk for $\{i,j,k\}$, we can take the representatives 012, 013, 014, 015, and 024. (That is, of each equivalence class, we take the lexicographical smallest sequence). Then, the adjacencies of 012 (the others are similar) in terms of the representatives are

$$\begin{aligned}
012 \sim\ & 013,\ 014,\ 015,\ 016 = 012-1,\\
& 123 = 012+1,\ 124 = 013+1,\ 125 = 014+1,\ 126 = 015+1,\\
& 023 = 015+2,\ 024,\ 025 = 024-2,\ 026 = 013-1.
\end{aligned}$$

The 5×5 polynomial matrix, indexed by the above representatives, is

$$\boldsymbol{B}(z) = \begin{pmatrix}
z+\frac{1}{z} & 1+z+\frac{1}{z} & 1+z & 1+z+z^2 & 1+\frac{1}{z^2}\\
1+z+\frac{1}{z} & 0 & 1+\frac{1}{z}+z^3 & 1+z^2+z^3 & z+\frac{1}{z}+z^3\\
1+\frac{1}{z} & 1+z+\frac{1}{z^3} & z^3+\frac{1}{z^3} & 1+\frac{1}{z}+z^3 & 1+\frac{1}{z^3}\\
1+\frac{1}{z}+\frac{1}{z^2} & 1+\frac{1}{z^2}+\frac{1}{z^3} & 1+z+\frac{1}{z^3} & 0 & z+\frac{1}{z^2}+z^3\\
1+z^2 & z+\frac{1}{z}+\frac{1}{z^3} & 1+z^3 & \frac{1}{z}+z^2+\frac{1}{z^3} & z^2+\frac{1}{z^2}
\end{pmatrix}, \tag{3.6}$$

giving, for $z = e^{i\frac{r2\pi}{7}}$ and $r = 0,1,\ldots,6$, the eigenvalues of $J(7,3) = F_3(K_7)$ shown in Table 3.

TABLE 3
All the eigenvalues of matrices $\boldsymbol{B}(\omega^r)$, which yield the eigenvalues of the Johnson graph $J(7,3) = F_3(K_7)$.

$\omega = e^{i\frac{2\pi}{7}}$, $z = \omega^r$	$\lambda_{r,1}$	$\lambda_{r,2}$	$\lambda_{r,3}$
$\mathrm{sp}(\boldsymbol{B}(\omega^0))$	12	$0^{[2]}$	$-3^{[2]}$
$\mathrm{sp}(\boldsymbol{B}(\omega^1)) = \mathrm{sp}(\boldsymbol{B}(\omega^6))$	5	$0^{[2]}$	$-3^{[2]}$
$\mathrm{sp}(\boldsymbol{B}(\omega^2)) = \mathrm{sp}(\boldsymbol{B}(\omega^5))$	5	$0^{[2]}$	$-3^{[2]}$
$\mathrm{sp}(\boldsymbol{B}(\omega^3)) = \mathrm{sp}(\boldsymbol{B}(\omega^4))$	5	$0^{[2]}$	$-3^{[2]}$

4. Token graphs of Cayley graphs. Let Γ be a group of n elements. For some k, with $1 \leq k \leq n$, consider the set $\Gamma^{(k)}$ whose elements are the $\binom{n}{k}$ k-subsets $\boldsymbol{\gamma} = \{g_1,\ldots,g_k\}$ with $g_i \in \Gamma$. A *k-set decomposition* $\mathcal{A}^{(k)}$ of Γ is a partition of $\Gamma^{(k)}$ into $r = \binom{n}{k}/n$ classes $A_1,\ldots,A_r$ with n elements each, such that each class A_i is a translation of some $\boldsymbol{\alpha}_i \in A_i$ (with $\boldsymbol{\alpha}_i$ being a 'representative' of the class) by Γ. That is, $A_i = \boldsymbol{\alpha}_i\Gamma$. In

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other words, the classes are the orbits (with the same order n) obtained when Γ acts on the representatives $\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_r$. Thus, a necessary condition for a group Γ to have a k-set decomposition is that n must divide $\binom{n}{k}$. For instance, this is the case when dealing with the Abelian group $\mathbb{Z}_{2\mu+1}\times\mathbb{Z}_{2\nu+1}$ and $k=2$ since, then, with $n=(2\mu+1)(2\nu+1)$, we have that the number of classes is $r=\binom{n}{2}/n=\frac{n-1}{2}=\mu\nu+\mu+\nu$. In this case, a possible choice of the r 2-sets representatives, of the form $\{(0,0),(i,j)\}=\{00,ij\}$, is as follows:

$$
\begin{array}{ccccccccc}
\{00,0n\} & \{00,1n\} & \{00,2n\} & \dots & \{00,mn\} & \{00,m+1,n\} & \dots & \{00,2m+1,n\} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\{\mathbf{00},\mathbf{01}\} & \{\mathbf{00},\mathbf{11}\} & \{\mathbf{00},\mathbf{21}\} & \dots & \{00,m1\} & \{00,m+11\} & \dots & (00,2m+1,1\} \\
 & \{\mathbf{00},\mathbf{10}\} & \{00,20\} & \dots & \{00,m0\} & & &
\end{array}
$$

Thus, in particular, the group $\Gamma=\mathbb{Z}_3\times\mathbb{Z}_3$ has the following 2-set decomposition with four orbits having representatives $\boldsymbol{\alpha}=\{00,01\}$, $\boldsymbol{\beta}=\{00,10\}$, $\boldsymbol{\gamma}=\{00,11\}$, and $\boldsymbol{\delta}=\{00,21\}$ (above, in boldface). The elements of the same orbit are indicated by the same Greek letter as their representative. For instance, the orbit of $\boldsymbol{\alpha}=(00,10)$ has elements $(00,20)$, $(10,20)$, $(01,11)$, ... (see Table 4).

Table 4
The elements of the same orbit are indicated by the same Greek letter as their representative in the group $\Gamma=\mathbb{Z}_3\times\mathbb{Z}_3$. The boldface indicates the representatives.

	00	10	20	01	11	21	02	12	22
00		$\boldsymbol{\alpha}$	α	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\delta}$	β	δ	γ
10			α	δ	β	γ	γ	β	δ
20				γ	δ	β	δ	γ	β
01					α	α	β	γ	δ
11						α	δ	β	γ
21							γ	δ	β
02								α	α
12									α
22									

Theorem 4.1. *Let Γ be a group with order n. For some generating set $S=\{\pm a_1,\dots,\pm a_s\}$ of Γ, consider the Cayley graph $G=\mathrm{Cay}(\Gamma,S)$, and its k-token graph $F_k(G)$ for some k. If Γ has a k-set decomposition, then $F_k(G)$ can be obtained as the lift of a base graph H with $r=\binom{n}{k}/n$ vertices and voltages on the group Γ.*

Proof. As in the case of cyclic groups, the vertices of H are labeled with the r representatives of the k-set decomposition. Now to fix the voltages, we reason as before: If vertex $\boldsymbol{\alpha}=\{\alpha_1,\alpha_2,\dots,\alpha_k\}$ is adjacent to, say, $\boldsymbol{\alpha}'=\{\alpha_1 a_i,\alpha_2,\dots,\alpha_k\}$ for some $a_i\in S$, there must be some $g\in\Gamma$ and $\boldsymbol{\beta}\in\mathcal{A}^{(k)}$ such that $\boldsymbol{\alpha}'=\boldsymbol{\beta}g$. Then, the arc in H from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$ has voltage g. □

Let us show an example with the 2-token graph of $\mathrm{Cay}(\Gamma,S)$ with $\Gamma=\mathbb{Z}_3\times\mathbb{Z}_3$ and $S=\{\pm(1,0),\pm(0,1)\}$, see Fig. 4.

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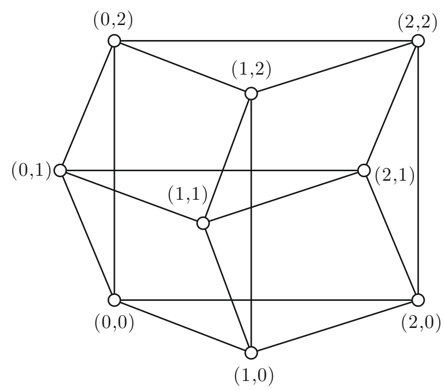

FIGURE 4. *The Cayley graph* $\mathrm{Cay}(\Gamma,S)$ *with* $\Gamma=\mathbb{Z}_3\times\mathbb{Z}_3$ *and* $S=\{\pm(1,0),\pm(0,1)\}$.

TABLE 5
The eigenvalues of the 2*-token graph of* $\mathrm{Cay}(\Gamma,S)$ *with* $\Gamma=\mathbb{Z}_3\times\mathbb{Z}_3$ *and* $S=\{\pm 10,\pm 01\}$.

$r\backslash s$	0	1	2
0	7.12, 1.34,−1,−3.13	3.78, 1.54, −1, −3.13	3.78, 1.54, −1, −3.13
1	3.78, 1.54, −1, −3.13	2, 0.56, −1, −3.56	2, 0.56, −1, −3.56
2	3.78, 1.54, −1, −3.13	2, 0.56, −1, −3.56	2, 0.56, −1, −3.56

$$\begin{aligned}
\boldsymbol{\alpha}=\{00,10\}\sim{}&\{00,20\}=\boldsymbol{\alpha}+20,\{00,11\}=\boldsymbol{\gamma},\{00,12\}=\boldsymbol{\delta}+12,\\
&\{20,10\}=\boldsymbol{\alpha}+01,\{01,10\}=\boldsymbol{\delta}+10,\{02,10\}=\boldsymbol{\gamma}+02.\\
\boldsymbol{\beta}=\{00,01\}\sim{}&\{00,11\}=\boldsymbol{\gamma},\{00,21\}=\boldsymbol{\delta},\{00,02\}=\boldsymbol{\beta}+02,\\
&\{10,01\}=\boldsymbol{\delta}+10,\{20,01\}=\boldsymbol{\gamma}+20,\{02,01\}=\boldsymbol{\beta}+01.\\
\boldsymbol{\gamma}=\{00,11\}\sim{}&\{00,21\}=\boldsymbol{\delta},\{00,01\}=\boldsymbol{\beta},\{00,12\}=\boldsymbol{\delta}+12,\{00,10\}=\boldsymbol{\alpha},\\
&\{10,11\}=\boldsymbol{\beta}+10,\{20,11\}=\boldsymbol{\delta}+20,\{01,11\}=\boldsymbol{\alpha}+01,(02,11)=\boldsymbol{\delta}+11.\\
\boldsymbol{\delta}=\{00,21\}\sim{}&\{00,01\}=\boldsymbol{\beta},\{00,11\}=\boldsymbol{\gamma},\{00,22\}=\boldsymbol{\gamma}+22,\{00,20\}=\boldsymbol{\alpha}+20,\\
&\{10,21\}=\boldsymbol{\gamma}+10,\{20,21\}=\boldsymbol{\beta}+20,\{01,21\}=\boldsymbol{\alpha}+21,\{02,21\}=\boldsymbol{\gamma}+21.
\end{aligned}$$

Then, the (two variables) polynomial matrix of the 2-token of the 'toroidal mesh' $G=\mathrm{Cay}(\Gamma,\{\pm 01,\pm 10\})$ is

$$\boldsymbol{B}(y,z)=\begin{pmatrix} y+y^2 & 0 & 1+z^2 & y+yz^2\\ 0 & z+z^2 & 1+y^2 & 1+y\\ 1+z & 1+y & 0 & 1+y^2+yz+yz^2\\ y^2+y^2z & 1+y^2 & 1+y+y^2z+y^2z^2 & 0\end{pmatrix}.$$

With the 9 possible pairs of values (y,z), for $y=e^{i\frac{r2\pi}{3}}$, $z=e^{i\frac{s2\pi}{3}}$, and $r,s=0,1,2$, the obtained (approximated) eigenvalues are shown in Table 5.

In the general case, when the group Γ is not necessarily Abelian, we proceed as follows. If $G=(V,E)$ is the base graph with voltage assignment $\alpha:E\to\Gamma$, its 'base matrix' $\boldsymbol{B}(G)$ has entries $\boldsymbol{B}(G)_{uv}=$

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$\alpha(a_1)+\cdots+\alpha(a_r)\in\mathbb{C}[\Gamma]$, where $a_1,\ldots,a_r$ are the arcs from u to v, and $\boldsymbol{B}(G)=0$ if $(u,v)\notin E$. Then, if $\rho\in\mathrm{Irep}(\Gamma)$ is a unitary irreducible representation of Γ, with dimension $d_\rho=\dim(\rho)$, the $d_\rho n\times d_\rho n$ matrix $\rho(\boldsymbol{B}(G))$ (where ρ acts on the elements of Γ) plays the same role as the polynomial matrix $\boldsymbol{B}(z)$ with a given value of $\boldsymbol{z}$. To find the spectrum of the lift G^α, we apply the following result, which can be seen as a generalization of Theorem 2.1.

THEOREM 4.2 ([10]). *Let $G=(V,E)$ be a base graph (or digraph) on n vertices, with a voltage assignment α in a group Γ. For every irreducible representation $\rho\in\mathrm{Irep}(\Gamma)$, let $\rho(\boldsymbol{B})$ be the complex matrix whose entries are given by $\rho(\boldsymbol{B}(G)_{u,v})$. Then,*

$$\mathrm{sp}\,G^\alpha=\bigcup_{\rho\in\mathrm{Irep}(\Gamma)} d_\rho\cdot\mathrm{sp}(\rho(\boldsymbol{B}(G))).$$

5. Further results. The results of the previous sections can be extended in the following directions.

5.1. Johnson graphs $J(n,k)$ with even n. Theorem 3.1 can be extended to Johnson graphs with n even, but, in this case, we need a generalization of voltage and lift graphs. The authors have studied this in [9] by introducing the concepts of combined voltage graphs and factored lifts. The key idea is to also give 'voltages' to the vertices. More precisely, each vertex of the combined voltage graph is endowed with a coset of a given subgroup. See Fig. 5 for the case of $J(4,2)$ (octahedral graph).

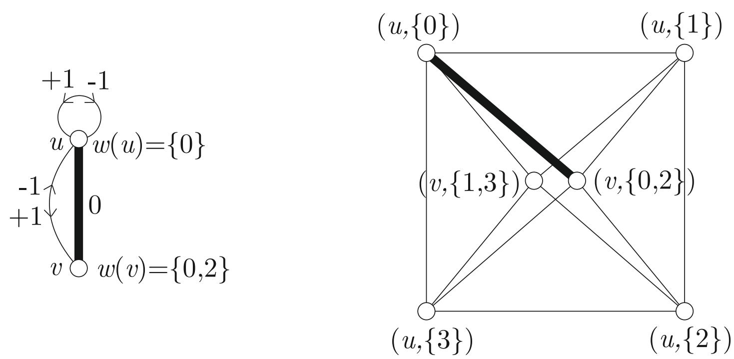

FIGURE 5. *A combined voltage graph for the Johnson graph $J(4,2)$.*

5.2. Computing Eigenspaces. The results of Theorems 2.1 and 4.2 are based on the fact that every eigenvector of the base graph G yields an eigenvector of the lift G^α. Thus, although not shown in the examples, the eigenspaces of our constructions can also be computed.

5.3. Universal matrix. As it was shown by the authors in [8], from the base graph G with a voltage assignment, we can associate a polynomial-type matrix representing the so-called universal adjacency matrix of the lift G^α. The universal adjacency matrix $\boldsymbol{U}$ of a graph, with adjacency matrix $\boldsymbol{A}$, is a linear combination, with real coefficients, of $\boldsymbol{A}$, the diagonal matrix $\boldsymbol{D}$ of vertex degrees, the identity matrix $\boldsymbol{I}$, and the all-1 matrix $\boldsymbol{J}$; that is, $\boldsymbol{U}=c_1\boldsymbol{A}+c_2\boldsymbol{D}+c_3\boldsymbol{I}+c_4\boldsymbol{J}$, with $c_i\in\mathbb{R}$ and $c_1\neq 0$. Thus, as particular cases, $\boldsymbol{U}$ may be the adjacency matrix, the Laplacian, the signless Laplacian, and the Seidel matrix (see, for instance, Haemers and Omidi [15]). Then, our method also gives the eigenvalues and eigenspaces of such matrices.

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5.4. Dealing with digraphs. As commented in the preliminary section, a graph is a type of digraph in which two opposite arcs constitute each edge. In fact, this paper's results and argumentation can be generalized naturally to digraphs. For instance, the vertices of the k-token digraph $F_k(G)$ of a digraph $G=(V,A)$, on n vertices, can be represented by k indistinguishable tokens placed in different vertices of G. One vertex of $F_k(G)$ is adjacent to another one (forming an arc) if the latter can be obtained from the former by moving one token along an arc in A.

Of course, when dealing with digraphs, the eigenvalues and eigenvectors are not necessarily real, but this does not affect the procedure for obtaining them. For example, the 2-token digraph of the directed cycle C_5, shown in Fig. 6, can be constructed as a lift of the base digraph on $\mathbb{Z}_5$, with vertices 01 and 02 and three arcs: $(01,02)$ with voltage 0; $(02,01)$ with voltage $+1$; and the loop $(02,02)$ with voltage -2. Then, the polynomial matrix is

$$\boldsymbol{B}(z)=\begin{pmatrix} 0 & 1 \\ z & \frac{1}{z^2} \end{pmatrix}.$$

By taking $z=e^{i\frac{r2\pi}{5}}$, for $r=0,\ldots,4$, the eigenvalues of $\boldsymbol{B}(z)$ yield all the eigenvalues of $F_2(C_5)$. Namely,

$$1.618,\ 0.5\pm 1.540\,i,\ 0.5\pm 0.363\,i,\ -0.618,\ -0.191\pm 0.588\,i,\ -1.309\pm 0.951\,i.$$

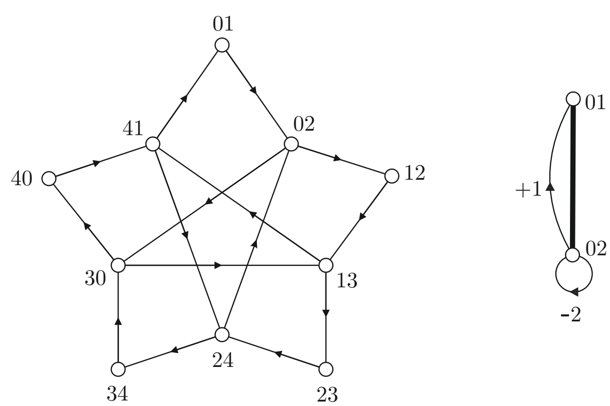

FIGURE 6. *The 2-token digraph of the directed cycle C_5 (with arcs $0\to 1\to 2\to 3\to 4\to 0$) and its base digraph. The thick line represents the edge with voltage* 0.

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