

CHARACTERIZATIONS OF JORDAN DERIVATIONS ON TRIANGULAR RINGS: ADDITIVE MAPS JORDAN DERIVABLE AT IDEMPOTENTS*

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Abstract. Let $\mathcal{T}$ be a triangular ring. An additive map δ from $\mathcal{T}$ into itself is said to be Jordan derivable at an element $Z \in \mathcal{T}$ if $\delta(A)B + A\delta(B) + \delta(B)A + B\delta(A) = \delta(AB + BA)$ for any $A, B \in \mathcal{T}$ with $AB + BA = Z$. An element $Z \in \mathcal{T}$ is called a Jordan all-derivable point of $\mathcal{T}$ if every additive map Jordan derivable at Z is a Jordan derivation. In this paper, we show that some idempotents in $\mathcal{T}$ are Jordan all-derivable points. As its application, we get the result that for any nest $\mathcal{N}$ in a factor von Neumann algebra $\mathcal{R}$, every nonzero idempotent element Q satisfying $PQ = Q, QP = P$ for some projection $P \in \mathcal{N}$ is a Jordan all-derivable point of the nest subalgebra $\text{Alg}\mathcal{N}$ of $\mathcal{R}$.

Key words. Jordan derivations, Triangular rings, Nest algebras.

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1. Introduction. Let $\mathcal{A}$ be a ring (or an algebra) with the unit I . Recall that an additive (or a linear) map δ from $\mathcal{A}$ into itself is called a derivation if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $A, B \in \mathcal{A}$. As well known that derivations are very important both in theory and applications, and were studied intensively [4, 5, 10]. The question under what conditions that a map becomes a derivation attracted much attention of mathematicians (for instance [5, 7, 8, 10, 12]). We say that a map $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is derivable at a point $Z \in \mathcal{A}$ if $\delta(A)B + A\delta(B) = \delta(AB)$ for any $A, B \in \mathcal{A}$ with $AB = Z$, and such Z is called a derivable point of $\mathcal{A}$. It is obvious that an additive (or a linear) map is a derivation if and only if it is derivable at all points. It is natural and interesting to ask the question whether or not an additive (or a linear) map is a derivation if it is derivable only at one given point. If such a point exists for a ring (or an algebra) $\mathcal{A}$, we call this point an all-derivable point of $\mathcal{A}$ as in [7, 8]. It is quite surprising that there do exist all-derivable points for some algebras. For instance, Xiong and Zhu in [12] proved that every strongly operator topology continuous linear

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map derivable at the unit operator I between nest algebras on complex separable Hilbert spaces is an inner derivation. Hou and Qi proved in [8] that the unit I is an all-derivable point of almost all Banach space nest algebras without the assumption of any continuity on the linear maps.

Besides derivations, sometimes one has to consider Jordan derivations. Recall that an additive (or a linear) map δ from a ring (or an algebra) $\mathcal{A}$ into itself is called a Jordan derivation if $\delta(A^2) = \delta(A)A + A\delta(A)$ for all $A \in \mathcal{A}$, or equivalently, if $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$ for all $A, B \in \mathcal{A}$ in the case that $\mathcal{A}$ is of characteristic not 2. In this paper, following [13, 14], we say that an element $Z \in \mathcal{A}$ is a Jordan derivable point, if $\delta(A)B + A\delta(B) + \delta(B)A + B\delta(A) = \delta(AB + BA)$ for any $A, B \in \mathcal{A}$ with $AB + BA = Z$; we say that an element $0 \neq Z \in \mathcal{A}$ is a *Jordan all-derivable point* of $\mathcal{A}$ if every additive (or linear) map from $\mathcal{A}$ into itself Jordan derivable at Z is a Jordan derivation. We mention that, in [9], δ is said to be Jordan derivable at Z if $\delta(A^2) = \delta(A)A + A\delta(A)$ whenever $A^2 = Z$. Clearly, this definition of Jordan derivable map at a point is weaker than the former one.

The following example shows that the zero point is not a Jordan all-derivable point of any ring.

Example 1.1. Let $\delta : \mathcal{A} \rightarrow \mathcal{A}$ be a Jordan derivation. Define an additive map $\varphi : \mathcal{A} \rightarrow \mathcal{A}$ by $\varphi(A) = \delta(A) + ZA$, where Z belongs to center of $\mathcal{A}$. It is obvious that φ is Jordan derivable at 0 but it is not a Jordan derivation. Thus 0 is not a Jordan all-derivable point of $\mathcal{A}$.

However, there do exist Jordan all-derivable points for some algebras. For instance, the unit I of a prime algebra is a Jordan all-derivable point (see [1]). In this paper, we discuss maps between triangular rings (or algebras) that are Jordan derivable at some given point.

The triangular rings were firstly introduced in [2] and then studied by many authors (see [3, 11]). Let $\mathcal{A}$ and $\mathcal{B}$ be two unital rings (or algebras) with unit I_1 and I_2 respectively, and let $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$ -bimodule, that is, $\mathcal{M}$ is a $(\mathcal{A}, \mathcal{B})$ -bimodule satisfying, for $A \in \mathcal{A}$, $A\mathcal{M} = \{0\} \Rightarrow A = 0$ and for $B \in \mathcal{B}$, $\mathcal{M}B = \{0\} \Rightarrow B = 0$. Recall that the ring (or algebra)

$$\mathcal{T} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{bmatrix} X & W \\ 0 & Y \end{bmatrix} : X \in \mathcal{A}, W \in \mathcal{M}, Y \in \mathcal{B} \right\}$$

under the usual matrix addition and formal matrix multiplication is called a *triangular ring (or algebra)*. Clearly, $\mathcal{T}$ is unital with the unit $I = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}$ and has a nontrivial idempotent element $P = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}$, which we'll call *the standard idem-*

tent.

Let $\mathcal{T} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular ring, and let $\delta : \mathcal{T} \rightarrow \mathcal{T}$ be an additive map. In Section 2, we show that, every additive map δ Jordan derivable at 0 has the form of $\delta(T) = \tau(T) + \delta(I)T$ if $\mathcal{T}$ is of characteristic not 2, and $\delta(I)$ belongs to the center of $\mathcal{T}$, where $\tau : \mathcal{T} \rightarrow \mathcal{T}$ is an additive derivation (see Theorem 2.2.). This reveals that, though zero is not a Jordan all-derivable point, in some situation, the converse of the fact presented in Example 1.1 is true. Particularly, this is the case for the additive (linear) maps between nest algebras $\text{Alg}\mathcal{N}$ on Banach space X that are Jordan derivable at zero if there is a $N \in \mathcal{N}$ is complemented in X (Corollary 2.3). In Section 3, under some mild assumptions on $\mathcal{T}$, we show that the standard idempotent P is a Jordan all-derivable point of triangular rings $\mathcal{T}$ (see Theorem 3.1). As a application of this result, we get that every nontrivial idempotent operator P with range in a Banach space nest $\mathcal{N}$ is an additive Jordan all-derivable point of the nest algebra $\text{Alg}\mathcal{N}$ (see Corollary 3.3). Section 4 is devoted to the study of the additive maps Jordan derivable at the unit I . Under some mild assumptions on $\mathcal{T}$, we show that, the unit I is an additive Jordan full-derivable point of $\mathcal{T}$ (see Theorem 4.1). As its application, we obtain that I is an additive Jordan all-derivable point of Banach space nest algebra $\text{Alg}\mathcal{N}$ if there is a $N \in \mathcal{N}$ is complemented in X (see Corollary 4.2). As a consequence of above results, we obtain that every nonzero idempotent element Q satisfying $PQ = Q$, $QP = P$ for some projection P in a nontrivial nest $\mathcal{N}$ of a factor von Neumann algebra $\mathcal{R}$ is an additive Jordan all-derivable point of the nest subalgebra $\text{Alg}\mathcal{N}$ of a factor von Neumann algebra $\mathcal{R}$ (see Corollary 4.4).

In this paper, the center of an algebra $\mathcal{A}$ will be denoted by $Z(\mathcal{A})$.

2. Additive maps Jordan derivable at 0. In this section, we characterize additive maps between triangular rings that are Jordan derivable at 0. To prove the main result, we need the following lemma.

LEMMA 2.1. *Let $\mathcal{T} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular ring, and assume that the characteristic of $\mathcal{T}$ is not 2. Let $\delta : \mathcal{T} \rightarrow \mathcal{T}$ be an additive map which is Jordan derivable at zero 0, then*

- (i) $\delta(E) = \delta(E)E + E\delta(E) - \delta(I)E$ and $\delta(I)E = E\delta(I)$ for any idempotent $E \in \mathcal{T}$;
- (ii) $\delta(N)N + N\delta(N) = 0$ for all $N \in \mathcal{T}$ with $N^2 = 0$.

Proof. It is enough to check (i), (ii) is obvious. For any idempotent $E \in \mathcal{T}$, from $E(I - E) + (I - E)E = 0$, we have $\delta(E(I - E) + (I - E)E) = \delta(E)(I - E) + E\delta(I - E) + \delta(I - E)E + (I - E)\delta(E) = 0$, thus $2\delta(E) = 2\delta(E)E + 2E\delta(E) - \delta(I)E - E\delta(I)$. Since the characteristic of $\mathcal{T}$ is not 2, by multiplying E in this equality from left and right sides respectively, we have $2E\delta(E)E - E\delta(I)E = \delta(I)E = E\delta(I)$. $\square$

The following is our main result in this section.

THEOREM 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings and $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$ -bimodule, and let $\mathcal{T} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular ring. Assume that, the characteristic of $\mathcal{T}$ is not 2, $\delta : \mathcal{T} \rightarrow \mathcal{T}$ is an additive map with $\delta(I) \in \mathcal{Z}(\mathcal{T})$. Then δ is Jordan derivable at 0 if and only if there is a derivation $\tau : \mathcal{T} \rightarrow \mathcal{T}$ such that $\delta(T) = \tau(T) + \delta(I)T$ for all $T \in \mathcal{T}$.*

Proof. The “if” part is obvious. Assume that δ is Jordan derivable at 0. Define $\tau(T) = \delta(T) - \delta(I)T$. Then τ is still additive and Jordan derivable at 0 as $\delta(I) \in \mathcal{Z}(\mathcal{T})$. Also, $\tau(I) = 0$. By Lemma 2.1, we get $\tau(E) = \tau(E)E + E\tau(E)$ for any idempotent $E \in \mathcal{T}$. Thus, without loss of generality, we may assume $\delta(I) = 0$ and $\delta(E) = \delta(E)E + E\delta(E)$ for any $E \in \mathcal{T}$ with $E^2 = E$. Next we show that δ is a derivation. Let $P = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}$ be the standard idempotent.

Because δ is additive, for any $X \in \mathcal{A}$, $W \in \mathcal{M}$, $Y \in \mathcal{B}$ we can write

$$\delta \begin{bmatrix} X & W \\ 0 & Y \end{bmatrix} = \begin{bmatrix} \delta_{11}(X) + \varphi_{11}(W) + \tau_{11}(Y) & \delta_{12}(X) + \varphi_{12}(W) + \tau_{12}(Y) \\ 0 & \delta_{22}(X) + \varphi_{22}(W) + \tau_{22}(Y) \end{bmatrix},$$

$\delta_{ij} : \mathcal{A} \rightarrow \mathcal{A}_{ij}$, $\varphi_{ij} : \mathcal{M} \rightarrow \mathcal{A}_{ij}$, $\tau_{ij} : \mathcal{B} \rightarrow \mathcal{A}_{ij}$, $1 \leq i \leq j \leq 2$, are additive maps with $\mathcal{A}_{11} = \mathcal{A}$, $\mathcal{A}_{12} = \mathcal{M}$ and $\mathcal{A}_{22} = \mathcal{B}$ (These notations will also be used in the proofs of Theorem 3.1 and Theorem 4.1).

Now $\delta(P) = \delta(P)P + P\delta(P)$ implies that

$$\begin{bmatrix} \delta_{11}(I_1) & \delta_{12}(I_1) \\ 0 & \delta_{22}(I_1) \end{bmatrix} = \delta(P) = \delta(P)P + P\delta(P) = \begin{bmatrix} 2\delta_{11}(I_1) & \delta_{12}(I_1) \\ 0 & 0 \end{bmatrix},$$

which forces that $\delta_{11}(I_1) = \delta_{22}(I_1) = 0$. Thus

$$\delta(P) = \begin{bmatrix} 0 & \delta_{12}(I_1) \\ 0 & 0 \end{bmatrix} \text{ and } \delta(I - P) = \begin{bmatrix} 0 & -\delta_{12}(I_1) \\ 0 & 0 \end{bmatrix}.$$

For any $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, let $T = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 0 \\ 0 & Y \end{bmatrix}$. Then $TS + ST = 0$ and

$$\begin{aligned} 0 &= \delta(TS + ST) = \delta(T)S + T\delta(S) + \delta(S)T + S\delta(T) \\ (2.1) \quad &= \begin{bmatrix} X\tau_{11}(Y) + \tau_{11}(Y)X & \delta_{12}(X)Y + X\tau_{12}(Y) \\ 0 & \delta_{22}(X)Y + Y\delta_{22}(X) \end{bmatrix}. \end{aligned}$$

Let $X = I_1$ in (2.1), we get

$$(2.2) \quad \tau_{11}(Y) = 0, \quad \tau_{12}(Y) = -\delta_{12}(I_1)Y \quad \text{for all } Y \in \mathcal{B}.$$

Similarly, let $Y = I_2$ in (2.1) we have

$$(2.3) \quad \delta_{22}(X) = 0, \quad \delta_{12}(X) = X\delta_{12}(I_1) \quad \text{for all } X \in \mathcal{A}.$$

For any $X \in \mathcal{A}, W \in \mathcal{M}$, let $T = \begin{bmatrix} X & XW \\ 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & W \\ 0 & -I_2 \end{bmatrix}$. Then $TS + ST = 0$. By (2.2)-(2.3), we have

$$(2.4) \quad \begin{aligned} 0 &= \delta(TS + ST) = \delta(T)S + T\delta(S) + \delta(S)T + S\delta(T) \\ &= \begin{bmatrix} \varphi_{11}(W)X + X\varphi_{11}(W) & G \\ 0 & -2\varphi_{22}(XW) \end{bmatrix}, \end{aligned}$$

$G = \delta_{11}(X)W + X\varphi_{12}(W) - \varphi_{12}(XW) + \varphi_{11}(XW)W + XW\varphi_{22}(W) + \varphi_{11}(W)XW + W\varphi_{22}(XW)$.

Taking $X = I_1$, we get

$$(2.5) \quad \varphi_{11}(W) = \varphi_{22}(W) = 0 \quad \text{for all } W \in \mathcal{M}.$$

Now by (2.4), it is easily seen that

$$(2.6) \quad \varphi_{12}(XW) = \delta_{11}(X)W + X\varphi_{12}(W) \quad \text{for all } X \in \mathcal{A}, W \in \mathcal{M}.$$

For any $X_1, X_2 \in \mathcal{A}$ and $W \in \mathcal{M}$, it follows from (2.6) that

$$\begin{aligned} \delta_{11}(X_1X_2)W + X_1X_2\varphi_{12}(W) &= \varphi_{12}(X_1X_2W) = \delta_{11}(X_1)X_2W + X_1\varphi_{12}(X_2W) \\ &= \delta_{11}(X_1)X_2W + X_1\delta_{11}(X_2)W + X_1X_2\varphi_{12}(W), \end{aligned}$$

and thus, using the fact $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$ -bimodule, we see that

$$(2.7) \quad \delta_{11}(X_1X_2) = \delta_{11}(X_1)X_2 + X_1\delta_{11}(X_2)$$

holds for all $X_1, X_2 \in \mathcal{A}$.

For any $W \in \mathcal{M}, Y \in \mathcal{B}$, considering $T = \begin{bmatrix} I_1 & W \\ 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & WY \\ 0 & -Y \end{bmatrix}$ we get

$$(2.8) \quad \begin{aligned} \varphi_{12}(WY) &= \varphi_{12}(W)Y + W\tau_{22}(Y), \\ \tau_{22}(Y_1Y_2) &= \tau_{22}(Y_1)Y_2 + Y_1\tau_{22}(Y_2) \end{aligned}$$

for all $W \in \mathcal{M}, Y_1, Y_2 \in \mathcal{B}$.

Now we are in a position to check that δ is a derivation. For any

$$T = \begin{bmatrix} X_1 & W_1 \\ 0 & Y_1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} X_2 & W_2 \\ 0 & Y_2 \end{bmatrix} \in \mathcal{T},$$

where $X_1, X_2 \in \mathcal{A}$, $W_1, W_2 \in \mathcal{M}$ and $Y_1, Y_2 \in \mathcal{B}$ by (2.2)-(2.5), on the one hand we have

$$\begin{aligned} \delta(TS) &= \delta \begin{bmatrix} X_1X_2 & X_1W_2 + W_1Y_2 \\ 0 & Y_1Y_2 \end{bmatrix} \\ &= \begin{bmatrix} \delta_{11}(X_1X_2) & X_1X_2\delta_{12}(I_1) - \delta_{12}(I_1)Y_1Y_2 + \varphi_{12}(X_1W_2 + W_1Y_2) \\ 0 & \tau_{22}(Y_1Y_2) \end{bmatrix}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\delta(T)S + T\delta(S) \\ &= \begin{bmatrix} \delta_{11}(X_1) & X_1\delta_{12}(I_1) - \delta_{12}(I_1)Y_1 + \varphi_{12}(W_1) \\ 0 & \tau_{22}(Y_1) \end{bmatrix} \begin{bmatrix} X_2 & W_2 \\ 0 & Y_2 \end{bmatrix} \\ &+ \begin{bmatrix} X_1 & W_1 \\ 0 & Y_1 \end{bmatrix} \begin{bmatrix} \delta_{11}(X_2) & X_2\delta_{12}(I_1) - \delta_{12}(I_1)Y_2 + \varphi_{12}(W_2) \\ 0 & \tau_{22}(Y_2) \end{bmatrix} \\ &= \begin{bmatrix} \delta_{11}(X_1)X_2 + X_1\delta_{11}(X_2) & G \\ 0 & \tau_{22}(Y_1)Y_2 + Y_1\tau_{22}(Y_2) \end{bmatrix}, \end{aligned}$$

where $G = X_1X_2\delta_{12}(I_1) - \delta_{12}(I_1)Y_1Y_2 + \delta_{11}(X_1)W_2 + X_1\varphi_{12}(W_2) + \varphi_{12}(W_1)Y_2 + W_1\tau_{22}(Y_2)$. By (2.6)-(2.8), it is obvious that $\delta(TS) = \delta(T)S + T\delta(S)$ holds for all $T, S \in \mathcal{T}$, that is, δ is a derivation. $\square$

COROLLARY 2.3. *Let $\mathcal{N}$ be a nest on a complex Banach space X such that there is a $N \in \mathcal{N}$ complemented in X , and let $\text{Alg}\mathcal{N}$ be the associated nest algebra. Assume that $\delta : \text{Alg}\mathcal{N} \rightarrow \text{Alg}\mathcal{N}$ is an additive map with $\delta(I) \in \mathbb{C}I$, then δ is Jordan derivable at 0 if and only if there exist a derivation τ and a scalar $\lambda \in \mathbb{C}$ such that $\delta(A) = \tau(A) + \lambda A$ for all $A \in \text{Alg}\mathcal{N}$.*

Proof. Since $N \in \mathcal{N}$ is complemented in X , there is a bounded idempotent operator P with range N . It is easy to check that $P \in \text{Alg}\mathcal{N}$. Denote $M = (I - P)(X)$, and let $\mathcal{A} = P\text{Alg}\mathcal{N}|_N$, $\mathcal{M} = P\text{Alg}\mathcal{N}|_M$ and $\mathcal{B} = (I - P)\text{Alg}\mathcal{N}|_M$. Then $\mathcal{M}$ is faithful $(\mathcal{A}, \mathcal{B})$ -bimodule, and $\text{Alg}\mathcal{N} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a triangular algebra. Thus this corollary follows immediately from Theorem 2.2. $\square$

In [6], Gilfeather and Larson introduced a concept of nest subalgebras of von Neumann algebras, which is a generalization of Ringrose's original concept of nest algebras. Let $\mathcal{R}$ be a von Neumann algebra acting on a complex Hilbert space H . A nest $\mathcal{N}$ in $\mathcal{R}$ is a totally ordered family of orthogonal projections in $\mathcal{R}$ which is closed in the strong operator topology, and which includes 0 and I . A nest is said to be non-trivial if it contains at least one non-trivial projection. If P is a projection, let $P^\perp$ denote $I - P$. The nest subalgebra of $\mathcal{R}$ associated to a nest $\mathcal{N}$, denoted by $\text{Alg}\mathcal{N}$, is the set of all elements $A \in \mathcal{R}$ satisfying $PAP = AP$ for each $P \in \mathcal{N}$. When $\mathcal{R} = \mathcal{B}(H)$, the algebra of all bounded linear operators acting on a complex Hilbert space H , $\text{Alg}\mathcal{N}$ is the usual one on the Hilbert space H .

If $\mathcal{N}$ is a nest in a factor von Neumann algebra $\mathcal{R}$, then $\mathcal{N}^\perp = \{P^\perp : P \in \mathcal{N}\}$ is also a nest, and $\text{Alg}\mathcal{N}^\perp = (\text{Alg}\mathcal{N})^*$. The von Neumann algebra $\text{Alg}\mathcal{N} \cap (\text{Alg}\mathcal{N})^*$ is the diagonal of $\text{Alg}\mathcal{N}$ and denoted by $\mathcal{D}(\mathcal{N})$. Let $\mathcal{R}(\mathcal{N})$ denote the norm closed algebra generated by $\{P\mathcal{R}P^\perp : P \in \mathcal{N}\}$. It follows from [6] that $\mathcal{D}(\mathcal{N}) + \mathcal{R}(\mathcal{N})$ is weakly dense in $\text{Alg}\mathcal{N}$, and that the commutant of $\text{Alg}\mathcal{N}$ is $\mathbb{C}I$.

Taking a non-trivial projection P in $\mathcal{N}$, it is easily seen that $\text{Alg}\mathcal{N}$ is a triangular algebra with the standard idempotent P . Thus from Theorem 2.2, we get a characterization of additive maps Jordan derivable at zero between nest subalgebras of factor von Neumann algebras.

COROLLARY 2.4. *Let $\mathcal{N}$ be a non-trivial nest in a factor von Neumann algebra $\mathcal{R}$, and let $\text{Alg}\mathcal{N}$ be the associated nest algebra. Assume that $\delta : \text{Alg}\mathcal{N} \rightarrow \text{Alg}\mathcal{N}$ is an additive map. Then δ is Jordan derivable at 0 if and only if there exist a derivation τ of $\text{Alg}\mathcal{N}$ and a scalar $\lambda \in \mathbb{C}$ such that $\delta(A) = \tau(A) + \lambda A$ for all $A \in \text{Alg}\mathcal{N}$.*

Proof. Let $\delta : \text{Alg}\mathcal{N} \rightarrow \text{Alg}\mathcal{N}$ be an additive map Jordan derivable at 0, it follows from Lemma 2.1 that

$$(2.9) \quad \delta(I)E = E\delta(I) \quad \text{for all idempotents } E \in \text{Alg}\mathcal{N}.$$

Since the set of finite linear combinations of projections in $\mathcal{D}(\mathcal{N})$ is norm dense in $\mathcal{D}(\mathcal{N})$, we have from (2.9) that

$$(2.10) \quad \delta(I)D = D\delta(I) \quad \text{for all } D \in \mathcal{D}(\mathcal{N}).$$

On the other hand, it is clear that $P + PSP^\perp$ is an idempotent in $\text{Alg}\mathcal{N}$ for all $P \in \mathcal{N}$ and $S \in \mathcal{R}$. Then by (2.9), we can obtain that $\delta(I)PSP^\perp = PSP^\perp\delta(I)$. Since the linear span of $\{PSP^\perp : S \in \mathcal{R}, P \in \mathcal{N}\}$ is norm dense in $\mathcal{R}(\mathcal{N})$, we have

$$(2.11) \quad \delta(I)R = R\delta(I) \quad \text{for all } R \in \mathcal{R}(\mathcal{N}).$$

Since $\mathcal{D}(\mathcal{N}) + \mathcal{R}(\mathcal{N})$ is weakly dense in $\text{Alg}\mathcal{N}$, we have from (2.10)-(2.11) that $\delta(I)A = A\delta(I)$ for all $A \in \text{Alg}\mathcal{N}$. Thus $\delta(I) = \lambda I$ for some complex number λ . Now this corollary follows from Theorem 2.2. $\square$

3. Additive maps Jordan derivable at the standard idempotent . In this section, we show that the standard idempotent P is a Jordan all-derivable point of the triangular ring (or algebra) $\mathcal{T}$.

THEOREM 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings with units I_1 and I_2 , respectively, and $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$ -bimodule. Let $\mathcal{T} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular ring and P be the standard idempotent of it. Assume $\frac{1}{2}I_1 \in \mathcal{A}$, and assume further for every $A \in \mathcal{A}$, there is some integer n such that $nI_1 - A$ is invertible. Then $\delta : \mathcal{T} \rightarrow \mathcal{T}$ is an additive map Jordan derivable at P if and only if δ is a derivation.*

Proof. Only the “only if” part needs to be checked. We use the same notations as that in Section 2. Since δ is Jordan derivable at P , we have

$$\begin{bmatrix} \delta_{11}(I_1) & \delta_{12}(I_1) \\ 0 & \delta_{22}(I_1) \end{bmatrix} = \delta(P) = \delta(P)P + P\delta(P) = \begin{bmatrix} 2\delta_{11}(I_1) & \delta_{12}(I_1) \\ 0 & 0 \end{bmatrix}.$$

Thus

$$(3.1) \quad \delta_{11}(I_1) = \delta_{22}(I_1) = 0.$$

For any invertible element $X \in \mathcal{A}$, let $S = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} X^{-1} & 0 \\ 0 & 0 \end{bmatrix}$. Then $ST + TS = 2P$, and so

$$\begin{bmatrix} 0 & 2\delta_{12}(I_1) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E & X\delta_{12}(X^{-1}) + X^{-1}\delta_{12}(X) \\ 0 & 0 \end{bmatrix},$$

where $E = \delta_{11}(X)X^{-1} + X^{-1}\delta_{11}(X) + \delta_{11}(X^{-1})X + X\delta_{11}(X^{-1})$. Therefore

$$(3.2) \quad \delta_{11}(X)X^{-1} + X^{-1}\delta_{11}(X) + \delta_{11}(X^{-1})X + X\delta_{11}(X^{-1}) = 0.$$

$$(3.3) \quad X\delta_{12}(X^{-1}) + X^{-1}\delta_{12}(X) = 2\delta_{12}(I_1).$$

Let $S = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} X^{-1} & 0 \\ 0 & I_2 \end{bmatrix}$. Then $ST + TS = 2P$, and by (3.2)-(3.3) so

$$\begin{bmatrix} 0 & 2\delta_{12}(I_1) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \tau_{11}(I_2)X + X\tau_{11}(I_2) & 2\delta_{12}(I_1) + \delta_{12}(X) + X\tau_{12}(I_2) \\ 0 & 2\delta_{22}(X) \end{bmatrix}.$$

Thus we get $\delta_{22}(X) = 0$, $\tau_{11}(I_2)X + X\tau_{11}(I_2) = 0$ and $\delta_{12}(X) + X\tau_{12}(I_2) = 0$. Letting $X = I_1$, we have

$$(3.4) \quad \tau_{11}(I_2) = 0, \quad \tau_{12}(I_2) = -\delta_{12}(I_1), \quad \delta_{12}(X) = X\delta_{12}(I_1) \quad \text{and} \quad \delta_{22}(X) = 0$$

for all invertible elements $X \in \mathcal{A}$. For every $X \in \mathcal{A}$, by assumptions $nI_1 - X$ is invertible for some integer n , thus it is easy to check (3.4) is true for all $X \in \mathcal{A}$.

For any $W \in \mathcal{M}$, let $S = \begin{bmatrix} I_1 & W \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} I_1 & -2W \\ 0 & I_2 \end{bmatrix}$. Then $ST + TS = 2P$, and by (3.1) and (3.4) so

$$\begin{bmatrix} 0 & 2\delta_{12}(I_1) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2\varphi_{11}(W) & -4\varphi_{11}(W)W - 4W\varphi_{22}(W) + W\tau_{22}(I_2) + 2\delta_{12}(I_1) \\ 0 & 2\varphi_{22}(W) \end{bmatrix}.$$

Thus

$$(3.5) \quad \varphi_{11}(W) = \varphi_{22}(W) = 0 \quad \text{and} \quad \tau_{22}(I_2) = 0.$$

For any $Y \in \mathcal{B}$, let $S = \begin{bmatrix} I_1 & 0 \\ 0 & Y \end{bmatrix}$ and $T = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $ST + TS = 2P$.

By (3.1) we get

$$\begin{bmatrix} 0 & 2\delta_{12}(I_1) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2\tau_{11}(Y) & 2\delta_{12}(I_1) + \delta_{12}(I_1)Y + \tau_{12}(Y) \\ 0 & 0 \end{bmatrix}.$$

Thus

$$(3.6) \quad \tau_{11}(Y) = 0, \quad \tau_{12}(Y) = -\delta_{12}(I_1)Y \quad \text{for all } Y \in \mathcal{B}.$$

For any $Y \in \mathcal{B}$ and $W \in \mathcal{M}$, let $S = \begin{bmatrix} I_1 & -W - WY \\ 0 & Y \end{bmatrix}$ and $T = \begin{bmatrix} I_1 & W \\ 0 & 0 \end{bmatrix}$.

Then $ST + TS = 2P$. By (3.1), (3.5) and (3.6),

$$\begin{bmatrix} 0 & 2\delta_{12}(I_1) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2\delta_{12}(I_1) + \varphi_{12}(W)Y + W\tau_{22}(Y) - \varphi_{12}(WY) \\ 0 & 0 \end{bmatrix}.$$

Thus we have

$$(3.7) \quad \varphi_{12}(WY) = \varphi_{12}(W)Y + W\tau_{22}(Y) \quad \text{for all } W \in \mathcal{M}, Y \in \mathcal{B}.$$

For any $Y_1, Y_2 \in \mathcal{B}$, by (3.7) and the fact $\mathcal{M}$ is a faithful $\{\mathcal{A}, \mathcal{B}\}$ -bimodule, it is easy to check that

$$(3.8) \quad \tau_{22}(Y_1Y_2) = \tau_{22}(Y_1)Y_2 + Y_1\tau_{22}(Y_2).$$

For any invertible element $X \in \mathcal{A}$ and $W \in \mathcal{M}$, let $S = \begin{bmatrix} X & W \\ 0 & 0 \end{bmatrix}$,

$T = \begin{bmatrix} X^{-1} & -X^{-1}W - X^{-2}W \\ 0 & I_2 \end{bmatrix}$. Then $ST + TS = 2P$, and by (3.1)-(3.5) so

$$\begin{bmatrix} 0 & 2\delta_{12}(I_1) \\ 0 & 0 \end{bmatrix} = \delta(2P) = \delta(S)T + S\delta(T) + \delta(T)S + T\delta(S) = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix},$$

where $F = -\delta_{11}(X)X^{-1}W - \delta_{11}(X)X^{-2}W + \varphi_{12}(W) - X\varphi_{12}(X^{-1}W) - X\varphi_{12}(X^{-2}W) + X^{-1}\varphi_{12}(W) + \delta_{11}(X^{-1})W$.

Thus we get

$$(3.9) \quad \begin{aligned} & \delta_{11}(X)X^{-1}W + \delta_{11}(X)X^{-2}W - \varphi_{12}(W) + X\varphi_{12}(X^{-1}W) + X\varphi_{12}(X^{-2}W) \\ & = X^{-1}\varphi_{12}(W) + \delta_{11}(X^{-1})W. \end{aligned}$$

Let $S = \begin{bmatrix} X & W \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} X^{-1} & -X^{-2}W \\ 0 & 0 \end{bmatrix}$. Then $ST + TS = 2P$, and by (3.2)-(3.5) so

$$\begin{bmatrix} 0 & 2\delta_{12}(I_1) \\ 0 & 0 \end{bmatrix} = 2\delta(P) = \delta(S)T + S\delta(T) + \delta(T)S + T\delta(S),$$

which is equal to

$$\begin{bmatrix} 0 & 2\delta_{12}(I_1) - X\varphi_{12}(X^{-2}W) - \delta_{11}(X)X^{-2}W + \delta_{11}(X^{-1})W + X^{-1}\varphi_{12}(W) \\ 0 & 0 \end{bmatrix}.$$

Thus $X\varphi_{12}(X^{-2}W) + \delta_{11}(X)X^{-2}W = \delta_{11}(X^{-1})W + X^{-1}\varphi_{12}(W)$, together with (3.9), we have

$$(3.10) \quad \varphi_{12}(X^{-1}W) = X^{-1}\varphi_{12}(W) - X^{-1}\delta_{11}(X)X^{-1}W.$$

For any invertible elements $X_1, X_2 \in \mathcal{A}$, from

$$\begin{aligned} & X_1X_2\varphi_{12}(W) - X_1X_2\delta_{11}(X_2^{-1}X_1^{-1})X_1X_2W = \varphi_{12}(X_1X_2W) \\ &= X_1\varphi_{12}(X_2W) - X_1\delta_{11}(X_1^{-1})X_1X_2W \\ &= X_1X_2\varphi_{12}(W) - X_1X_2\delta_{11}(X_2^{-1})X_2W - X_1\delta_{11}(X_1^{-1})X_1X_2W, \end{aligned}$$

hence $X_1X_2\delta_{11}(X_2^{-1}X_1^{-1})X_1X_2 = X_1X_2\delta_{11}(X_2^{-1})X_2 + X_1\delta_{11}(X_1^{-1})X_1X_2$ and

$$\delta_{11}(X_2^{-1}X_1^{-1}) = \delta_{11}(X_2^{-1})X_1^{-1} + X_2^{-1}\delta_{11}(X_1^{-1})$$

for all invertible elements $X_1, X_2 \in \mathcal{A}$, that is δ is a derivation on $\mathcal{A}$. Thus

$$\delta_{11}(X^{-1}) = -X^{-1}\delta_{11}(X)X^{-1}.$$

This equality together with (3.10) implies

$$(3.11) \quad \begin{aligned} \varphi_{12}(XW) &= X\varphi_{12}(W) + \delta_{11}(X)W, \\ \delta_{11}(X_1X_2) &= \delta_{11}(X_1)X_2 + X_2\delta_{11}(X_1). \end{aligned}$$

for all $X_1, X_2 \in \mathcal{A}$ and for all $W \in \mathcal{M}$.

Now by (3.1)-(3.11) and using similar arguments as that in the proof of Theorem 2.2, it is easily checked that δ is a derivation. $\square$

Similarly, we can prove $P_1 = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}$ is a full-derivable of $\mathcal{T}$, and hence we get the following theorem.

THEOREM 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital rings with units I_1 and I_2 , respectively, and $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$ -bimodule. Let $\mathcal{T} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the triangular ring.*

Assume that, $\frac{1}{2}I_2 \in \mathcal{B}$, and, for every $B \in \mathcal{B}$, there is some integer n such that $nI_2 - B$ is invertible. Then $\delta : \mathcal{T} \rightarrow \mathcal{T}$ is an additive map Jordan derivable at P_1 if and only if δ is a derivation.

From Theorem 3.1 and Theorem 3.2, we have the following corollary.

COROLLARY 3.3. *Let $\mathcal{N}$ be a non-trivial nest on a complex Banach space X and $\text{Alg}\mathcal{N}$ be the associated nest algebra, and let $\delta : \text{Alg}\mathcal{N} \rightarrow \text{Alg}\mathcal{N}$ be an additive map. Then δ is Jordan derivable at a nontrivial idempotent P with range $P(X) \in \mathcal{N}$ if and only if δ is a derivation.*

COROLLARY 3.4. *Let $\mathcal{N}$ be a non-trivial nest in a factor von Neumann algebra $\mathcal{R}$, and let $\text{Alg}\mathcal{N}$ be the associated nest algebra. Assume that $\delta : \text{Alg}\mathcal{N} \rightarrow \text{Alg}\mathcal{N}$ is an additive map. Then δ is Jordan derivable at an idempotent element Q satisfying $PQ = Q$ and $QP = P$ for some nontrivial projection $P \in \mathcal{N}$ if and only if δ is a derivation.*

4. Additive maps Jordan derivable at the unit . In this section, we will show that the unit I is a Jordan all-derivable point of $\mathcal{T}$.

THEOREM 4.1. *Let $\mathcal{T} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular ring, where $\mathcal{A}, \mathcal{B}$ are unital rings of characteristic not 3 with unit I_1 and I_2 respectively, and $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$ -bimodule. Assume that $\frac{1}{2}I_1 \in \mathcal{A}$, $\frac{1}{2}I_2 \in \mathcal{B}$, and for any $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, there are some integers n_1 and n_2 such that $n_1I_1 - X$ and $n_2I_2 - Y$ are invertible. Then $\delta : \mathcal{T} \rightarrow \mathcal{T}$ is an additive map Jordan derivable at the unit I if and only if δ is a derivation.*

Proof. The “if” part is obvious. To check the “only if” part, assume that δ is Jordan derivable at I . We use notations as that in Section 2. It is obvious from $I^2 = I$ that $\delta(I) = \delta(I^2) = \delta(I)I + I\delta(I) = 2\delta(I)$, hence $\delta(I) = 0$. Thus

$$(4.1) \quad \delta_{11}(I_1) + \tau_{11}(I_2) = 0, \quad \delta_{12}(I_1) + \tau_{12}(I_2) = 0 \quad \text{and} \quad \delta_{22}(I_1) + \tau_{22}(I_2) = 0.$$

For any invertible element $X \in \mathcal{A}$, $Y \in \mathcal{B}$, let $S = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$ and $T = \begin{bmatrix} X^{-1} & 0 \\ 0 & Y^{-1} \end{bmatrix}$. Then $ST + TS = 2I$, and so

$$0 = \delta(2I) = \delta(S)T + S\delta(T) + \delta(T)S + T\delta(S) = \begin{bmatrix} G_{11}(X, Y) & G_{12}(X, Y) \\ 0 & G_{22}(X, Y) \end{bmatrix},$$

where

$$\begin{aligned} G_{11}(X, Y) &= \delta_{11}(X)X^{-1} + X\delta_{11}(X^{-1}) + \delta_{11}(X^{-1})X + X^{-1}\delta_{11}(X) \\ &\quad + \tau_{11}(Y)X^{-1} + X\tau_{11}(Y^{-1}) + \tau_{11}(Y^{-1})X + X^{-1}\tau_{11}(Y); \\ G_{12}(X, Y) &= \delta_{12}(X)Y^{-1} + \tau_{12}(Y)Y^{-1} + X\tau_{12}(Y^{-1}) + X\delta_{12}(X^{-1}) \\ &\quad + \delta_{12}(X^{-1})Y + \tau_{12}(Y^{-1})Y + X^{-1}\tau_{12}(Y) + X^{-1}\delta_{12}(X); \\ G_{22}(X, Y) &= \tau_{22}(Y^{-1})Y + Y^{-1}\tau_{22}(Y) + \tau_{22}(Y)Y^{-1} + Y\tau_{22}(Y^{-1}) \\ &\quad + \delta_{22}(X)Y^{-1} + Y\delta_{22}(X^{-1}) + \delta_{22}(X^{-1})Y + Y^{-1}\delta_{22}(X). \end{aligned}$$

Letting $X = \frac{I_1}{2}$ and $Y = I_2$ and applying (4.1), we have $4\delta_{11}(I_1) + 4\tau_{11}(I_2) + \tau_{11}(I_2) = \tau_{11}(I_2) = 0$. Thus $\delta_{11}(I_1) = \tau_{11}(I_2) = 0$. Similarly, considering $G_{22}(X, Y) = 0$ and letting $Y = \frac{I_2}{2}$ and $X = I_1$, it is easily check that $\delta_{22}(I_1) = \tau_{22}(I_2) = 0$. Letting $Y = I_2$ in $G_{11}(X, Y) = 0$, we get

$$(4.2) \quad \delta_{11}(X)X^{-1} + X\delta_{11}(X^{-1}) + \delta_{11}(X^{-1})X + X^{-1}\delta_{11}(X) = 0$$

for all invertible elements $X \in \mathcal{A}$. Taking $X = I_1$ in $G_{22}(X, Y) = 0$, we get

$$(4.3) \quad \tau_{22}(Y)Y^{-1} + Y\tau_{22}(Y^{-1}) + \tau_{22}(Y^{-1})Y + Y^{-1}\tau_{22}(Y) = 0$$

for all invertible elements $Y \in \mathcal{B}$. From (4.2)-(4.3) and $G_{11}(X, Y) = G_{22}(X, Y) = 0$, we have

$$(4.4) \quad \begin{aligned} \tau_{11}(Y)X^{-1} + X\tau_{11}(Y^{-1}) + \tau_{11}(Y^{-1})X + X^{-1}\tau_{11}(Y) &= 0, \\ \delta_{22}(X)Y^{-1} + Y\delta_{22}(X^{-1}) + \delta_{22}(X^{-1})Y + Y^{-1}\delta_{22}(X) &= 0. \end{aligned}$$

Letting $X = \frac{I_1}{2}$ and $X = I_1$ respectively in (4.4), we get $\tau_{11}(Y) + \tau_{11}(Y^{-1}) = 0$ and $\tau_{11}(Y) + 4\tau_{11}(Y^{-1}) = 0$, thus $3\tau_{11}(Y) = 0$. As $\mathcal{A}$ is of characteristic not 3, we must have $\tau_{11}(Y) = 0$ for all invertible element $Y \in \mathcal{B}$. Similarly, we have $\delta_{22}(X) = 0$ for all invertible element $X \in \mathcal{A}$.

Taking $Y = \frac{1}{2}I_2$, $Y = I_2$ and $Y = 2I_2$ respectively in $G_{12}(X, Y) = 0$ and from

$$\begin{aligned} &2\tau_{12}(I_2) + X\delta_{12}(X^{-1}) + X^{-1}\delta_{12}(X) \\ &= -2\delta_{12}(X) - 2X\tau_{12}(I_2) - \frac{1}{2}\delta_{12}(X^{-1}) - \frac{1}{2}X^{-1}\tau_{12}(I_2), \\ &= -\delta_{12}(X) - X\tau_{12}(I_2) - \delta_{12}(X^{-1}) - X^{-1}\tau_{12}(I_2) \\ &= -\frac{1}{2}\delta_{12}(X) - \frac{1}{2}X\tau_{12}(I_2) - 2\delta_{12}(X^{-1}) - 2X^{-1}\tau_{12}(I_2), \end{aligned}$$

and thus we get

$$(4.5) \quad \delta_{12}(X) = X\delta_{12}(I_1) \quad \text{for all invertible element } X \in \mathcal{A}.$$

Similarly, Letting $X = 2I_1$, $X = I_1$ and $X = \frac{1}{2}I_1$ in $G_{12}(X, Y) = 0$, we have

$$(4.6) \quad \tau_{12}(Y) = -\delta_{12}(I_1)Y \quad \text{for all invertible elements } Y \in \mathcal{B}.$$

Now, by replacing X and Y with invertible elements $n_1 I_1 - X$ and $n_2 I_2 - Y$, it is easily checked (4.5)-(4.6) and

$$(4.7) \quad \delta_{22}(X) = 0 \quad \text{and} \quad \tau_{11}(Y) = 0$$

hold for all $X \in \mathcal{A}$ and $Y \in \mathcal{B}$.

For any invertible element $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, and for any $W \in \mathcal{M}$, let $S = \begin{bmatrix} X & XW \\ 0 & Y \end{bmatrix}$ and $T = \begin{bmatrix} X^{-1} & -WY^{-1} \\ 0 & Y^{-1} \end{bmatrix}$. Then $ST + TS = 2I$. By (4.2)-(4.3) and (4.5)-(4.6) we get

$$0 = \delta(2I) = \delta(S)T + S\delta(T) + \delta(T)S + T\delta(S) = \begin{bmatrix} H_{11}(X, W, Y) & H_{12}(X, W, Y) \\ 0 & H_{22}(X, W, Y) \end{bmatrix},$$

where

$$\begin{aligned} H_{11}(X, W, Y) &= \varphi_{11}(XW)X^{-1} - X\varphi_{11}(WY^{-1}) - \varphi_{11}(WY^{-1})X + X^{-1}\varphi_{11}(XW) \\ H_{22}(X, W, Y) &= \varphi_{22}(XW)Y^{-1} - Y\varphi_{22}(WY^{-1}) - \varphi_{22}(WY^{-1})Y + Y^{-1}\varphi_{22}(XW) \\ H_{12}(X, W, Y) &= -\delta_{11}(X)WY^{-1} + \varphi_{12}(XW)Y^{-1} + \delta_{12}(X)Y^{-1} + \tau_{12}(Y)Y^{-1} \\ &\quad - \varphi_{12}(WY^{-1})Y + \delta_{12}(X^{-1})Y + \tau_{12}(Y^{-1})Y + X^{-1}\varphi_{12}(XW) \\ &\quad + X^{-1}\delta_{12}(X) + X^{-1}\tau_{12}(Y) - WY^{-1}\tau_{22}(Y) - \varphi_{11}(XW)WY^{-1} \\ &\quad - XW\varphi_{22}(WY^{-1}) - \varphi_{11}(WY^{-1})XW - WY^{-1}\varphi_{22}(XW) \\ &\quad + \delta_{11}(X^{-1})XW + X\tau_{12}(Y^{-1}) - X\varphi_{12}(WY^{-1}) \\ &\quad + X\delta_{12}(X^{-1}) + XW\tau_{22}(Y^{-1}) - XW\varphi_{22}(WY^{-1}). \end{aligned}$$

Taking $X = 2I_1$ and $Y = I_2$ in $H_{11}(X, W, Y) = 0$ and $H_{22}(X, W, Y) = 0$ we get

$$(4.8) \quad \varphi_{11}(W) = 0 \quad \text{and} \quad \varphi_{22}(W) = 0.$$

Letting $Y = \frac{I_2}{2}$ and $Y = I_2$ in $H_{12}(X, W, Y) = 0$ and by (4.5) and (4.8), we have

$$\begin{aligned} \varphi_{12}(W) - X^{-1}\varphi_{12}(XW) - \delta_{11}(X^{-1})XW &= -2\delta_{11}(X)W + 2\varphi_{12}(XW) - 2X\varphi_{12}(W) \\ &= -\delta_{11}(X)W + \varphi_{12}(XW) - X\varphi_{12}(W) \end{aligned}$$

Thus

$$(4.9) \quad \varphi_{12}(XW) = X\varphi_{12}(W) + \delta_{11}(X)W$$

for all invertible elements $X \in \mathcal{A}$ and any $W \in \mathcal{M}$. Similarly,

$$(4.10) \quad \varphi_{12}(WY) = \varphi_{12}(W)Y + W\tau_{22}(Y)$$

for all invertible elements $Y \in \mathcal{B}$ and any $W \in \mathcal{M}$. Considering $n_1 I_1 - X$ and $n_2 I_2 - Y$, we get (4.9) and (4.10) hold for all $X \in \mathcal{A}$, $W \in \mathcal{M}$ and $Y \in \mathcal{B}$. By the same arguments in section 3, we get that

$$(4.11) \quad \begin{aligned} \delta_{11}(X_1 X_2) &= \delta_{11}(X_1) X_2 + X_1 \delta_{11}(X_2), \\ \tau_{22}(Y_1 Y_2) &= \tau_{22}(Y_1) Y_2 + Y_1 \tau_{22}(Y_2). \end{aligned}$$

for all $X_1, X_2 \in \mathcal{A}$ and for all $Y_1, Y_2 \in \mathcal{B}$.

From (4.1)-(4.11) and from the proofs of Theorem 3.1, we obtain δ is a derivation. $\square$

From Theorem 4.1 and by similar arguments as that in section 3, we obtain

COROLLARY 4.2. *Let $\mathcal{N}$ be a nest on a complex Banach space X such that there is a $N \in \mathcal{N}$ complemented in X , and let $\text{Alg}\mathcal{N}$ be the associated nest algebra. Then $\delta : \text{Alg}\mathcal{N} \rightarrow \text{Alg}\mathcal{N}$ is an additive map Jordan derivable at I if and only if δ is a derivation.*

COROLLARY 4.3. *Let $\mathcal{N}$ be a non-trivial nest in a factor von Neumann algebra $\mathcal{R}$ and $\text{Alg}\mathcal{N}$ be the associated nest algebra. Then $\delta : \text{Alg}\mathcal{N} \rightarrow \text{Alg}\mathcal{N}$ is an additive map Jordan derivable at I if and only if δ is a derivation.*

By Corollary 3.4 and Corollary 4.3, we have

COROLLARY 4.4. *Let $\mathcal{N}$ be a non-trivial nest in a factor von Neumann algebra $\mathcal{R}$, and let $\text{Alg}\mathcal{N}$ be the associated nest algebra. Assume that $\delta : \text{Alg}\mathcal{N} \rightarrow \text{Alg}\mathcal{N}$ is an additive map. Then δ is Jordan derivable at a nonzero idempotent element Q satisfying $PQ = Q$ and $QP = P$ for some projection $P \in \mathcal{N}$ if and only if δ is a derivation.*

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